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A Noncooperative Foundation of Progressive Taxation*

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Abstract

We present a model of n-person noncooperative bargaining for dividing taxes among members of a society, each of whom is characterized by his/her income level. Our bargaining model is a modified version of Hart and Mas-Colell's (*Econometrica* **64**, 1996, 357-380) model. The key feature of our model is the bargaining procedure in which only the responders drop out with equal probability. We show that our bargaining procedure generates equal after-tax income allocation when the cost of delay is low in both a nontransferable utility case and a transferable utility case. Thus, progressive income taxation is supported in a noncooperative manner.

JEL Classification Numbers: C72, C78, D78, H24.

Key words: tax assignment, bankruptcy problem, noncooperative bargaining, progressive taxation.

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1 Introduction

Why do most democratic countries employ progressive income taxation? Economists have tried to answer the question and offered various explanations. The most common one is the utilitarian approach, which considers that the tax structure is chosen by a benevolent planner, whose objective is to maximize the sum of individual utilities, what is called the social welfare. If all individuals have the same utility function defined over after-tax income and there is no disincentive effect of taxation, the maximization of the sum of individual utilities leads to tax structure such that after-tax incomes are equalized. Because the maximization of the social welfare corresponds to the equalization of marginal sacrifices of all individuals for taxation, the utilitarian approach has often been interpreted as the equal sacrifice approach. Such a support of progressive taxation has long been offered by Edgeworth (1897), Pigou (1947), and others. Musgrave (1959) summarized these arguments. Moreover, Young (1988) has succeeded in an axiomatization of equal sacrifice taxation. Mirrlees (1971) has generated the optimal income tax approach by introducing the disincentive effect of taxation and the informational constraints to the utilitarian approach. After his seminal work, enormous studies of the optimal tax approach has been undertaken. Unfortunately, this approach led to an unclear conclusion regarding the shape of the optimal income tax function. Myles (2000) showed that every qualitative pattern of marginal tax rates, both regressive and progressive, may be achieved as an optimal tax schedule by appropriate selection of the skill distribution. Diamond (1998) and Saez (2001) also examined the shape of the optimal marginal income tax rate and the progressivity of the income tax.

Both the utilitarian and equal sacrifice approaches are commonly based on a social justice that transcends the determinants of self-interested individual behavior. These approaches are often called the normative approach. In contrast to the normative approach, several studies has proposed a positive approach. The positive approach has tried to explain progressive income taxation as an endogenous consequence of the democratic political process – for example, majority voting, political parties' competition and so on. It is stressed that players are self-interested. Aumann and Kurz (1977, 1978) have assumed that taxation policies are determined by majority voting, and they adopted the NTU Harsanyi-Shapley value as a solution concept. Thus, their analysis was classified as a cooperative game approach. They then showed that the marginal income tax rate is always between 50 percent and 100 percent, but that the income tax structure can be progressive, regressive or neutral in its economic-political equilibrium. Roemer (1999) has considered that each of two political parties, left and right parties, consists of reformists, militants and opportunists. He introduced to the two parties game a Nash equilibrium that required an intra-party consensus among reformists, militants and opportunists. Then, he showed that both parties propose progressive income taxation schemes in such equilibria. De-Donder and Hindriks (2000) presented sufficient conditions for progressive income taxation to emerge as a majority-voting outcome.

Over the last few years, the progressivity of income tax has been animatedly discussed from both normative and positive perspectives. This paper employs the positive approach. Thus, we will investigate whether progressive taxation appears as an outcome of the behavior of self-interested individuals. In contrast to previous studies, we do not suppose that political decision processes such as majority voting select a tax structure. We present a noncooperative bargaining model for apportioning taxes among members of a society. In our bargaining model, it is assumed that all players are self-interested and that the tax structure is determined by voluntary negotiations among players. Especially, we consider an n-person noncooperative bargaining model based on a coalitional form game (N, V), which is similar to the model of Hart and Mas-Colell (1996).

Our bargaining model has the following features. We define a coalitional form game (N, V) according to the method of Aumann and Maschler's (1985) analysis of the bankruptcy problem. The bankruptcy problem in Aumann and Maschler (1985) is to consider how the remaining estate should be divided among creditors with various claims. Our problem, however, is to divide a given total tax among individuals, each of whom is endowed with a different income. Our taxation problem, however, is reduced to the problem of how total after-tax income should be divided among the individuals with different incomes. Thus, the problem of tax assignments has the same structure as the bankruptcy problem, with the total after-tax income and the endowment of income corresponding to the remaining estate and the claim, respectively[1]. For this reason, we will employ a coalitional form game such as Aumann and Maschler have introduced. Furthermore, we extend the game to a nontransferable utility case, though Aumann and Maschler considered only a transferable utility case.

Our bargaining procedure runs as follows. In each round, one player is selected as a proposer with equal probability among all existing players. The player can propose a pair of taxes paid by the active players that is sufficient to provide a given total tax revenue. The requirement for agreement is

[1]Young (1988) has also recognized the close relationship between the tax assignment problem and the bankruptcy problem.

unanimity. The key feature of our bargaining procedure is the rule regarding what happens if there is no agreement. Hart and Mas-Colell (1996) suppose that only the proposer may cease to be an active player at next round after the proposal is rejected. On the contrary, we assume that only the responders drop out, all with equal probability. This modification has an important effect on our results. In addition, we focus on a stationary subgame perfect equilibrium in the bargaining game.

We obtained the following results. First our bargaining procedure generated, in a equilibrium, an equal after-tax income allocation, i.e., an allocation in which all individuals have the same after-tax income, when the probability of the responders' dropping out is close to zero. In other words, all members of the society come to an agreement on an extremely progressive taxation in a noncooperative way. The result does not depend on whether the coalitional form game is the game with nontransferable utility or that with transferable utility. Next, we considered the pure bargaining case. In this case, only the grand coalition has worth to negotiate the tax assignment and other coalitions have no value; thus, the payoff for every player becomes zero, even if one player drops out. Then, we showed that our equilibria do indeed yield a solution to a Nash social welfare maximization problem (a Nash bargaining solution) as the cost of delay becomes small. The result in the pure bargaining case establishes a noncooperative foundation for the Nash social welfare maximization framework as, for example, in the analysis of an optimal income tax schedule conducted by Kaneko (1981, 1982).

This paper is organized as follows. Section 2 introduces our noncooperative bargaining model for dividing taxes among the members of society and defines our equilibrium concept. Section 3 characterizes the equilibrium payoff configurations in a nontransferable utility, a transferable utility and a pure bargaining case. Concluding remarks are gathered in Section 4.

2 Noncooperative Bargaining Model

2.1 Coalitional form game

Let $N \stackrel{\text{def}}{=} \{1, 2, \ldots, n\}$ be a finite set of players and a nonempty subset S of N is called a coalition of players. We define the family of coalitions S by

$$\mathscr{P}_0(N) \stackrel{\text{def}}{=} \{S \mid S \subset N,\ S \neq \emptyset\}.$$

Let denote by y_i a (taxable) income of player i and by $y \stackrel{\text{def}}{=} (y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$ an income profile of the n-person society. We assume that each player $i \in N$

is characterized only by an income level y_i. We denote an aggregate income of the society N with an income profile y by $Y(N) \overset{\text{def}}{=} \sum_{i \in N} y_i$ and an aggregate income of a coalition S by $Y(S) \overset{\text{def}}{=} \sum_{i \in S} y_i$.

We assume that members of the society must collect a given tax revenue $T > 0$ and that the aggregate income of the society exceeds the required tax revenue, i.e., $Y(N) - T > 0$. The family of coalitions that can finance the total tax from their aggregate income is defined by

$$\mathscr{S} \overset{\text{def}}{=} \left\{ S \in \mathscr{P}_0(N) \;\middle|\; Y(S) - T > 0 \right\}.$$

We assume that $y_i < T$ for all $i \in N$. Thus, $\{i\} \notin \mathscr{S}$ for all $i \in N$.

