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# A Constructive Proof of Existence and a Characterization of the Farsighted Stable Set in a Price-Leadership Cartel Model under the Optimal Pricing\*

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## Abstract

Diamantoudi (2005, *Economic Theory*) has proved the existence of the (unique) stable set of cartels in a price-leadership cartel model, in which firms are assumed to be farsighted and the dominant cartel adopts the optimal pricing policy. In this note, we give an alternative, elementary proof based on a constructive algorithm. With this, we can fully characterize the stable set of cartels: it contains at least one Pareto-efficient cartel and, in particular, the largest stable cartel in it is Pareto-efficient. By using a simple example, we also show that there can be some stable, but not Pareto-efficient cartels and some Pareto-efficient, but not stable cartels. *JEL classification:* C71, D43, L13. *Keywords:* price leadership model, cartel stability, foresight, stable set

## 1 Introduction

D'Aspremont et al. [1] have examined a price-leadership cartel model in which a dominant cartel sets price at the level that maximizes the profits of the member firms and each firm is assumed to be able to enter or exit from

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the dominant cartel freely. They have shown that if the number of firms is finite, there always exists a stable cartel—a cartel such that if once formed, no member firm wants to exit from it and no outside firm wants to enter it.

As pointed out by Diamantoudi [2], the analysis by d’Aspremont et al. [1] exhibited some inconsistency in firms’ attitudes or perspectives toward other firms’ reactions. In the d’Aspremont et al. [1] model, it is assumed that a cartel firm contemplating exiting from the current cartel compares the current profit (as a member of the cartel) with the prospective profit (as a fringe firm) that can be gained under a new price set by the new (smaller) cartel of the remaining cartel firms, and the firm will actually deviate from the current cartel if the latter is higher than the former; a similar argument applies to a fringe firm contemplating entering the current cartel as well. That is, in the d’Aspremont et al. [1] model, each firm contemplating a deviation (entry or exit) is assumed to have an ability to recognize the reaction of readjusting price by the members in the new cartel established after its deviation; in a sense, each firm is assumed to be farsighted to some extent. Despite this farsightedness, each firm in the d’Aspremont et al. [1] model ignores possible reactions of entry-exit by other firms subsequent to its own deviation. Firms are assumed to be farsighted on one hand, but myopic on the other.

Diamantoudi [2] has modified the d’Aspremont et al. [1] model so that each firm is farsighted enough to recognize not only the reaction of readjusting price by the new cartel members but also the reactions of entry-exit by other firms subsequent to its own deviation. Adopting (a variant of) the von Neumann-Morgenstern stable set as the solution concept, she has shown the existence of a unique set of stable cartels.<sup>1</sup> Her proof of the existence and uniqueness of the stable set heavily depends upon a well-known theorem due to von Neumann and Morgenstern [6] that an abstract system with a *strictly acyclic* relation admits a unique stable set. Her argument is essentially *existential* and the characteristics of the stable set for the price-leadership cartel model have not been fully investigated.<sup>2</sup>

In this paper, we investigate the characteristics of the stable set for the price-leadership cartel model. To this end, we first give an alternative, constructive proof of the existence of the stable set. The method of our proof is based on an algorithm to find and construct the stable set. After proving

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<sup>1</sup>Kamijo and Muto [3] and Kamijo and Nakanishi [4] have modified the Diamantoudi [2] model in two different directions. Kamijo and Muto [3] have examined the case where *coalitional* entry-exit is allowed. Kamijo and Nakanishi [4] have examined the case where the dominant cartel can set price at any (nonnegative) level it wants (i.e., the dominant cartel adopts the flexible pricing policy). These studies have shown that, in each model, every *individually rational* and *Pareto-efficient* cartel structure constitutes a (farsighted) stable set.

<sup>2</sup>Diamantoudi [2] has examined the relationship between the stable set of cartels and the *myopic* core of cartels and shown that the intersection of them contains the smallest cartel belonging to the stable set (Diamantoudi [2], Theorem 5).

its existence by construction, we show that the stable set of cartels and the set of Pareto-efficient cartels have a nonempty intersection; in particular, we show that *the largest cartel* in the stable set is Pareto-efficient. Further, by using a simple example, we also show that neither of the stable set and the Pareto-efficient set can include the other as its subset.

## 2 Model

### 2.1 Price-leadership Cartel

We consider an industry composed of  $n \geq 2$  identical firms, which produce a homogeneous good. The demand for the good is represented by a continuous, monotonically decreasing function  $d(p)$ , where  $p$  is the price of the good. Every firm has an identical cost function  $c(q)$ , where  $q$  is the output level of a firm.

When a firm does not participate in a cartel, it behaves competitively. We denote the supply function of a (competitive) fringe firm by  $q_f(p)$ , which is derived from the usual, price-equal-marginal cost condition. If  $k \geq 1$  firms decide to form a cartel, then the cartel becomes able to exercise a power to determine the market price of the good. (As in d'Aspremont et al. [1] and Diamantoudi [2], we assume that there can be only one dominant cartel.) We denote the cartel consisting of  $k$  firms as  $C(k)$  and the set of all possible cartels as  $\mathbb{C} \equiv \{C(0), C(1), \dots, C(n)\}$ .<sup>3</sup> Although  $C(0)$  actually represents a situation with *no* cartel, we include  $C(0)$  in the set of all possible cartels for notational convenience.

