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A geometric description of the elliptic Painlevé equation

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Abstract

This note is an introduction to a geometric aspect of the elliptic Painlevé equation which is described as a non-autonomous deformation of the addition formula on cubic curves.

1 Introduction

Discrete analogs of the Painlevé equations have attracted much interest over the years. This is largely because, by considering the differential and discrete Painlevé equations together, their common algebraic/geometric aspect, such as the affine Weyl group symmetry, becomes more transparent.

Extending Okamoto's pioneering work, Sakai[9] clarified that one origin of the affine Weyl group symmetry of Painlevé equations is the Cremona isometry on a rational surface (the nine point blown-up of the projective plane)[1]. In this formulation, the discrete and differential Painlevé equations appear according to various configurations of the nine points. Among them, the equation corresponding to the most generic configuration is unique since it is an "elliptic" difference equation.

In this note, we give an introduction to this elliptic Painlevé equation based on our recent work[2, 3]. In this formulation, the "elliptic" nature of the equation appears intrinsically. Application to construction of the hypergeometric solutions is also discussed.

2 Elliptic Painlevé equation

2.1 A geometric formulation

Let $P_1, \ldots, P_{10}$ be points on the projective plane $\mathbb{P}^2$. In the context of Painlevé equations, the first nine points play the role of parameters and the last point P_{10} is the dependent variable. For $1 \leq i \neq j \leq 9$, define a map $T_{ij} : P_k \mapsto \overline{P}_k$ as follows: (**i**) Parameters $P_1, \ldots, P_9$ are transformed as

$$\begin{aligned} &\overline{P}_k = P_k, \quad (k \neq i, j), \\ &P_1 + \cdots + P_{j-1} + \overline{P}_j + P_{j+1} + \cdots + P_9 = 0, \\ &\overline{P}_i + \overline{P}_j = P_i + P_j, \end{aligned} \tag{1}$$

with respect to the addition on the cubic curve C_0 passing through $P_1, \ldots, P_9$. (**ii**) Dependent variable P_{10} is transformed as

$$\overline{P}_j + \overline{P}_{10} = P_i + P_{10}, \tag{2}$$

with respect to the addition on the cubic C passing through $P_1, \ldots, \check{P}_j, \ldots, P_{10}$, where ˇ means deletion. We note that this is not the original definition of the elliptic Painlevé equation based on the Cremona transformation but a consequence of it[9, 2]. Here, we simply start from the above formulation to make the geometric aspect clear.

2.2 A Numerical Example

It would be instructive to give an explicit numerical example. We put the initial configuration of the ten points in the inhomogeneous coordinates (x,y) as $P_1=(2,1)$, $P_2=(\frac{7}{4},\frac{5}{4})$, $P_3=(\frac{1}{5},1)$, $P_4=(\frac{2}{5},\frac{4}{5})$, $P_5=(0,1)$, $P_6=(1,0)$, $P_7=(\frac{1}{3},\frac{4}{3})$, $P_8=(\frac{3}{2},\frac{5}{2})$, $P_9=(3,\frac{3}{7})$ and $P_{10}=(\frac{1}{4},\frac{7}{4})$. We evaluate their transformations $\overline{P}_i=T_{12}(P_i)$.

The equations of the relevant cubics C_0 and C are given by

$$\begin{aligned} C_0:\quad & 36300+25210x-79050x^2+17540x^3-83708y+8859xy\\ & +40462x^2y+52707y^2-27053xy^2-5299y^3=0,\\ C:\quad & 847524-651870x-315774x^2+120120x^3-1753090y+924983xy\\ & +51510x^2y+1171181y^2-225065xy^2-265615y^3=0, \end{aligned} \tag{3}$$