Each of the players has a preference represented by a utility function $u : \mathbb{R}_+ \to \mathbb{R}$ which depends only on his/her after-tax income. Later in this paper we will give an alternative interpretation of the utility function depending only on income. The after-tax income of player i is defined by $x_i \overset{\text{def}}{=} y_i - t_i$, where t_i is the tax paid by player i. We assume that each utility function of the players satisfies the following assumption.

ASSUMPTION 1. (i) All players have the same utility function $u : \mathbb{R}_+ \to \mathbb{R}$. (ii) The utility function u is continuous, strictly concave, strictly increasing and C^2. Furthermore, the function u is bounded, i.e., for some $M > 0$, $u(x) < M$ for all $x \in \mathbb{R}_+$. In addition, $u(0) = 0$.

Here, we consider the *tax assignment problem* (T, y) that an aggregate tax burden T is distributed among members $i \in N$ of the society with different incomes y_i. The problem (T, y) is equivalent to the problem (E, y) that the net aggregate income $E \overset{\text{def}}{=} Y(N) - T$ is allocated among members of the society with an income profile y, because, if a pair of after-tax incomes $x = (x_1, \ldots, x_n)$ with $E = \sum_{j \in N} x_j$ is given, then the tax burden of each player $t_i = y_i - x_i,\ i \in N$ is uniquely determined. We should notice that (E, y) is a bankruptcy problem from the Talmud in Aumann and Maschler (1985) if E is the estate owned by a died man and $(y_1, \ldots, y_n)$ is his leaving debts. The bankruptcy problem considers how the estate E should be divided among the creditors with their claims y. We will call the vector $x = (x_1, \ldots, x_n)$ *an allocation of after-tax income* and call the vector $t = (t_1, \ldots, t_n)$ satisfying $T = \sum_{j \in N} t_j$ *a tax assignment.*

Let us define the characteristic function corresponding to the tax assignment problem. The set of feasible allocations of after-tax income for a coalition S is defined by

$$X^S \overset{\text{def}}{=} \left\{ (x_i)_{i \in S} \;\middle|\; \sum_{i \in S} x_i \leq \max(0, Y(S) - T),\ x_i \in \mathbb{R}_+,\ \forall\, i \in S \right\}.$$

Using the definition of X^S, we define the characteristic function as

$$V(S) = \left\{ z^S \in \mathbb{R}^S_+ \mid \exists (x_i)_{i \in S} \in X^S,\ \forall i \in S,\ z^S_i \leq u(x_i) \right\} \text{ for } \forall S \subset N. \quad (1)$$

By definition, $V(S)$ represents the attainable set of utilities for members in a coalition S after every member j in the complementary coalition $N \backslash S$ gets his/her full income y_j. This is a natural extension of the characteristic function for the bankruptcy problem defined by Aumann and Maschler (1985). In their definition, the payoff for player $i \in N$ was the amount x_i itself and the characteristic functions were defined as

$$v(S) \stackrel{\text{def}}{=} \max(0, E - \sum_{j \in N \backslash S} y_j) = \max(0, Y(S) - T), \quad \text{for } \forall S \subset N. \quad (2)$$

In our definition, the payoff for player i with after-tax income x_i is given by his/her utility $u(x_i)$. As a result, our characteristic functions are represented by (1). The game (N, V) is an n-person coalitional form game with non-transferable utility. On the other hand, the game (N, v) is a coalitional form game with transferable utility, where the characteristic function v is defined by equation (2).

We call the coalitional form game (N, V) a *tax assignment game*. We will also consider the case in which the payoff for player i is an after-tax income x_i itself later. In this case, the tax assignment game is described by (N, v).

The characteristic function defined by (1) has the following properties.

THEOREM 1. *For any coalition $S \subset N$, the set $V(S)$ is closed, convex and comprehensive, i.e., if $z^S \in V(S)$ and $\hat{z}^S_i \leq z^S_i$ for all $i \in S$, then $\hat{z}^S \in V(S)$. Moreover, $0 \in V(S)$ and $V(S) \cap \mathbb{R}^S_+$ is bounded.*

THEOREM 2. *For any coalition $S \in \mathscr{S}$, the boundary $\partial V(S) \cap \mathbb{R}^S_{++}$, where $\partial V(S)$ is the boundary of $V(S)$, is smooth and nonlevel. Note that the boundary $\partial V(S) \cap \mathbb{R}^S_{++}$ is smooth if and only if at each $z^S \in \partial V(S) \cap \mathbb{R}^S_{++}$, there exists a single outward normal direction. In addition, the boundary $\partial V(S) \cap \mathbb{R}^S_{++}$ is nonlevel if and only if the outward normal vector at any point of $\partial V(S) \cap \mathbb{R}^S_{++}$ is positive in all coordinates.*

THEOREM 3. *The characteristic function V is monotone, i.e., $V(S) \times \{0^{H \backslash S}\} \subset V(H)$ whenever $S \subset H$.*

Proofs of Theorem 1, 2 and 3. See Appendix. □

Theorems 1, 2 and 3 correspond to assumptions (A-1), (A-2) and (A-3), which are imposed on the game (N, V) by Hart and Mas-Colell (1996).

In Theorem 2, the smoothness and nonlevelness are proved only for vectors in $\partial V(S) \cap \mathbb{R}^S_{++}$. But, we will need the smoothness and nonlevelness conditions for vectors in $\partial V(S) \cap \mathbb{R}^S_{+} \backslash \mathbb{R}^S_{++}$ in order to prove Proposition 6 later. Therefore, the smoothness and nonlevelness conditions must be extended from $\partial V(S) \cap \mathbb{R}^S_{++}$ to $\partial V(S) \cap \mathbb{R}^S_{+}$. For this purpose, we define an outward normal vector at each $\hat{z}^S$ in $\partial V(S) \cap \mathbb{R}^S_{+} \backslash \mathbb{R}^S_{++}$ as a limit of the sequence of outward normal vectors at $z^S_\nu, \nu = 1, 2, \ldots,$ where a sequence $\{z^S_\nu\}^\infty_{\nu=1}$ converges to $\hat{z}^S$. Then, we shall add the following assumption.

ASSUMPTION 2. For each coalition $S \in \mathscr{S}$, we assume that there exists a single outward normal direction at any point $\hat{z}^S$ in $\partial V(S) \cap \mathbb{R}^S_{+} \backslash \mathbb{R}^S_{++}$ and that the outward normal vector at $\hat{z}_S$ is positive in all coordinates.

Thus, $\partial V(S) \cap \mathbb{R}^S_{+}$ is smooth and nonlevel for all $S \in \mathscr{S}$.

2.2 Noncooperative game

We describe a noncooperative bargaining procedure based on a coalitional form game (N, V). Let $0 \leq \rho < 1$ be a fixed parameter. Then the n-person noncooperative bargaining model runs as follows. At every round $t = 1, 2, \ldots,$ there is a set N^t of all active players, and a proposer $i \in N^t$. In the first round $N^1 = N$.

(i) One player is selected as a proposer with equal probability among all players in N^t. The selected player $i \in N^t$ proposes a payoff vector in $V(N^t)$.

(ii) All other players in N^t either accept or reject the proposal sequentially. We assume that the responses are made according to a predetermined order over N^t. If all members of $N^t \backslash \{i\}$ accept, then the game ends with these payoffs[2]. If some members of N^t reject, then the game moves to the next round. With probability ρ, the set of active players at round $t+1$ is unchanged, i.e., $N^{t+1} = N^t$, and, with probability $1-\rho$, one player j drops out with equal probability among all players in $N \backslash i$, and the set of active players becomes $N^t \backslash j$, i.e., $N^{t+1} = N^t \backslash j$. In latter case, the player j who dropped out gets a final payoff of 0.