Taking account of the responses by the non-cartel firms, firms in  $C(k)$  derive the residual demand and divide it equally among them. Then, production per firm in  $C(k)$  can be written as a function of the number of firms in the cartel,  $k$ , and price  $p$ : For  $k = 1, \dots, n$ ,

$$r(k, p) \equiv \frac{\max \{d(p) - (n - k)q_f(p), 0\}}{k}. \quad (1)$$

The optimal price for cartel  $C(k)$ , in the sense that it maximizes the joint profit of its members, can be written as a function of  $k$ : For  $k = 1, \dots, n$ ,

$$p^*(k) \equiv \arg \max_{p \geq 0} p \cdot r(k, p) - c(r(k, p)). \quad (2)$$

Then, the profits of a fringe firm and a cartel firm evaluated at the optimal price  $p^*(k)$  can be written as functions of the cartel size  $k$ :

$$\pi_f^*(k) \equiv p^*(k)q_f(p^*(k)) - c(q_f(p^*(k))), k = 1, \dots, n - 1, \quad (\text{fringe firm}) \quad (3)$$

$$\pi_c^*(k) \equiv p^*(k)r(k, p^*(k)) - c(r(k, p^*(k))), k = 1, \dots, n. \quad (\text{cartel firm}) \quad (4)$$

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<sup>3</sup>Actually,  $\mathbb{C}$  is the set of possible cartel sizes (in terms of the number of firms). We regard two different cartels of the same size as the same. In this paper, as in d'Aspremont et al. [1] and Diamantoudi [2], cartels are identified by their respective sizes.

We assume that the competitive equilibrium prevails if there is no actual cartel (i.e.,  $k = 0$ ). We denote the competitive equilibrium price by  $p^{\text{comp}}$ , which is derived from the market clearing condition  $d(p) = nq_f(p)$ . Therefore, we have  $\pi_f^*(0) \equiv p^{\text{comp}}q_f(p^{\text{comp}}) - c(q_f(p^{\text{comp}}))$ . Under certain regularity conditions on the demand and cost functions, we can show the following results:

**Proposition 1.**  $\pi_c^*$  and  $\pi_f^*$  satisfy the following properties:

- (i)  $\pi_c^*(k)$  is increasing in  $k$ ,
- (ii)  $\pi_c^*(k) > \pi_f^*(0)$  for all  $k = 1, \dots, n$ ,
- (iii)  $\pi_f^*(k) > \pi_c^*(k)$  for all  $k = 1, \dots, n - 1$ .

We omit the proof of the above proposition; see d'Aspremont et al. [1]. Property (i) means entry of a new cartel member is beneficial to each incumbent cartel member. Property (ii) means that a situation with *no* cartel is the worst for every firm. Property (iii) means that for a given cartel size, it is preferable for a firm to stay outside the cartel than to be inside the cartel.

Given a cartel  $C(k)$ , we can specify (i) members of the cartel (i.e.,  $i \in C(k)$ ), (ii) fringe firms (i.e.,  $j \in N \setminus C(k)$ ), and (iii) the price level  $p^*(k)$  set by the cartel (or the competitive price  $p^{\text{comp}}$  when  $k = 0$ ). Therefore, we can regard a cartel  $C(k)$  itself as a description of the current state of the economy. With this understanding, we say that “ $C(k)$  Pareto-dominates  $C(m)$ ” if all firms’ profits under  $C(k)$  are not lower than under  $C(m)$  and some firms’ profits are strictly higher than under  $C(m)$ . A cartel  $C(k)$  is said to be Pareto-efficient if there is no other cartel  $C(m)$  that Pareto-dominates  $C(k)$ . Let  $P \subset \mathbb{C}$  be the set of Pareto-efficient cartels. The following proposition characterizes  $P$ .

**Proposition 2.** *The set of Pareto-efficient cartels is characterized as follows:*

$$P = \{C(k) \in \mathbb{C} \mid C(k) = C(n) \text{ or } \pi_f^*(k) > \pi_c^*(n)\}. \quad (5)$$

*Proof.* Let  $P'$  be the set appeared in the right-hand-side of Eq. (5). We first show that  $P \supset P'$ . Let us consider  $C(n) \in P'$  and take an arbitrary cartel  $C(m)$ ; note that  $m < n$  by definition. Under  $C(n)$ , all firms receive  $\pi_c^*(n)$ . If  $C(m) \neq \emptyset$ , members of the cartel  $C(m)$  receive  $\pi_c^*(m)$ . By Proposition 1, we have  $\pi_c^*(m) < \pi_c^*(n)$ . If  $C(m) = \emptyset$ , all firms receive  $\pi_f^*(0)$ . Again, by Proposition 1,  $\pi_f^*(0) < \pi_c^*(n)$ .  $C(m)$  cannot Pareto-dominate  $C(n)$ .