respectively. Then the point $\overline{P}_2$ is the intersection of these cubics other than the trivial ones $P_1,P_3,\ldots,P_9$, namely $\overline{P}_2=(\frac{33410885}{46521159},\frac{100479044}{46521159})$. The point $\overline{P}_1=(\frac{32228531258}{92184280557},\frac{77552262590}{92184280557})\in C_0$ is such that the lines $\ell_{\overline{P}_1\overline{P}_2}$ and $\ell_{P_1P_2}$ meet on C_0. Finally, the point $\overline{P}_{10}=(\frac{393389319323583}{663005087997780},-\frac{99340628463911}{33150254399890})\in C$ is determined so that the lines $\ell_{\overline{P}_2\overline{P}_{10}}$ and $\ell_{P_1P_{10}}$ meet on C.

If the initial points are rational, so is the transformed points since the transformation is birational. Note that the cubic C moves according to the time evolution, while the cubic C_0 is always fixed.

For the readers convenience, we give a sample *Mathematica* program for the above computation. (The following part in this TEXsource file will be directly used in *Mathematica*).

```
Clear[m1,m3,c1,c3,thd,cbc,cbc0,np,T];
m1[{x_,y_}]:={x,y,1};
m3[{x_,y_}]:={x^3,x^2 y,x y^2,y^3,x^2,x y,y^2,x,y,1};
c1[p_]:=Det[Map[m1[#]&,Append[p,{x,y}]]];
c3[p_]:=Det[Map[m3[#]&,Append[p,{x,y}]]];
thd[cubic_,p_]:=Complement[{x,y}/.Solve[{cubic,c1[p]}==0,{x,y}],p][[1]];
cbc[p_,i_]:=cbc[p,i]=c3[Drop[p,{i}]];
cbc0:=cbc[pp,10];
np[p_,i_,j_,j_]:=np[p,i,j,j]=Complement[{x,y}/.Solve[{cbc[p,j],cbc0}==0,{x,y}],p][[1]];
np[p_,i_,j_,10]:=thd[cbc[p,j],{thd[cbc[p,j],{p[[i]],p[[10]]}],np[p,i,j,j]}];
np[p_,i_,j_,i_]:=thd[cbc0,{thd[cbc0,{p[[i]],p[[j]]}],np[p,i,j,j]}];
np[p_,i_,j_,k_]:=p[[k]]/;(k=!=i||k=!=j||k=!=10);
T[{i_,j_},p_]:=T[{i,j},p]=Table[np[p,i,j,k],{k,1,10}];
(* T[{1,2},pp] gives the above example for the input data:
pp={{2,1},{7/4,5/4},{1/5,1},{2/5,4/5},{0,1},{1,0},{1/3,4/3},
{3/2,5/2},{3,3/7},{1/4,7/4}}; *)
```

3 Commutativity

It is easy to see that $T_{ji}=T_{ij}^{-1}$, but the following is non-trivial.

Theorem 3.1 *The transformations T_{ij}'s are mutually commutative:*

$$[T_{ij}, T_{kl}] = 0. \tag{4}$$

We will give a direct geometric proof of this.

3.1 Proof: autonomous case

Let us first consider the case when the nine points $P_1, \ldots, P_9$ are in the *special configuration* such that they are the intersection points of a pencil (=one parameter family) of cubic curves $\lambda f(P) + \mu g(P) = 0$. We call this case *autonomous* since $\overline{P}_k = P_k$ ($k = 1, \ldots, 9$). We note that the discrete dynamical systems obtained from this case is nothing but the QRT map[11].

Let X be the blown-up of $\mathbb{P}^2$ at the nine points $P_1, \ldots, P_9$. Then X has an elliptic fibration: $\pi : X \to \mathbb{P}^1$,

$$\pi(P) = (\lambda : \mu), \quad P \in X, \quad \lambda f(P) + \mu g(P) = 0. \tag{5}$$

The transformation $P_{10} \mapsto \overline{P}_{10}$ is nothing but the fiber-wise application of the addition formula and their commutativity is obvious.

