

PDF issue: 2025-06-07

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(Citation) Kobe University Economic Review,61:1-21

(Issue Date) 2015

(Resource Type) departmental bulletin paper

(Version) Version of Record

(JaLCDOI) https://doi.org/10.24546/81009163

(URL) https://hdl.handle.net/20.500.14094/81009163





# TWO STEADY STATES UNDER THE BALANCED BUDGET RULE—AN ECONOMY WITH DIVISIBLE LABOR\*<sup>†</sup>

# **By FUJIO TAKATA**

When governments levy taxes on labor income on the basis of a balanced budget rule, this rule causes a nonlinear system. Thus, two steady states in an economy appear. This article deals with the existence of these two steady states. Many articles in the literature discuss this issue from the viewpoint of indivisible labor. On a more general assumption of divisible labor, however, we demonstrate that there can be two steady states in the economy. Our analyses explicitly derive the labor supply function and include the case of indivisible case as a special case.

#### 1. Introduction

In this paper we demonstrate that there exist two steady states in an economy governed by a balanced budget rule, in which labor income taxation is assumed.

There is a lot of literature on the issue of indeterminacy. We can list Black (1974), Brock (1974), and Benhabib and Farmer (1994) as classical papers. Furthermore, Kamihigashi (2002) deals with this issue in terms of externality and nonlinear discounting with respect to utility. Guo and Lansing (1998) studies the problem in terms of externality and taxation and Wen (2001) of externality and depreciation.<sup>1</sup>

In indeterminacy, some non-fundamental factors such as expectations, which can be called sunspots or animal spirits, can determine a real economic path. Thus an expectation can be self-fulfilling. In this sense, indeterminacy is a hot issue. Many economists have located the causes of this indeterminacy externally in production or monopolistic competition. Obviously, these authors mainly seek the causes of indeterminacy in economic conditions, that is, in naturally produced economic outcomes. On the other hand, we would like to discuss indeterminacy from another viewpoint, that is, in relation to fiscal policy, as a product only of artificial determinants. In this context, fiscal policy specifically means labor income taxation and a balanced budget principle.

Schmitt-Grohé and Uribe (1997) examines this issue in an infinite horizon model and on the basis of a government balanced budget rule. They conclude with an affirmation that indeterminacy can be caused within a certain range of tax rates on labor income in steady states, but not within other tax rates. Their model framework is fundamentally based on Ramsey (1928), but it does not deal with planned economies only, as Ramsey (1928) did, instead extending the scope to equilibria in markets. Therefore, their discussion in relation to indeterminacy or other movement patterns depends intensively upon the existence and properties of steady states, which should be derived in

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<sup>†</sup> I have benefitted from discussions with Professor Jun Iritani. Moreover, I thank Professor Kazuo Mino for his sharp and relevant criticism. Naturally, I accept full responsibility for any errors.

These papers discuss a one-sector model. On the other hand, Benhabib and Farmer (1996) and Guo and Harrison (2001) discuss indeterminacy in a two-sector model.

their model.

Furthermore, following Hansen (1985) and Rogerson (1988), they assume indivisible labor. This assumption implies that households choose to work or do not, which leads to a recognition that a change in labor is caused only by entering or quitting the labor market. This implies that the elasticity of labor with regard to real wages is infinite, and that the household feels no risk aversion. In short, the linear utility function with regard to labor is derived by this situation. Kamihigashi (2000) and Angnostopoulos and Giannitsarou (2013) analyses are also based on this postulation.

However, this seems unnatural and too restrictive. In fact, Schmitt-Grohé and Uribe (1997) and Angnostopoulos and Giannitsarou (2013) both assume that households feel certainty in relation to consumption but not in relation to labor supply, thus do not feel risk aversion.

A more reasonable idea is that all households choose their labor supply according to adjustments along an intensive margin, that is, variations in utilization. Given this assumption, all households work the same amount, have an increasing marginal disutility with regard to labor, as well as an elasticity of labor with regard to real wages which is continuously positive, and in this situation there is no sense of risk.

In light of the argument above, we deal only with the properties of the steady states based on a new model, which can include an indivisible labor type such as Schmitt-Grohé and Uribe (1997) as a special case.

In this article, we deal with labor income taxation, in which the tax rates on labor income are flexible and, correspondingly, government expenditure is exogenously constant, under the balanced budget rule. This system causes nonlinear dynamics.<sup>2)</sup>

As a result, we conclude that there are two steady states. There can be double steady states, and we demonstrate some conditions which result in two steady states, which are established from the viewpoint of labor supply incentives in relation to labor tax rates in *Theorem 1*. More precisely, *Theorem 1* implies that there exist two steady states when government expenditure is less than or equal to the maximum revenue from labor taxation. (See the first subsection in section 3.) We also maintain that there are clear conditions under which maximum government revenues are decided, and these conditions are explicitly shown by fixed parameters alone, which indicate fundamentals in an economy. Furthermore, we demonstrate that the tax revenue is inversely proportional to the elasticity of the labor supply with regard to wages.

We organize this article as follows: In the next section, we introduce the framework and approach of the paper, in which the labor supply is deduced explicitly. And, it is affected by labor income tax rates and the elasticity. Then we offer a dynamic system. In what follows, in section 3, we demonstrate that there can be two steady states in our system. Subsequently, section 4 is devoted to a conclusion.

<sup>2)</sup> On the other hand, there is an another balanced budget rule in which tax rates remain constant while government expenditure is flexible. This system causes linear dynamics (See Guo and Harrison (2004) and Takata (2006)).

#### 2. The model

#### 2.1 Assumptions

An economy consists of three kinds of agents, namely, households, firms and a government. Households and firms are each assumed to be represented by an agent.

