Diversity among banks may increase systemic risk

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August 2012

Abstract

The problem of how to stabilize the financial system has attracted considerable attention since the global financial crisis of 2007-2009. Recently, Beal et al. (2011, "Individual versus systemic risk and the regulator's dilemma", *Proc Natl Acad Sci USA* 108: 12647-12652) demonstrated that higher portfolio diversity among banks would reduce systemic risk by decreasing the likelihood of simultaneous defaults. Here, I show that this result is overturned once a financial network comes into play. In a networked financial system, the failure of one bank can bring about a contagion of failure. The optimality of individual risk diversification, as opposed to economy-wide risk diversification, is thus restored. I also present a new method to quantify how the diversity of bank size affects the stability of a financial system. It is shown that a higher diversity of bank size itself makes the financial system more fragile even if external risk exposure is controlled for. The main reason for this is that larger banks are more likely to become a "super spreader" of infectious defaults. In this situation the social cost of letting a bank fail is not uniform and depends on the size of the failing bank. This strongly implies that larger banks are systemically more important than smaller banks, and preventing large banks from being exposed to high external risks would therefore be the most effective vaccine against financial crisis.

JEL Classification: C63, D85, G01.

Keywords: Systemic risk, financial crisis, financial network.

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1 Introduction

Over the past decade, simulation of financial crisis has been conducted by many researchers in various fields (Beal et al., 2011, Upper, 2011, Gai et al., 2011, Gai and Kapadia, 2010, May and Arinaminpathy, 2010). One reason for this heavy use of simulation is the lack of data on interbank trading. While such data are, at least partially, available in some countries such as Austria, Italy and the U.S., data accessibility is usually limited to the central bankers (Söramaki et al., 2007, Elsinger et al., 2006, Iori et al., 2008). One of the main objectives of simulation-based research is to gain theoretical understanding of how the topology of a financial network affects the likelihood of financial crisis. Employing concepts developed in network theory, such as degree distribution, assortativity, modularity, etc., those studies attempt to extract the necessary conditions for a stable financial system by conducting simulations under various patterns of network topology (Lenzu and Tedeschi, 2012, Gai and Kapadia, 2010, Nier et al., 2007, Kyriakopoulos et al., 2009).

On the other hand, while the main focus of those simulation studies is on the network structure, the portfolio structure of individual banks is another key to understanding systemic stability. Interestingly, Beal et al. (2011) recently pointed out that higher portfolio diversity among banks would reduce systemic risk by decreasing the likelihood of simultaneous defaults. This puts into effect the "regulator's dilemma", because it implies that minimizing the risk of individual failure does not add up to minimizing systemic risk.

In this paper, I show that this result is overturned once a financial network comes into play. In a networked financial system, the failure of one bank may bring about a contagion of failure. The optimality of individual risk diversification, as opposed to economy-wide risk diversification, is thus restored.

It is also shown that the diversity of the portfolio structure of interbank assets, as well as the portfolio structure of external assets, can be an important source of systemic fragility. In this model, the amount of funds a bank lends to bank i relative to bank j depends only on the relative size of bank i and bank j. This means that the difference between banks with respect to interbank risk exposure exclusively reflects the heterogeneity of bank size. The virtue of this property is that the pure effect of bank-size heterogeneity, or of diversity of interbank assets, can be extracted without altering external risk exposure. Although

some of the recent works on systemic risk also explore the role of portfolio diversity in the context of a financial network (Gai and Kapadia, 2010, Gai et al., 2011), to the best of my knowledge, this paper is the first to provide a methodology that can quantify the pure effects of bank-size heterogeneity or of the diversity of interbank assets. I formally show that diversity among banks in terms of interbank asset allocation is inefficient from the point of view of systemic risk minimization as well as individual risk minimization.

The simulation suggests that an extremely large bank is prone to become a "super spreader" of infectious defaults. I propose two measures to quantify the systemic importance of a bank: infectivity and susceptibility. It is shown that a very large bank is highly infective in the sense that a failure of the bank will cause contagious defaults with high probability. By contrast, a highly infective bank is not necessarily susceptible to contagion in the sense that a failure of the bank is in most cases caused by a "fundamental default". This strongly implies that larger banks are systemically more important than smaller banks, and that preventing large banks from being exposed to high external risks will be the most effective vaccine against contagious defaults.

2 Model 1: Uniform bank size

There are N interconnected banks whose balance-sheet sizes are identical. The asset side of bank i's balance sheet consists of risky external assets, a_i, interbank assets, l_i, and the riskless asset, b_i. In general each bank holds K kinds of risky external assets and N interbank assets, where $a_i = \sum_{k=1}^{K} a_{i,k}$ and $l_i = \sum_{j\neq i}^{N} l_{i,j}$. The liability side of bank i's balance sheet consists of interbank liability, $\bar{p}_i$, deposits, d_i and networth, w_i. The balance sheet condition implies that $a_i + b_i + l_i = \bar{p}_i + d_i + w_i$, $\forall\, i \in [1, N]$. The relative proportions of a_i, b_i and l_i are common to all banks, while the portfolio structure of external risk assets, $(a_{i,1}, \ldots, a_{i,K})$, and the portfolio structure of interbank assets, $(l_{i,1} \ldots, l_{i,N})$ are allowed to vary from bank to bank.

The amount of bank i's borrowings from bank j is expressed as $\pi_{i,j}\bar{p}_i$, where $\pi_{i,j}$ denotes the relative weight of bank i's borrowings from j, and thereby $\sum_{j\neq i} \pi_{i,j} = 1$, $\forall\, i \in [1, N]$.