For our later use, let us determine the action of T_{ij} on the Picard lattice of X : $\mathrm{Pic}X = \mathbb{Z}\mathcal{E}_0 \oplus \mathbb{Z}\mathcal{E}_1 \oplus \cdots \oplus \mathbb{Z}\mathcal{E}_9$. Here, $\mathcal{E}_0$ is the class of line and $\mathcal{E}_i$ ($1 \le i \le 9$) is the exceptional divisor. The intersection pairing is given by $\mathcal{E}_i \cdot \mathcal{E}_j = \mathrm{diag}(1, -1, \ldots, -1)$. The following results were obtained by Manin[7]. For $1 \le i \le 9$, let M_i be the transformation on X such that $M_i(P) + P + P_i = 0$ on the cubic passing through $P_1, \ldots, P_9$ and P. Then we have

$$\begin{aligned} M_i(\mathcal{E}_j) &= \mathcal{E}_0 - \mathcal{E}_i - \mathcal{E}_j, \quad (j \neq 0, i) \\ M_i(\mathcal{E}_i) &= \delta + \mathcal{E}_0 - 2\mathcal{E}_i, \\ M_i(\mathcal{E}_0) &= \delta + 2\mathcal{E}_0 - 3\mathcal{E}_i, \end{aligned} \tag{6}$$

where $\delta = 3\mathcal{E}_0 - \mathcal{E}_1 - \cdots - \mathcal{E}_9$ ($M_i(\delta) = \delta$). Since $T_{ij} = M_j M_i$, one has

$$\begin{aligned} T_{ij}(\mathcal{E}_k) &= \mathcal{E}_k + (\mathcal{E}_i - \mathcal{E}_j) + \delta, \quad (k \neq 0, i, j) \\ T_{ij}(\mathcal{E}_j) &= \mathcal{E}_j + (\mathcal{E}_i - \mathcal{E}_j), \\ T_{ij}(\mathcal{E}_i) &= \mathcal{E}_i + (\mathcal{E}_i - \mathcal{E}_j) + 2\delta, \\ T_{ij}(\mathcal{E}_0) &= \mathcal{E}_0 + 3(\mathcal{E}_i - \mathcal{E}_j) + 3\delta. \end{aligned} \tag{7}$$

The commutativity (4) also follows from (7).

Remark. We give an interpretation of (7) in view of the affine Weyl group and the Cremona transformation. Put $Q = \{\alpha \in \mathrm{Pic}X | \alpha \cdot \delta = 0\}$. A basis of Q is chosen as $\alpha_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3$ and $\alpha_i = \mathcal{E}_i - \mathcal{E}_{i+1}$ ($1 \le i \le 8$). The intersection matrix $(\alpha_i \cdot \alpha_j)_{0 \le i,j \le 8}$ is the minus of the Cartan matrix of type $E_8^{(1)}$. The simple reflections $s_i(\lambda) = \lambda + (\lambda \cdot \alpha_i)\alpha_i$ generate the affine Weyl group $W(E_8^{(1)})$ on $\mathrm{Pic}X$. Note that the reflection s_0 is the standard Cremona transformation with center P_1, P_2, P_3. The translation T_β along $\beta \in Q$ is given by the Kac formula

$$T_\beta(\lambda) = \lambda + \beta(\delta \cdot \lambda) - \Big[\frac{\beta^2}{2}(\delta \cdot \lambda) + (\beta \cdot \lambda)\Big]\delta. \tag{8}$$

Comparing with (7), we conclude that $T_{ij} = T_{\mathcal{E}_i - \mathcal{E}_j}$. [1]

[1] The translations of type $T_{\mathcal{E}_i - \mathcal{E}_j}$ generate an index three subgroup of the full translations $\mathbb{Z}^8 \subset W(E_8^{(1)})$. Simple geometric description for the remaining part is expected.