A representative household possesses capital and loans it to a firm in exchange for rental fees, and additionally offers its labor to a firm. The labor population is constant and households are identical. A household faces a dynamic problem in which it determines an infinite scenario of work and consumption under budget constraints over time.

#### 2.2 Optimization

First, we assume an instantaneous utility function of a representative household,  $U_l$ , as follows:

$$U_t = \log c_t - \frac{H_t^{1-\chi}}{1-\chi}.$$

Here,  $c_t$  denotes consumption and  $H_t$  working hours, respectively. A household is assumed to live forever and to maximize lifetime utility. The above utility function increases marginal disutility with respect to  $H_t$ , unlike the models in which the utility function is linear in relation to labor, such as Schmitt-Grohé and Uribe (1997).<sup>3)</sup> The latter model shows that the representative household is not risk averse in relation to employment, while the former has no uncertainty in relation to employment. Here,  $\chi$  indicates a negative constant, so that  $-1/\chi$  shows the elasticity of the labor supply with respect to wages, which will be proved later.

For simplicity, we postulate that the duration of capital is infinite, and as a result we omit depreciation.

In this situation, a household considers labor tax rates,  $\tau_t$ , real wage rates,  $w_t$ , and real rental rates of capital,  $u_t$ , as given, and solves the following dynamic problem:

$$\max_{c_t,H_t} \int_0^\infty e^{-\rho t} \left[ \log c_t - \frac{H_t^{1-\chi}}{1-\chi} \right] dt,$$

subject to

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - c_t, \quad t \in \Re_+.$$

$$\tag{1}$$

Here,  $\Re_+$  is the set of positive reals. Additionally,  $\rho$  indicates a subjective discount rate of utility.

Second, the government observes a balanced budget rule. Since the government can observe a tax base,  $w_t H_t$ , through the labor market, it determines a tax rate,  $\tau_t$ , in order to make the labor

<sup>3)</sup> In Schmitt-Grohé and Uribe (1997), instantaneous utility is expressed as  $U_t = \log c_t - AH_t$ , which obviously shows a constant marginal disutility of labor, corresponding to  $\chi = 0$  in our utility function. Furthermore, the utility regarding labor described by this linear form is expected utility.

taxes levied equal to constant government spending, G, which a balanced budged rule requires. This can be formulated as

$$G = \tau_t w_t H_t. \tag{2}$$

Third, we discuss firms.

We introduce a production function *F*, a Cobb-Douglas type, as  $F(K_t, H_t) = K_t^{\alpha} H_t^{1-\alpha}$ , where  $\alpha$  is a constant satisfying  $0 < \alpha < 1$  and implies the share of capital in output. We assume that the markets for labor, capital, and output are fully competitive, and that a firm aims to maximize its profit. As a result, the following relationships hold:

$$w_t = \frac{\partial F}{\partial H} = F_H(K_t, H_t) = (1 - \alpha) \left(\frac{K_t}{H_t}\right)^{\alpha},$$

and

$$u_t = \frac{\partial F}{\partial K} = F_K(K_t, H_t) = \alpha \left(\frac{K_t}{H_t}\right)^{\alpha - 1}$$

Here, output price is assumed to be 1.

At this stage, we define a Hamiltonian as

$$R = e^{-\rho t} \left[ \log c_t - \frac{H_t^{1-\chi}}{1-\chi} \right] + \mu_t [u_t K_t + (1-\tau_t) w_t H_t - c_t].$$

We obtain the following necessary conditions for optimization:

$$\frac{\partial R}{\partial c_t} = e^{-\rho t} \frac{1}{c_t} - \mu_t = 0,$$

which can be transformed as

$$\mu_t = \frac{e^{-\rho t}}{c_t}.$$
(3)

On the other hand,

$$\frac{\partial R}{\partial H_t} = e^{-\rho t} \left[ -H_t^{-\chi} \right] + \mu_t (1 - \tau_t) w_t = 0$$

holds. Our present maximization determines the supply of labor. The above equality, together with (3), enables us to know:

$$H_t = \left[\frac{(1-\tau_t) w_t}{c_t}\right]^{-\frac{1}{\chi}}.$$
(4)

Based on (4), we can easily confirm that the elasticity of this labor supply with respect to wages, *e*, equals  $-1/\chi$ , and this results in *e* being independent from  $\alpha$ . This shows that  $H_t$  increases when  $w_t$  does.

Furthermore, the following holds concerning the adjoint variable  $\mu_t$ :

$$\frac{d\mu_t}{dt} = -\frac{\partial R}{\partial K_t} = -\mu_t u_t,$$

which yields

$$\frac{\dot{\mu}_t}{\mu_t} = -u_t$$

Based on the above relationship and (3), we obtain an Euler equation:

$$\frac{\dot{c}_t}{c_t} = u_t - \rho$$

Here, we should notice that the Euler equation does not include tax rate  $\tau_t$ .

Moreover, for the household to project its budget constraints over time, the transversality condition needs to be satisfied,<sup>4)</sup> which can be expressed as  $\lim_{t\to\infty} \mu_t k_t = 0$ . This can be further transformed as

$$\lim_{t\to\infty}\frac{k_t}{c_t}\;e^{-\rho\;t}=0$$

Here, we define the capital intensity of labor as  $k_t = K_t/H_t$ .

Of course, the transversality condition means that the present value of the marginal utility of consumption at infinity in terms of capital stock, which can be transformed into consumption through perfect substitution, must be null.<sup>5</sup>)

In addition, since R is concave with respect to  $c_t$ ,  $H_t$ ,  $\mu_t$  and  $K_t$ , the conditions sufficient for the objective function, are satisfied. In short, solutions satisfying necessary conditions are solutions for optimization.

