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# Human Capital Accumulation, Home Production and Equilibrium Dynamics

Yunfang Hu \*

## Abstract

This paper constructs a human-capital accumulation propelled endogenous growth model in which a home production sector is modeled explicitly. We confirm first the basic equilibrium properties of a unique balanced-growth path, then conduct comparative statics and comparative dynamics exercises analytically. The implications of home production are explored in both the long and short runs, hence some economic realities can be explained. For example, it is shown that the present structure can shed light on understanding the rise trend of women's market work hours observed in recent decades.

*JEL classification:* D13, D91, O41

*Keywords:* Home production; Endogenous growth; Equilibrium dynamics

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# 1 Introduction

One characteristic of the labor market in recent decades is that more and more married women (with higher education) do not quit their market occupations after marriages while make a balance between the home and market work. The data in the United States labor market reveal a clear trend that married women worked much longer at the market than before (Jones, Manuelli and MaGrattan, 2003). The statistics of the Japanese Cabinet Office reveal a similar trend in Japan's female labor supply too. At the same period, an observation on the education sector reveals that the enrollment rate of female students in college has increased steadily. For example, according to the statistics of the Japanese Cabinet Office, the percentage of female students entering junior college or university has exceeded that of the male students since 1989.

It is natural to connect the above two increments in women's education years and market hours together. In fact, empirical studies have confirmed the existence of a positive relation between education years and the attitude towards market work. The present paper therefore constructs an endogenous growth model with home production, in which human capital accumulation is the engine of permanent economic growth. In the original Uzawa (1965) and Lucas (1988) models, resources are allocated to market goods production and education activities. In the absence of externalities, this kind of two-sector growth model generally involves a unique balanced-growth path, which is determined solely by the aggregate physical/human capital ratio<sup>1</sup>. In this paper, we assume that households can derive utility from both market goods consumption and homemade goods enjoyment. That is, in addition to market goods production and human capital accumulation, resources can also be spent at home for non-market production.

The explicit introduction and analysis of the home sector has by now become a fairly standard feature of economic models in business cycle research: see, for example,

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<sup>1</sup>This is illustrated in Bond, Wang and Yip (1996), Caballé and Santos (1993), and Mino (1996), for example.

Benhabib, Rogerson and Wright (1991), Greenwood and Hercowitz (1991) and Perli (1998). The good fitness to evidences of the predicted results by the models with home production suggests that home activities are relevant to macroeconomic analyses<sup>2</sup>. By contrast, in the field of (endogenous) growth theory, with a few exceptions<sup>3</sup>, most work is silent on the role of home production. This neglect is based on the conjecture that home activity is isolated to market ones. This conjecture is, however, groundless both empirically and intuitively. For example, the results in McGattan, Rogerson and Wright (1997) suggest that there is a significant elasticity of substitution between home and market goods. Intuitively, on the other hand, home activity should be relevant to growth process when home production needs investment from market sector. This is because the inclusion of home sector affects resource allocation between distinct production sectors, while this allocation is crucial to the growth process in the Uzawa-Lucas framework.

There are some developments in the literature that are related to ours if leisure can be interpreted as special form of home production. With constant returns to scale technologies, Ladron-de-Guevara, Ortigueira and Santos (1997,1999) analyze the equilibrium dynamics of a pure-leisure-time version of the Lucas model. They reveal that the inclusion of leisure leads to the possible existence of multiple balanced growth paths. Ortigueira (2000), on the other hand, shows that the Lucas model with human capital adjusted quality leisure displays well behaved dynamics in the sense that the balanced growth path is uniquely determined and it satisfies global saddle-path stability. These distinctive results show the sensitivity of dynamic properties to different specifications of home production activity.

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<sup>2</sup>The intuition behind this good fitness can be understand as follows. The encompass of home sector to the standard one-sector real business cycle model brings about possibility of substitution between market and nonmarket production over time. Therefore, relative productivity differentials between the two sectors can give arise volatility in market activity. Furthermore, the substitution between home and market commodities at a given date, not just at different dates, affects the size of fluctuations induced by productivity shocks also.

<sup>3</sup>In an endogenous growth framework, Einarsson and Marquis (2001) examine the consequences for an economy when moving to a less distortion tax regime. Parente, Rogerson and Wright (2000) assess the impact of distortion policies in a neoclassical growth model with home production.

Different from these existing contributions, we assume that home activity is, just like market production, to use machines and time to produce homemade goods. That is, it is assumed that households combine at-home time and home capital (which is constituted with consumer durable goods and residence) to produce non-tradable goods and services. This kind of specification of home sector can be justified by considering that households value their at-home time because of what they can do with it. Therefore valued leisure is not the residual time unoccupied by production. Instead, it is the output from a home production function, in which home time, home capital, and home technology appear just as market time, market capital and market technology do in the market production function. Some numerical examples may also help to justify our specification of home sector. Greenwood and Hercowitz (1991) point out that household activities involve approximately as much capital as market sector. Eisner's summary (1988) suggests an estimate of home-produced output relative to measured gross national production in the range of 20-50 percent in the US data<sup>4</sup>. The evidences are more striking in the developing economies. For examples, it is said more than 80 percent of income goes unreported in Argentina (Easterly, 1993).

Main findings of the papers are as follows. Under mild conditions, there exists a unique balanced growth path, which is locally saddle-point stable. In the long run, the growth rate of the present model economy is determined by the productivity of the education sector, as in the standard model. Comparative statics analysis illustrates also that an increase in the home capital share or the propensity towards market consumption goods do not affect the education time while can lead to less home work time hence more labor time in market. Furthermore, the market work time predicted the present model is shorter than the Lucas (1988) model, but longer than the case without home capital.

Transitional dynamics analysis is conducted in the case that a positive shock in physical capital occurs. We first derive the shape of the stable arm of the balanced

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<sup>4</sup>Two of the main items of the household product are service of household capital (e.g., owner-occupied housing) and unpaid household labor (e.g., housework, care of family and others).

growth path, then investigate the moving pattern of the endogenous variables during the transitional process. For example, a sudden increase in the physical capital (this can be caused by the coming of an international transfer) causes the home work time, the market good consumption, and the home good consumption to jump up at first and then decrease along the transitional process. The effects of this change on the market work time, the home capital share, and the education time are general ambiguous, for which the separating rules are given. At last, numerical computations present intuitive examples for the obtained results.