(iii) Every player has perfect information about the history of the game play whenever he/she makes a decision.

We do not consider a time discount. Instead of a time discount, the probability ρ represents the cost of delay in agreement in our bargaining model. If $\rho \to 1$, the cost of delay is very low.

[2]We will write $S \backslash i$ for $S \backslash \{i\}$.

Our bargaining procedure is similar to that of Hart and Mas-Colell (1996). They have presented a noncooperative bargaining procedure whose stationary subgame perfect equilibrium coincides with the Shapley value in the transferable utility (TU) case and coincides with the consistent value (introduced by Maschler and Owen (1989, 1992)) in the nontransferable utility (NTU) case. They are mainly interested in providing a noncooperative foundation for the cooperative solution concepts. On the other hand, we have an interest in considering the tax assignment problem in a noncooperative bargaining framework. In addition, our procedure itself is different from that of Hart and Mas-Colell (1996). In our procedure, only the responders, not the proposer, drop out with equal probability after the i's proposal is rejected by one member of $S \backslash i$. On the other hand, only the proposer drops out with probability ρ after the proposal is rejected in the procedure of Hart and Mas-Colell. Note that Hart and Mas-Colell (1996) also presented a more general bargaining procedure which contains our procedure as a special case in Section 6 of their paper. They did not, however, discuss the bargaining model in the context of tax assignment. Furthermore, they did not fully characterize the equilibrium payoffs under the general procedure in an NTU case.

Before giving a formal definition of our solution concept, let us examine our bargaining procedure in the context of the tax assignment problem. First, we will discuss the procedural fairness in our bargaining. In our model, every member of the society can participate in the bargaining that determines each tax-burden, and every member has an equal opportunity to be selected as a proposer in the first round. Furthermore, unanimous agreement among all active players is required to determine the tax-burden on each member of the society. We can say that our bargaining procedure is very close to the ideal decision-making process of a direct democracy. Our bargaining procedure also contains some undesirable parts. In particular, the rule in which one member is compulsorily dropped out with probability $1-\rho$ after the proposal is rejected undoubtedly goes against the democratic rules. For this reason we will focus on the properties of equilibria in the bargaining model for the tax assignment as the probability that a player drops out converges to zero, i.e., $\rho \to 1$.

Next, let us focus on the relationship between the payoff and the after-tax income for each player in this bargaining. Suppose that, for example, a proposal $(z_i^*)_{i \in S} \in V(S)$ is accepted by all players in a coalition S. Then, an agreement is reached on the tax assignment among them, and the payoff for player $i \in S$ results in z_i^*. For any player $i \in S$, it is satisfied that $z_i^* = u(x_i^*) = u(y_i - t_i^*)$. By definition of $V(S)$, the vector $(x_i^*)_{i \in S}$ belongs to X^S. Therefore, we obtain $\sum_{i \in S} t_i^* \geq T$. Thus, every player $j \in N \backslash S$

pays no tax and obtains his/her full income y_j. Recall, however, that the payoff for any dropped-out player, which in this example would be the player $j \in N \backslash S$, was assumed to be zero. This assumption can be interpreted as follows. A state or a government is formed by members of the coalition S and provides essential public goods or infrastructures G by using the tax revenue T. Thus, every player outside the coalition S is unable to benefit from the public good. In other words, players outside a state cannot free-ride on the public good. The public good does not have a property of non-excludability here. As a result, each player $j \in N \backslash S$ receives the payoff of 0. In other words, every player has a utility function that depends on two variables; an after-tax income, i.e., consumption of a private good and a public good, $\tilde{u} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$. The payoff for each player $i \in S$ is represented by $\tilde{u}(x_i^*, G) \overset{\text{def}}{=} u(x_i^*) = z_i^*$ and that for each player $j \in N \backslash S$ is defined as $\tilde{u}(y_i^*, 0) = 0$. The utility function u of after-tax income is a reduced form of the function $\tilde{u}$.

Formally, our bargaining model can be represented as an extensive form game with perfect information and with chance moves. We assume that the rule of the game is a common knowledge for all players. For every coalition $S \in \mathscr{P}_0(N)$ and for a parameter ρ, we denote by $G^S(\rho)$ the bargaining model with the set of active players S. A pure strategy for player i in $G^N(\rho)$ is a sequence $\sigma_i = \{\sigma_i^t\}_{t=1}^{\infty}$ of mappings, where σ_i^t is the tth round strategy. The tth round strategy σ_i^t, $t = 1, 2, \ldots$, prescribes (i) a proposal $a_{N^t,i} \in V(N^t)$ and (ii) a response function assigning "accept" or "reject" to all possible proposals by other players. σ_i^t may depend on the history of the game play up to round t. For a strategy combination $\sigma = (\sigma_1, \ldots, \sigma_n)$, the expected payoffs for the players in $G^N(\rho)$ are determined in the usual manner.

Let us define our solution concept of the game $G^N(\rho)$.

DEFINITION 1. A strategy combination $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ of the game $G^N(\rho)$ is said to be a *stationary subgame perfect equilibrium point* (**SSPE**) if it is a subgame perfect equilibrium point, at which for every $t = 1, 2, \ldots,$ the tth round strategy of every player depends only on the set N^t of all active players at round t and on the current proposer i.

In our model, a subgame of the whole game $G^N(\rho)$, having the set of active players S and starting with a chance move to select a proposer at round t is identical to the bargaining game $G^S(\rho)$, regardless of the history preceding the subgame. Under the stationary assumption of a strategy combination, all players employ identical strategies in every two identical subgames.

For an SSPE $\sigma = (\sigma_1, \ldots, \sigma_n)$ of the game $G^N(\rho)$ and for each coalition $S \in \mathscr{P}_0(N)$, we will denote by $v^S = (v_i^S)_{i \in S} \in \mathbb{R}_+^S$ the expected payoff vector of players for σ in the subgame $G^S(\rho)$. We call the collection $\{v^S \mid S \in \mathscr{P}_0(N)\}$ the *payoff configuration* of the SSPE σ.

2.3 Characterization of SSPE

We will characterize the SSPE of our bargaining model $G^N(\rho)$ corresponding to the tax assignment problem. Let us denote the proposal when the set of active players is S and the proposer is i in the SSPE σ by $a_{S,i} = (a^1_{S,i}, a^2_{S,i}, \ldots, a^{|S|}_{S,i}) \in \mathbb{R}^{|S|}$ for $i \in S \subset N$. Let also $a_S \stackrel{\text{def}}{=} (1/|S|) \sum_{i\in S} a_{S,i}$ be their average. By definition, it is clear that the jth component of the vector a_S becomes $a^j_S = (1/|S|) \sum_{i\in S} a^j_{S,i}$.

First, we shall prove the basic proposition about an SSPE. This proposition and the next proposition correspond to Proposition 1 and to the Corollary in Hart and Mas-Colell's (1996) paper. Since the rule of our bargaining game is different from that of the game of Hart and Mas-Colell (1996), the equations in the propositions are modified.

PROPOSITION 4. *In every SSPE σ of the game $G^N(\rho)$, the corresponding proposals are always accepted, and they are characterized by:*

$$a_{S,i} \in \partial V(S), \quad \text{for } \forall i \in S \subset N, \text{ and} \tag{3}$$

$$a^j_{S,i} = \rho a^j_S + (1-\rho) \sum_{k\in S\setminus i} \frac{1}{|S|-1} a^j_{S\setminus k} \quad \text{for } \forall i,j \in S \subset N,\ i \neq j. \tag{4}$$

Moreover, these proposals are nonnegative, i.e., $a_{S,i} \in \mathbb{R}^{|S|}_+$ for all $S \in \mathscr{P}_0(N)$ and for all $i \in S$.