#### 2.3 Dynamics

Now, we need to derive the dynamic system of our economy. First, in light of  $k_t = K_t/H_t$  we obtain the following, from (1):

$$\dot{k_t} = u_t \frac{K_t}{H_t} + (1 - \tau_t) w_t - \frac{c_t}{H_t} - k_t \frac{\dot{H}_t}{H_t}.$$
(5)

<sup>4)</sup> We assume that a household eventually dies at infinity, and that capital stock can be transformed into consumption with perfect substitution.

<sup>5)</sup> See Blanchard and Fischer (1989).

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Second, based on (4), we can express the rate of change in the labor supply as

$$\frac{\dot{H}_{t}}{H_{t}} = \frac{1}{\chi} \frac{\frac{\dot{c}_{t}}{c_{t}} \frac{1 - \tau_{t}}{\tau_{t}} + \frac{\dot{\tau}_{t}}{\tau_{t}} - \frac{1 - \tau_{t}}{\tau_{t}} \frac{\dot{w}_{t}}{w_{t}}}{\frac{1 - \tau_{t}}{\tau_{t}}}.$$
(6)

Here, let us consider government behavior, which is clearly described by (2), and which can be transformed as

$$\frac{\dot{\tau}_t}{\tau_t} = -\frac{\dot{w}_t}{w_t} - \frac{\dot{H}_t}{H_t}.$$
(7)

Substituting (7) into (6) and rearranging the result, we eventually obtain the rate of change in the labor supply as

$$\frac{\dot{H}_t}{H_t} = \frac{1}{(1-\chi) \ \tau_t + \chi} \left( \frac{\dot{c}_t}{c_t} \ (1-\tau_t) - \frac{\dot{w}_t}{w_t} \right). \tag{8}$$

Additionally, we of course postulate that  $(1 - \chi) \tau_t + \chi \neq 0$ .

Now, let us begin to calculate  $\dot{k_t}$ , which is described in (5). Then we can obtain a dynamic equation with regard to  $k_t$  as

$$\frac{\dot{k_t}}{k_t} = \frac{(\alpha + (1 - \alpha)(1 - \tau_t))((1 - \chi) \tau_t + \chi) - \alpha (1 - \tau_t)}{(1 - \chi) \tau_t + \chi - \alpha} k_t^{-(1 - \alpha)} - \frac{c_t^{1 - \frac{1}{\chi}} (1 - \tau_t)^{\frac{1}{\chi}} (1 - \alpha)^{\frac{1}{\chi}} ((1 - \chi) \tau_t + \chi)}{(1 - \chi) \tau_t + \chi - \alpha} k_t^{\frac{\alpha}{\chi} - 1} + \frac{\rho (1 - \tau_t)}{(1 - \chi) \tau_t + \chi - \alpha}.$$
(9)

Additionally, we can confirm  $(1 - \chi)\tau_t + \chi - \alpha \neq 0$ .

We offer a proof in Appendix A. Next, we explore a dynamic equation with regard to  $\tau_t$ , which plays a critical role in our system, as will be revealed later.

Based on (7) and (8), we obtain the following:

$$\frac{\dot{\tau}_t}{\tau_t} = -\frac{\dot{w}_t}{w_t} - \frac{1}{(1-\chi)\tau_t + \chi} \left(\frac{\dot{c}_t}{c_t}(1-\tau_t) - \frac{\dot{w}_t}{w_t}\right) \\
= \frac{1-\tau_t}{(1-\chi)\tau_t + \chi} \left[\alpha (1-\chi)\frac{\dot{k}_t}{k_t} - \frac{\dot{c}_t}{c_t}\right].$$
(10)

In light of (9) and the Euler equation, (10) can be transformed into (11):

$$\frac{\dot{\tau}_{t}}{\tau_{t}} = \frac{1 - \tau_{t}}{(1 - \chi)\tau_{t} + \chi} \left[ \alpha \frac{(\chi + (1 - \chi)\tau_{t})((\alpha - \chi) - (1 - \chi)(1 - \alpha)\tau_{t})}{(1 - \chi)\tau_{t} + \chi - \alpha} k_{t}^{\alpha - 1} - \frac{\alpha(1 - \chi)c_{t}^{1 - \frac{1}{\chi}}(1 - \tau_{t})^{\frac{1}{\chi}}(1 - \alpha)^{\frac{1}{\chi}}((1 - \chi)\tau_{t} + \chi)}{(1 - \chi)\tau_{t} + \chi - \alpha} k_{t}^{\frac{\alpha}{\chi} - 1} + \frac{\rho(1 - \alpha)((1 - \chi)\tau_{t} + \chi)}{(1 - \chi)\tau_{t} + \chi - \alpha} \right].$$
(11)

As a consequence, we can establish our dynamic system, consisting of (9), the Euler equation and (11).

At this stage, we introduce new variables:  $\log k_t = \lambda_t$ ,  $\log c_t = \delta_t$  and  $\log \tau_t = \eta_t$ . In this situation, our system can be expressed as follows:

$$\dot{\delta}_t = \alpha e^{-(1-\alpha)\lambda_t} - \rho. \tag{12}$$

From this we can extrapolate the following:

$$\begin{aligned} \dot{\lambda}_{t} &= \frac{(\alpha + (1 - \alpha)(1 - e^{\eta_{t}}))((1 - \chi)e^{\eta_{t}} + \chi) - \alpha(1 - e^{\eta_{t}})}{(1 - \chi)e^{\eta_{t}} + \chi - \alpha} e^{(\alpha - 1)\lambda_{t}} \\ &- \frac{e^{\frac{\chi - 1}{\chi}\delta_{t}}(1 - e^{\eta_{t}})^{\frac{1}{\chi}}(1 - \alpha)^{\frac{1}{\chi}}((1 - \chi)e^{\eta_{t}} + \chi)}{(1 - \chi)e^{\eta_{t}} + \chi - \alpha} e^{(\frac{\alpha}{\chi} - 1)\lambda_{t}} \\ &+ \frac{\rho(1 - e^{\eta_{t}})}{(1 - \chi)e^{\eta_{t}} + \chi - \alpha}, \end{aligned}$$
(13)