Let the N-by-N matrix $\mathbf{\Pi}$ have $\pi_{i,j}$ as its (i, j)-th elements. The diagonal elements

are all zero because no banks lend to themselves. The amount of bank i's total interbank assets, l_i, can be given as

$$l_i = \sum_{j \neq i} l_{i,j} = \sum_{j \neq i} \pi_{j,i} \bar{p}_j,$$

In vector form,

$$\boldsymbol{l} = \boldsymbol{\Pi}' \bar{\boldsymbol{p}}.$$

All the off-diagonal elements of Π are assumed to be $1/(N-1)$ in Model 1.

Bank i tries to pay back the full amount of interbank liability $\bar{p}_i$ after the external asset values are realized, but the funds available at that time are expressed as

$$\sum_{j \neq i} \pi_{j,i} p_j + \tilde{a}_i + b_i - d_i, \tag{1}$$

where $\tilde{a}$ denotes the ex-post external assets. It should be pointed out that deposits d_i are reserved because deposits are senior to interbank assets. I also assume the following three conditions: (a) limited liability; (b) the priority of debt claims; and (c) the proportionality of the amount of repayment after default (Eisenberg and Noe, 2001). The actual amount of repayment, p_j, is not necessarily equal to its face value, $\bar{p}_j$, since bank j's loss of assets might be greater than its networth, w_j. This also means that the solvency of a bank depends on the debtors' solvency. Specifically, the market clearing vector, $\boldsymbol{p}^*$, is expressed as a fixed point of the map, $\boldsymbol{\Phi} : [\boldsymbol{0}, \bar{\boldsymbol{p}}] \to [\boldsymbol{0}, \bar{\boldsymbol{p}}]$ defined as

$$\boldsymbol{\Phi}(\boldsymbol{p}) = \bar{\boldsymbol{p}} \wedge (\boldsymbol{\Pi}' \boldsymbol{p} + \tilde{\boldsymbol{a}} + \boldsymbol{b} - \boldsymbol{d}), \tag{2}$$

where $\wedge$ denotes the meet operator, such that $\boldsymbol{x} \wedge \boldsymbol{y} = (min(x_1, y_1), \ldots, min(x_n, y_n))$. Under the reasonable conditions (a), (b) and (c), Eisenberg and Noe prove that a fixed point of (2) exists and is unique if the financial network is a complete graph and $\sum_i (\tilde{a}_i + b_i - d_i) > 0$. They also show that the unique fixed point is the solution of the following problem:

$$\max_{\boldsymbol{p} \in [\boldsymbol{0}, \bar{\boldsymbol{p}}]} f(\boldsymbol{p}) \text{ s.t. } \boldsymbol{p} \leq \boldsymbol{\Pi}' \boldsymbol{p} + \tilde{\boldsymbol{a}} + \boldsymbol{b} - \boldsymbol{d}$$

for any strictly increasing function $f(\cdot) : \mathbb{R}^N \to \mathbb{R}$.

The simulation procedure is as follows. First, the size and the composition of the balance sheet are chosen. Given the networth and the asset-capital ratio, the size of the

balance sheet is automatically determined. In model 1, it is assumed that the networth is identical across banks. Interbank assets and liabilities are determined simultaneously as one assigns $\mathbf{\Pi}$ and $\bar{\boldsymbol{p}}$. The ratio of the total amount of external risk asset to networth is fixed and common to all banks. Deposits and safe assets are determined residually so that the LHS and the RHS of the balance sheet are equal. Second, the random returns of K external risk assets are drawn. Let n_{nc} be the number of defaults at this stage (i.e., "fundamental defaults"). Third, given the funds available for the repayment of interbank loans, $\tilde{\boldsymbol{a}} + \boldsymbol{b} - \boldsymbol{d}$, the unique market-clearing vector, $\boldsymbol{p}^*$, is calculated. Let n_c be the number of defaults after contagion takes place. Note that $n_c - n_{nc}$ represents the number of defaults caused purely by contagion.

Following Beal et al., I assume that social costs, C, depend on the number of bank defaults, n: $C(n) = n^s, s > 0$. The expected cost is thus

$$E[C] = \sum_{n=1}^{N} q(n) n^s,$$

where $q(\cdot)$ is the probability function. The case of $s > 1$ means that simultaneous multiple bank defaults will be more costly than the sum of single bank defaults.

3   Results of Model 1

Figure 1 illustrates how the expected costs depend on the distance between banks, D, and the distance from the optimal portfolio, G, which are defined as

$$D \equiv \frac{1}{2N(N-1)} \sum_i \sum_j \sum_k |a_{i,k} - a_{j,k}|, \tag{3}$$

$$G \equiv \frac{1}{N} \sum_k \left| \sum_i (a_{i,k} - 1/K) \right|. \tag{4}$$

Beal et al. demonstrate that when $s > 1$, the social costs tend to be reduced by increasing D. This is because the greater the distance between banks, the lower the probability of simultaneous defaults. This is reconfirmed in the simulation without contagion (Figures 1A, 1C). On the other hand, it turns out that this outcome does not hold true once the possibility of contagion is taken into account (Figures 1B, 1D). In the latter case a bank's

default undermines the lenders' balance sheets, which would give rise to further defaults. In the presence of such a negative spillover effect, the optimal combination of (D, G) moves toward the origin compared to the case without contagion. It follows that the stability of an interconnected financial system will be achieved by ensuring the stability of individual banks.