## 2 The model

We consider a closed economy with competitive markets where many identical, rational households dwell in. The number of population is fixed, and is normalized to one. Therefore, we can express the aggregate economy with a representative agent model in which the per-capita variables express the aggregate ones as well. Assume the representative agent derives utility from consumption goods acquired in the market,  $C_m(t)$ , and produced at home,  $C_n(t)$ . Its lifetime utility is

$$\int_0^{\infty} U(C(t))e^{-\rho t} dt, \quad \rho > 0, \quad (1)$$

where  $\rho$  is the rate of time preference,  $C(t)$  is a composite of market consumption and homemade goods. The relation between  $C(t)$ ,  $C_m(t)$  and  $C_n(t)$  is

$$C(t) = C_m(t)^\gamma C_n(t)^{1-\gamma}, \quad 0 \leq \gamma \leq 1. \quad (2)$$

that is, the intratemporal elasticity of substitution between market goods and home goods is one. We should point out that this specification of  $C(t)$  is by no mean the best

one, but it is helpful for illustrating the equilibrium paths explicitly.<sup>5</sup> The instantaneous utility  $U(C(t))$  is assumed to have the usual form of

$$\begin{aligned} U(C(t)) &= \frac{C(t)^{1-\sigma} - 1}{1-\sigma} && \text{for } \sigma > 0, \quad \text{and } \sigma \neq 1 \\ U(C(t)) &= \log C(t) && \text{for } \sigma = 1, \end{aligned} \tag{3}$$

where  $\sigma$  is the inverse of the intertemporal elasticity of substitution for consumption.

Notice that raw leisure time does not enter the utility function directly. Instead, non-market time affects utility only by being an input in the production of home goods. We emphasize that households value their leisure time only when they can do something useful with it. Therefore, valued leisure is not the residual time unoccupied by production. Accordingly, we follow Greenwood and Hercowitz (1991) in which households derive utility from market and home goods consumption alone.

Each agent in the economy is endowed with one unit of time per period which can be assigned to market work, home work and schooling. Time allocation fractions to these three activities are  $u(t)$ ,  $l(t)$  and  $1 - u(t) - l(t)$ , respectively. Following Lucas (1988), we omit physical capital input in education sector in view of the fact that education sector is a relative human capital intensive one. That is, physical capital is allocated between market and home sectors. Suppose  $s(t) \in [0, 1]$  and  $1 - s(t)$  are the corresponding fractions allocated to the market and home sectors, respectively.

The accumulation equation of human capital is

$$\dot{H}(t) = B[1 - u(t) - l(t)]H(t) - \eta H(t), \quad u(t), l(t) \in [0, 1], \tag{4}$$

where  $B$  is a positive technology parameter,  $\eta \in [0, 1)$  the constant depreciation rate of

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<sup>5</sup>For more general specification of  $C(t)$  in CES function form, see, for example, numerical experiments in Benhabib et al. (1991).

human capital. Suppose both market and home sectors own Cobb-Douglas technologies:<sup>6</sup>

$$\begin{aligned} Y_m(t) &= A[s(t)K(t)]^{\beta_1}[u(t)H(t)]^{1-\beta_1}, \quad 0 < \beta_1 < 1, \quad A > 0 \\ Y_n(t) &= [(1-s(t))K(t)]^{\beta_2}[l(t)H(t)]^{1-\beta_2}, \quad 0 < \beta_2 < 1 \end{aligned} \quad (5)$$

where  $Y_m(t)$ ,  $Y_n(t)$  are market and home outputs, respectively.

Note that home production equals home consumption period by period, that is,  $Y_n(t) = C_n(t)$ . If market capital,  $s(t)K(t)$ , and home capital,  $(1-s(t))K(t)$ , depreciate in the same rate of  $\delta \in [0, 1)$ , then the accumulation of aggregate physical capital  $K(t)$  of the economy is

$$\dot{K}(t) = Y(t) - C_m(t) - \delta K(t). \quad (6)$$

Since there is no distortion in the economy, we are considering a optimal growth model with an unbounded horizon. Hence, solution of the planner's problem coincides with the competitive equilibrium achieved in a decentralized way through competition among firms and optimizing behavior of the agents. Thus, without loss of generality, we focus on the analysis of the planner's problem in the following.

The optimization problem of the social planner is to maximize the representative agent's life utility in (1) under the resources constraints (4), (5) and (6) (problem  $(P)$  thereafter). The current-value Hamiltonian function is

$$\begin{aligned} &\mathcal{H}\{K(t), H(t), p_1(t), p_2(t), C_m(t), C_n(t), s(t), u(t), l(t)\} \\ &= \frac{1}{1-\sigma} [C_m(t)^{\gamma(1-\sigma)} C_n(t)^{(1-\gamma)(1-\sigma)} - 1] + p_1 [A(s(t)K(t))^{\beta_1} (u(t)H(t))^{1-\beta_1} \\ &\quad - C_m(t) - \delta K(t)] + p_2 [B(1-u(t)-l(t))H(t) - \eta H(t)], \end{aligned}$$

where  $p_1(t)$  and  $p_2(t)$  are co-state variables for  $K(t)$  and  $H(t)$ , respectively.

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<sup>6</sup>When specifying the home production as using physical capital and raw labor only as inputs, we can show that multiple balanced growth paths are possible (Hu, 2003).



If omit the time argument, by the maximum principle, we know the interior optimal solution satisfies

$$\gamma C_m^{\gamma(1-\sigma)-1} C_n^{(1-\gamma)(1-\sigma)} = p_1, \quad (7)$$

$$\beta_2(1-\gamma)C_m(t)^{\gamma(1-\sigma)} C_n^{(1-\gamma)(1-\sigma)} [(1-s)K]^{-1} = p_1 A \beta_1 k_m^{\beta_1-1}, \quad (8)$$

$$(1-\beta_2)(1-\gamma)C_m^{\gamma(1-\sigma)} C_n^{(1-\gamma)(1-\sigma)} (lH)^{-1} = p_2 B, \quad (9)$$

$$p_1 A(1-\beta_1)k_m^{\beta_1} = p_2 B, \quad (10)$$

$$\dot{p}_1 = p_1[\rho + \delta - A\beta_1 k_m^{\beta_1-1}], \quad (11)$$

$$\dot{p}_2 = p_2[\rho + \eta - B]. \quad (12)$$

with the feasibility conditions (4), (6), and boundary conditions:

$$K(0) = K_0, \quad H(0) = H_0, \quad (13)$$

$$\lim_{t \rightarrow \infty} p_1 K e^{-\rho t} = 0, \quad \lim_{t \rightarrow \infty} p_2 H e^{-\rho t} = 0, \quad (14)$$

where  $k_m(t) \equiv s(t)K(t)/u(t)H(t)$  is the capital-labor ratio in the market sector.