and

$$\begin{split} \dot{\eta}_{t} &= \frac{1 - e^{\eta_{t}}}{(1 - \chi)e^{\eta_{t}} + \chi} \left[ \frac{\alpha(\chi + (1 - \chi)e^{\eta_{t}})((\alpha - \chi) - (1 - \chi)(1 - \alpha)e^{\eta_{t}})}{(1 - \chi)e^{\eta_{t}} + \chi - \alpha} e^{(\alpha - 1)\lambda_{t}} \right. \\ &- \frac{\alpha(1 - \alpha)^{\frac{1}{\chi}}(1 - \chi)e^{\frac{\chi - 1}{\chi}\delta_{t}}(1 - e^{\eta_{t}})^{\frac{1}{\chi}}((1 - \chi)e^{\eta_{t}} + \chi)}{(1 - \chi)e^{\eta_{t}} + \chi - \alpha} e^{(\frac{\alpha}{\chi} - 1)\lambda_{t}} \\ &+ \frac{\rho(1 - \alpha)((1 - \chi)e^{\eta_{t}} + \chi)}{(1 - \chi)e^{\eta_{t}} + \chi - \alpha} \right]. \end{split}$$
(14)

This system, (12) through (14), consists of three variables,  $\delta_t$ ,  $\lambda_t$  and  $\eta_t$  and the three corresponding differential equations. Notice that the system does not include government spending *G*.

#### 3. Existence of two equilibria

#### 3.1 Two equilibria

Assume that there is a steady state in our system, denoted by  $\delta^*$ ,  $\lambda^*$  and  $\eta^*$ .

First, by making  $\dot{\delta}_t = 0$  in (12), we obtain

$$k^* = e^{\lambda^*} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1-\alpha}},\tag{15}$$

which shows that the capital intensity of labor in steady states is unique and determined by two parameters alone,  $\alpha$  and  $\rho$ , independently of labor tax rates.

Second, in a similar fashion, as  $\dot{\delta}_t = 0$ , by making  $\dot{\lambda}_t = 0$  in (13) and  $\dot{\eta}_t = 0$  in (14), we obtain the following:

When  $\dot{\lambda}_t = 0$ ,

$$[(\alpha + (1 - \alpha)x)((\chi - 1)x + 1) - \alpha x] \left(\frac{\rho}{\alpha}\right) - x^{\frac{1}{\chi}}(1 - \alpha)^{\frac{1}{\chi}}((\chi - 1)x + 1) q e^{\frac{\alpha - \chi}{\chi}\lambda^{*}} + \rho x = 0, \quad (16)$$

and when  $\dot{\eta}_t = 0$ ,

$$\alpha(\chi + (1-\chi)(1-\chi))((\alpha - \chi) - (1-\chi)(1-\alpha)(1-\chi))\left(\frac{\rho}{\alpha}\right) - \alpha (1-\alpha)^{\frac{1}{\chi}}(1-\chi) q x^{\frac{1}{\chi}}((\chi - 1) x + 1) e^{\frac{\alpha - \chi}{\chi}\lambda^{*}} + \rho (1-\alpha)((\chi - 1)x + 1) = 0.$$
(17)

Additionally, to have a clear perspective, we have transformed variables as  $1 - e^{\eta^*} = x$  and  $e^{\frac{\chi - 1}{\chi}\delta^*} = q$ . Here, because  $0 < e^{\eta^*} = 1 - x = \tau^* < 1$ , 0 < x < 1. In this situation, we consider x and q as variables.

We consider the above system, (16) and (17), as a simultaneous equation system concerning x and q.

However, we can easily confirm that  $(16) \times \alpha(1 - \chi) = (17)$ . This implies that (16) and (17) are not independent of each other. Therefore, we seek another relationship including *x* and *q*.

We need to focus on the balanced budget in a steady state, which is described as

$$G = \tau^* w^* H^* = \tau^* w^* \left[ \frac{(1 - \tau^*) w^*}{c^*} \right]^{-\frac{1}{\chi}}.$$
 (18)

Notice that (17) is originally deduced from the balanced budget rule, and therefore the dynamic equation concerning  $\tau$  is not independent (See (10)).

Rearranging (18), we obtain the following:

$$q = \left[\frac{G^{\chi}}{(w^*)^{\chi - 1}} \frac{x}{(1 - x)^{\chi}}\right]^{\frac{\chi - 1}{\chi}}.$$
(19)

Substituting (19) into (16), we obtain an equation only in terms of x as

$$[(\alpha + (1 - \alpha)x)((\chi - 1)x + 1) - \alpha x]\left(\frac{\rho}{\alpha}\right) + \rho x$$
  
-  $(1 - \alpha)^{\frac{1}{\chi}} e^{\frac{\alpha - \chi}{\chi}\lambda^*}((\chi - 1)x + 1)$   
 $\times x^{\frac{1}{\chi}} \left[\frac{G^{\chi}}{(w^*)^{\chi - 1}}\right]^{\frac{\chi - 1}{\chi}} \left[\frac{x}{(1 - x)^{\chi}}\right]^{\frac{\chi - 1}{\chi}} = 0,$  (20)

which can further be transformed as

$$[(\alpha + (1 - \alpha)x)((\chi - 1)x + 1) - \alpha x] \left(\frac{\rho}{\alpha}\right) + \rho x$$
  
-  $G^{\chi - 1} (1 - \alpha)^{2 - \chi} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(2 - \chi) - 1}{1 - \alpha}} ((\chi - 1)x + 1)(1 - x)^{1 - \chi} x = 0.$  (21)