Proof. See Appendix. □

Remark that $a^j_{S\setminus j} = 0$. Equation (4) implies that player i will propose the expected payoff that player j would obtain in the continuation of the game if the proposal is rejected by j. Furthermore, Proposition 4 shows that no delay of the agreement will occurs; thus, an agreement will be achieved in the first round, in the SSPE σ.

The next proposition shows the relationship between the proposals $a_{N,i}$ for $i \in N$ and the average a_N.

PROPOSITION 5. *(i) $(M, \ldots, M) \in \mathbb{R}^n_+$ is an upper bound for the set $V(N) \cap \mathbb{R}^n_+$. (ii) $|a^j_{N,i} - a^j_N| \leq M(1-\rho)$ for all $i,j \in N$.*

Proof. (i) By Assumption 1, $M > 0$ is an upper bound for the range of utility function u. Then, it is straightforward that $(M, \dots, M)$ is an upper bound for $V(N) \cap \mathbb{R}^n_+$.

(ii) By using equation (4) in Proposition 4, we have

$$\begin{aligned} a^j_{N,i} - a^j_N &= (1-\rho) \sum_{k \in N \setminus i} \frac{1}{n-1} a^j_{N \setminus k} - (1-\rho) a^j_N, \\ &= (1-\rho) \big[\sum_{k \in N \setminus i} \frac{1}{n-1} a^j_{N \setminus k} - a^j_N \big]. \end{aligned}$$

It is clear that

$$0 \leq \sum_{k \in N \setminus i} \frac{1}{n-1} a^j_{N \setminus k} \leq \frac{n-2}{n-1} M < M, \text{ and, } 0 \leq a^j_N \leq M.$$

Then, we obtain $|a^j_{N,i} - a^j_N| \leq M(1-\rho)$. □

It follows from Proposition 5 (ii) that if ρ is close to 1, i.e., if the cost of delay is low, then there is little distance between the proposal $a_{N,i}$ of player i and the average a_N for all $i \in N$. This implies that dispersion among individual proposals would vanish as $\rho \to 1$. Thus, it does not matter who makes a proposal in the first round of the game if ρ is close to 1.

3 Equal Income Allocation

3.1 Nontransferable utility case

Let us study the equilibria of the noncooperative game. We begin with the nontransferable utility case, i.e., the case in which the payoff function for each player is given by a strictly concave utility function of after-tax income satisfying Assumption 1.

PROPOSITION 6. *Let (N, V) be an NTU coalitional form. For each $\rho, 0 \leq \rho < 1$, there exists an SSPE. Moreover, as $\rho \to 1$, the every limit point of SSPE payoff configurations $(a_S)_{S \subset N}$ is, for all $i \in S \in \mathscr{S}$,*

$$a^i_S = \sum_{k \in S \setminus i} \frac{a^i_{S \setminus k}}{|S|} + \sum_{k \in S \setminus i} \frac{1}{|S|(|S|-1)} \left[\sum_{j \in S} \lambda^j_S a^j_S - \sum_{j \in S \setminus k} \lambda^j_S a^j_{S \setminus k} \right] \Big/ \lambda^i_S, \quad (5)$$

where $(\lambda^j_S)_{j \in S} \in \mathbb{R}^S_{++}$ is the unique supporting normal to the boundary of $V(S)$ at a_S, and, for all $S \in \mathscr{P}_0(N) \setminus \mathscr{S}$, $a_S = \{0^S\}$.

Proof. See Appendix. □

Rewriting equation (5), we obtain that for every $i \in S$ and for every $S \in \mathscr{S}$,

$$\left[\frac{1}{|S|}\sum_{j\in S}\lambda_S^j a_S^j - \lambda_S^i a_S^i\right] + \sum_{k\in S\backslash i}\frac{1}{|S|}\left[\lambda_S^i a_{S\backslash k}^i - \frac{1}{|S|-1}\sum_{j\in S\backslash k}\lambda_S^j a_{S\backslash k}\right] = 0. \tag{6}$$

The system of equations (6) has a (unique) solution

$$a_S = (a_S^1, a_S^2, \ldots, a_S^{|S|}) = (\bar{a}_S, \bar{a}_S, \ldots, \bar{a}_S), \quad \text{for } \forall S \in \mathscr{S},$$

where $\bar{a}_S$ is a positive real variable. Thus, the equal payoff allocation is a solution to (6) for each coalition $S \in \mathscr{S}$. Combining this fact and Proposition 5, we have:

COROLLARY 1. *Let (N, V) be an NTU form. Then, as $\rho \to 1$, the SSPE proposals $a_{S,i}$ converge to the equal payoff allocation for all $i \in S \in \mathscr{P}_0(N)$.*

By Proposition 4, we know that the proposal $a_{N,i}$ defined by (3) and (4) belongs to the boundary of $V(N)$ and is accepted by every responder $j \in N\backslash i$ at the first round in the SSPE. Therefore, as $\rho \to 1$, the equal payoff allocation, i.e., the payoff vector $a_N = (a_N^1, \ldots, a_N^n)$ such that $a_N \in \partial V(N)$ and $a_N^1 = a_N^2 = \cdots = a_N^n$ is realized on the equilibrium path, irrespective of who makes a proposal at the first round. Because every player has the same utility function, the equal payoff allocation implies equal after-income allocation;

$$x_1^* = x_2^* = \cdots = x_n^*, \text{ and, } \sum_{i\in N} t_i^* \stackrel{\text{def}}{=} \sum_{i\in N}(y_i - x_i^*) = T.$$

We have thus obtained equal income allocation in a noncooperative manner. In other words, *perfectly progressive* income taxation has been unanimously agreed upon in noncooperative bargaining with respect to the choice of tax systems when ρ is close to 1. Here, we say that an income tax is *perfectly progressive* if income taxation leads to equal income allocation.

3.2 Transferable utility case

Next, we consider the case in which the payoff function for each player i is his/her after-tax income x_i itself. In this case, the basic data for the bargaining are represented by the game in coalitional form with transferable utility

(N, v) defined as equation (2). It is well-known that the law of a decreasing marginal utility of income plays a central role in supporting progressive taxation in the context of utilitarian social welfare maximization and equal sacrifice taxation (see Musgrave (1959), Young (1988)). Although we define the payoff function as the after-tax income of each player (that is, there is no decreasing marginal utility of income here), we can show that (perfectly) progressive income taxation, i.e., equal income allocation, is obtained as an outcome of noncooperative bargaining when ρ is close to 1. Note that the payoff for every dropped-out player is assumed to be zero for the same reason as in the nontransferable utility case.

We can establish the following proposition in the transferable utility (TU) case.

PROPOSITION 7. *Let (N, v) be a TU form. If $0 \leq \rho < 1$, then there exists an unique SSPE, whose payoff configurations $(a_S)_{S \subset N}$ satisfy*

$$a_S^i = \sum_{k \in S \backslash i} \frac{1}{|S|} a_{S \backslash k}^i + \sum_{k \in S \backslash i} \frac{1}{|S|(|S|-1)} [v(S) - v(S \backslash k)] \tag{7}$$

for any $i \in S \in \mathscr{S}$. Moreover, $a_S^i = 0$ for all $i \in S \in \mathscr{P}_0(N) \backslash \mathscr{S}$.

Proof. See Appendix. □

COROLLARY 2. *Let (N, v) be a TU form. Then, as $\rho \to 1$, any SSPE proposals $a_{S,i}$ converge to the vector $(v(S)/|S|, \ldots, v(S)/|S|)$ for all $S \in \mathscr{P}_0(N)$.*

Proof. Rearranging equation (7), we can obtain that for all $S \in \mathscr{S}$ and for all $i \in S$,

$$\frac{v(S)}{|S|} - a_S^i + \sum_{k \in S \backslash i} \frac{1}{|S|} (a_{S \backslash k}^i - \frac{1}{|S|-1} v(S \backslash k)) = 0. \tag{8}$$

The system of equations (8) has a (unique) solution $a_S = (v(S)/|S|, \ldots, v(S)/|S|)$ for all $S \in \mathscr{S}$. For $S \in \mathscr{P}_0(N) \backslash \mathscr{S}$, $a_S = (0, \ldots, 0)$. Taking into account of Proposition 5 and $\rho \to 1$, we can obtain the result. □

This implies that the SSPE payoffs also coincide with equal income allocation in the TU case;

$$(v(N)/n, \ldots, v(N)/n) = (E/n, \ldots, E/n).$$

Thus, all players come to unanimous agreement to employ perfectly progressive income taxation in the bargaining for the tax assignment when ρ is close to 1.