Next, let us consider a cartel  $C(k) \in P'$  that satisfies  $\pi_f^*(k) > \pi_c^*(n)$ . Take an arbitrary cartel  $C(m)$  with  $m \neq k$ . If  $m < k$ , then, similar to the above paragraph,  $C(m)$  cannot Pareto-dominate  $C(k)$ . Suppose  $m > k$ . Then, there must exist a firm that is a fringe firm under  $C(k)$ , but a cartel

firm under  $C(m)$ . Such a firm receives  $\pi_f^*(k)$  under  $C(k)$  and  $\pi_c^*(m)$  under  $C(m)$ . By assumption and by the monotonicity of  $\pi_c^*$ , we have  $\pi_f^*(k) > \pi_c^*(n) \geq \pi_c^*(m)$ .  $C(m)$  cannot Pareto-dominate  $C(k)$ . Hence,  $P \supset P'$ .

Lastly, we show that  $P \subset P'$ . Take an arbitrary cartel  $C(m)$  not in  $P'$ . That is,  $C(m)$  satisfies  $\pi_f^*(m) \leq \pi_c^*(n)$ . By Proposition 1, we have  $\pi_c^*(m) < \pi_f^*(m)$  if  $m \neq 0$  and  $\pi_f^*(0) < \pi_c^*(n)$  if  $m = 0$ .  $C(n)$  Pareto-dominates  $C(m)$ . Then,  $C(m) \notin P$ . Hence,  $P \subset P'$ .  $\square$

## 2.2 The Diamantoudi Model

A cartel is said to be stable if, once it is established, no member firm wants to exit from it and no fringe firm wants to enter it. For example, let us consider a cartel firm, say firm  $i$ , in  $C(m)$ . As a member of  $C(m)$ , firm  $i$  receives the profit  $\pi_c^*(m)$ . If firm  $i$  exits from  $C(m)$ , then the cartel changes to  $C(m-1)$  and firm  $i$  receives the profit  $\pi_f^*(m-1)$  as a fringe firm. A *myopic* firm will actually exit from  $C(m)$  if  $\pi_f^*(m-1) > \pi_c^*(m)$ . On the other hand, a *farsighted* firm, anticipating reactions by other firms subsequent to its own exit, may decide not to exit from  $C(m)$  even if  $\pi_f^*(m-1) > \pi_c^*(m)$ . A farsighted firm looks forward and it decides to deviate from the current state only if the *ultimate* outcome can give rise to a higher profit.

To incorporate the farsightedness of firms into the model, Diamantoudi [2] has defined the following dominance relation, denoted by  $\succ$ , on  $\mathbb{C}$ .

**Definition 1 (Indirect domination).** For  $C(k), C(m) \in \mathbb{C}$ , we have  $C(k) \succ C(m)$  if

$$\begin{cases} k > m \text{ and } \pi_c^*(k) > \pi_f^*(m+j) \ \forall j = 0, \dots, k-1-m, \text{ or} \\ k < m \text{ and } \pi_f^*(k) > \pi_c^*(m-j) \ \forall j = 0, \dots, -k-1+m. \end{cases} \quad (6)$$

The first line means that there is an increasing sequence of cartels  $C(m), C(m+1), \dots, C(k-1), C(k)$  such that each entering firm will be made better-off as a member of the final cartel  $C(k)$ . The second line means that there is a decreasing sequence of cartels  $C(m), C(m-1), \dots, C(k+1), C(k)$  such that each exiting firm will be made better-off as a fringe firm outside the final cartel  $C(k)$ . When  $C' \succ C$  for  $C, C' \in \mathbb{C}$ , we simply say “ $C'$  indirectly dominates  $C$ .” The pair of the set of all possible cartels  $\mathbb{C}$  and the binary relation  $\succ$  defines an abstract system associated with the price-leadership cartel under the optimal pricing policy:  $(\mathbb{C}, \succ)$ . The dominance relation  $\succ$  on  $\mathbb{C}$  is said to be *strictly acyclic* if there is no infinite sequence of elements  $C, C', C'', \dots \in \mathbb{C}$  such that  $C \prec C' \prec C'' \prec \dots$  (*ad infinitum*).<sup>4</sup>

The von Neumann-Morgenstern stable set (simply, the stable set) for  $(\mathbb{C}, \succ)$  is defined as follows:

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<sup>4</sup>For  $C, C' \in \mathbb{C}$ , we write  $C \succ C'$  and  $C' \prec C$  interchangeably.

**Definition 2 (The stable set).** A set  $K \subset \mathbb{C}$  is said to be a (von Neumann-Morgenstern) stable set for  $(\mathbb{C}, >)$  if it satisfies the following two conditions:

- For any  $C \in K$ , there does not exist  $C' \in K$  such that  $C' > C$ ,
- For any  $C \in \mathbb{C} \setminus K$ , there exists  $C' \in K$  such that  $C' > C$ .

These conditions are called “internal stability” and “external stability,” respectively.