4 Model 2: Variable bank size

Previous studies on financial stability usually assume that a financial system consists of identical banks in terms of the size of balance sheets. This assumption is ubiquitous because introducing difference in bank size generally entails heterogeneity on the asset-side of the balance sheets. Note that an increase of one bank's interbank borrowing is always accompanied by a rise in the lenders' interbank assets. The relative amounts of a_i, b_i and l_i thus generally vary with bank-size distribution, which generates heterogeneity of risk exposure. This heterogeneity prohibits the researcher from extracting the pure effect of bank-size diversity on financial stability.

Here, I show a way to examining the relation between bank-size diversity and financial stability, keeping the relative size of a_i, b_i and l_i fixed for all $i \in [1, N]$. The idea is that the elements of $\mathbf{\Pi}$ and $\bar{\boldsymbol{p}}$, which were exogenous in the previous section, are adjusted to ensure that bank-size difference is reflected only in the portfolio of interbank assets. Specifically, I assume the following:

$$\frac{\sum_{j \neq i+1} \pi_{j,i+1} \bar{p}_j}{\sum_{j \neq i} \pi_{j,i} \bar{p}_j} = \frac{\omega_{i+1}}{\omega_i}, \quad \forall\, i \in [1, N-1], \tag{5}$$

$$\frac{\pi_{j+1,i} \bar{p}_{j+1}}{\pi_{j,i} \bar{p}_j} = \left(\frac{\omega_{j+1}}{\omega_j}\right)^{\alpha}, \\ \forall\quad i \in [1, N],\, j \in [1, N-1],\, i \neq j, \tag{6}$$

$$\sum_{j \neq i} \pi_{i,j} = 1, \quad \forall\, i \in [1, N], \tag{7}$$

where ω_i denotes the size of bank i's balance sheet. Eq.(5) requires that the ratio of l_{i+1} to l_i be equal to the relative size of bank $i + 1$ to bank i. This also ensures that the fraction of interbank assets in the balance sheet is identical between banks. Eq.(6) states that the amount of bank i's interbank lending to $j+1$ relative to j, $l_{i,j+1}/l_{i,j}$, is equal to the relative

size of bank $j + 1$ to the power α. With this condition bank-size differences are directly reflected in the portfolio of interbank assets. The case of $\alpha > 1$ means that the effect of bank-size heterogeneity is magnified.

Note that there are $(N^2 - 1)$ restrictions in total. Since the number of off-diagonal elements of Π is $N(N - 1)$ and the vector $\hat{\boldsymbol{p}} = (\bar{p}_2, \ldots \bar{p}_N)$ has $N - 1$ elements, all the $(N^2 - 1)$ conditions may be satisfied by appropriately choosing the elements of Π and $\hat{\boldsymbol{p}}$. $\bar{p}_1$ is used as the numéraire, which standardizes the amounts of interbank loans and interbank assets. I find that the solution for $\boldsymbol{\Pi}$ and $\hat{\boldsymbol{p}}$ respectively satisfy $0 < \pi_{i,j} < 1$, $\forall\ (i,j) \in \{(i,j) : i \in [1,N], j \in [1,N], i \neq j\}$ and $\bar{p}_i > 0,\ \forall\ i \in [2,N]$. Details of the derivation are shown in the Appendix. Note that the Eisenberg and Noe's sufficient condition can be applied since the financial network is a complete graph.

As noted above, a virtue of this method is that heterogeneity of bank size is reflected exclusively in the portfolio of interbank assets, so that a_i/ω_i, b_i/ω_i and l_i/ω_i are common to all $i \in [1, N]$ regardless of the size of the bank.

This property allows us to evaluate the influence of bank-size distribution on systemic risk without affecting the probability of fundamental default. It should be pointed out that such irrelevancy of bank-size distribution may not hold true if some of the elements of $\boldsymbol{p}^*$ are zero, in which case the liability structure could affect the likelihood of defaults. To avoid this situation, $\bar{p}_1$ should be set at a sufficiently large value. The parameter configuration is shown in the appendix.

5   Results of Model 2

Figures 2A, 2B and 2C illustrate the total number of defaults under uniform bank size and under diverse bank size. The index of the bank is in ascending order with respect to bank size. The distribution of bank size is illustrated in Figure 4A. The most notable feature is that a failure of the largest bank is less frequent under diverse bank size than under uniform size, while the opposite is true for smaller banks. When one of the banks in a financial system is very large, the largest bank is unlikely to fail unless it fails by fundamental default. In contrast, higher diversity of bank size increases the likelihood of a

failure of smaller banks because smaller banks are more susceptible to contagious defaults.

Figure 3 shows the extent of "infectivity" and "susceptibility". By the infectivity of bank i, I mean the total probability of generating contagious defaults conditional on a single fundamental default of bank i. By the susceptibility of bank i, I mean the total probability of suffering from infectious defaults conditional on a single fundamental default of bank $j \neq i$. As the figure shows, a failure of the largest bank is highly infective while its susceptibility is rather low. This is because the smaller banks have a large risk exposure to the largest bank while the largest bank has a relatively more uniform risk exposure. The portfolio of interbank assets, as well as the portfolio of external assets, is quite important in evaluating systemic stability. Other things being equal, more heterogeneous exposure in the interbank market leads to a more unstable financial system.