3.3 Pure bargaining case

Finally, we consider the pure bargaining case. We call the case in which bargaining has no value if the number of active players is less than n, i.e., $V(S) = \{0^S\}$ for all $S \subsetneq N$, the *pure bargaining case*. So far we have assumed that every player has the same utility function. Let us relax this assumption. We allow the possibility that each individual has a different utility function. Thus, we abandon part (i) of Assumption 1 and replace the utility function u with u_i in part (ii) of Assumption 1. Then, the characteristic function in the pure bargaining case is defined as

$$\begin{aligned} &V(N) = \left\{ z \in \mathbb{R}^n_+ \mid \exists (x_i)_{i\in N} \in X^N, \forall i \in N, z_i \leq u_i(x_i) \right\}, \\ &V(S) = \{0^S\} \text{ for all } S \subsetneq N. \end{aligned}$$

Even if different utility functions allowed, the contents of Proposition 6 are similarly satisfied as long as the noncooperative bargaining procedure is maintained. (As a matter of fact, Theorems 1, 2 and 3, Propositions 4 and 5, and Corollary 1 also hold without the assumption of the identical utility function. The result of perfectly progressive income taxation, however, is not obtained.) In the pure bargaining case, $a^i_S = 0$ for all $i \in S$ and for all $S \subsetneq N$. Then, the system of equations (5) in Proposition 6 becomes reduced to the following one:

$$\lambda^i_N a^i_N - \frac{1}{n} \sum_{j \in N} \lambda^j_N a^j_N = 0, \ \forall i \in N, \tag{9}$$

where $(\lambda^j_N)_{j\in N} \in \mathbb{R}^S_{++}$ is the unique supporting normal to the boundary of $V(N)$ at a_N. This implies that a solution to the system of equations satisfies

$$\lambda^1_N a^1_N = \lambda^2_N = \cdots = \lambda^n_N a^n_N. \tag{10}$$

We denote by x^*_i the after-tax income corresponding to the solution a^i_N for each $i \in N$. Thus, $a^i_N = u_i(x^*_i)$ for all $i \in N$. Moreover, the vector $(\lambda^1_N, \ldots, \lambda^n_N)$ is in proportion to the vector

$$(1/u'_1(x^*_1), \ldots, \ldots, 1/u'_n(x^*_n)).$$

Therefore, equation (10) is expressed as

$$\frac{u'_1(x^*_1)}{u_1(x^*_1)} = \cdots = \frac{u'_n(x^*_n)}{u_n(x^*_n)}. \tag{11}$$

When ρ is close to 1, the after-tax income allocation $(x^*_1, x^*_2, \ldots, x^*_n)$ is realized in an SSPE of the noncooperative bargaining game for the tax assignment.

Let us consider the problem of maximizing a Nash social welfare function under the resource constraint, that is, under the set of feasible after-tax income allocations X^N. Namely, we consider the problem

$$\max_x \prod_{i=1}^{n} u_i(x_i) \ \text{ subject to } \ \sum_{i \in N} x_i \leq E. \tag{12}$$

This problem is equivalent to the tax assignment problem of maximizing a Nash social welfare:

$$\max_t \prod_{i=1}^{n} u_i(y_i - t_i) \ \text{ subject to } \ \sum_{i \in N} t_i \geq T.$$

The Nash social welfare maximization problem has often been employed in the context of optimal income taxation (see, for example, Kaneko (1981, 1982)). Note that optimal taxation has been discussed in a general equilibrium model and the disincentive effect of taxation on work has been taken into account.

It is easy to verify that the first-order necessary condition for problem (12) can be written as (11). Thus, a solution to the Nash social welfare maximization problem also satisfies condition (11). Since a solution to the maximization problem is unique (by the strict concavity and monotonicity of u_i), the vector $(x_1^*, \ldots, x_n^*)$ coincides with the solution to the problem of maximizing a Nash social welfare function under the resource constraint. Thus, we have implemented a solution to a Nash social welfare maximization problem in a noncooperative manner.

4 Concluding Remarks

In this paper, we presented a noncooperative bargaining model of tax assignment which realized equal (after-tax) income allocation in an SSPE as $\rho \to 1$. This result implies that members of the society come to an agreement on a perfect progressive income tax schedule in order to collect a given tax revenue. Our bargaining game proceeds in an extremely democratic manner, where everyone has equal opportunities for making a proposal and the tax assignment is determined by the unanimity rule among the active players. Because of the noncooperative framework, binding commitments among the players of the group are not allowed. Furthermore, each player has no altruistic preference and makes a decision about his/her strategies so as to maximize his/her own payoff. Up to now, progressive taxation has been regarded as means for achieving a social justice, e.g., utilitarianism, or has been obtained through some voting mechanism. In our bargaining for the tax assignment, all players are primarily self-interested and there is no social mechanism planner. Therefore, our analysis has established a noncooperative foundation for progressive taxation.

In the pure bargaining case, we showed that our bargaining procedure implements a solution to the Nash social welfare maximization problem (a Nash bargaining solution) in an SSPE. This implies that a Nash social welfare maximization problem, for example, the theory of optimal taxation, is supported in a noncooperative manner, as long as the pure bargaining case is considered.

The possibility of alternative bargaining procedures exists in our model. In particular, we have to discuss the procedure by which a players drops out after the proposal is rejected. In our bargaining model, only the responders, not the proposer, drop out with equal probability. This procedure seems to be arbitrary. On the other hand, Hart and Mas-Colell (1996) considered and focused on the case in which only the proposer drops out with some probability, and they obtained the Shapley value and the Maschler-Owen consistent value in a TU case and an NTU case, respectively. For procedural fairness, or, for the sake of equal opportunity in bargaining, it would be appropriate to establish a rule in which all players drop out with equal probability. In such a procedure, the resulting solution is not characterized as either the Shapley value or equal income allocation. As the number of players n, however, is very large, the solutions converge to our solution in this paper. This implies that our solution, namely equal income allocation, is a close approximation of the bargaining solution in which all players drop out with equal probability.

There are some limitations in our bargaining model for the tax assignment problem. First, our noncooperative bargaining game does not allow for strategic coalition formation as in the models of Selten (1981), Chatterjee, Dutta, Ray and Sengupta (1993), Perry and Reny (1994), Moldovanu and Winter (1995) and Okada (1996). Secondly, we ignored the possibility that players act to evade their taxes. For example, we should allow the members to behave strategically by forming a coalition in which income is rearranged among the members in order to evade full taxation, as Nakayama (1976) has considered. Finally, we assumed that all players supply labor inelastically. It is well-known that a progressive income tax schedule would reduce the incentive to supply labor. Taking into consideration that our result has supported perfectly progressive taxation, introducing work incentives to our bargaining model would be a very interesting next direction for our research.