When we denote the left side in the above equation as h(x), that can be expressed as

$$\begin{cases} h(x) = (\chi - 1) \left( x - \frac{1}{1 - \chi} \right) (1 - \alpha) \varphi(x), \\ \varphi(x) = x \left[ 1 - B(1 - x)^{1 - \chi} \right] + \frac{\alpha}{1 - \alpha}, \\ B = (1 - \alpha)^{1 - \chi} G^{\chi - 1} \left( \frac{\alpha}{\rho} \right)^{\frac{\alpha(1 - \chi)}{1 - \alpha}} = \left( \frac{w^*}{G} \right)^{1 - \chi} > 0. \end{cases}$$

$$(22)$$

The solution in the above equation, h(x) = 0, shows the steady state which we are investigating. Of course, 0 < x < 1 must be satisfied. Proof is offered in Appendix B.

In (22),  $x = 1/(1-\chi)$  seems to be a solution on appearance. However, if  $x = 1/(1-\chi)$  holds,  $(1-\chi)\tau^* + \chi = 0$  is satisfied, which we have rejected (See (8)).

At this stage, it is obvious that the steady state is a solution satisfying  $\varphi(x) = 0$ . Therefore, we need to examine how  $\varphi(x)$  behaves in the range of 0 < x < 1. We know that the curve has a minimum point at  $0 < x^* < 1$ . So, if the curve at the minimum point is negative, our system can have two steady states.<sup>6</sup> Based on Appendix C, we offer an example in Figure 1.

As we have discussed so far, assuming that all the households participate in the labor market, we can offer some conditions for two steady states as a theorem:

Let  $x^*$  be a solution to the following equation, as is demonstrated in Appendix C:

$$1 - (2 - \chi)x^* = \frac{1}{B(1 - x^*)^{-\chi}}.$$
(23)

In this situation, we can show the existence of two steady states.

<sup>6)</sup> For details, see Appendix C. Additionally, there is a possibility that the system has a double steady state. We deal with this issue in the second subsection of this section.



Figure 1 Two xs can exist; if  $\varphi(x^*) < 0$ , two steady states exist; in the case of  $\varphi(x^*) = 0$ , double steady states exist.

**Theorem 1 (The existence of two steady states)** *There exist two steady states if the following conditions are satisfied:* 

$$\begin{cases} (a) \ B > 1, \\ (b) \ x^* [1 - B(1 - x^*)^{1 - \chi}] + \frac{\alpha}{1 - \alpha} \leq 0, \\ (c) \ K^{\alpha} H^{1 - \alpha} = c + G. \end{cases}$$

Additionally, condition (c) reflects Walras' law and shows the values in the steady states.

When a unique G is given, in the case of inequality in (b), there can correspondingly be two steady states, while in the case of equality in (b), a double root in relation to  $x^*$  appears. This implies that two similar steady states can coexist. *Theorem 1* is established based on a particular  $x^*$  satisfying (23). However, we can demonstrate the same relationship as in *Theorem 1* for an arbitrary range in which 0 < x < 1.

**Corollary 1 (The existence of two steady states)** *There exist two steady states if the following conditions are satisfied:* 

$$\begin{cases} (d) \ B = \frac{x + \frac{\alpha}{1 - \alpha}}{x(1 - x)^{1 - \chi}}, \\ (e) \ K^{\alpha} H^{1 - \alpha} = c + G. \end{cases}$$

Since the right hand side of (d) in *Corollary 1* is larger than unity for any value of 0 < x < 1, the condition B > 1 in (a) in *Theorem 1* is necessarily satisfied in *Corollary 1*. (A proof is given in

Appendix D.) It is not clear that Schmitt-Grohé and Uribe demonstrate or admit the equality in (b) in *Theorem 1*, or the existence of a double steady state. However, the discussions in (d) and in Appendix D are applicable to the case of  $\chi = 0$ , as in Schmitt-Grohé and Uribe (1997). This is why our model is a generalization which includes Schmitt-Grohé and Uribe (1997) as a special case.

In this situation, we need to elucidate (d) in more detail, from an economic viewpoint. Transforming (d), we obtain

$$G = w^* (1 - x) \left(\frac{x}{x + \frac{\alpha}{1 - \alpha}}\right)^{\frac{1}{1 - \chi}}.$$
(24)

The above equation determines x, i.e.  $\tau^*$ . Of course, once  $\tau^*$  is determined, it is indisputable that the values of other variables in the steady state are all determined. The right side in (24) shows labor tax revenue in terms of x alone<sup>7)</sup> and is independent of G. Now, let us focus on the shape of this labor tax revenue function  $\zeta(\tau^*) = w^* \tau^* h(\tau^*)$ , where  $h(\tau^*)$  is deduced as

$$h(\tau^*) = \left(\frac{1-\tau^*}{1-\tau^*+\frac{\alpha}{1-\alpha}}\right)^{\frac{1}{1-\chi}}$$

which indicates the labor supply in terms of  $\tau^*$  alone, and  $\alpha$  and  $\chi$  are a parameter of  $h(\tau^*)$ . Here, we should notice that in the above relationship, the value of the content in the bracket is less than one. At this stage, let us explore  $h(\tau^*)$  more intensively. Obviously,  $\alpha/(1-\alpha)$  indicates the ratio of capital to that of labor in output. This ratio implies that the income from working is comparatively more advantageous than that from renting capital, when  $\alpha$  is small, and vice versa.  $h(\tau^*)$  reflects this kind of incentive in relation to labor supply. Of course, if  $\alpha$  increases, the labor supply shifts downward, and vice versa. Furthermore, when  $\chi$  becomes large, the labor supply decreases drastically, if real wage rates change. Then,  $h(\tau^*)$  shifts downwards.