### 3 Steady state analysis

Denote  $k \equiv K/H$ ,  $p \equiv p_2/p_1$  as the aggregate capital-labor ratio and the price of human capital in units of market goods, respectively. From (7), (8) and (9), the per unit efficiency labor market good consumption  $C_m/H$ , home capital share  $1-s$ , and market work time ratio  $u$  can be expressed by  $(k, k_m, l)$  as

$$\frac{C_m}{H} = A \left( \frac{1-\beta_1}{1-\beta_2} \right) \left( \frac{\gamma}{1-\gamma} \right) k_m^{\beta_1} l, \quad (15)$$

$$1-s = \left( \frac{1-\beta_1}{\beta_1} \right) \left( \frac{\beta_2}{1-\beta_2} \right) \frac{k_m l}{k}. \quad (16)$$

$$u = \frac{k}{k_m} - \left( \frac{1 - \beta_1}{\beta_1} \right) \left( \frac{\beta_2}{1 - \beta_2} \right) l. \quad (17)$$

Rearranging (10)-(12), (4), (6) and by use of the above results, the dynamic equations of the market capital-labor ratio  $k_m$  and the aggregate capital-labor ratio  $k$  are

$$\frac{\dot{k}_m}{k_m} = Ak_m^{\beta_1-1} - \frac{1}{\beta_1}(B + \delta - \eta), \quad (18)$$

$$\begin{aligned} \frac{\dot{k}}{k} &= Ak_m^{\beta_1-1} - A \left( \frac{1 - \beta_1}{1 - \beta_2} \right) \left( \frac{\beta_2}{\beta_1} + \frac{\gamma}{1 - \gamma} \right) k_m^{\beta_1} \frac{l}{k} - (B + \delta - \eta) \\ &+ B \frac{k}{k_m} + Bl \left[ 1 - \left( \frac{1 - \beta_1}{\beta_1} \right) \left( \frac{\beta_2}{1 - \beta_2} \right) \right]. \end{aligned} \quad (19)$$

Working out  $\dot{l}/l$  needs some algebraic calculations. Taking *logs* in (7)-(9), and eliminating  $\log C_m$  and  $\log[(1 - s)K]$ ,  $\log(lH)$  can be expressed by  $\log k_m$ ,  $\log p_1$  and  $\log p_2$ . Then differentiating the two sides of this relation with respect to time  $t$ , we obtain the growth rate of  $l$

$$\begin{aligned} \frac{\dot{l}}{l} &= B \frac{k}{k_m} - \frac{\beta_2(1 - \gamma)(1 - \sigma)}{\sigma} \left( \frac{1 - \beta_1}{\beta_1} \right) \left( \frac{\beta_2}{1 - \beta_2} \right) (B + \delta - \eta) - \frac{\rho + (1 - \sigma)\delta}{\sigma} \\ &+ B \left[ 1 - \left( \frac{1 - \beta_1}{\beta_1} \right) \left( \frac{\beta_2}{1 - \beta_2} \right) \right] l + [\beta_2 + (\beta_1 - \beta_2)\gamma] Ak_m^{\beta_1-1} \end{aligned} \quad (20)$$

In summary, differential equations (18), (19) and (20) constitute a complete dynamic system with respect to capital/labor ratio in the market sector  $k_m$ , aggregate capital/labor ratio of the economy  $k$ , and working effort allocated to home sector  $l$ . By investigating this reduced-form dynamic system we can understand how these variables changing over time for any given initial conditions. Once we understand the behaviors of  $k_m$ ,  $k$ , and  $l$ , the dynamics of other controls, that is, working effort in the market sector  $u$ , fraction of physical capital invested to market  $s$ , and market goods consumption  $C_m$ , can be derived by inspecting the relations (15), (16) and (17), respectively.

### 3.1 Steady state equilibrium

Under the constant-returns-to-scale technology, it is straightforward that,  $K$ ,  $H$ ,  $C_m$  and hence the market output grow in a common rate along the steady growth path. Therefore,  $k_m$ ,  $k$  and  $l$  will have zero growth rate at the steady state. From (18), the steady state  $k_m^*$  satisfies

$$A\beta_1 k_m^{*\beta_1-1} = B + \delta - \eta. \quad (21)$$

Noticing that  $(1-s)K$ ,  $lH$  and  $C_m$  grow at the same rate at the steady state, from (7), (11) and (21), we derive this growth rate as

$$\nu^* = \frac{1}{\sigma}(B - \eta - \rho). \quad (22)$$

Notice that, the home sector does not contribute to the long-run growth of the economy. Like the standard model, it is the return to capital which determine the long-run growth rate of the economy.

In order for this economy to have permanent growth and to ensure interior solutions, the following conditions are needed:

$$B - \eta > \rho > (1 - \sigma)\nu^*. \quad (23)$$

It is worth noting that this last condition is a sufficient condition for the transversality conditions (14), therefore the above necessary conditions of an interior equilibrium, to be satisfied.

**Proposition 1.** *Given that (23) holds, there exists a unique (interior) steady growth path in the economy, which, in the case of  $\beta_1 \neq \beta_2$ <sup>7</sup>, is described as*

$$k_m^{*\beta_1-1} = \frac{1}{A\beta_1}(B + \delta - \eta),$$

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<sup>7</sup> For the case of  $\beta_1 = \beta_2$ , see Appendix 1.

$$\begin{aligned}
k^* &= \frac{\Delta_2}{\Delta_1 + \Delta_2} \left( \frac{B - \eta - \nu^*}{B} \right) k_m^*, \\
l^* &= \frac{\phi_2}{B(\phi_2 - \phi_1)} \left[ (B - \eta - \nu^*) - B \frac{k^*}{k_m^*} \right], \\
u^* &= \left( \frac{B - \eta - \nu^*}{B} \right) \left[ \frac{(\phi_2 - \phi_1)\Delta_2 - \phi_1\Delta_1}{(\phi_2 - \phi_1)(\Delta_1 + \Delta_2)} \right], \\
s^* &= \left( 1 + \frac{\phi_1}{\phi_2} \frac{l^*}{u^*} \right)^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 &\equiv (\beta_1^{-1} - 1)(B - \eta + \delta) + (B - \eta - \nu^*), \quad \phi_1 \equiv \frac{1 - \beta_1}{\beta_1}, \quad \phi_2 \equiv \frac{1 - \beta_2}{\beta_2} \\
\Delta_2 &\equiv \frac{(B + \delta - \eta)}{\beta_1} \left[ \frac{\phi_1}{\phi_2} + \left( \frac{1 - \beta_1}{1 - \beta_2} \right) \left( \frac{\gamma}{1 - \gamma} \right) \right] \left( \frac{\phi_2}{\phi_2 - \phi_1} \right),
\end{aligned} \tag{24}$$

and  $k_m^*, k^* > 0$ , and  $0 < s^*, u^*, l^* < 1$ .