It is a well-known theorem due to von Neumann and Morgenstern [6] that an abstract system with a strictly acyclic dominance relation admits a unique stable set. Therefore, to show the existence and uniqueness of the stable set for  $(\mathbb{C}, >)$ , it suffices to show that  $>$  is strictly acyclic. Diamantoudi [2] has proved this fact:<sup>5</sup>

**Fact 1 (Diamantoudi [2]).** The binary relation  $>$  on  $\mathbb{C}$  is strictly acyclic.

Fact 1 and the von Neumann-Morgenstern theorem together imply the following theorem immediately:

**Theorem 1 (Diamantoudi [2]).** There exists a unique, non-empty stable set of cartels for  $(\mathbb{C}, >)$ .

Although the stable set for  $(\mathbb{C}, >)$  is determined uniquely, this does not imply the stable set itself is a singleton. That is, the unique stable set may contain some different sizes of cartels. We call a cartel in the stable set for  $(\mathbb{C}, >)$  as a “stable cartel under the optimal pricing.”

Theorem 1 is essentially *existential*. Unfortunately, it does not provide us with much information about the shape or the characteristics of the stable set. The von Neumann-Morgenstern theorem is so general that it makes us unable to extract some useful information from a specific model. In the next section, we give an alternative, constructive proof of Theorem 1, with which we can fully characterize the unique stable set of cartels under the optimal pricing.

### 3 Results

Our alternative proof of Theorem 1 is elementary and constructive; it does not rely on Fact 1 nor on the von Neumann-Morgenstern theorem. To prove the theorem, we first define an algorithm that determines a certain subset of cartels, which is a candidate for the stable set; then, we show that this subset is actually the unique stable set for  $(\mathbb{C}, >)$ .

Before stating our main results, we show one useful lemma:

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<sup>5</sup>In Diamantoudi [2], Fact 1 is mentioned only in the proof of her Theorem 3.

**Lemma 1.** For  $C(k), C(m) \in C$  such that  $C(k) \succ C(m)$ , we have the following results:

- (i) if  $k > m$ , then  $C(k) \succ C(\ell)$  for any  $\ell$  with  $k > \ell \geq m$ ; and
- (ii) if  $k < m$ , then  $C(k) \succ C(\ell)$  for any  $\ell$  with  $k < \ell \leq m$ .

*Proof.* Because a similar argument can be applied to (ii), we only prove (i). By definition,  $C(k) \succ C(m)$  implies  $\pi_f^*(m') < \pi_c^*(k)$  for all  $m' = m, m + 1, \dots, k - 1$ . In particular, because  $k > \ell \geq m$ , we have  $\pi_f^*(m') < \pi_c^*(k)$  for all  $m' = \ell, \ell + 1, \dots, k - 1$ . Therefore,  $C(k) \succ C(\ell)$ .  $\square$

### 3.1 Algorithm for constructing the stable set

Let us define a sequence of integers,  $h_1, h_2, \dots$ , recursively, as follows:<sup>6</sup>

$$\begin{aligned} h_1 &\equiv 1, \\ h_{j+1} &\equiv \min \{k \in Z \mid \pi_c^*(k) \geq \pi_f^*(h_j)\}, \quad j = 1, 2, \dots, \end{aligned} \quad (7)$$

where  $Z \equiv \{1, 2, \dots, n\}$ . Because  $\pi_f^*(k) > \pi_c^*(k)$  for all  $k = 1, \dots, n - 1$  and  $\pi_c^*$  is increasing, the above recursive procedure is defined well and it determines a finite sequence of integers:  $h_1, h_2, \dots, h_J$ . Let  $H \equiv \{h_1, h_2, \dots, h_J\}$  be the set of such integers. We can easily verify that  $h_j < h_{j+1}$  for all  $j = 1, 2, \dots, J - 1$ . Define a subset of cartels as follows:

$$D \equiv \{C(k) \in C \mid k \in H\}. \quad (8)$$

By construction, we have  $C(h_j) \not\succ C(h_{j+1})$  for all  $j = 1, 2, \dots, J - 1$ . This implies  $C(h_r) \not\succ C(h_s)$  for any  $h_r, h_s \in H$  with  $r < s$ . However, it is possible to have  $C(h_j) < C(h_{j+1})$  for some  $j$ .

Next, we construct a new subset of cartels by deleting some elements from  $D$  according to the following recursive deletion procedure:

- Let  $D^{(0)} \equiv D$ .
- From  $D^{(0)}$ , delete all cartels that are indirectly dominated by  $C(h_J)$ , which is the largest cartel in  $D^{(0)}$ , and let  $D^{(1)}$  be the resulting set of cartels;
- From  $D^{(1)}$ , delete all cartels that are indirectly dominated by the *second* largest cartel in  $D^{(1)}$  (the largest one is  $C(h_J)$ ) and let  $D^{(2)}$  be the resulting set of cartels;
- In general, from  $D^{(\ell-1)}$ , delete all cartels that are indirectly dominated by the  $\ell$ th largest cartel in  $D^{(\ell-1)}$  and let  $D^{(\ell)}$  be the resulting set of cartels.

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<sup>6</sup>An analogous technique has been utilized in Nakanishi [5] to prove the existence of the purely noncooperative farsighted stable set for an  $n$ -player prisoners' dilemma.



