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The optimal and acceptable sizes of social security under uncertainty

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Abstract

This paper investigates a choice of social security plans under uncertainty with the analogy of portfolio allocation among assets that yield different and uncertain rates of return. It is shown that even if wage income growth is lower than the interest rate, there can be a case in which a PAYG plan is acceptable and more desirable than a funded plan. Also, our numerical simulations illustratively derive the optimal and acceptable sizes of a PAYG plan, and find that the results depend heavily on the mean-variance structure of interest rate and wage income growth, as well as the degree of people's risk aversion.

Key words: social security, defined-contribution plan, defined-benefit plan

JEL classification: H55, H21

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1. Introduction

Population aging is creating major problems for public pension plans. Under an unfunded (PAYG: pay-as-you-go) plan, either tax rates must be raised or the replacement rate must be reduced when the ratio of the elderly to the young rises. Hence, some researchers have been proposing a shift to a funded plan, which is considered to be less exposed to demographic pressures. However, no social security plan can be free from uncertainty and risk. For example, a plan with defined contributions (DC)—whether funded or PAYG—exposes pension beneficiaries to uncertainty about future benefits. Also, a PAYG plan with defined benefits (DB) makes workers face uncertainty about taxes to finance already committed benefits.

In fact, there is a body of literature that incorporates uncertainty into social security. For example, demographic uncertainty and risk sharing among different generations via PAYG plans are addressed by Brandts and de Bartolome (1992), Bohn (2001), and Sánchez-Marcos and Sánchez-Martín (2006) for example. Also, uncertainty about portfolio returns is one of the central issues of investment-based social security, as explicitly discussed by Feldstein and Rangelova (2001) and Feldstein, Rangelova, and Samwick (2001). The optimal shape of social security must depend on assumptions about uncertainty, as well as the degree of people's risk aversion.

In this paper, we discuss a choice of social security plans with the analogy of portfolio allocation under uncertainty over rates of return. It is reasonable to expect that there is an optimal structure of social security, as well an acceptable range. We attempt to compare DC funded, DC PAYG, and DB PAYG plans based on a simplified lifetime model under uncertainty, and to illustrate the optimal and acceptable sizes of a PAYG plan based on numerical simulations.

2. Basic model

Consider a simple two-period life-cycle model, in which an individual maximizes expected lifetime utility:

$$EU = u(c_1) + E[u(c_2)] / (1 + \rho), \quad (1)$$

where $u(c)$ is a utility function ($u'(c) > 0$ and $u''(c) < 0$), c_1 and c_2 are consumption when young and retired, respectively, and ρ is the discount rate. An individual earns wage income and pays social security tax when young, and receives social security benefits when retired and leaves no bequest. An individual does not know his wage income when he sets up his lifetime consumption plan, but we normalize his wage income as one, and incorporate the uncertainty about his wage income into the uncertainty about wage growth over the period between current and future generations. If social security tax is wage-proportional and denoted by t , a DC PAYG benefit is given by $(1+g)t$, where g is the growth rate of total wage income; that is, the sum of the growth rate of population and the growth rate of per-capita wage.

Then, an individual's lifetime budget constraint is given by

$$c_2 = (1+r)(1-c_1-t) + (1+g)t, \quad (2)$$

where r is the interest rate. The tax rate, t , can be interpreted as the replacement rate (relative to wage income of the current worker), and the plan of $t=0$ corresponds to a DC funded plan.

Equation (2) means that the choice of social security plans is equivalent to portfolio allocation: to allocate $(1-c_1-t)$ to an asset that yields the rate of return, r , and allocate the remaining t to an asset that yields the rate of return, g . Both of these rates of return are uncertain, and are further assumed so that in each period:

$$r = R + \varepsilon_r; \quad E(\varepsilon_r) = 0, \quad \text{var}(\varepsilon_r) = \sigma_r^2,$$

$$g = G + \varepsilon_g; \quad E(\varepsilon_g) = 0, \quad \text{var}(\varepsilon_g) = \sigma_g^2,$$

and

$$\text{cov}(\varepsilon_r, \varepsilon_g) = \eta \sigma_r \sigma_g,$$

where η is the coefficient of correlation between r and g .