Appendix

Proof of Theorem 1: (Closedness) We will prove that the set $V(S)$ is closed. Let $\{z_\nu^S\}_\nu^\infty$ be any convergent sequence, where $z_\nu^S \in V(S)$ for all $\nu = 1, 2, \dots$, and let z^S be the limit point of the sequence. By definition of $V(S)$ and $z_\nu^S \in V(S)$, there exists $x_\nu^S \overset{\text{def}}{=} (x_{i\nu}^S)_{i\in S} \in X^S$ such that $z_{i\nu}^S \leq u(x_{i\nu}^S)$ for any $i \in S$. Since there exists x_ν^S for each $\nu = 1, 2, \dots$, the sequence $\{x_\nu^S\}_{\nu=1}^\infty$ is constructed. By the compactness of X^S, the sequence $\{x_\nu^S\}_{\nu=1}^\infty$ contains a convergent subsequence $\{x_{\nu_j}^S\}_{j=1}^\infty$. Let denote the limit point of this subsequence by x^S.

(i) Because a utility function u is continuous, $x_{i\nu_j}^S \to x_i^S$ implies $u(x_{i\nu_j}^S) \to u(x_i^S)$ for all $i \in S$.

(ii) Furthermore, since $\{x_{\nu_j}^S\}_{j=1}^\infty$ is a subsequence of $\{x_\nu^S\}_{\nu=1}^\infty$, it is satisfied that $z_{i\nu_j}^S \leq u(x_{i\nu_j}^S), \forall i \in S$ for each number ν_j.

Let define $z_{\nu_j}^S \overset{\text{def}}{=} (z_{i\nu_j}^S)_{i\in S} \in \mathbb{R}^S$.

(iii) Since $z_\nu^S \to z^S$, the subsequence $\{z_{\nu_j}^S\}_{j=1}^\infty$ also converges to z^S.

From (i), (ii) and (iii), we can obtain $z_i^S \leq u(x_i^S)$ for all $i \in S$. This implies that $z^S \in V(S)$. $\square$

(Convexity) Let $z^S, z'^S \in V(S)$. Then, there exists $(x_i)_{i\in S} \in X^S$ such that $z_i^S \leq u(x_i)$ for all $i \in S$, and there exists $(x'_i)_{i\in S} \in X^S$ such that $z'^S \leq u(x'_i)$ for all $i \in S$.

Therefore, it is satisfied that for any $0 \leq t \leq 1$,

$$tz_i^S + (1-t)z_i'^S \leq tu(x_i) + (1-t)u(x'_i) \text{ for all } \ i \in S.$$

Since the utility function u is strictly concave, then, for any $0 \leq t \leq 1$,

$$tz_i^S + (1-t)z_i'^S \leq u(tx_i + (1-t)x'_i) \text{ for all } \ i \in S.$$

This implies that $tz_i^S + (1-t)z'^S \in V(S)$. $\square$

(Comprehensiveness) Let $z^S \in V(S)$ and take $\hat{z}^S$ such that $\hat{z}_i^S \leq z_i^S$ for all $i \in S$. Then, there exists $(x_i)_{i\in S} \in X^S$ such that $z_i^S \leq u(x_i)$ for all $i \in S$. Since $\hat{z}_i^S \leq z_i^S$ for all $i \in S$, it holds that $\hat{z}_i^S \leq u(x_i)$ for all $i \in S$. Thus, $\hat{z}^S \in V(S)$. $\square$

($0 \in V(S)$) Because $0 \leq \max(0, Y(S) - T)$, then $\{0^S\} \in X^S$. Furthermore, $0 \leq u(0)$. Hence, $\{0^S\} \in V(S)$. $\square$

(Boundedness) It is trivial that $\{0^S\}$ is a lower bound for $V(S) \cap \mathbb{R}_+^S$. We assume that u is a bounded function, and we denote the upper bound by M. So, it holds that for any $x_i \in \mathbb{R}_+$, $u(x_i) \leq M$ for all $i \in S$. Hence, for any

$(z_i^S)_{i\in S} \in V(S)\cap\mathbb{R}^S_+$, $z_i^S \leq u(x_i) \leq M$ for all $i \in S$. Thus, $(M,\ldots,M) \in \mathbb{R}^S_+$ is an upper bound for $V(S)\cap\mathbb{R}^S_+$. $\square$

Proof of Theorem 2: Let $\hat{z}^S \in \partial V(S)\cap\mathbb{R}^S_{++}$. That is, the vector $\hat{z}^S$ belongs to the boundary of $V(S)\cap\mathbb{R}^S_+$. In addition, the set $V(S)\cap\mathbb{R}^S_+$ is convex. Then, by the supporting hyperplane theorem, there exists a (unique) vector $\hat{\lambda}^S \neq \{0^S\}$ such that $\hat{\lambda}^S \cdot \hat{z}^S \geq \hat{\lambda}^S \cdot z^S$ for all $z^S \in V(S)\cap\mathbb{R}^S_+$. Moreover, since $V(S) - \mathbb{R}^S_+ \subset V(S)$, we must have $\hat{\lambda}^S \geq 0$.

In addition, if $\hat{z}^S \in \partial V(S)\cap\mathbb{R}^S_{++}$, then there exists $(\hat{x}_i)_{i\in S} \in \mathbb{R}^S_+$ satisfying the following system of equations,

$$\sum_{i\in S} \hat{x}_i = Y(S) - T > 0, \text{ and, } \hat{z}_i^S = u(\hat{x}_i), \quad \forall i \in S.$$

The above system of equations reduces to

$$\sum_{i\in S} u^{-1}(\hat{z}_i^S) - Y(S) + T = 0.$$

Let define a function $g : \mathbb{R}^S_{++} \to \mathbb{R}$ as

$$g(z^S) \stackrel{\text{def}}{=} \sum_{i\in S} u^{-1}(z_i^S) - Y(S) + T.$$

It is obvious that the function g is continuous. Thus, we show that there exists a continuous function $g : \mathbb{R}^S_{++} \to \mathbb{R}$ such that

$$\partial V(S)\cap\mathbb{R}^S_{++} = \{z^S \in \mathbb{R}^S_{++} \mid g(z^S) = 0\}.$$

Therefore, $\hat{\lambda}^S$ is represented by the gradient of g at each $\hat{z}^S \in \partial V(S)\cap\mathbb{R}^S_{++}$. The gradient of g is the vector $(1/u'(u^{-1}(\hat{z}_1^S)),\ldots,1/u'(u^{-1}(\hat{z}_{|S|}^S)))$, which is uniquely defined and positive in all its coordinates at $\hat{z}^S$. We can take this gradient as the outward normal direction (vector) at each $\hat{z}^S \in \partial V(S)\cap\mathbb{R}^S_{++}$. $\square$

Proof of Theorem 3: Let $S \subset H$ and $z^H \in V(S)\times\{0^{H\setminus S}\}$. We denote the $V(S)$'s coordinate of the point z^H by $z^H|_S$. Thus, $z^H|_S \in V(S)$ and $z^H = (z^H|_H, 0^{H\setminus S})$. Therefore, there exists $(x_i)_{i\in S} \in X^S$ such that $z_i^H|_S \leq u(x_i)$ for all $i \in S$. Furthermore, we choose $x_i = 0$ for all $i \in H\setminus S$. Then, it is satisfied that $0 \leq u(0) = u(x_i)$ for all $i \in H\setminus S$. Using x_i obtained above, we can define $x^H \stackrel{\text{def}}{=} (x_1,\ldots,x_{|S|},0,\ldots,0)$. The vector x^H satisfies the inequality:

$$\sum_{i\in H} x_i = \sum_{i\in S} x_i \leq \max(0, Y(S) - T) \leq \max(0, Y(H) - T).$$

Therefore, $x^H \in X^H$. Moreover, it holds that $z_i^H \leq u(x_i^H)$ for all $i \in H$. This implies $z^H \in V(H)$. $\square$

Proof of Proposition 4: The proof is by induction. The proposition holds trivially for the 1-player case. Assume that it holds for the less than n-players case. Let $a_{S,i}$ for $i \in S \subset N$ be the proposals of an SSPE. We will show that equations (3) and (4) are satisfied. We denote the expected payoff vector for the members of S by $v_S \in \mathbb{R}^S$ in the subgames where S is the set of active players. Since $V(S)$ is convex we obtain $v_S \in V(S)$. By induction hypothesis, $a_S = v_S$. Then, equations (3) and (4) are satisfied for $S \neq N$.