Because the set of cartels is finite, the above procedure will stop within some finite steps. We denote the final set obtained by the above procedure as  $D^*$ . By construction,  $C(h_J)$  is never deleted and must remain in  $D^*$ . Accordingly,  $D^*$  is non-empty. We write the subset of  $H$  that corresponds to  $D^*$  as  $H^* \equiv \{h_1^*, \dots, h_t^*, \dots, h_T^*\}$ . Without loss of generality, we can set  $h_1^* < h_2^* < \dots < h_T^*$ . By definition, we have  $h_J = h_T^*$ ,  $T \leq J$ , and  $h_1 \leq h_1^*$ . (It is possible to have  $h_1 < h_1^*$ ; this implies that  $h_1$  has been deleted in the above deletion procedure.) In the following, we prove that  $D^*$  is the unique stable set for  $(\mathbb{C}, \succ)$ .

### 3.2 Alternative proof of Theorem 1

**[External Stability]:** Take an arbitrary cartel  $C(k) \in \mathbb{C} \setminus D^*$ . Consider the case where  $k < h_1^*$ . We show that  $C(k)$  is dominated by  $C(h_1^*)$ . If  $k = 0$ , we have  $\pi_f^*(k) = \pi_f^*(0) < \pi_c^*(1) \leq \pi_c^*(h_1^*)$  by Property 1; and if  $h_1 = 1 \leq k < h_1^*$ , we have  $C(h_1) < C(h_1^*)$  by the construction of  $h_1^*$ , which implies  $C(k) < C(h_1^*)$  by Lemma 1. In any case, we obtain  $C(k) < C(h_1^*) \in D^*$ .

Next let us consider the case where  $h_t^* < k < h_{t+1}^*$  for some  $h_t^* \in H^*$  (or  $h_T^* < k$ ). Note that  $h_t^* = h_r$  for some  $h_r \in H$ . We distinguish two subcases: (a)  $h_t^* = h_r < k < h_{r+1}$  and (b)  $h_{r+1} \leq k < h_{t+1}^*$ . In subcase (a), we have  $\pi_f^*(h_t^*) = \pi_f^*(h_r) > \pi_c^*(k)$  by the definition of  $h_{r+1}$ . Since  $\pi_c^*$  is increasing, we have  $\pi_f^*(h_t) > \pi_c^*(k) > \pi_c^*(k-1) > \dots > \pi_c^*(h_t+1)$ . Therefore,  $C(k) < C(h_t^*) \in D^*$ . In subcase (b), we must have  $C(h_{r+1}) < C(h_s^*)$  for some  $s$  with  $s \geq t+1$  by definition. (Note that  $C(h_{r+1})$  had been deleted *before*  $C(h_t^*)$  was reached in the recursive deletion procedure.) If  $s > t+1$ ,  $C(h_{r+1}) < C(h_s^*)$  implies  $C(h_{t+1}^*) < C(h_s^*)$  by Lemma 1. This contradicts the definition of  $C(h_{t+1}^*)$ . Then, we have  $s = t+1$ . In turn,  $C(h_{r+1}) < C(h_{t+1}^*)$  implies  $C(k) < C(h_{t+1}^*) \in D^*$  by Lemma 1. Hence,  $D^*$  is externally stable.

**[Internal Stability]:** Take arbitrary  $h_t^*, h_s^* \in H^*$  with  $t < s$ . Note that we have  $h_t^* = h_r$  for some  $h_r \in H$ . We cannot have  $C(h_r) > C(h_{r+1})$  by the definition of  $h_{r+1}$ . Then, *a fortiori*, we cannot have  $C(h_t^*) = C(h_r) > C(h_s^*)$ ; otherwise, Lemma 1 will be violated. Further, by definition, we cannot have  $C(h_t^*) < C(h_s^*)$ . Hence,  $D^*$  is internally stable.

**[Uniqueness]:** We first show that there is *no* cartel that indirectly dominates  $C(h_1^*)$ ; this means that  $C(h_1^*)$  is in the core of  $(\mathbb{C}, \succ)$ .<sup>7</sup> Suppose, in negation, that there exists a cartel  $C(k)$  that indirectly dominates  $C(h_1^*)$ . Because we have  $k \notin H^*$  by the internal stability of  $D^*$ , we only have to distinguish the following three cases: case 1 where  $k < h_1^*$ ; case 2 where  $h_s^* < k$  and  $C(h_s^*) > C(k)$  for some  $s \geq 1$ ; and case 3 where  $k < h_s^*$  and  $C(k) < C(h_s^*)$  for some  $s \geq 2$ .

<sup>7</sup>Note that  $C(h_1^*)$  is the smallest cartel in  $D^*$ ; hence, this result corresponds to Theorem 5 in Diamantoudi [2].

Case 1. Similar to the first part of the proof of the external stability, we have  $C(k) \leq C(h_1^*)$ , which implies  $\pi_f^*(k) < \pi_c^*(h_1^*)$ . Therefore,  $C(k) \succ C(h_1^*)$  cannot be true.