Based on the definition of  $\zeta(\tau^*)$ , we obtain the following:

$$rac{d\zeta}{d au^*} = w^*h( au^*)\left(1+rac{ au^*}{h( au^*)}rac{dh( au^*)}{d au^*}
ight),$$

where we define  $\varepsilon(\tau_*) = \frac{\tau_*}{h(\tau_*)} \frac{dh(\tau_*)}{d\tau^*}$ .  $\varepsilon(\tau^*)$  indicates the elasticity of labor supply in relation to labor income tax rates and has the following properties:  $\varepsilon'(\tau_*) = \frac{-\frac{\alpha}{1-\alpha}\tau^*}{(1-\tau^*)(1-\chi)(1-\tau^*+\frac{\alpha}{1-\alpha})} < 0$ ;  $\varepsilon(0) = 0$ ;  $\varepsilon(1) \to -\infty$ . This implies that as  $\tau^*$  increases, the labor supply decreases drastically. In this situation, in ranges of comparatively small  $\tau^*$ , since the increment ratio in  $\tau^*$  is larger than the rate of decrease in  $w^*h(\tau^*)$ , i.e., the tax base, tax revenues increase. On the other hand, in ranges of comparatively larger  $\tau^*$ , since the rate of increment in  $\tau^*$  is smaller than the the rate of

<sup>7)</sup> Note that  $w^*$  is equal to the marginal product of labor in our two steady states and is constant.

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decrease in the tax base, tax revenues decrease. Therefore, when  $0 < \tau^* \leq \tau^{**}, \varepsilon + 1 \ge 0$ , while in ranges of  $\tau^{**} < \tau^* < 1, \varepsilon + 1 < 0$ . This indicates that when  $0 < \tau^* \leq \tau^{**}, \frac{d\zeta}{d\tau^*} \ge 0$ , and when  $\tau^{**} < \tau^* < 1, \frac{d\zeta}{d\tau^*} < 0$ . In this context, we can see that the less  $\chi$  is, the more  $\varepsilon'(\tau^*)$  is. This implies the response of the labor supply to the change in tax rates.

Additionally,  $\tau^{**}$  shows the labor income tax rate corresponding to maximum tax revenue. Furthermore,  $\zeta(\tau^*)$  has the following properties:  $\zeta(0) = 0$ ;  $\zeta(1) = 0$ ,  $\zeta > 0$ , and  $\frac{d^2\zeta}{d(\tau^*)^2} < 0$ ,<sup>8)</sup> for  $0 < \tau^* < 1$ . And  $\zeta$  is a continuous function with respect to  $\tau^*$ .

From what we have described so far, it is obvious that the shape of  $\zeta$  is like that of a bell curve.<sup>9)</sup> This bell curve is independent of *G*. Therefore, we can say that there are two  $\tau^*$ s when G is given. In other words, there are two alternatives in relation to  $\tau^*$  which correspond to the government spending financed by this tax revenue. Theorem 1 shows that there is a bell curve in our system, which causes nonlinear dynamics. In this context, G must naturally be less than or equal to the maximum tax revenue, which is determined by fundamental parameters alone. (See the second subsection.) This is what *Corollary 1* shows. In short, G is restricted by the fundamentals in the system.

### 3.2 Maximum G

At this stage, let us focus on the relationship between G and our two xs. We can easily know from the above analyses that if G increases,  $\varphi(x)$  tends to go upwards, which is caused by a decrement of B. This leads to a double root or no root with regard to x. This implies an upper limitation with regard to G in our context. Of course, maximum G corresponds to maximum labor income tax revenue.

We can find the maximum *G* as follows: First, since  $\varphi'(x^*) = 0$  must be satisfied, we obtain

$$B = \frac{(1-x^*)^{\chi}}{1-(2-\chi)x^*}.$$
(25)

Second, under this condition of *B*,  $\varphi(x^*) = 0$  must be also satisfied. Therefore, the following relationship holds:

$$x^* \left[ 1 - \frac{(1-x^*)^{\chi} (1-x^*)^{1-\chi}}{1-(2-\chi)x^*} \right] + \frac{\alpha}{1-\alpha} = 0,$$

which can be transformed as

$$(\boldsymbol{\chi}-1)(\boldsymbol{x}^*)^2 - \frac{\alpha(2-\boldsymbol{\chi})}{1-\alpha}\boldsymbol{x}^* + \frac{\alpha}{1-\alpha} = 0.$$

<sup>8)</sup> A proof is omitted.

<sup>9)</sup> Schmitt-Grohé and Uribe deduce this kind of bell curve avoiding H itself, instead considering K, i.e. savings, which is a function of  $\tau^*$ . In an indivisible labor type model, such as Schmitt-Grohé and Uribe (1997) or Anagnostopoulos and Giannitsarou (2013), an economic system can be described in terms of the capital-labor ratio. Once the capital intensity of labor is determined, what follows is the determination of capital, thus capital is in fact predetermined. That is, labor is automatically determined by capital intensity.

Solving the above quadratic equation, we eventually obtain the solution required as

$$x^{*} = \frac{\alpha(2-\chi) - \sqrt{\alpha(\alpha\chi^{2} - 4\chi + 4)}}{2(1-\alpha)(\chi - 1)} > 0.$$
 (26)

Substituting (26) into (25), *B* is determined by parameters such as  $\alpha$  and  $\chi$  alone, and this shows the maximum *G*. Thus,

$$G = (1 - \alpha) \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1 - \alpha}} \left[\frac{1 - (2 - \chi)x^*}{(1 - x^*)^{\chi}}\right]^{\frac{1}{1 - \chi}}$$

is derived. Moreover, in this context, we can demonstrate that the values in this equilibrium of c,  $\tau$  and H are explicitly determined by fundamental parameters alone, such as  $\alpha$ ,  $\rho$  and  $\chi$  as G.