This point is also confirmed in Figure 4 in terms of expected costs. This figure illustrates the relation between the expected social costs and the Herfindahl-Hirschman Index (HHI), defined as

$$HHI = \sum_{i=1}^{N} S_i^2,$$

where $S_i \in [0, 1]$ is the market share of bank i. The higher the HHI, the more concentrated the market is. Figure 4 implies that the probability of a severe contagion becomes higher as the financial market becomes more concentrated. Although the network structure always takes a form of a complete graph, the introduction of bank-size diversity will make it easier for infectious defaults to be transmitted.

The question is how we could prevent a large bank from being a "super spreader" of infectious defaults. I here suggest that the large bank's portfolio of external risk assets should be diversified preferentially. One measure useful for this purpose is a correlation between bank size and the l_1 distance from the optimal portfolio, $\boldsymbol{GI} = (GI(1), \ldots, GI(N))$, where

$$GI(i) \equiv \sum_k |a_{i,k} - 1/K|, \quad \forall\, i \in [1, N]. \tag{8}$$

Figure 5 indicates that the expected social cost is the lowest when there is a strong negative correlation between bank size and GI. This corresponds to a situation in which larger banks have less risky portfolios of external risk assets, i.e., $(a_{i,1}, \ldots a_{i,K})$ is more uniformly distributed around $1/K$ than smaller banks. The systemic risk of an interbank

market can be effectively lowered by having the "too-big-too-fail" banks diversify their external risks.

6 Discussion

Over the past 5 years the difference between micro- and macro-prudential policies has been emphasized by many academic economists, central bankers and other regulators. Since the failure of Bear Stearns and Lehman Brothers in 2008, there seems to have been wide agreement that micro-prudential policies do not necessarily add up to a macro-prudential policy. An ongoing challenge for regulators is to search for the best macro-prudential policy to stabilize the financial system as a whole.

One extreme of the arguments is that the portfolio of risk assets should be diversified from bank to bank so that simultaneous multiple failures would not occur. The negative side of this proposal is that systemic stability requires the self-sacrifice of individual banks. It is important to note that such an argument misses a crucial aspect of actual financial systems: interconnectivity. Almost by construction, a financial system is a networked system in which individual banks are connected with each other via the interbank market. It is widely recognized that a crash in the interbank market was one of the most important factors leading Bear Stearns and Lehman to fail.

In this paper I show that diversity among banks should be minimized in order to enjoy maximum financial stability. The main implications can be summarized as follows: First, every individual bank should be encouraged to minimize its own external risk. In an interconnected financial system, independently minimizing the risk of individual default will also minimize systemic risk by reducing the probability of contagious defaults. Therefore, high diversity among banks in terms of their portfolios of external assets will not be optimal. Second, diversity of bank size itself tends to make the financial system more fragile. This is because an extremely large bank is prone to become a "super spreader''. While the balance sheets of large banks are not so sensitive to the failure of other banks, smaller banks are highly susceptible to the failure of large banks. In such a "skewed'' financial system, preventing large banks from being exposed to high external risks is the most effective

vaccine against systemic turmoil. At the same time, this also implies that the regulators should supervise the network structure thoroughly. Systemic risk increases as the extent of heterogeneity in the interbank asset allocation increases, given the risk exposure of external assets. Heterogeneity of interbank trading itself may be an important source of financial crisis. A more concrete investigation into the relationship between network structure and financial stability is left for future research.

7 Appendix

7.1 Baseline parameters

Throughout the paper, the number of banks, N, and assets, K, are 5 and 3, respectively. The mean bank size is 100 for both Model 1 and Model 2. $\bar{p}_i/w_i = 4 \ \forall \ i \in [1, N]$ in Model 1, and $\bar{p}_1/w_1 = 4$ in Model 2. $a_i/w_i = 2$ and $w_i/\omega_i = 0.1, \ \forall \ i \in [1, N]$ in both models. The parameter α in Eq.(6) is 1.2. The systemic cost parameter, s, is set at 1 or 4.

7.2 Generating the loss of external risk assets

The portfolio of external risk assets is chosen as follows: first, the fraction of asset 1 held by bank i, $a_{i,1}^f \equiv a_{i,1}/a_i$, is set at $1/K$ for all i. Second, bank $i's$ relative holding of asset 2, $a_{i,2}^f$, is taken from uniform distribution on $[0, 1 - a_{i,1}^f]$. The fraction of asset 3 is then taken from uniform distribution on $[0, 1 - a_{i,1}^f - a_{i,2}^f]$, and so forth. Finally, the relative amount of asset K must satisfy $a_{i,K}^f = 1 - \sum_{k=1}^{K-1} a_{i,k}^f$.

I assume that the loss of each asset follows a normal distribution and the probability that the asset loss takes a value greater than x_0 is $1 - p_0$ if the bank invests in a single asset. Let $p_0 \equiv F(x_0 | \mu, \sigma_0)$ denote the CDF of a normal distribution with mean μ and standard deviation σ_0. If $\mu = 0$, the CDF can be written as

$$F(x_0 | 0, \sigma_0) = \frac{1}{2} \left[1 + \frac{2}{\sqrt{\pi}} \int_0^{\frac{x_0}{\sigma_0 \sqrt{2}}} e^{-t^2} dt \right].$$

Let $p_w \equiv F(w | 0, \sigma_w)$ be the probability that the asset loss is lower than the networth, w. To ensure that the probability of fundamental default takes an arbitrary value $1 - p_0$, we need to have $p_w = p_0$, or $w/\sigma_w = x_0/\sigma_0$. It follows that $\sigma_w = w\sigma_0/x_0$. The *ex post* asset

value thus leads to

$$\tilde{a}_{i,k} = \max\left(0,\ a_{i,k} - a_{i,k}\frac{\theta\sigma_0}{x_0}x_k\right),\ \forall\, i \in [1, N],\ k \in [1, K].$$

where the capital-asset ratio, $\theta = w_i/a_i$, is common to all i and $x_k \sim i.i.n.(0, 1)$, $\forall\, k \in [1, K]$. This implies that the rate of return of asset k is $\max(-1,\ \frac{\theta\sigma_0}{x_0}x_k)$. Note that if $a_{i,k} = a_i$, $\forall\, k \in [1, K]$, then the probability of bank i's default is $1 - p_0$. p_0 is set at 0.9 in the simulation.