In the case of a DB PAYG plan, we assume that the benefit is fixed at $(1+G)t$, where G is the mean of the wage growth rate, meaning that each worker in the following generation will be required to pay $(1+G)t/(1+g)$ to finance the social security benefits paid to his parents' generation. When an individual establishes his consumption plan, we assume that he knows the previous generation's wage income but not his own. Hence, he faces an uncertain wage growth rate, which is equal to g^1 . Normalizing his wage as one as mentioned above, his lifetime budget constraint is given by

$$c_2 = (1+r)[1 - c_1 - [(1+G)/(1+g)]t] + (1+G)t,$$

Using the linear approximation: $1/(1+g) \approx 1/(1+G) - \varepsilon_g/(1+G)^2$, we rewrite this budget constraint to its approximate version:

$$c_2 = (1+r)(1 - c_1 - t) + [1+g + [(r-G)/(1+G)]\varepsilon_g]t, \quad (2)'$$

Defining

$$g_{DB} = g + [(r-G)/(1+G)]\varepsilon_g = G + [(1+R+\varepsilon_r)/(1+G)]\varepsilon_g$$

and interpreting it as a rate of return in a DB PAYG plan, we can treat a DB PAYG plan almost in the same way as a DC PAYG plan by replacing g with g_{DB} .² It should be noted, however, that the uncertainty about the rate of return in a DB PAYG plan reflects both uncertainty about interest rate and wage growth rate, as well as their interaction.

With no uncertainty about both interest rate and wage growth rate, DC PAYG and

¹ Bohn (2001) emphasizes that a DB PAYG plan is more efficient than a DC PAYG or funded plan because smaller cohorts enjoy favorable factor prices. This argument corresponds to the case of a high g in our model of a DB PAYG plan.

² Borgmann (2005) compares DC and DB PAYG plans based on a different lifetime utility function, and considers not only a linear but also a quadratic approximation of the variance.

DB PAYG plans are equivalent. And, if we additionally assume that $\rho=R$, we obtain the optimal consumption levels:

$$c_1=c_2=[(1+R)/(2+R)][1+[(G-R)/(1+R)]t] \quad (3)$$

for both plans. Hence, a PAYG plan—whether DC and DB—makes individuals worse-off than a DC funded plan if $R>G$. Introduction of a PAYG plan is desirable otherwise, but equation (3) suggests just that the higher the tax rate is, the better-off an individual will be. However, the optimal or acceptable size of a PAYG plan cannot be derived without assumptions about uncertainty.

3. Incorporating uncertainty

We now incorporate uncertainty into the model, starting with a DC PAYG plan. The first-order conditions for lifetime utility maximization are given by

$$u'(c_1)=E[(1+r)u'(c_2)]/(1+\rho) \quad (4)$$

and the budget constraint (2). To make the model tractable and the calculations simple, we assume a quadratic utility function (to be replaced with a CRRA one in Section 4):

$$u(c)=c-\alpha c^2/2, \alpha>0, u'(c_1)=1-\alpha c>0 \quad (5)$$

Assuming that $\rho=R$, from (2), (4) and (5) we have:

$$c_1=E(c_2)+\text{cov}(r, c_2)/(1+R)=E(c_2)+[(1-c_1-t)\sigma_r^2+t\eta\sigma_r\sigma_g]/(1+R) \quad (6)$$

Plugging this into (2) we obtain

$$c_1=[(1+R)^2+\sigma_r^2+[(1+R)(G-R)-\sigma_r^2+\eta\sigma_r\sigma_g]t]/[(1+R)(2+G)+\sigma_r^2]. \quad (7)$$