We offer only the results of this determination as follows:

$$\begin{cases} \tau^* = \frac{\chi(2-\alpha) - 2 + \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1-\alpha)(\chi-1)}, \\ G = (1-\alpha)\left(\frac{\alpha}{\rho}\right)^{\overline{1-\alpha}} \left[\frac{(\tau^*)^{\chi}}{1-(2-\chi)(1-\tau^*)}\right]^{-\frac{1}{1-\chi}}, \\ c^* = (1-\alpha)\left(\frac{\alpha}{\rho}\right)^{\overline{1-\alpha}} (1-\tau^*) \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)}\right]^{-\frac{\chi}{1-\chi}}, \\ H^* = \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)}\right]^{-\frac{1}{1-\chi}}. \end{cases}$$

From what we have analyzed so far, it becomes obvious that  $w^*$ ,  $c^*$ , G and  $H^*$  can be expressed by parameters alone. Of course, so can  $k^*$ . Thus, we can easily describe  $K^*$ ,  $u^*$  and  $Y^*$  by parameters alone as well. See Appendix E for details.

#### 4. Conclusion

We have explored the possibility of two steady states under a balanced budget rule, assuming divisible labor. This implies that all households offer the same amount of labor and participate in the labor market, and that they have increasing marginal disutility in relation to labor. In this situation, there is no uncertainty in relation to employment. On the other hand, Schmitt-Grohé and Uribe (1997) postulates indivisible labor. Therefore, they assume that some households participate in the labor market and others do not, and that all households are not risk aversive to employment. We have unambiguously demonstrated a crucial issue, that there are two steady states in our economy. These phenomena characteristically occur under economic conditions in which fundamental factors such as production technologies, utility functions and attitudes with regard to the labor supply remain unchanged.

We postulate a utility function which has a property of increasing marginal disutility with regard to labor supply, while in Schmitt-Grohé and Uribe (1997) and Anagnostopoulos and Giannitsarou

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(2013), a representative household posits constant marginal disutility with regard to labor supply, in which an individual chooses whether to work or not. Our model generalizes Schmitt-Grohé and Uribe (1997). Due to our postulation, we can specifically deduce a labor supply function, and it becomes obvious that the shape of the tax revenue function is like a bell curve with an independent argument regarding tax rates at steady states. Moreover, this shows that our system is nonlinear. With the help of global analysis, we demonstrate that this causes the existence of two steady states. This offers the government two alternatives regarding tax rate policy. Higher or lower rates can be imposed in order to levy similar tax revenues.

However, the intensity of capital to labor is the same in both steady states. This leads to the fact that labor productivity is the same in both equilibria, and that real wage rates and real rental rates are also the same in the two equilibria. This is why labor income taxation does not affect the relative prices of consumption and labor in either equilibrium. However, this taxation affects savings through income effects.

Moreover, we demonstrate that in our economy, there is a maximum tax revenue which is determined by fundamental structures alone, such as indications of the share of capital in output,  $\alpha$ , plus  $\chi$ , which shows attitudes regarding the labor supply when wages change, and  $\rho$ , which shows a subjective discount rate of utility. We have demonstrated that at this maximum tax revenue, double steady states appear. We should remember that one of these steady states exists in a smaller region , the other in a larger one, according to the position of the maximum tax revenue rate in terms of  $\tau^*$ .

Finally, the tax revenue in the steady state obviously depends on  $\chi$ , that is, the value decreases when  $\chi$  increases. Since  $-1/\chi$  indicates the elasticity of labor supply with regard to wages, tax revenues in the steady state is high when that elasticity is small. In this case, correspondingly, G can be larger.

#### Appendix A

Here, we explain step by step how we can reach  $\dot{k_t}/k_t$ , that is,  $(9^*)^{10}$  on page 6.

At this stage, we have already established  $H_t$  and  $\dot{H}_t/H_t$  in terms of (4<sup>\*</sup>) and (8<sup>\*</sup>), respectively. When we rewrite them, they are expressed as follows:

$$H_t = \left[\frac{c_t}{(1-\tau_t) w_t}\right]^{\frac{1}{\chi}},$$
$$\frac{\dot{H}_t}{H_t} = \frac{1}{(1-\chi) \tau_t + \chi} \left(\frac{\dot{c}_t}{c_t} (1-\tau_t) - \frac{\dot{w}_t}{w_t}\right).$$

Here, we intend to express  $\dot{k}_t/k_t$ , which shows a part of our dynamic system.

First, we have described  $\dot{k}_t$  in (5<sup>\*</sup>) as

$$\dot{k_t} = u_t \frac{K_t}{H_t} + (1 - \tau_t) w_t - \frac{c_t}{H_t} - k_t \frac{\dot{H}_t}{H_t}$$

<sup>10)</sup> Numbers with \* regarding mathematical equations in the Appendices below all refer to those in the body.

Substituting  $H_t$  and  $\dot{H}_t/H_t$  into the above relationship, we can express this as follows:

$$\begin{split} \dot{k_t} &= u_t \ k_t + (1 - \tau_t) \ w_t - c_t \times \left[ \frac{c_t}{(1 - \tau_t) \ w_t} \right]^{-\frac{1}{\chi}} \\ &- k_t \times \frac{1}{(1 - \chi) \ \tau_t + \chi} \ \left( \frac{\dot{c_t}}{c_t} \ (1 - \tau_t) - \frac{\dot{w_t}}{w_t} \right), \end{split}$$