7.3   Generating bank-size distribution

In order to capture the influences purely caused by bank-size heterogeneity, a sequence of bank-size distribution needs to be generated. Here, I consider a growth process that allows bank size to deviate gradually from the uniform distribution and converge to a power-law distribution. The basic process of the bank-size growth is given by

$$\omega_{i,t+1} = \zeta_{t+1}^i \omega_{i,t}, \tag{9}$$

where ζ_{t+1}^i, $i =\in [1, N]$ are independently and identically distributed growth rates. It is important to note that ζ_{t+1}^i is independent of the bank size. I assume that banks of size 100 are uniformly distributed at the beginning, and then randomly chosen banks grow by the rate of $\zeta \in \{.95, 1.1\}$ at each step. As Gabaix pointed out, this simple process produces a log-normal distribution without a steady state. Some frictions need to be added for a power-law distribution to arise in a steady state (Gabaix, 1999). Here, I let the lower bound of w^i be 5. Throughout the growth process the average size of banks is kept unchanged. I repeat the growth process 1000 times.

7.4   Derivation of Π and $\hat{p}$

From Eqs.(5), (6) and (7), the following conditions can be obtained:

$$\pi_{13} = \pi_{12}\left(\frac{\omega_3}{\omega_2}\right)\frac{1 + \left(\frac{\omega_3}{\omega_1}\right)^\alpha + \left(\frac{\omega_4}{\omega_1}\right)^\alpha + \ldots + \left(\frac{\omega_N}{\omega_1}\right)^\alpha}{1 + \left(\frac{\omega_2}{\omega_1}\right)^\alpha + \left(\frac{\omega_4}{\omega_1}\right)^\alpha + \ldots + \left(\frac{\omega_N}{\omega_1}\right)^\alpha}, \tag{10}$$

$$\pi_{14} = \pi_{12}\left(\frac{\omega_4}{\omega_2}\right)\frac{1 + \left(\frac{\omega_3}{\omega_1}\right)^\alpha + \left(\frac{\omega_4}{\omega_1}\right)^\alpha + \ldots + \left(\frac{\omega_N}{\omega_1}\right)^\alpha}{1 + \left(\frac{\omega_2}{\omega_1}\right)^\alpha + \left(\frac{\omega_3}{\omega_1}\right)^\alpha + \left(\frac{\omega_5}{\omega_1}\right)^\alpha + \ldots + \left(\frac{\omega_N}{\omega_1}\right)^\alpha}, \tag{11}$$

10

$$\vdots$$

Inserting these equations into Eq.(7) yields

$$\pi_{12}\left[1+\left(\frac{\omega_3}{\omega_2}\right)\frac{1+\left(\frac{\omega_3}{\omega_1}\right)^\alpha+\left(\frac{\omega_4}{\omega_1}\right)^\alpha+\ldots+\left(\frac{\omega_N}{\omega_1}\right)^\alpha}{1+\left(\frac{\omega_2}{\omega_1}\right)^\alpha+\left(\frac{\omega_4}{\omega_1}\right)^\alpha+\ldots+\left(\frac{\omega_N}{\omega_1}\right)^\alpha}\right.$$
$$\left.+\left(\frac{\omega_4}{\omega_2}\right)\frac{1+\left(\frac{\omega_3}{\omega_1}\right)^\alpha+\left(\frac{\omega_4}{\omega_1}\right)^\alpha+\ldots+\left(\frac{\omega_N}{\omega_1}\right)^\alpha}{1+\left(\frac{\omega_2}{\omega_1}\right)^\alpha+\left(\frac{\omega_3}{\omega_1}\right)^\alpha+\left(\frac{\omega_5}{\omega_1}\right)^\alpha+\ldots+\left(\frac{\omega_N}{\omega_1}\right)^\alpha}+\ldots\right]=1 \qquad (12)$$

Thus π_{12}, $\pi_{13},\ldots,\pi_{1N}$ are obtained from Eqs.(10), (11) and (12). From Eq.(7), we have

$$\pi_{2,1}\bar{p}_2+\pi_{2,3}\bar{p}_2+\pi_{2,4}\bar{p}_2+\ldots+\pi_{2,N}\bar{p}_2=\bar{p}_2. \qquad (13)$$

It follows that

$$\pi_{2,1}\bar{p}_2+\left(\frac{\omega_2}{\omega_1}\right)^\alpha\pi_{1,3}\bar{p}_1+\left(\frac{\omega_2}{\omega_1}\right)^\alpha\pi_{1,4}\bar{p}_1+\ldots+\left(\frac{\omega_2}{\omega_1}\right)^\alpha\pi_{1,N}\bar{p}_1=\bar{p}_2. \qquad (14)$$