Case 2.  $C(k) \succ C(h_1^*)$  implies  $\pi_f^*(m) < \pi_c^*(k)$  for all  $m = h_1^*, h_1^* + 1, \dots, k$ . In particular, we have  $\pi_f^*(h_s^*) < \pi_c^*(k)$ . This contradicts  $C(h_s^*) \succ C(k)$ . Then, we cannot have  $C(k) \succ C(h_1^*)$ .

Case 3.  $C(k) \leq C(h_s^*)$  implies  $\pi_f^*(m) < \pi_c^*(h_s^*)$  for all  $m = k, k+1, \dots, h_s^* - 1$ . If  $C(h_1^*) \leq C(k)$ , then  $\pi_f^*(m) < \pi_c^*(k)$  for all  $m = h_1^*, h_1^* + 1, \dots, k - 1$ . These facts together imply  $C(h_1^*) \leq C(h_s^*)$ . This, however, contradicts the construction of  $h_1^*$ . Again,  $C(k) \succ C(h_1^*)$  cannot hold. None of cases 1 through 3 can be true. Hence, we obtain the desired result.

Now let  $K$  be an arbitrary stable set for  $(\mathbb{C}, \succ)$ . In order to prove the uniqueness, it suffices to show that  $K = D^*$ . Note that, by the result just obtained above, we must have  $C(h_1^*) \in K$ ; otherwise the external stability of  $K$  will be violated. Note also that, by applying a similar argument as case 1 above, we have  $C(k) \notin K$  for all  $k < h_1^*$ ; otherwise the internal stability of  $K$  will be violated. The rest of the proof is divided into several steps. In step 1, we show that any cartel  $C(k)$  such that  $h_1^* < k < h_2^*$  cannot be in  $K$ ; in step 2, we show that  $C(h_2^*) \in K$ ; then in step 3, repeatedly applying the same arguments as steps 1 and 2, we show that any cartel  $C(k)$  such that  $h_j^* < k < h_{j+1}^*$  or  $h_T^* < k$  cannot be in  $K$  and  $C(h_j^*) \in K$  for  $j = 1, \dots, T$ .

**Step 1.** Note that  $h_1^* = h_r$  for some  $r$ . We distinguish two cases: case 1 where  $h_1^* = h_r < k < h_{r+1} \leq h_2^*$  and case 2 where  $h_{r+1} \leq k < h_2^*$ . Suppose, in negation, that  $C(k) \in K$ .

Case 1. Because, by the definitions of  $h_r$  and  $h_{r+1}$ ,  $C(h_1^*) \in K$  indirectly dominates  $C(k)$ ; this contradicts the internal stability of  $K$ . Case 1 is not possible.

Case 2. Because  $C(h_2^*) \succ C(h_{r+1})$  by definition, we have  $C(h_2^*) \succ C(k)$  by Lemma 1. By the internal stability of  $K$ ,  $C(h_2^*)$  cannot be in  $K$ . Then, by the external stability of  $K$ , there exists a cartel  $C(m) \in K$  that indirectly dominates  $C(h_2^*)$ . We consider three subcases: (i)  $h_1^* < m < k$ , (ii)  $h_{r+1} \leq k < m < h_2^*$ , and (iii)  $h_2^* < m$ .

In case 2-(i),  $C(m) \succ C(h_2^*)$  implies  $C(m) \succ C(k)$  by Lemma 1; this, however, violates the internal stability of  $K$ .

In case 2-(ii), because  $C(h_2^*) \succ C(h_{r+1})$  by definition, then we have  $C(h_2^*) \succ C(m)$  by Lemma 1. Then, we have  $\pi_f^*(m) < \pi_c^*(h_2^*)$ . This contradicts  $C(m) \succ C(h_2^*)$ .

In case 2-(iii),  $C(m)$  not only indirectly dominates  $C(h_2^*)$  but also Pareto-dominates  $C(h_2^*)$ . Then, by simply connecting the sequence realizing  $C(m) \succ C(h_2^*)$  to the one realizing  $C(h_2^*) \succ C(k)$ , we obtain an appropriate sequence that realizes  $C(m) \succ C(k)$ . This contradicts the internal stability of  $K$ . Case 2 is not possible, either. Hence, we can conclude that  $C(k) \notin K$  for

any  $k$  with  $h_1^* < k < h_2^*$ .

**Step 2.** Suppose, in negation, that  $C(h_2^*) \notin K$ . Then, by the external stability of  $K$ , there must exist a cartel  $C(m) \in K$  that indirectly dominates  $C(h_2^*)$ . By the results obtained in step 1, we must have  $m > h_2^*$ .<sup>8</sup> We distinguish four cases: case 1 where  $m = h_j^* > h_2^*$  for some  $j$ ; case 2 where  $h_2^* \leq h_j^* \leq h_s < m < h_{s+1} \leq h_{j+1}^*$  for some  $j$  and  $s$ ; case 3 where  $h_2^* \leq h_j^* < h_s = m < h_{j+1}^*$  for some  $j$  and  $s$ ; and case 4 where  $h_2^* \leq h_T^* < m$ .

Case 1.  $C(m) = C(h_j^*) > C(h_2^*)$  contradicts the definition of  $H^*$ .