In addition, we, recall the certainty-equivalence relationship:

$$E[u(c_2)]=u[E(c_2)-\pi],$$

where π is the risk premium regarding retirement consumption, and is approximately calculated by

$$\pi = -[u''(c_2)/u'(c_2)]\text{var}(c_2)/2. \quad (8)$$

Hereafter, we define

$$c_2^* = E(c_2) - \pi = (1+R)(1-c_1-t) + (1+G)t - \pi \quad (9)$$

then we calculate the expected lifetime utility as

$$EU = u(c_1) + u(c_2^*)/(1+R). \quad (10)$$

To highlight the importance of uncertainty, consider a special case: $R=G=0$, where a DC PAYG plan is equivalent to a DC funded plan if there is no uncertainty. We can show that $dEU/dt|_{t=0} > 0$ as far as $\eta < \sigma_r/\sigma_g$ (see **Appendix**). This means that introducing a DC PAYG plan improves social welfare as far as $\eta < \sigma_r/\sigma_g$ even if $R=G(=0)$. This also points to the possibility that a DC PAYG plan is desirable, unless R is not much larger than G .

We can also calculate the optimal tax rate that maximizes the expected lifetime utility. To illustrate this, consider the extreme case of $R=G=\eta=0$, $\sigma_r=1$, $\sigma_g=0$, and $\alpha=1$, which means that there is uncertainty only about the interest rate. Then, we have

$$EU = (2-t)/3 - (2-t)^2/18 + (1+2t-2t^2)/[2(2-t)] - (1+2t-2t^2)^2/[8(2-t)^2],$$

which is maximized at $t=1/2$. In contrast, if we assume that $\sigma_r=0$ and $\sigma_g=1$, we have

$$EU = 7/8 - t^2/2 - (1/2 - t^2/2)^2,$$

which is maximized at $t=0$, meaning that introducing a DC PAYG plan is not desirable. These two extreme examples underline the fact that the optimal size of social security depends heavily on the structure of uncertainty.

We can treat a DB PAYG plan in almost the same way as a DC PAYG plan by replacing g with g_{DB} . It should be noted, however, that

$$G_{DB} = E(g_{DB}) = E[G + [(1+R+\varepsilon_r)/(1+G)]\varepsilon_g] = G + \eta\sigma_r\sigma_g/(1+G),$$

which suggests that the average rate of return of a DB PAYG plan is higher (lower) than that of a DC PAYG plan if the interest rate and the growth rate of wage income are positively (negatively) correlated. Also, we obtain

$$\sigma_{gDB}^2 \quad \text{var}(g_{DB}) = [(1+R)^2 \sigma_g^2 + 2(1+R) \text{cov}(\varepsilon_g, \varepsilon_r \varepsilon_g) + \text{var}(\varepsilon_r \varepsilon_g)] / (1+G)^2,$$

and

$$\text{cov}(r, g_{DB}) = [(1+R) \eta \sigma_r \sigma_g + \text{cov}(\varepsilon_g, \varepsilon_r \varepsilon_g)] / (1+G).$$

Unfortunately, we cannot clearly state anything in general about the relative performance of a DB PAYG plan. However, if there is no uncertainty about the wage growth rate— $G_{DB}=G$ and $\sigma_{gDB}^2=0$ and also $\text{cov}(r, g_{DB})=0$ —then DB and DC PAYG plans are equivalent. On the other hand, if there is no uncertainty about the interest rate— $G_{DB}=G$ and $\sigma_{gDB}=[(1+R)/(1+G)]\sigma_g$ and also $\text{cov}(r, g_{DB})=0$ —then a DB PAYG plan is superior (inferior) to a DC PAYG one if $R < G$ ($R > G$) because the former has a larger (smaller) variance with the same mean.