Using the relation

$$\pi_{2,1}\bar{p}_2=\pi_{1,2}\bar{p}_1\left(\frac{\omega_1}{\omega_2}\right)\frac{1+\left(\frac{\omega_3}{\omega_1}\right)^\alpha+\left(\frac{\omega_4}{\omega_1}\right)^\alpha+\ldots+\left(\frac{\omega_N}{\omega_1}\right)^\alpha}{1+\left(\frac{\omega_3}{\omega_2}\right)^\alpha+\left(\frac{\omega_4}{\omega_2}\right)^\alpha+\ldots+\left(\frac{\omega_N}{\omega_2}\right)^\alpha}, \qquad (15)$$

we obtain $\pi_{2,1},\ldots,\pi_{2,N}$ and $\bar{p}_2$. In the same manner, $\pi_{3,1},\ldots\pi_{3,N}$ and $\bar{p}_3$ can be derived by noting that

$$\pi_{3,1}\bar{p}_3+\pi_{3,2}\bar{p}_3+\pi_{3,4}\bar{p}_3+\ldots+\pi_{3,N}\bar{p}_3=\bar{p}_3 \qquad (16)$$

and

$$\left(\frac{\omega_3}{\omega_2}\right)^\alpha\pi_{2,1}\bar{p}_2+\left(\frac{\omega_3}{\omega_1}\right)^\alpha\pi_{1,2}\bar{p}_1+\left(\frac{\omega_3}{\omega_1}\right)^\alpha\pi_{1,4}\bar{p}_1+\ldots+\left(\frac{\omega_3}{\omega_1}\right)^\alpha\pi_{1,N}\bar{p}_1=\bar{p}_3. \qquad (17)$$

In this way, the other elements of $\mathbf{\Pi}$ and $\hat{\boldsymbol{p}}$ can also be computed recursively. A Matlab program for solving these equations is available from the author upon request.