*Proof.* See Appendix 2.

Note that the home work time,  $l^*$ , will be zero if  $\gamma = 1$ , that is, the agent has no interest in homemade goods. In this case, the three-sector model degenerates to the usual Lucas model in which no home sector is considered. Therefore, to guarantee the interiority of the equilibrium, (since it is empirically plausible as aforementioned), we focus on the case of  $0 < \gamma < 1$ .

The proof of the above proposition gives out the following side results that will be used later.

**Lemma 1.**  $\Delta_1 > 0$ ,  $\Delta_2 = \text{sign}\{\beta_1 - \beta_2\}$ , and  $\Delta_1 + \Delta_2 = \text{sign}\{\beta_1 - \beta_2\}$ .

### 3.2 Stability of the balanced-growth path

Linearizing the dynamic system(18), (19) and (20) around the steady-state  $(k_m^*, k^*, l^*)$ ,

we obtain:

$$\begin{pmatrix} \dot{k}_m \\ \dot{k} \\ \dot{l} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} k_m - k_m^* \\ k - k^* \\ l - l^* \end{pmatrix},$$

where

$$\begin{aligned} a_{11} &= A(\beta_1 - 1)k_m^{*\beta_1-1}, \\ a_{21} &= -A(1 - \beta_1)k_m^{*\beta_1-2}k^* - B\frac{k^{*2}}{k_m^{*2}} \\ &\quad - A\beta_1k_m^{*\beta_1-1}\left[\frac{\phi_1}{\phi_2} + \left(\frac{1 - \beta_1}{1 - \beta_2}\right)\left(\frac{\gamma}{1 - \gamma}\right)\right]l^* \\ a_{22} &= Ak_m^{*\beta_1-1}\left[\frac{\phi_1}{\phi_2} + \left(\frac{1 - \beta_1}{1 - \beta_2}\right)\left(\frac{\gamma}{1 - \gamma}\right)\right]\frac{k_m^*l^*}{k^*} + B\frac{k^*}{k_m^*} \\ a_{23} &= \left(\frac{\phi_2 - \phi_1}{\phi_2}\right)\left[B - \Delta_2\frac{k_m^*}{k^*}\right]k^* \\ a_{31} &= -B\frac{k^*l^*}{k_m^{*2}} - (1 - \beta_1)\left(\frac{1 - \sigma}{\sigma}\right)[\beta_1\gamma + (1 - \gamma)\beta_2]Ak_m^{*\beta_1-2}l^* \\ a_{32} &= B\frac{l^*}{k_m^*}, \quad a_{33} = B\left(\frac{\phi_2 - \phi_1}{\phi_2}\right)l^*. \end{aligned}$$

Notice that, there are two jumpable variables,  $k_m$  and  $l$ , and a state one,  $k$ , in the above system. If denote the coefficient matrix as  $J^*$ , to conclude a locally unique saddle-path stable equilibrium result, we need to show the characteristic equation of  $J^*$  has one stable and two unstable roots. Then by examining the elements of  $J^*$ :  $a_{ij}$  ( $i, j = 1, 2, 3$ ), the saddle-path stability result in Proposition 2 can be obtained. Before that, we present Lemma 2 firstly for future use.

**Lemma 2.**  $a_{11} < 0$ ,  $a_{21} < 0$ ,  $a_{22} > 0$ ,  $a_{23} < 0$ ,  $a_{32} > 0$ , and  $a_{33} = \text{sign}\{\beta_1 - \beta_2\}$ .

*Proof.* The above results can be obtained immediately when noticing the ranges which related variables and parameters lie in except  $a_{23}$ . In fact, when  $\beta_1 < \beta_2$ , by Lemma 1,

$a_{23} < 0$  can be readily obtained. In the case of  $\beta_1 > \beta_2$ , by transforming  $a_{23}$  as

$$a_{23} = B \left( \frac{\phi_2 - \phi_1}{\phi_2} \right) \frac{(1 - \frac{1}{\beta_1})(B - \eta + \delta) - \Delta_2}{B - \eta - \nu^*},$$

the result can be achieved easily as well.  $\square$

**Proposition 2.** *Under conditions (23), the unique steady-state,  $(k_m^*, k^*, l^*)$ , is locally saddle-path stable.*

*Proof.* See Appendix-3.

We, therefore, have shown that there does exist a saddlepath stable steady growth path in the home production model, which is a desire property from the perspective of economy reality. That is, the introduction of a new home production sector, making the model goes nearer to the real economy, risks nothing of losing any plausibility comparing with the standard Uzawa-Lucas framework. The saddle-path stability of the stead state equilibrium remains as in the original Uzawa-Lucas model.

### 3.3 Comparative statics

Based on the previous steady-state analysis, we conduct several exercises of comparative statics in this subsection. The main findings are as follows.

(I) *An increase in productivity of the education sector raises the balanced growth rate of the economy.*

This is a straightforward result when noticing of the expression of  $\nu^*$ . Since an increase in productivity of education sector raises the wage rate per efficiency unit of labor, and a higher wage rate leads to a larger amount of labor input into education sector. Hence, in the long run, a higher balanced growth rate,  $\nu^*$ , can be realized.

(II) *An increase in home capital share in home production technology, or increase of propensity to market consumption goods lowers the time of household work.*

These are obtained by calculating  $\partial l^*/\partial \beta_2$  and  $\partial l^*/\partial \gamma$ , respectively. The implications behind these results can be understood as follows. An increase in  $\beta_2$  means a larger level of home capital is employed in home production, which substitutes home work time. On the other hand, a rise in  $\gamma$  (expenditure share of market goods) shifts consumption spending from home goods to market goods. Both of these two forces lead to more labor input to the market sector, hence a higher rate of market participation of female labors, a coincident result with the observation.

*(III) In comparison with the existing work, the present model predicts a lower market work time than Lucas (1988) model, while a higher one than the setting of pure-leisure home production. However the education time is the same.*

Our model includes Lucas (1988) as a special case, i.e., the case of  $\gamma = 1$ . On the other hand, if  $\beta_2 = 0$ , we obtain a pure-leisure time version of the home production economy. To see the relation between those different settings, we can calculate the related derivatives respectively. The above statements are based on the calculations. Similarly, other comparative statics exercises can also be conducted, e.g., effect on wage rate, labor supply, or effect on welfare.

## 4 Transitional dynamics

In the previous section we have shown that under mild conditions a unique steady state exists, which is locally saddlepoint stable. In this section, we suppose the saddlepoint stability is assured, while concentrate on investigating the transitional dynamics. Following the experiment conducted by Caballe and Santos (1993) among others, we consider the consequences of a sudden positive shock in physical capital to an economy which is initially in the steady growth path.