3.2 Proof: non-autonomous case

Consider the nine points $P_1, \dots, P_9$ in general position. In this case, the points P_i, P_j also move under the action of T_{ij} as well as the dependent variable P_{10}. For the parameters, the transformation is given by the addition on the fixed cubic C_0, hence the commutativity is clear. For the dependent variable P_{10}, however, the action of T_{ij} is defined on *moving* cubic C, and the commutativity is not obvious. Nevertheless the argument of the autonomous case can also be applied in this situation. The key points are the following: (**i**) The new point $\overline{P}_{10}$ is independent of P_j. (**ii**) The points $\overline{P}_j$ and $P_1, \dots, \check{P}_j, \dots, P_9$ are in the special position. Due to these properties, by replacing P_j with $\overline{P}_j$, one can compute the action of T_{ij} on Pic$X$ as in equation (7). Hence the commutativity follows.

Remark. As we have seen above, cubic pencils are important in the geometric formulation of the discrete Painlevé equations. It is interesting to note that a certain cubic pencil also plays an essential role in the differential Painlevé equations [4].

4 Applications

4.1 Relation to the bilinear formalism

For $\lambda = d\mathcal{E}_0 - m_1\mathcal{E}_1 - \cdots - m_9\mathcal{E}_9 \in \mathrm{Pic}X$, a family of curves $|\lambda|$ (called the linear system) is defined as follows: $C \in |\lambda| \Leftrightarrow$ the degree of C is d and the multiplicity of C at P_i is m_i. The dimension of the family and the genus of the curve is given by

$$2\mathrm{dim}|\lambda| = \lambda^2 + \lambda \cdot \delta, \quad 2g - 2 = \lambda^2 - \lambda \cdot \delta. \tag{9}$$

We put $\Lambda = \{\lambda \in \mathrm{Pic}X | \lambda^2 = -1, \quad \lambda \cdot \delta = 1\}$. For $\lambda \in \Lambda$, $|\lambda|$ is a unique rational curve ($\simeq \mathbb{P}^1$). Let $\phi_\lambda = 0$ be the defining equation of the curve. For any $\mu \in \mathrm{Pic}X$ such that $\mu^2 = 0$ and $\delta \cdot \mu = 2$, the $|\mu|$ is one-parameter family of rational curves. Simple geometric argument (counting of Euler number) shows that there exist eight decompositions $\mu = \lambda_i + \tilde{\lambda}_i$, such that $\lambda_i, \tilde{\lambda}_i \in \Lambda$, $(1 \le i \le 8)$. Then for any $\{i, j, k\} \subset \{1, \dots, 8\}$, we have the following bilinear relation

$$c_i \phi_{\lambda_i} \phi_{\tilde{\lambda}_i} + c_j \phi_{\lambda_j} \phi_{\tilde{\lambda}_j} + c_k \phi_{\lambda_k} \phi_{\tilde{\lambda}_k} = 0, \tag{10}$$

with some constants c_i, c_j, c_k. This is the geometric origin of the bilinear equations given in [8]. To make the bilinear equation explicit, one need to fix the normalization of the polynomials ϕ. Appropriately normalized polynomials ϕ are called the τ functions. Detailed discussion on the τ functions and their bilinear equations is given in [5].

4.2 Hypergeometric solutions

For (discrete) Painlevé equations, special solutions of hypergeometric type appear on reflection hyperplanes in the parameter space. In our case, each hyperplane corresponds to a root $\alpha \in Q$ such that $\alpha^2 = -2$. Typical examples are as follows:

α	condition
$\mathcal{E}_i - \mathcal{E}_j$	$P_i = P_j$
$\mathcal{E}_0 - \mathcal{E}_i - \mathcal{E}_j - \mathcal{E}_k$	P_i, P_j, P_k are collinear
$2\mathcal{E}_0 - \mathcal{E}_i - \mathcal{E}_j - \mathcal{E}_k - \mathcal{E}_l - \mathcal{E}_m - \mathcal{E}_n$	$P_i, P_j, \dots, P_n$ are on a conic