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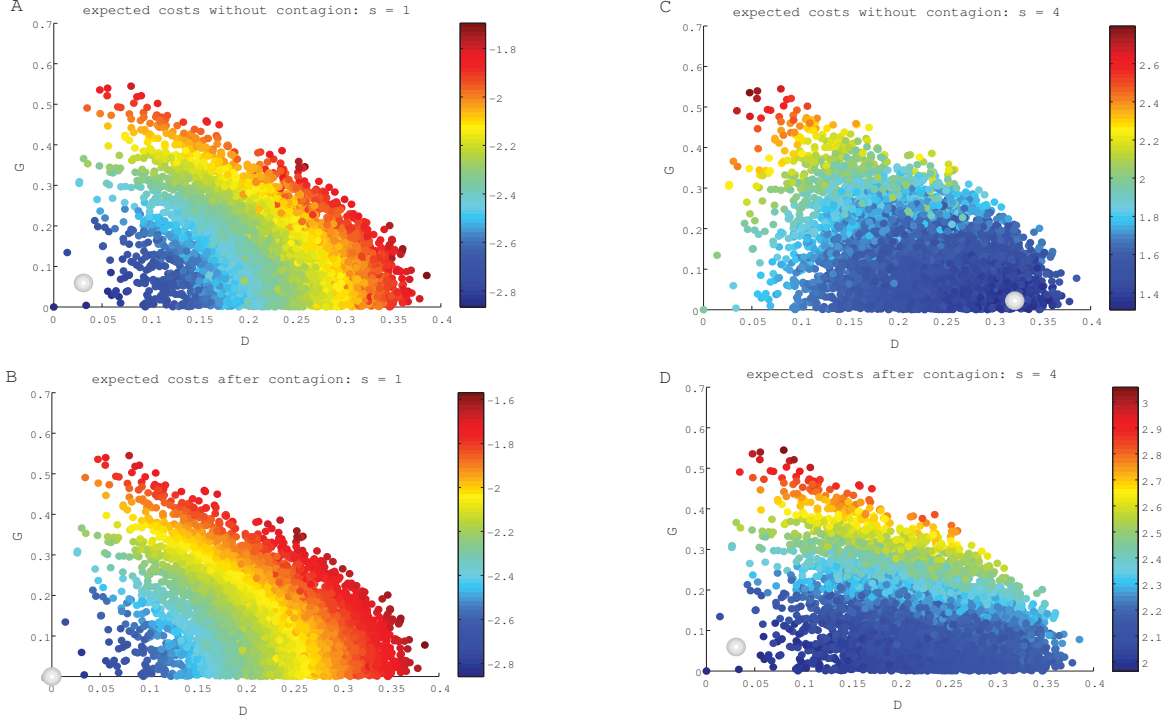
Fig. 1: The logged expected costs as a function of D and G. The optimal combination of (D, G) is shown by the gray sphere. Random draws are conducted 20000 times for 5000 patterns of external asset portfolio. The size of banks is identical. (A) Expected social costs without a financial network: $s = 1$. (B) Expected social costs with a financial network: $s = 1$. (C) Expected social costs without a financial network: $s = 4$. (D) Expected social costs with a financial network: $s = 4$. The introduction of an interbank network moves the optimal distance between banks, D, toward the origin. This is because the failure of one bank may generate contagious defaults. In the presence of a financial network, saving individual banks will save the financial system as a whole.
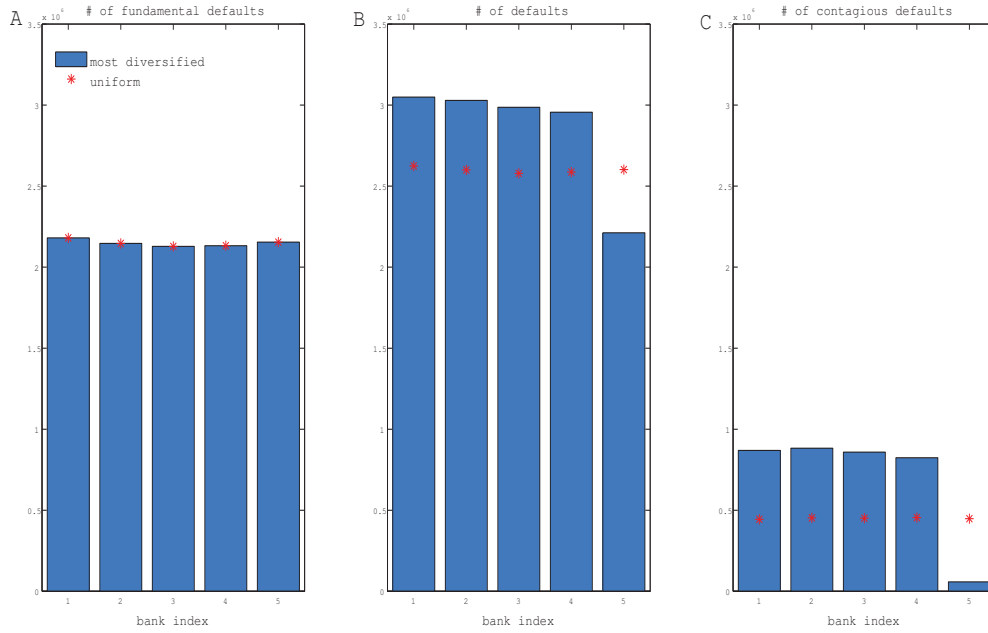
Fig. 2: The effects of bank-size distribution on the frequency of defaults. (A) Total number of fundamental defaults. (B) Total number of defaults after contagion takes place. (C) Total number of contagious defaults. The failure of an extremely large bank is less frequent than that of smaller banks.
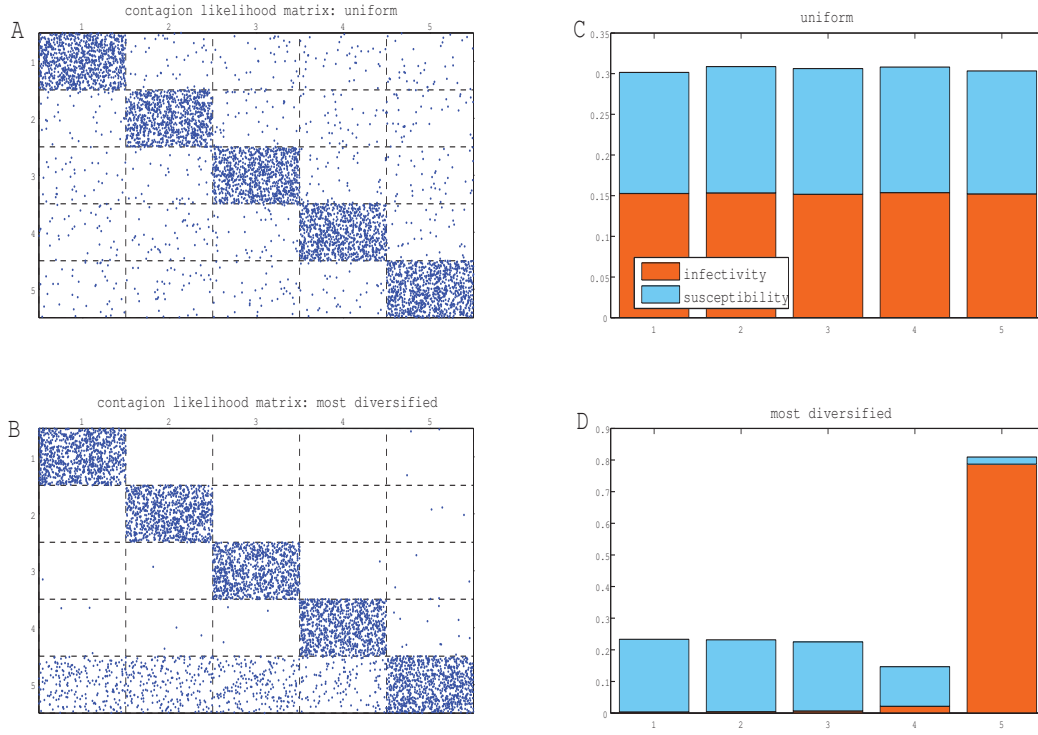
Fig. 3: Susceptibility and infectivity. (A) The contagion likelihood matrix under uniform bank size. The number of plots in the (i, j)-th element indicates the expected number of defaults of bank j per 1000-time single defaults of bank i. (B) The