4. The optimal and acceptable sizes of a PAYG plan

Finally, we illustratively derive the optimal and acceptable sizes of a PAYG plan, based on numerical simulations. We replace the tentative quadratic utility function (5) with a CRRA utility function,

$$u(c) = (c^{1-\nu} - 1) / (1-\nu), \quad \nu \geq 0,$$

where ν is a degree of relative risk aversion ($u(c) = \log c$ for $\nu=1$). Then, the risk aversion, π , around $E(c_2)$ is calculated as

$$\pi = \nu \text{var}(c_2) / [2E(c_2)],$$

so we have

$$\begin{aligned} c_2^* \quad E(c_2) - \pi &= (1+R)(1-c_1-t) + (1+G)t - \nu \text{var}(c_2) / [2E(c_2)] \\ &= (1+R)(1-c_1-t) + (1+G)t \\ &\quad - \nu [(1-c_1-t)^2 \sigma_r^2 + t^2 \sigma_g^2 + 2(1-c_1-t)t \eta \sigma_r \sigma_g] / [2[(1+R)(1-c_1-t) + (1+G)t]] \end{aligned}$$

The expected lifetime utility is obtained by plugging this into (2) given c_1 . However, it is impossible to algebraically solve the optimal consumption levels that maximize

the expected lifetime utility. Hence, we conduct numerical calculations in the following steps: (i) put some assumed values for the fixed parameters R , G , σ_r , σ_g , η , and v ; (ii) put some value for t (starting with zero); (iii) under a given value of t , gradually raise c_1 from zero to one, and correspondingly calculate c_2^* and EU to search for the maximized level of expected lifetime utility; (iv) repeat (iii) with different values of t to find the value that maximizes the maximized expected lifetime utility for each t .

For parameter values, we start with $R=1$, $G=0$, $\sigma_r=1$, $\sigma_g=1$, $\eta=0$, and $v=4$, where $R=1$ corresponds to about a 2.3 percent annual interest rate (assuming that one period is equivalent to thirty years) and $G=0$ (compared to $R=1$) reflects declining population growth. **Figure 1** depicts how the maximized level of expected lifetime utility corresponds to each given tax rate under a DC PAYG plan. Under no uncertainty, the assumption that $R>G$ leads to the claim that a DC funded plan (with $t=0$) is more desirable than a DC PAYG plan. As illustrated by this figure, however, this is not necessarily the case under uncertainty. By raising the tax rate from zero, we can improve expected lifetime utility. Our simulation finds that the optimal tax rate that maximizes expected lifetime utility is equal to 0.078, which is denoted by $t_{optimal}$ in the figure. Also, a DC PAYG plan can achieve the same or higher expected lifetime utility than a DC funded plan provided the tax rate does not exceed 0.154, which is denoted by t_{max} in the figure. The values of $t_{optimal}$ and t_{max} indicate the optimal and maximum sizes of a DC PAYG plan, respectively, under uncertainty in this case.

Of course, these simulation results depend heavily on parameter values. **Table 1** summarizes selected simulation results, which help us to understand how sensitive the results are to different assumptions about parameter values. The upper part compares cases with $R=1$ and $G=0$ —including the above-mentioned case illustrated

in Figure 1 as the baseline case (A)—under different values of σ_r , σ_g , and η . We find that (i) a negative correlation between interest rate and wage growth tends to make a DC PAYG plan more acceptable (by comparing (B) and (C)), (ii) more uncertainty about the interest rate tends to make a DC PAYG plan more acceptable (by comparing (D) and (E)), and (iii) more risk-averse individuals tend to accept the DC PAYG plan, which leads to a more diversified portfolio (by comparing (A)-(E) (with $\nu=4$) to (F)-(J) (with $\nu=2$)). The bottom part of the table shows the same comparisons in the case of $R=G=1$, where DC PAYG and funded plans are equivalent under no uncertainty. We find that a DC PAYG plan is more acceptable and is optimized at a higher tax rate than in the case of $R>G$.