#### 4.1 Stable arm of the balanced growth path

Recall dynamic system (31), there two jumpable and one state variables involve in. Saddle-path stability of the system ensures it has only one stable eigenvalue. In our case, this stable root is  $\lambda \equiv a_{11}$ . Therefore the generic form of the stable solution is:

$$\begin{aligned} k_m(t) - k_m^* &= x_1 e^{\lambda t} \\ k(t) - k^* &= x_2 e^{\lambda t} \\ l(t) - l^* &= x_3 e^{\lambda t} \end{aligned} \tag{25}$$

where  $x_1, x_2, x_3$  represent the elements of eigenvector corresponding to  $a_{11}$ . Hence

$$\begin{cases} a_{21}x_1 + (a_{22} - a_{11})x_2 + a_{23}x_3 = 0 \\ a_{31}x_1 + a_{32}x_2 + (a_{33} - a_{11})x_3 = 0 \end{cases}$$

from the definition of eigenvalue. To see the relation between  $k_m(t), k(t)$  and  $l(t)$ , we only need to know the signs of  $x_i$  ( $i = 1, 2, 3$ ). For example, to see the relation between  $k(t)$  and  $l(t)$ , eliminating  $x_1$  from the above system, which gives the following relation:

$$[a_{31}(a_{22} - a_{11}) - a_{21}a_{32}]x_2 + [a_{31}a_{23} - a_{21}(a_{33} - a_{11})]x_3 = 0.$$

Similarly, if the relation between  $k_m$  and  $k$  is of interest, eliminate  $x_3$  from the above simultaneous equations and get

$$[a_{21}(a_{33} - a_{11}) - a_{31}a_{23}]x_1 + [(a_{22} - a_{11})(a_{33} - a_{11}) - a_{32}a_{23}]x_2 = 0.$$

Defining

$$\begin{aligned} b_3 &\equiv a_{21}(a_{33} - a_{11}) - a_{31}a_{23}, \\ b_2 &\equiv a_{31}(a_{22} - a_{11}) - a_{21}a_{32}, \end{aligned}$$



$$b_1 \equiv a_{32}a_{23} - (a_{22} - a_{11})(a_{33} - a_{11}),$$

then the above two equations can be simplified as

$$b_2x_2 = b_3x_3, \quad b_1x_2 = b_3x_1.$$

For showing the transitional process clearly, we confine to the case of  $\sigma = 1$ . In addition, we assume  $\beta_1 > \beta_2$ , the case that we think is of evidently plausible. Under these specifications, definitely

$$b_1 < 0, \quad b_3 < 0.$$

These can be derived directly by inspecting the results in Lemma 2. In addition, we have:

**Lemma 3.** *In the case of log utility,  $b_2 < 0$ .*

*Proof.* See Appendix-4.

Therefore, we conclude that the stable arm is positive sloped, since

$$x_1 > 0, \quad x_2 > 0, \quad x_3 > 0.$$

## 4.2 Behavior of the endogenous variables

Based on the shape of the stable arm, the following moving pattern of  $k_m$ ,  $k$ , and  $l$  can be obtained.

**Aggregate capital/labor ratio  $k$ , capital/labor ratio in the market sector  $k_m$ , home work time  $l$ .** A sudden increase in physical capital  $K$ , causes  $k$ ,  $k_m$ , and  $l$  to jump up at first, and decrease along the transitional process then.

Starting from the steady state  $(k_m^*, k^*, l^*)$ , a positive shock in  $K$  means a jump up in  $k$ , since human capital cannot change instantly. When  $k > k^*$ , we have  $k_m > k_m^*$  and  $l > l^*$  in view of the shape of the stable arm. That is, the shock results jumping

up in  $k_m$  and  $l$  firstly, but after that all three variables move back to  $(k_m^*, k^*, l^*)$  with  $\dot{k} < 0$ ,  $\dot{k}_m < 0$ , and  $\dot{l} < 0$ . Intuition behind these dynamics can be described as follows: an increase in  $K$ , leads to the relative price of  $H$  in units of  $K$ ,  $p$ , to rise up, while the relation in (10) tells us this rising in  $p$  corresponds to an increase of  $k_m$ . The jumping up of  $l$  can also be understood as a consequence of rising in  $p$ , since comparing with expensive education spending, working at home is a more preferable choice.

To induce the moving pattern of other variables, we rely on the optimal conditions again. In the case of  $\sigma = 1$ , we have the following first-order conditions, which are the special expressions of (7)-(12):<sup>8</sup>

$$\frac{\gamma}{C_m} = p_1, \quad (26)$$

$$\frac{1}{1-s}\beta_2(1-\gamma) = p_1 A \beta_1 K k_m^{\beta_1-1}, \quad (27)$$

$$p_1 A (1 - \beta_1) k_m^{\beta_1} = p_2 B, \quad (28)$$

$$\frac{1}{l}(1-\gamma)(1-\beta_2) = p_2 B H, \quad (29)$$

$$\dot{p}_1 = p_1(\rho + \nu - A \beta_1 k_m^{\beta_1-1}), \quad (30)$$

$$\dot{p}_2 = p_2(\rho + \nu - B), \quad (31)$$

and the resource constraint conditions:

$$\frac{\dot{K}}{K} = A m k_m^{\beta_1-1} - \frac{C_m}{K} - \nu, \quad (32)$$

$$\frac{\dot{H}}{H} = B(1 - u - l) - \nu. \quad (33)$$

Here, we have set the depreciation rate  $\delta = \eta = 0$ , for simplicity.

**Market good consumption  $C_m$ , home-good consumption  $C_n$ .** A sudden increase in  $K$ , causes  $C_m$  to jump up firstly, and decreases along the transition path

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<sup>8</sup>To abuse the notations, from now on, we assume the variables have been reformulated, by scaling down their steady growth rates, so that all variables are costants along the balanced growth path.

thereafter. The homemade good consumption  $C_n$  also experiences a similar positive jump when the shock occurs.

Since during the transition process,  $k_m > k_m^*$ , then  $\dot{p}_1 > 0$  from (30), while this needs  $p_1$  to jump down at the first place. In spirit of (26), the jump down of  $p_1$  corresponds to a jump up in  $C_m$  and then  $\dot{C}_m < 0$  before arriving the BGP. The result on  $C_n$  can be derived by examining (27), which reveals a jump up in  $(1-s)K$ . Since both  $(1-s)K$  and  $l$  jump up, we can conclude  $C_n$  must jump up as well.