For such α, there exists a rational curve $C_\alpha \in |\alpha|$ if the corresponding condition is satisfied. For $\beta \in Q$ such that $\alpha \cdot \beta = 0$, we have $T_\beta(\alpha) = \alpha$. This means that if $P_{10} \in C_\alpha$ then $T_\beta(P_{10}) \in C_\alpha$. Hence, in this case, the elliptic Painlevé equation can be reduced to the discrete Riccati equation $T_\beta(u) = (au + b)/(cu + d)$ on $C_\alpha \simeq \mathbb{P}^1$ which can be easily linearized.[2]

Example. Consider the case where P_5, P_6, P_7 are on a line ℓ and $P_{10} \in \ell$. Under a suitable normalization of homogeneous coordinates of $P_{10}, \overline{P}_{10}, \underline{P}_{10}$, we have

$$\frac{d_{jk9}d_{ki8}d_{ij\overline{9}}d_{568}}{d_{i8\overline{9}}}\Big(\frac{d_{ik\overline{9}}}{d_{ik8}}\overline{F} - F\Big) + \frac{d_{jk8}d_{ki9}d_{ij\underline{8}}d_{569}}{d_{i\underline{8}9}}\Big(\frac{d_{ik\underline{8}}}{d_{ik9}}\underline{F} - F\Big) = d_{ijk}d_{k89}d_{56j}F, \tag{11}$$

where $\{i, j, k\} = \{1, 2, 3\}$, $\overline{x} = T_{89}(x)$, $\underline{x} = T_{98}(x)$, $d_{abc} = \det[P_a, P_b, P_c]$ and $F = \det[P_j, P_k, P_{10}]$.

In [2], (11) was identified with the three-term relation for the elliptic hypergeometric function ${}_{10}E_9$[10]. An advantage of the geometric method is that it can be applied also for the degenerate cases such as q-Painlevé equations [3, 6] to provide a "good" coordinate for identifying the linearized equation with three-term relation of appropriate q-hypergeometric function.

References

[1] I. Dolgachev and D. Ortland, *Point sets in projective spaces and the theta functions*, Astérisque **165** (1998) Soc. Math. de France.

[2] K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada, ${}_{10}E_9$ *solution to the elliptic Painlevé equation*, J. Phys. A: Math. Gen. **36** (2003) L263–L272.

[3] -. *Hypergeometric solutions to the q-Painlevé equation*, Internat. Math. Res. Notices **47** (2004) 2497–2521.

[4] -. *Cubic pencils and Painlevé Hamiltonians*, nlin.SI/0403009, to appear in Funkcial. Ekvac. **48** (2005) 147–160.

[5] -. *Point configurations, Cremona transformations and the elliptic difference Painlevé equation*, nlin.SI/0411003.

[6] -. *Construction of hypergeometric solutions to the q-Painlevé equations*, nlin.SI/0501051, to appear in Internat. Math. Res. Notices.

[7] Ju. I. Manin, *The Tate height of points on an abelian variety, its variants and applications*, Izv. Akad. Nauk SSSR Ser. Mat. **28** (1964), 1363–1390. Amer. Math. Soc. Transl. (2) **59** (1966), 82–110.

[8] Y. Ohta, A. Ramani and B. Grammaticos, *An affine Weyl group approach to the eight-parameter discrete Painlevé equation*, J. Phys. A: Math. Gen. **34** (2001) 10523–10532.

[9] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations*, Commun. Math. Phys. **220** (2001) 165–229.

[2]The curve C_α itself may move due to the shift of the reference points $P_1, \ldots, P_9$.

[10] V. P. Spiridonov and A. S. Zhedanov, *Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids*, Special Functions 2000, Kluwer, 2001, pp.365-388.

[11] T. Tsuda, *Integrable mappings via rational elliptic surfaces*, J. Phys. A: Math. Gen. **37** (2004) 2721–2730.

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