It is impossible to calculate the optimal tax rate for a DB PAYG plan in general, unless we specify $\text{cov}(\varepsilon_g, \varepsilon_r \varepsilon_g)$ and $\text{var}(\varepsilon_g, \varepsilon_g)$. If there is no uncertainty about interest rate or wage growth rate, however, we can apply the same methodology to obtain the optimal tax rate. If $\sigma_g=0$, a DB PAYG plan is equivalent to a DC PAYG one, so we can apply the results for cases (D), (D)', (I) and (I)' with no change. If $\sigma_r=0$, a DB PAYG plan has $G_{DB}=G$ and $\sigma_{gDB}=[(1+R)/(1+G)]\sigma_g$ so we replace 1 with 2 for σ_g in cases (E) and (J) and leave cases (E') and (J') unchanged. However, the results do not change; i.e., introduction of a PAYG plan is not desirable, because a DB PAYG plan is inferior to a DC PAYG plan due to the wider variance with the same mean rate of return.

5. Conclusions

We have investigated a choice of social security plans under uncertainty with the analogy of portfolio allocation among assets that yield different and uncertain rates of return. We have confirmed that even if wage income growth is lower than the

interest rate, there can be a case in which a PAYG plan is acceptable and more desirable than a funded plan. Also, our numerical simulations calculate the optimal and acceptable sizes of a PAYG plan, which depend heavily on the mean-variance structure of interest rate and wage income growth, as well as the degree of people's risk aversion.

Appendix: Proof of $dEU/dt|_{t=0} > 0$ as far as $\eta < \sigma_r/\sigma_g$ if $R=G=0$

Putting $R=G=0$ into (7), (8), and (9), we obtain

$$\begin{aligned} c_1 &= [1 + \sigma_r^2 - (\sigma_r - \eta\sigma_g)\sigma_r t] / (2 + \sigma_r^2), \\ c_2^* &= [1 + (\sigma_r - \eta\sigma_g)\sigma_r t] / (2 + \sigma_r^2) - \pi, \\ \pi &= \alpha[(1 - c_1 - t)^2 \sigma_r^2 + \sigma_g^2 t^2 + 2(1 - c_1 - t)t\eta\sigma_r\sigma_g] / [2[1 - \alpha(1 - c_1)]]. \end{aligned}$$

Differentiating the expected lifetime utility with respect to the tax rate

$$\begin{aligned} dEU/dt &= (1 - \alpha c_1)dc_1/dt + (1 - \alpha c_2^*)dc_2^*/dt \\ &= \alpha[(\sigma_r - \eta\sigma_g)\sigma_r / (2 + \sigma_r^2)](c_1 - c_2^*) - (1 - \alpha c_2^*)d\pi/dt \end{aligned}$$

and evaluating this at $t=0$, we have

$$dEU/dt|_{t=0} = \alpha[(\sigma_r - \eta\sigma_g)\sigma_r / (2 + \sigma_r^2)](c_1 - c_2^*)|_{t=0} - (1 - \alpha c_2^*)d\pi/dt|_{t=0},$$

where

$$(c_1 - c_2^*)|_{t=0} = \sigma_r^2 / (2 + \sigma_r^2) + \pi > 0$$

and

$$d\pi/dt|_{t=0} = \alpha(\sigma_r - \eta\sigma_g)[4(\alpha - 2) + \sigma_r^2(\alpha - 4)] / [2[(2 + \sigma_r^2) - \alpha]^2(2 + \sigma_r^2)] < 0$$

as far as $\eta < \sigma_r/\sigma_g$, because $c_1 = (1 + \sigma_r^2)/(2 + \sigma_r^2) \geq 1/2$ and we assume $1 - \alpha c_1 < 1$ so $\alpha < 2$.

Hence, we conclude $dEU/dt|_{t=0} > 0$ as far as $\eta < \sigma_r/\sigma_g$.