Denote  $V(K, H)$ , the value function of the maximization problem (P). That is

$$V(K_0, H_0) = \max_{c, l, u, s} \int_0^\infty \log C(t) e^{-\rho t} dt,$$

subject to resources constraints, (4), (5) and (6), and initial condition, (13). By the property of value function,<sup>9</sup> we have  $\partial V / \partial K = p_1$  and  $\partial V / \partial H = p_2$ . And notice also  $V(K, H)$  is homogenous of degree 0 in the case of  $\sigma = 1$ , hence

$$\frac{\partial p_1}{\partial K} \frac{K}{p_1} + \frac{\partial p_2}{\partial K} \frac{H}{p_1} = -1.$$

Taking log-derivatives in (27)-(28), after some algebra, we get

$$\frac{K}{u} \frac{\partial u}{\partial K} = \frac{1}{s\beta_1} [\beta_1 - 1 - (\frac{p}{k} + 1 - \beta_1 + s\beta_1) \frac{K}{p_2} \frac{\partial p_2}{\partial K}], \quad (34)$$

$$\frac{K}{s} \frac{\partial s}{\partial K} = -\frac{(1-s)}{s\beta_1} [(1-\beta_1) + (\frac{p}{k} + 1 - \beta_1) \frac{K}{p_2} \frac{\partial p_2}{\partial K}], \quad (35)$$

$$\frac{K}{l} \frac{\partial l}{\partial K} = -\frac{K}{p_2} \frac{\partial p_2}{\partial K}. \quad (36)$$

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<sup>9</sup>For general properties of value function, see the original work of Benveniste and Scheinkman (1982). For application of it, see Caballe and Santos (1993), or Ladron-de-Guevara et al. (1997), for example.

$$\begin{aligned} \frac{K}{(u+l)} \frac{\partial(u+l)}{\partial K} &= \frac{u}{u+l} \frac{\beta_1 - 1}{s\beta_1} \\ &- \frac{1}{u+l} \left[ \frac{u}{s\beta_1} \left( \frac{p}{k} + 1 - \beta_1 + s\beta_1 \right) + l \right] \frac{K}{p_2} \frac{\partial p_2}{\partial K}. \end{aligned} \quad (37)$$

Notice that, around the BGP, approximately, we have  $\partial l / \partial k = b_2 / b_3$ , where

$$\begin{aligned} b_2 &= B^2 \psi \frac{l^*}{k_m^*} l^* (-\phi_1), \\ b_3 &= B^2 \left( \frac{k^*}{k_m^*} \right)^2 \left[ (l^* - 1) \left( \frac{\phi_2 - \phi_1}{\phi_2} \right) + B\phi_1 \right] - B\psi l^* \left( \frac{\phi_2 - \phi_1}{\phi_2} + B\phi_1 \right) \\ &+ B \frac{k^*}{k_m^*} \left[ (\Delta_2 l^* - \phi_1) \left( \frac{\phi_2 - \phi_1}{\phi_2} \right) - B\phi_1^2 \right], \end{aligned}$$

which can be expressed by parameters only, through the following relations:

$$\begin{aligned} k_m^* &= \left( \frac{B}{A\beta_1} \right)^{\frac{1}{\beta_1-1}}, \quad k^* = \frac{\Delta_2}{\Delta_1 + \Delta_2} \frac{\rho}{B} k_m^*, \quad l^* = \frac{\rho}{B} \frac{\phi_2}{\phi_2 - \phi_1} \frac{\Delta_1}{\Delta_1 + \Delta_2}, \\ \Delta_1 &= \rho + B\phi_1, \quad \Delta_2 = \frac{B}{\beta_1} \left( \frac{\phi_2}{\phi_2 - \phi_1} \right) \psi, \quad \psi = \frac{\phi_1}{\phi_2} + \left( \frac{1 - \beta_1}{1 - \beta_2} \right) \left( \frac{\gamma}{1 - \gamma} \right). \end{aligned}$$

On the other hand, suppose  $H^* = 1$ , since  $H$  is unjumpable variable, we have

$$\frac{K^*}{l^*} \frac{\partial l}{\partial K} = \frac{k^*}{l^*} \frac{\partial l}{\partial k} = \frac{k^*}{l^*} \frac{b_2}{b_3} \equiv \varepsilon.$$

Therefore,  $\varepsilon$  can be expressed by parameters as  $\varepsilon = (\varepsilon' - 1)^{-1}$ , where

$$\begin{aligned} \varepsilon' &= \frac{\Delta_1}{B\psi\phi_1} \left[ \frac{\phi_1}{l^*} + \psi \frac{k_m^*}{k^*} \right] \\ &= \frac{(\phi_2 - \phi_1)(\Delta_1 + \Delta_2)}{\psi\rho\phi_2} + \frac{\Delta_1(\Delta_1 + \Delta_2)}{\rho\phi_1\Delta_2}. \end{aligned}$$

Then we have the followings:

**Market work time  $u$ :** After a positive shock of  $K$ , the transitional property of  $u$

is ambiguous. Specifically,

$$\frac{\partial u}{\partial k} < (=, >) 0 \Leftrightarrow \varepsilon < (=, >) \frac{1 - \beta_1}{p^*/k^* + s^*\beta_1 + (1 - \beta_1)} \equiv \Omega_u.$$

This can be readily deducted by inspecting (34). Similar work can be conducted, by examining (35) and (37), to get the following result:

**Market sector capital share  $s$ :**

$$\frac{\partial s}{\partial k} < (=, >) 0 \Leftrightarrow \varepsilon < (=, >) \frac{1 - \beta_1}{p^*/k^* + (1 - \beta_1)} \equiv \Omega_s.$$

**Education time  $1 - u - l$ :**

$$\frac{\partial(u + l)}{\partial k} < (=, >) 0 \Leftrightarrow \varepsilon < (=, >) \Omega,$$

where

$$\Omega = \frac{1 - \beta_1}{p^*/k^* + 1 - \beta_1 + s^*\beta_1 + s^*\beta_1 l^*/u^*}.$$

## 5 Numerical computations

To show the achieved results concretely, we present some numerical examples in this section. In accordance with Ortigueira (2000), we set  $A = 3$ ,  $B = 0.07$ ,  $\beta_1 = 0.4$ ,  $\rho = 0.05$ , and  $\gamma = 0.45$ . Under the specification of  $\sigma = 1$  and  $\eta = \delta = 0$ , for different values of  $\beta_2$ , we get corresponding steady-state values in Table 1.