Case 2.  $C(m) > C(h_2^*)$  implies  $C(m) > C(h_s)$  by Lemma 1. On the other hand, by the definition of  $h_{s+1}$  and the monotonicity of  $\pi_c^*$ , we have  $\pi_f^*(h_s) > \pi_c^*(m)$ ;  $C(m)$  cannot indirectly dominate  $C(h_s)$ —a contradiction.

Case 3. By the construction of  $H^*$ ,  $C(h_{j+1}^*)$  indirectly dominates  $C(m)$  ( $= C(h_s)$ ). In addition,  $C(h_{j+1}^*)$  Pareto-dominates  $C(m)$  by construction. Then, similar to case 2-(iii) in step 1, we have  $C(h_2^*) < C(h_{j+1}^*)$ . This contradicts the definition of  $H^*$ .

Case 4.  $C(m) > C(h_2^*)$  implies  $C(m) > C(h_T^*)$  by Lemma 1. However, by the definition of  $h_T^*$  ( $= h_J$ ), we have  $\pi_f^*(h_T^*) > \pi_c^*(m)$ ;  $C(m)$  cannot indirectly dominate  $C(h_T^*)$ . This is a contradiction. All cases 1 through 4 in step 2 are not possible. Hence, we have  $C(h_2^*) \in K$ .

**Step 3.** Repeatedly applying the same arguments as step 1 and step 2, that any cartel  $C(k)$  such that  $h_{j-1}^* < k < h_j^*$  or  $h_T^* < k$  cannot be in  $K$  and  $C(h_j^*) \in K$  for  $j = 1, \dots, T$ . Hence, we can conclude  $K = D^*$ .  $\square$

### 3.3 Characterization of the stable set

The stable set for  $(\mathbb{C}, >)$  is nothing but  $D^*$ , which is obtained through the algorithm we have shown. With this result, we can effectively compare the set of stable cartels under the optimal pricing and the set of Pareto-efficient cartels.

**Theorem 2.** *The set of stable cartels under the optimal pricing and the set of Pareto-efficient cartels have a non-empty intersection. In particular, the largest cartel in the set of stable cartels under the optimal pricing is Pareto-efficient. That is,*

$$C(h_T^*) \in D^* \cap P.$$

*Proof.* By the definition of the recursive deletion procedure, we have  $C(h_T^*) = C(h_J) \in D^*$ . Then, it remains to show that  $C(h_T^*) = C(h_J) \in P$ . Remember that  $h_J$  is the last and the largest integer generated from the recursive

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<sup>8</sup>Remember that  $C(h_1^*)$  cannot indirectly dominate  $C(h_2^*)$  by definition and that  $C(k) \notin K$  for any  $k$  with  $k < h_1^*$  or  $h_1^* < k < h_2^*$ .

equation (7). If  $h_J = n$ , then the proof ends. Suppose  $h_J < n$  and suppose, in negation, that  $C(h_J) \notin P$ . By Proposition 2 and the properties of  $\pi_f^*$  and  $\pi_c^*$ , we have  $\pi_c^*(n) \geq \pi_f^*(h_J) > \pi_c^*(h_J)$ . Then, by Eq. (7), another integer  $h_{J+1}$  must be generated after  $h_J$ —a contradiction. Hence,  $C(h_J) \in P$ .  $\square$

## 4 Example

Our Theorem 2 is all that we can say about the relationship between the set of stable cartels under the optimal pricing and the set of Pareto-efficient cartels *in a general setting*. To get more insights, we construct a concrete example. By this example, we show that neither of  $D^*$  and  $P$  can contain the other as a subset.

We specify the demand function and the cost function as follows:

$$d(p) \equiv a - bp, \quad a, b > 0, \quad (9)$$

$$c(q) \equiv \frac{q^2}{2}. \quad (10)$$

Then, the supply function of a fringe firm becomes  $q_f(p) \equiv p$ . The competitive equilibrium price  $p^{\text{comp}}$  can be derived from  $d(p) = nq_f(p)$ . Then, the per-firm residual demand for a cartel firm becomes as follows:

$$r(k, p) \equiv \frac{a - (b + n - k)p}{k}. \quad (11)$$

Then, the profits of a fringe firm and a cartel firm can be written as functions of  $p$  and  $k$ :

$$\pi_f(p) \equiv \frac{p^2}{2}, \quad (12)$$

$$\pi_c(k, p) \equiv pr(k, p) - \frac{1}{2} \{r(k, p)\}^2. \quad (13)$$

Through a usual procedure, we obtain the optimal price for the size  $k$  cartel:

$$p^*(k) \equiv \frac{a(b + n)}{(b + n)^2 - k^2}, \quad k = 1, 2, \dots, n. \quad (14)$$

By substituting  $p^*(k)$  and  $p^{\text{comp}}$  into  $\pi_f$  and  $\pi_c$ , we obtain  $\pi_f^*$  and  $\pi_c^*$ :

$$\pi_f^*(k) \equiv \frac{a^2(b + n)^2}{2 \{(b + n)^2 - k^2\}^2}, \quad k = 0, 1, \dots, n - 1, \quad (15)$$

$$\pi_c^*(k) \equiv \frac{a^2}{2 \{(b + n)^2 - k^2\}}, \quad k = 1, 2, \dots, n. \quad (16)$$