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Figure 1. DC PAYG tax rate and expected lifetime utility

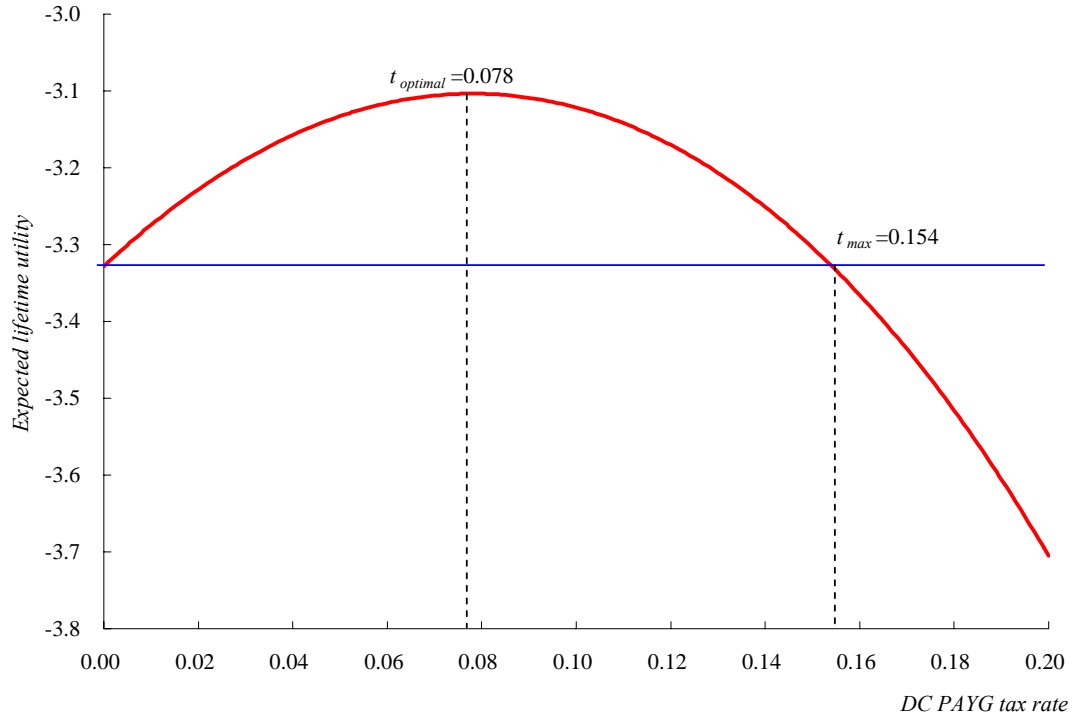


Table 1. Optimal and maximum tax rates of a DC PAYG plan

Case	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	(I)	(J)
R	1	1	1	1	1	1	1	1	1	1
G	0	0	0	0	0	0	0	0	0	0
σ_r	1	1	1	1	0	1	1	1	1	0
σ_g	1	1	1	0	1	1	1	1	0	1
η	0	-1	1	0	0	0	-1	1	0	0
v	4	4	4	4	4	2	2	2	2	2
$t_{optimal}$	0.078	0.149	0.000	0.250	0.000	0.000	0.096	0.000	0.000	0.000
t_{max}	0.154	0.353	0.000	0.477	0.000	0.000	0.191	0.000	0.000	0.000
Case	(A')	(B')	(C')	(D')	(E')	(F')	(G')	(H')	(I')	(J')
R	1	1	1	1	1	1	1	1	1	1
G	1	1	1	1	1	1	1	1	1	1
σ_r	1	1	1	1	0	1	1	1	1	0
σ_g	1	1	1	0	1	1	1	1	0	1
η	0	-1	1	0	0	0	-1	1	0	0
v	4	4	4	4	4	2	2	2	2	2
$t_{optimal}$	0.383	0.333	-	0.333	0.000	0.174	0.333	-	0.333	0.000
t_{max}	0.486	0.486	1.000	0.486	0.000	0.394	0.412	1.000	0.486	0.000

Note: In cases (C') and (H'), expected lifetime utility is the same for any tax rate.