Comparing these results with the ones of Ortigueira (2000)'s, we find that, depending on the scale of  $\beta_2$ , we can get larger or smaller steady state values than his. However, when take  $\beta_2$  in the range of  $0.05 - 0.3$ , suggested by Greenwood and Hercowitz (1991) and Parente et.al. (2000), we notice that, when home capital is included, a longer marketwork time, shorter homework time and about the same education time result is obtained. The less of  $\beta_2$ , the smaller are the difference. This means the present result

Table 1: Steady state values

$\beta_2$	$k^*$	$l^*$	$u^*$	$1 - s^*$
0.30	65.90	0.400	0.330	0.450
0.20	52.10	0.410	0.300	0.336
0.10	40.00	0.437	0.277	0.208
0.05	34.19	0.450	0.264	0.118
0.01	30.12	0.457	0.257	0.026

coincides with Ortigueira (2000)'s, if home capital takes a trivial share. Compare with the result without home capital considered, the present paper reaches a nearer result to the empirical work (See, for example, Greenwood et al., 1995).

## 6 Concluding remarks

The present paper showed that introducing a home sector to the standard two-sector model leads to new findings. In the long run, the present home production model gives insights in several observations, e.g., the increasing trend of the female labor-market participation. In the short run, for an endowment shock, we show how the economy move along the transitional path. Although the saddle-path stability of the balanced growth equilibrium is still available, different transitional dynamics results occur. For the future work, two issues stand out. The one is to consider the dynamic properties of this home production model when some distortions, e.g. taxation, are introduced. The other is to explore the open economy version of the present setting.

## Appendices

### Appendix-1: The steady-state results when $\beta_1 = \beta_2$ (for referee, not for publication)

If the relations in (23) are assumed, then there is a unique interior steady-state equilibrium in the economy with the steady-state values:

$$\begin{aligned} k_m^{*\beta_1-1} &= \frac{1}{A\beta_1}(B + \delta - \eta), \\ k^* &= \frac{k_m^*}{B}(B - \eta - \nu^*), \\ l^* &= \frac{(1 - \gamma)(B - \eta - \nu^*)(Ak_m^{*\beta_1-1} - \nu^* - B)}{ABk_m^{*\beta_1-1}}, \\ 1 - s^* &= \left(\frac{1 - \beta_1}{\beta_1}\right)\left(\frac{\beta_2}{1 - \beta_2}\right)\frac{k_m^* l^*}{k^*} = \frac{k_m^* l^*}{k^*}, \end{aligned}$$

where  $k_m^*, k^* > 0$ ,  $0 < s^*, u^*, l^* < 1$ .

*Proof.* Eq. (21) tells us that

$$k_m^{*\beta_1-1} = \frac{1}{A\beta_1}(B + \delta - \eta).$$

When (23) is assumed, we can get a unique steady state  $k_m^* > 0$  from the above relation.

Letting  $\dot{l}/l = 0$  yield

$$B \frac{k^*}{k_m^*} = (B - \eta) - \left(\frac{B - \eta - \rho}{\sigma}\right)$$

or

$$k^* = \frac{k_m^*}{B}[B - \eta - \nu^*]$$

by Eq. (22). In order to confirm the existence of  $k^*$ , we need to prove

$$B - \eta - \nu^* > 0. \tag{38}$$

This relation is trivial for the case of  $\sigma > 1$  when notice that  $B - \eta = \rho + \sigma\nu^*$ .

When  $0 < \sigma < 1$ , we have

$$B - \eta = \rho + \sigma\nu^* > (1 - \sigma)\nu^* + \sigma\nu^* = \nu^*$$

by (23). That is, (38) is true always.

Next, from  $\dot{k}/k = 0$ , we get

$$l^* = \frac{(1 - \gamma)(B - \eta - \nu^*)(Ak_m^{\beta_1-1} - \nu^* - \delta)}{ABk_m^{\beta_1-1}}.$$

To prove  $l^* > 0$ , we need to show

$$Ak_m^{\beta_1-1} > \nu^* + \delta. \quad (39)$$

From (23),

$$(\frac{\sigma}{\beta_1} - 1)\nu^* + \frac{\rho}{\beta_1} + (\frac{1}{\beta_1} - 1)\delta > 0$$

since  $(\frac{1}{\beta_1} - 1)\delta \geq 0$ . Note that  $\sigma\nu^* + \rho = B - \eta$  from (22), then rearranging the above equation, we get

$$\frac{B - \eta + \delta}{\beta_1} > \nu^* + \delta,$$

which means (39) is true. Combining (38) with (39), we can say there exist a  $l^* > 0$  which satisfies the steady growth conditions:  $\dot{l}/l = \dot{k}/k = \dot{k}_m/k_m = 0$ .

From

$$u^* = \frac{k^*}{k_m^*} - (\frac{1 - \beta_1}{\beta_1})(\frac{\beta_2}{1 - \beta_2})l^*.$$

In the case of  $\beta_1 = \beta_2$ , it takes the form of

$$u^* = \left(\frac{B - \eta - \nu^*}{B}\right) \left[\frac{\nu^* + \delta + \gamma(Ak_m^{\beta_1-1} - \nu^* - \delta)}{Ak_m^{\beta_1-1}}\right].$$



(38) and (39) imply  $u^* > 0$ . On the other hand, since the relation,  $u = sk/k_m$  is true always, then  $u^*, k^*, k_m^* > 0$  imply  $s^* > 0$  as well. Notice also that

$$1 - s^* = \left( \frac{1 - \beta_1}{\beta_1} \right) \left( \frac{\beta_2}{1 - \beta_2} \right) k_m^* l^* \frac{1}{k^*} > 0,$$

thus  $0 < s^* < 1$ .

From (4), we have

$$\nu^* + \eta = B(1 - u^* - l^*).$$

Under the endogenous growth framework,  $\nu^* > 0$ , thus  $1 - u^* - l^* > 0$  or  $u^* + l^* < 1$ . Since we have had  $u^*, l^* > 0$ , that is,  $u^*, l^* \in (0, 1)$ .

## Appendix-2: Proof of Proposition 1.

Except for some additional tedious calculations, the logic in this proof is similar to the one in Proposition 1. Firstly, let  $\dot{l}/l = 0$ , to derive the following:

$$B \frac{k^*}{k_m^*} + B l^* \left( 1 - \frac{\phi_1}{\phi_2} \right) = (B - \eta) - \nu^*, \quad (40)$$

thus

$$l^* = \frac{1}{B} \left( \frac{\phi_2}{\phi_2 - \phi_1} \right) \left[ (B - \eta - \nu^*) - B \frac{k^*}{k_m^*} \right]. \quad (41)$$

Then, from  $\dot{k}/k = 0$  yield:

$$\frac{k^*}{k_m^*} = \frac{\Delta_2}{\Delta_1 + \Delta_2} \left( \frac{B - \eta - \nu^*}{B} \right), \quad (42)$$

where  $\Delta_1$  and  $\Delta_2$  are expressed in (24). Note that,  $\Delta_1 > 0$  always by (38) and (23).