contagion likelihood matrix under heterogeneous bank size. (C) The degree of infectivity and susceptibility under uniform bank size. The infectivity of bank i is calculated as the sum of the off-diagonal elements of the i-th row of the contagion likelihood matrix. The susceptibility of bank i is calculated as the sum of the off-diagonal elements of the i-th column of the contagion likelihood matrix. (D) The degree of infectivity and susceptibility under heterogeneous bank size. Note that an extremely large bank is highly infective but not very susceptible.
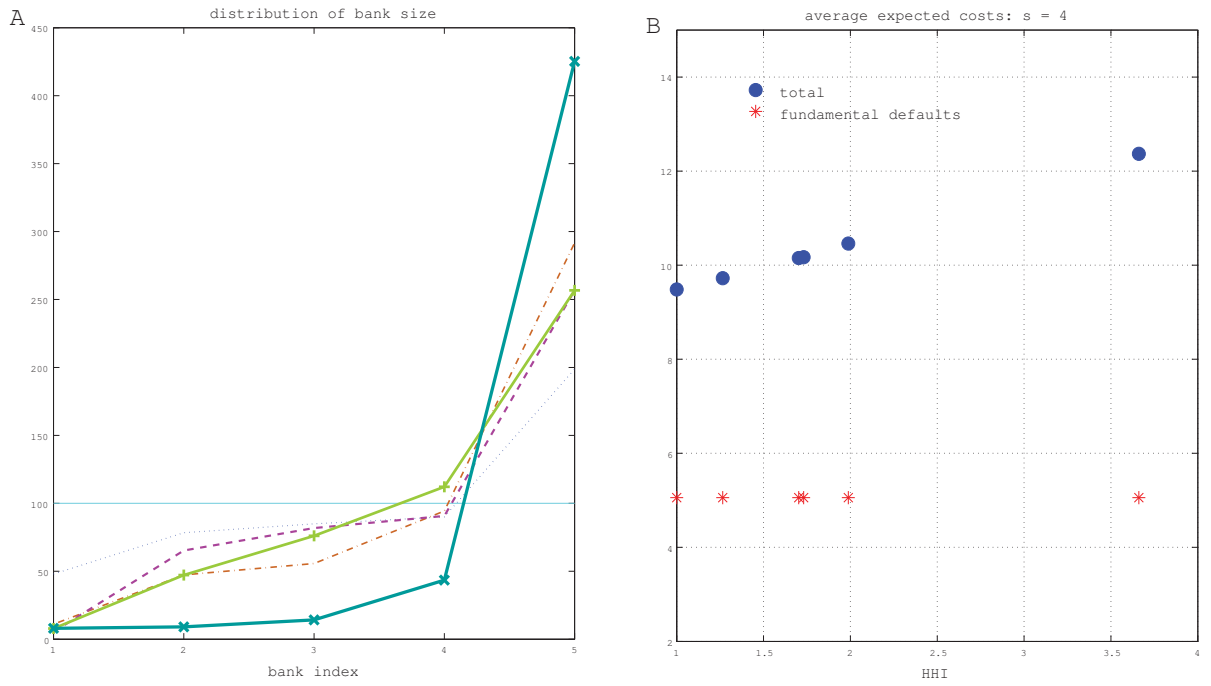
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Fig. 4: The effect of bank-size diversity on expected costs. (A) The sequence of generated bank-size distributions. (B) The relation between expected costs and the Herfindahl-Hirschman Index (HHI). Systemic risk will increase as the interbank market becomes more concentrated.
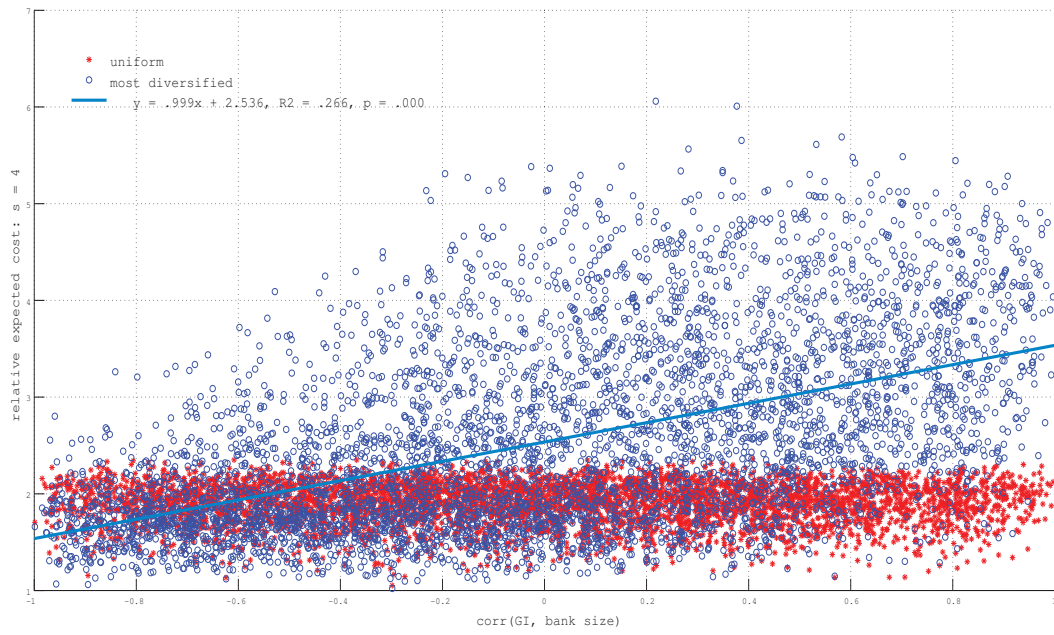
Fig. 5: The expected relative costs and the correlation between GI and bank size. GI is the distance from the optimal portfolio of external risk assets. The relative costs are defined as the ratio of total costs to the costs of fundamental defaults. The upward-sloping line is the best fit under the OLS using the simulated data under diversified bank size. This indicates that expected cost of contagion will be reduced as larger banks' external risks are decreased.

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