By the convexity of $V(S)$, it holds that $v_N \in V(N)$. Since for any $k \in N$, $a_{N\setminus k} \in V(N\setminus k)$, it follows from the monotonicity of V that $(a_{N\setminus k}, 0) \in V(N)$, where 0 is the kth coordinate. Now, we denote by $a^k_{N\setminus k}$ the kth coordinate, and then $(a_{N\setminus k}, 0)$ is represented by $(a_{N\setminus k}, a^k_{N\setminus k})$. The convexity of $V(N)$ implies that for all $i \in N$

$$\frac{1}{n-1}\sum_{k\in N\setminus i}(a_{N\setminus k}, a^k_{N\setminus k}) = (\frac{1}{n-1}\sum_{k\in N\setminus i} a^1_{N\setminus k}, \ldots, \frac{1}{n-1}\sum_{k\in N\setminus i} a^n_{N\setminus k}) \in V(N).$$

Hence, the convex combination $\rho v_N + (1-\rho)(1/(n-1))\sum_{k\in N\setminus i}(a_{N\setminus k}, a^k_{N\setminus k}) \in V(N)$ for all $i \in N$. Let increase in the ith coordinate of the vector $\rho v_N + (1-\rho)(1/(n-1))\sum_{k\in N\setminus i}(a_{N\setminus k}, a^k_{N\setminus k})$ until reaching the boundary $\partial V(N)$ for all i. We denote by d_i the induced vector, which satisfies $d_i^j = \rho v_N^j + (1-\rho)(1/(n-1))\sum_{k\in N\setminus i} a^j_{N\setminus k}$ for $j \neq i$ and $d_i^i \geq \rho v_N^i + (1-\rho)(1/(n-1))\sum_{k\in N\setminus i} a^i_{N\setminus k}$. For $j \neq i$, the amount d_i^j is the expected payoff of j when player j would rejects i's proposal. Therefore, d_i is the best proposal for i among the proposals that will be accepted if i is the proposer. Furthermore, if i makes any proposal that is rejected, then i obtains at most $\rho v_N^i + (1-\rho)(1/(n-1))\sum_{k\in N\setminus i} a^i_{N\setminus k}$, which is less than d_i^i. Hence, player i will propose $a_{N,i} = d_i$ and the proposal will be accepted. Then, we have $v_N = a_N$.

Next, let us show that $a_{N,i} \geq 0$. Consider the following strategy: in the case of the responder, accept only if offered at least 0 and, in the case of the proposer, propose $0 \in V(N)$. This strategy will guarantee a payoff of at least 0 to player i. Therefore, $a_{N,i} \geq 0$.

Let prove the converse. We show that proposals $(a_{S,i})_{S\subset N, i\in N}$ satisfying equations (3) and (4) can be supported as an SSPE. Let us show that all proposals are nonnegative. Since $a_{N,i} \in V(N)$ and $V(N)$ is convex, it holds that the average $a_N \in V(N)$. Furthermore, $(1/(n-1))\sum_{k\in N\setminus i}(a_{N\setminus k}, a^k_{N\setminus k}) \in V(N)$ for all $i \in N$. It follows that $b_i \overset{\text{def}}{=} \rho a_N + (1-\rho)\sum_{k\in N\setminus i}(1/(n-1))(a_{S\setminus k}, a^k_{N\setminus k}) \in V(N)$. It is obvious that $a_{N,i} \geq b_i$, $\forall i \in N$. By the induction hypothesis, $a_{N\setminus k} \geq 0$. Then, $a_{N,i} \geq \rho a_N$ for all $i \in N$. Taking an average with respect to i, it holds that $a_N \geq \rho a_N$. Because $0 \leq \rho < 1$, we must have $a_N \geq 0$. Therefore, the all $a_{N,i}$ are nonnegative.

Finally, let us verify that the strategies corresponding to these proposals do form an SSPE. By the induction hypothesis, it holds for any subgame after one of the players has dropped out. The strategies of the other players do not allow player i to increase in his/her payoff from proposals that are accepted. The only possibility to gain remains by managing to drop out. This, however, gives a payoff of 0. The suggested strategy yields nonnegative payoffs. □

Proof of Proposition 6: The proof will be demonstrated by proving the following three lemmas. These lemmas will be proved in the same line as the proofs of Proposition 6, 7, and 8 in Hart and Mas-Colell's (1996) paper.

First we prove the existence of SSPE.

LEMMA 1. *Let (N, V) be an NTU form. Then, there exists an SSPE for each $0 \leq \rho < 1$.*

Proof. We proceed by induction. It is trivial for $n = 1$. Assume that there exists a_S for all $S \neq N$ such that for any $H \subset N$, $H \neq N$, $(a_S)_{S \subset H}$ is an SSPE payoff configuration for the game (H, V). By Proposition 4, $a_S \geq 0$ for all S. Let us define n functions $\alpha_i(b)$ by:

$$\alpha_i(b) \in \partial V(N), \text{ and, } \alpha_i^j(b) \stackrel{\text{def}}{=} \rho b^j + (1-\rho) \sum_{k \in N \setminus i} \frac{1}{n-1} a_{N\setminus k}^j, \ j \neq i.$$

We assume that the domain of the functions is defined as the compact convex set $V(N) \cap \mathbb{R}_+^N$. Then it induces that the range of the functions is also $V(N) \cap \mathbb{R}_+^N$. By the nonlevelness in Theorem 2 and Assumption 2, we can apply the implicit function theorem. Then, the functions are well-defined and continuous. By the convexity of the domain, $(1/n)\sum_{i \in N} \alpha_i(b)$ maps also into $V(N) \cap \mathbb{R}_+^N$. By an application of Brouwer's fixed point theorem, there exists a vector $a_N \in V(N) \cap \mathbb{R}_+^N$ satisfying $a_N = (1/n)\sum_{i \in N} \alpha_i(a_N)$. Proposition 4 implies that a_N are equilibrium payoffs for N and that $a_{N,i} = \alpha_i(a_N)$, $\forall i$. By the induction hypothesis, $(a_S)_{S \subset N}$ are the payoffs of an overall SSPE for the game (N, V). □

Next, we introduce the notion of a *hyperplane coalitional form game* $(N, \tilde{V})$. In this game, each $\tilde{V}(S)$ is defined as a half space in $\mathbb{R}_+^S$. Thus, the set $\tilde{V}(S)$ is represented by for some $\lambda_S \in \mathbb{R}_{++}^S$,

$$\tilde{V}(S) \stackrel{\text{def}}{=} \left\{ c \in \mathbb{R}^S \mid \sum_{i \in S} \lambda_S^i c^i \leq w_S \right\}.$$

LEMMA 2. *Let $(N, \tilde{V})$ be a hyperplane coalitional form. Then for each $0 \le \rho < 1$ there exist an unique SSPE. Moreover, the SSPE payoff configuration $(a_S)_{S \subset N}$ satisfies that for all $i \in S \subset N$,*

$$a_S^i = \sum_{k \in S \setminus i} \frac{a_{S \setminus k}^i}{|S|} + \sum_{k \in S \setminus i} \frac{1}{|S|(|S|-1)} \left[\sum_{j \in S} \lambda_S^j a_S^j - \sum_{j \in S \setminus k} \lambda_S^j a_{S \setminus k}^j \right] \Big/ \lambda_S^i. \quad (13)$$

Proof. We proceed by induction. Assume the statement is correct for the less than n-players case. Let $\lambda_N \in \mathbb{R}_{++}^n$ and

$$\tilde{V}(N) \stackrel{\text{def}}{=} \left\{ c \in \mathbb{R}^n \mid \sum_{i \in N} \lambda_N^i c^i \le w_N \right\}.$$