*Case 1.*  $\beta_1 > \beta_2$ , that is  $\phi_2 > \phi_1$ .

In this case,  $\Delta_2 > 0$ . Then we have  $k^*/k_m^* > 0$  by  $B - \eta - \nu^* > 0$ .  $\Delta_1 > 0$  and  $\Delta_2 > 0$

imply

$$B - \eta - \nu^* > \frac{\Delta_2}{\Delta_1 + \Delta_2}(B - \eta - \nu^*),$$

which means  $B - \eta - \nu^* > Bk^*/k_m^*$ . Therefore,  $l^* > 0$ . Substitute  $l^*$  and  $k^*/k_m^*$  into  $u^* = k^*/k_m^* - l^*\phi_1/\phi_2$ , and rearrange the terms, we get

$$u^* = \left(\frac{B - \eta - \nu^*}{B}\right) \left[\frac{(\phi_2 - \phi_1)\Delta_2 - \phi_1\Delta_1}{(\phi_2 - \phi_1)(\Delta_1 + \Delta_2)}\right].$$

Based on an obvious fact:

$$Ak_m^{*\beta_1-1} \left(\frac{1-\beta_1}{1-\beta_2}\right) \left(\frac{\gamma}{1-\gamma}\right) > -(\nu^* + \delta) \frac{\phi_1}{\phi_2},$$

we can show  $\Delta_2(\phi_2 - \phi_1) > \Delta_1\phi_1$ , which implies  $u^* > 0$ . Similarly with the proof of Proposition 1, we can then show  $s^*, u^*, l^* \in (0, 1)$ .

*Case 2.*  $\beta_1 < \beta_2$ , that is  $\phi_2 < \phi_1$ .

In this case, we have  $\Delta_2 < 0$ . Rearranging the term  $\Delta_1 + \Delta_2$ , the following can be achieved:

$$\Delta_1 + \Delta_2 = \left(\frac{\phi_1}{\phi_2 - \phi_1}\right) \frac{(B - \eta + \pi)}{\beta_2(1 - \gamma)} + (B - \eta - \nu^*).$$

In order for  $k^*/k_m^* > 0$ ,  $\Delta_1 + \Delta_2 < 0$  must hold. Since

$$(1 - \beta_1)(\delta + \nu^*) > (B - \eta - \nu^*)[(\beta_2 - 1) - \gamma(\beta_2 - \beta_1)].$$

that is

$$\left(\frac{\phi_1}{\phi_2 - \phi_1}\right) \frac{(B - \eta + \delta)}{\beta_2(1 - \gamma)} > B - \eta - \nu^*.$$

If  $\beta_1 < \beta_2$ , the above relation means  $\Delta_1 + \Delta_2 < 0$ .  $l^* > 0$  can be easily checked by noticing that

$$\frac{\Delta_2}{\Delta_1 + \Delta_2} \cdot (B - \eta - \nu^*) > B - \eta - \nu^*.$$

Since  $\Delta_1 > 0$ ,  $\Delta_2 < 0$ , then  $B - \eta - \nu^* < Bk^*/k_m^*$  by Eq. (42). Thus

$$l^* = \frac{1}{B} \left( \frac{\phi_2}{\phi_2 - \phi_1} \right) \left[ (B - \eta - \nu^*) - B \frac{k^*}{k_m^*} \right] > 0.$$

Since  $\Delta_2(\phi_2 - \phi_1) > \Delta_1\phi_1$  always, no matter  $\beta_1 > \beta_2$  or not,  $u^* > 0$  is true universally. Similarly as before, we can conclude that there is still a unique steady value  $(s^*, l^*, u^*) \in (0, 1) \times (0, 1) \times (0, 1)$  under the preconditions (23).

### Appendix-3. Proof of Proposition 2

Since  $a_{11} < 0$ , the adjustment process of  $k_m(t)$  is stable always in the vicinity of the steady-state. Therefore a stable eigenvalue  $\lambda \equiv a_{11}$  is obtained already. Given  $k_m(t)$  is stable, to investigate the stability property of dynamic system constituted by  $(k_m, k, l)$ , we only need to consider the following sub-matrix of  $J^*$ :

$$J_2^* \equiv \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

Based on Lemma 1, it is clearly that the steady-state is saddle-path stable by noticing that  $Trace(J_2^*) > 0$  and  $Det(J_2^*) > 0$  when  $\beta_1 \geq \beta_2$ . In the case of  $\beta_1 < \beta_2$ , substituting  $l^*$  with the relation in Eq. (??), we can rewrite  $a_{22}$  as  $a_{22} = \Delta_1 + Bk^*/k_m^*$ . Therefore

$$\begin{aligned} Trace(J_2^*) &= a_{22} + a_{33} \\ &= \Delta_1 + B \frac{k^*}{k_m^*} + Bl^* \left( \frac{\phi_2 - \phi_1}{\phi_2} \right) = \Delta_1 + (B - \eta - \gamma). \end{aligned}$$

Where the last equality is derived from (40). Hence  $Trace(J_2^*) > 0$  due to  $\Delta_1 > 0$  and (38). On the other hand,  $Det(J_2^*)$  can be transformed as

$$Det(J_2^*) = k^* l^* B \left( \frac{\phi_2 - \phi_1}{\phi_2} \right) \left( \frac{\Delta_1 + \Delta_2}{k^*} \right).$$

In the case of  $\beta_1 > \beta_2$ , i.e.  $\phi_2 > \phi_1$ , we have  $\Delta_1 > 0$ ,  $\Delta_2 > 0$  (Lemma 1). Thus  $\text{Det}(J_2^*) > 0$ . In the case of  $\beta_1 < \beta_2$ , i.e.  $\phi_2 < \phi_1$ , we have  $\Delta_1 > 0$  and  $\Delta_1 + \Delta_2 < 0$  (Lemma 1). So that  $\text{Det}(J_2^*) > 0$ . This completes the proof.

#### **Appendix-4. Proof of Lemma 3 (for referee, not for publication)**

$$b_2 = a_{31}(a_{22} - a_{11}) - a_{21}a_{32} = B^2\psi \frac{l^*}{k_m^*} l^*(-\phi_1) < 0,$$

where

$$\psi = \frac{\phi_1}{\phi_2} + \left( \frac{1 - \beta_1}{1 - \beta_2} \right) \left( \frac{\gamma}{1 - \gamma} \right) > 0.$$

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