Both  $\pi_f^*$  and  $\pi_c^*$  is monotonically increasing in  $k$ . As can be seen easily, we have

$$\pi_f^*(k) = \frac{(b + n)^2}{(b + n)^2 - k^2} \cdot \pi_c^*(k), \quad k = 1, \dots, n. \quad (17)$$

For all  $k \geq 1$ , the *multiplier* to  $\pi_c^*$  in the right-hand-side of the above equation is greater than unity. Therefore, we have  $\pi_f^*(k) > \pi_c^*(k)$  for all  $k \geq 1$ . By simple calculation, we can show that

$$\pi_c^*(k+1) \geq \pi_f^*(k) \Leftrightarrow k^4 \geq (b+n)^2(k^2 - 2k - 1). \quad (18)$$

Then, assuming that  $(b+n)$  is sufficiently large, we obtain the following facts:

$$\pi_c^*(k+1) > \pi_f^*(k), \quad k = 0, 1, 2, \quad (19)$$

$$\pi_c^*(k+1) < \pi_f^*(k), \quad k = 3, 4, \dots, n. \quad (20)$$

These facts imply that  $C(k) < C(k+1)$  for  $k = 0, 1, 2$  and that  $C(k-1) > C(k)$  for  $k = 4, 5, \dots, n$ .

Now, let us construct the set  $D$ . To be more concrete, we henceforth assume  $b = 1$  and  $n = 20$ . Given  $h_j$ , the integer  $h_{j+1}$  is the minimum integer  $k$  satisfying  $\pi_f^*(h_j) \leq \pi_c^*(k)$ . By simple calculation, we can show that

$$\pi_f^*(h_j) \leq \pi_c^*(k) \Leftrightarrow \left[ \frac{2(b+n)^2(h_j)^2 - (h_j)^4}{(b+n)^2} \right]^{1/2} \leq k. \quad (21)$$

By definition,  $h_1 = 1$ . Then, by repeatedly applying the above equation, we obtain the following results:

$$h_1 = 1, \quad h_2 = 2, \quad h_3 = 3, \quad h_4 = 5, \quad h_5 = 7, \quad h_6 = 10, \quad h_7 = 14, \quad h_8 = 18. \quad (22)$$

That is,  $J = 8$ . Hence,

$$D = \{C(1), C(2), C(3), C(5), C(7), C(10), C(14), C(18)\}. \quad (23)$$

Next, let us consider the recursive deletion procedure and construct  $D^*$ . Let  $D^{(0)} = D$ . The largest cartel in  $D^{(0)}$  is  $C(18)$ ; but, it does not indirectly dominate any other cartel in  $D^{(0)}$ ; therefore,  $D^{(1)} = D^{(0)}$ . The second largest cartel in  $D^{(1)}$  is  $C(14)$ ; but, it does not indirectly dominate any other cartel in  $D^{(1)}$ ; therefore,  $D^{(2)} = D^{(1)}$ . Similarly, we have  $D^{(0)} = D^{(1)} = \dots = D^{(5)}$ . The sixth largest cartel in  $D^{(5)}$  is  $C(3)$ . In this case,  $C(3)$  indirectly dominates both  $C(1)$  and  $C(2)$ . By deleting  $C(1)$  and  $C(2)$  from  $D^{(5)}$ , we obtain  $D^{(6)} = D^{(5)} \setminus \{C(1), C(2)\}$  and then the deletion procedure stops. Consequently, we have

$$D^* = D^{(6)} = \{C(3), C(5), C(7), C(10), C(14), C(18)\}. \quad (24)$$

In this case, we have  $h_1^* = h_3 = 3$ ,  $h_2^* = h_4 = 5$ ,  $h_3^* = h_5 = 7$ ,  $h_4^* = h_6 = 10$ ,  $h_5^* = h_7 = 14$ , and  $h_6^* = h_8 = 18$  (i.e.,  $T = 6$ ).

Let us examine the set  $P$  of Pareto-efficient cartels. Clearly, we have  $C(n) = C(20) \in P$ . For  $C(k)$  with  $k \neq n$  to be included in  $P$ , we must

have  $\pi_f^*(k) > \pi_c^*(n)$ . By simple calculation, we obtain a necessary-sufficient condition for this inequality:

$$k > \left[ (b+n)^2 - [(b+n)^2 \{(b+n)^2 - n^2\}]^{1/2} \right]^{1/2} \quad (25)$$

By substituting  $b = 1$  and  $n = 20$ , we obtain  $k > 17.508 \dots$ . Hence,

$$P = \{C(18), C(19), C(20)\}. \quad (26)$$

As our Theorem 2 indicates, we have  $C(18) \in D^* \cap P$ . In fact, in this specific example,  $C(18)$  is the only stable cartel included in both  $D^*$  and  $P$ . Furthermore, we have both  $D^* \setminus P \neq \emptyset$  and  $P \setminus D^* \neq \emptyset$ . Therefore, neither  $D^* \subset P$  nor  $P \subset D^*$  can be true.

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