By definition of a_N^i and by Proposition 4, it holds that for every $i \in N$

$$\begin{aligned} n\lambda_N^i a_N^i &= \lambda_N^i a_{N,i}^i + \sum_{j \in N \setminus i} \lambda_N^i a_{N,j}^i \\ &= (w_N - \sum_{j \in N \setminus i} \lambda_N^j a_{N,i}^j) + \sum_{j \in N \setminus i} \lambda_N^i a_{N,j}^i \\ &= w_N - \sum_{j \in N \setminus i} \lambda_N^j (\rho a_N^j + (1-\rho) \sum_{k \in N \setminus i} \frac{a_{N \setminus k}^j}{n-1}) \\ &\qquad + \sum_{j \in N \setminus i} \lambda_N^i (\rho a_N^i + (1-\rho) \sum_{k \in N \setminus j} \frac{a_{N \setminus k}^i}{n-1}). \end{aligned}$$

Since $w_N = \sum_{j \in N} \lambda_N^j a_N^j$, the above equality is rewritten by

$$\begin{aligned} n\lambda_N^i a_N^i = (1-\rho) \sum_{j \in N} \lambda_N^j a_N^j + \lambda_N^i \rho a_N^i - \frac{1-\rho}{n-1} \sum_{k \in N \setminus i} \sum_{j \in N \setminus i} \lambda_N^j a_{N \setminus k}^j \\ + (n-1)\lambda_N^i \rho a_N^i + \frac{1-\rho}{n-1} \sum_{j \in N \setminus i} \sum_{k \in N \setminus k} \lambda_N^i a_{N \setminus k}^i. \end{aligned}$$

Then, it reduces to the following equality:

$$\begin{aligned} n(1-\rho)\lambda_N^i a_N^i = (1-\rho) \sum_{k \in N \setminus i} \lambda_N^i a_{N \setminus k}^i + (1-\rho) \sum_{j \in N} \lambda_N^j a_N^j \\ - \frac{1-\rho}{n-1} \sum_{k \in N \setminus i} \sum_{j \in N \setminus i} \lambda_N^j a_{N \setminus k}^j. \end{aligned}$$

By dividing the last equality by $n(1-\rho)\lambda_N^i$, we obtain

$$a_N^i = \sum_{k\in N\setminus i} \frac{a_{N\setminus k}^i}{n} + \sum_{k\in N\setminus i} \frac{1}{n(n-1)} \left[\sum_{j\in N} \lambda_N^j a_N^j - \sum_{j\in N\setminus k} \lambda_N^j a_{N\setminus k}^j \right] \Big/ \lambda_N^i.$$

Let us choose a_S satisfying equation (13) and define $a_{S,i}$ by equations (3) and (4) in order to show the existence of the SSPE. Then, it holds that $a_S = (1/|S|)\sum_{j\in S} a_{S,j}$. Therefore, these proposals form an SSPE by Proposition 4. □

LEMMA 3. *Let (N,V) be an NTU form and $a(\rho)$ be an SSPE payoff configuration for each ρ. If $a=(a_S)_{S\subset N}$ is a limit point of $a(\rho)$ as $\rho\to 1$, then a satisfies the equation (13), where $\lambda_S \in \mathbb{R}^S_{++}$ for each $S\in\mathscr{S}$ is the outward unit normal vector to $\partial V(S)$ at a_S.*

Proof. Let $\lambda_S(\rho)$ be the outward unit normal to the hyperplane passing through the vector $\{a_{S,i} \mid i\in S\}$, and let $\tilde{V}_\rho(S)$ be the half-space below the hyperplane. Then, we have a hyperplane coalitional form game $(N,\tilde{V}_\rho)$ for each ρ. By Proposition 5, $a_{S,i}(\rho)\to a_S$ as $\rho\to 1$. Furthermore, it follows from Theorem 2 and Assumption 2 that the boundary $\partial V(S)\cap\mathbb{R}^{|S|}_+$ is smooth and nonlevel. Hence, we have $\lambda_S(\rho)\to\lambda_S$. Therefore, we obtain

$$\tilde{V}_\rho(S) \to \tilde{V}'(S) \stackrel{\text{def}}{=} \{c\in\mathbb{R}^S \mid \lambda_S c \le \lambda_S a_S\}.$$

It is clear from the fact of Proposition 4 that the payoff configuration $a(\rho)$ remains an SSPE payoff configuration for the hyperplane coalitional form $(N,\tilde{V}_\rho)$. By Lemma 2, $a(\rho)$ satisfies the equation (13) of $(N,\tilde{V}_\rho)$. Therefore, as $\rho\to 1$, the limit point $(a_S)_{S\subset N}$ of the SSPE payoff configuration of the game (N,V) is precisely the payoff configuration of the hyperplane form $(N,\tilde{V}')$, which satisfies the equation (13). This completes the proof. □

Proof of Proposition 7: For the proof of this proposition, it suffices to repeat the similar arguments to those of Lemma 2. Here we will only show that the SSPE payoff configuration satisfies equation (7). A proposer $i\in S$ proposes to each other player $j\in S\setminus i$ the expected payoff of next round, i.e.,

$$a_{S,i}^j = \rho a_S^j + (1-\rho)\sum_{k\in S\setminus i} \frac{1}{|S|-1} a_{S\setminus k}^j, \tag{14}$$

where $a_{S\setminus j}^j = 0$. Then, the proposer takes all the surplus

$$a_{S,i}^i = v(S) - \sum_{j\in S\setminus i} a_{S,i}^j. \tag{15}$$

By the substitution of equation (15) to (14), we obtain that

$$\begin{aligned} a^i_{S,i} &= v(S) - \rho \sum_{j\in S\backslash i} a^j_S - (1-\rho) \sum_{j\in S\backslash i} \sum_{k\in S\backslash i} \frac{1}{|S|-1} a^j_{S\backslash k} \\ &= v(S) - \rho(\sum_{j\in S} a^j_S - a^i_S) - (1-\rho) \sum_{k\in S\backslash i} \frac{1}{|S|-1} [\sum_{j\in S} a^j_{S\backslash k} - a^i_{S\backslash k}]. \end{aligned} \quad (16)$$

In the transferable utility case, the characteristic function v is represented by

$$v(S) = \sum_{j\in S} a^j_S, \text{ and } v(S\backslash k) = \sum_{j\in S\backslash k} a^j_{S\backslash k}. \quad (17)$$

Taking account of $a^k_{S\backslash k} = 0$ and equalities (17), we can reduce equation (16) to

$$a^i_{S,i} = \rho a^i_S + (1-\rho) v(S) + \sum_{k\in S\backslash i} \frac{1-\rho}{|S|-1} (a^i_{S\backslash k} - v(S)).$$

Since $1-\rho = \sum_{k\in S\backslash i} (1-\rho)/(|S|-1)$, then

$$a^i_{S,i} = \rho a^i_S + \sum_{k\in S\backslash i} \frac{1-\rho}{|S|-1} (a^i_{S\backslash k} + v(S) - v(S\backslash k)). \quad (18)$$

Fix i in S, and take expectation of $a^i_{S,j}$ over j. Then, by definition of a^i_S, we obtain

$$a^i_S = \frac{1}{|S|} \sum_{j\in S} \rho a^i_S + \sum_{k\in S\backslash i} \frac{1-\rho}{|S|} a^i_{S\backslash k} + \sum_{k\in S\backslash i} \frac{1-\rho}{|S|(|S|-1)} [v(S) - v(S\backslash k)].$$

Thus,

$$a^i_S = \sum_{k\in S\backslash i} \frac{1}{|S|} a^i_{S\backslash k} + \sum_{k\in S\backslash i} \frac{1}{|S|(|S|-1)} [v(S) - v(S\backslash k)].$$

We have equation (7) as the payoff configuration of (N, v). $\square$

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