which yields

$$\begin{split} \dot{k_t} &= \alpha k_t^{-(1-\alpha)} k_t + (1-\tau_t) w_t - c_t^{1-\frac{1}{\chi}} (1-\tau_t)^{\frac{1}{\chi}} w_t^{\frac{1}{\chi}} \\ &- \frac{1-\tau_t}{(1-\chi)\tau_t + \chi} k_t \left(\frac{\dot{c_t}}{c_t}\right) + \frac{k_t}{(1-\chi)\tau_t + \chi} \left(\frac{\dot{w_t}}{w_t}\right) \\ &= \alpha k_t^{\alpha} + (1-\tau_t)(1-\alpha) k_t^{\alpha} - c_t^{1-\frac{1}{\chi}} (1-\tau_t)^{\frac{1}{\chi}} (1-\alpha)^{\frac{1}{\chi}} k_t^{\frac{\alpha}{\chi}} \\ &- \frac{1-\tau_t}{(1-\chi)\tau_t + \chi} k_t (\alpha k_t^{-(1-\alpha)} - \rho) + \frac{k_t}{(1-\chi)\tau_t + \chi} \alpha \frac{\dot{k_t}}{k_t} \end{split}$$

We should describe optimal conditions for the firm as

$$\begin{cases} w_t = \boldsymbol{\eta}(k_t), \\ \boldsymbol{\eta}(k_t) \stackrel{\text{def}}{=} (1 - \boldsymbol{\alpha}) k_t^{\boldsymbol{\alpha}}, \end{cases}$$

and

$$\begin{cases} u_t = \phi(k_t), \\ \phi(k_t) \stackrel{\text{def}}{=} \alpha k_t^{\alpha - 1}. \end{cases}$$

However, we omit such expressions for simplicity. Hereafter,  $w_t$  and  $u_t$  are both equilibrium prices. In the above procedures, we utilize the following:

$$\begin{cases} c_t \left[ \frac{c_t}{(1-\tau_t) w_t} \right]^{-\frac{1}{\chi}} = c_t^{1-\frac{1}{\chi}} \left[ (1-\tau_t) w_t \right]^{\frac{1}{\chi}}, \\ \frac{\dot{w}_t}{w_t} = \alpha \frac{\dot{k}_t}{k_t}, \\ w_t = (1-\alpha) k_t^{\alpha}, \\ \frac{\dot{c}_t}{c_t} = u_t - \rho = \alpha k_t^{-(1-\alpha)} - \rho. \end{cases}$$

Then, transforming the above formula, beginning with  $\dot{k}$ , we can reach (9<sup>\*</sup>).

# **Appendix B**

Here, we intend to deduce a relationship in which solutions satisfy a necessary condition for the existence of two equilibria.

First, by making  $\dot{\lambda}_t = 0$  in (13<sup>\*</sup>), we obtain

$$[(\alpha + (1 - \alpha)(1 - e^{\eta^*}))((1 - \chi)e^{\eta^*} + \chi) - \alpha(1 - e^{\eta^*})]\left(\frac{\rho}{\alpha}\right) - [e^{\frac{\chi - 1}{\chi}\delta^*}(1 - e^{\eta^*})^{\frac{1}{\chi}}(1 - \alpha)^{\frac{1}{\chi}}((1 - \chi)e^{\eta^*} + \chi)]\left(\frac{\alpha}{\rho}\right)^{\frac{\alpha - \chi}{(1 - \alpha)\chi}} + \rho(1 - e^{\eta^*}) = 0.$$

Second, in a similar fashion, by making  $\dot{\eta}_t = 0$  in (14<sup>\*</sup>), the following holds:

$$\begin{aligned} & [\alpha \ (\chi + (1-\chi)e^{\eta^*})((\alpha-\chi) - (1-\chi)(1-\alpha)e^{\eta^*})](\frac{\rho}{\alpha}) \\ & - [\alpha(1-\alpha)^{\frac{1}{\chi}}(1-\chi)e^{\frac{\chi-1}{\chi}\delta^*}(1-e^{\eta^*})^{\frac{1}{\chi}}((1-\chi)e^{\eta^*}+\chi)](\frac{\alpha}{\rho})^{\frac{\alpha-\chi}{(1-\alpha)\chi}} \\ & + \rho(1-\alpha)((1-\chi)e^{\eta^*}+\chi) = 0. \end{aligned}$$

Additionally, we utilize the following relationships in the above processes:

$$\begin{cases} x^{\frac{1}{\chi}} \left[ \frac{x}{(1-x)^{\chi}} \right]^{\frac{\chi-1}{\chi}} = x(1-x)^{1-\chi}, \\ (1-\alpha)^{\frac{1}{\chi}} e^{\frac{\alpha-\chi}{\chi}\lambda^*} \left[ G^{\chi} (w^*)^{-(\chi-1)} \right]^{\frac{\chi-1}{\chi}} = G^{\chi-1} (1-\alpha)^{2-\chi} \left( \frac{\alpha}{\rho} \right)^{\frac{\alpha(2-\chi)-1}{1-\alpha}} \end{cases}$$

,

and  $w^* = (1 - \alpha) \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1 - \alpha}}$  and  $e^{\lambda^*} = \left(\frac{\alpha}{\rho}\right)^{\frac{1}{1 - \alpha}}$ .

At this stage, we define the above as  $E = G^{\chi-1} (1-\alpha)^{2-\chi} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(2-\chi)-1}{1-\alpha}}$ . Here, consider the function h(x), which is defined in (22\*):

$$h(x) = (1-\alpha)(\chi-1)x^2 + (\alpha\chi+1-2\alpha)x + \alpha$$
$$-\left(\frac{\alpha}{\rho}\right)E((\chi-1)x+1)(1-x)^{1-\chi}x,$$

which can be transformed as

$$h(x) = (\chi - 1)(x - \frac{1}{1 - \chi}) \left[ (1 - \alpha) \left( x + \frac{\alpha}{1 - \alpha} \right) - \left( \frac{\alpha}{\rho} \right) E(1 - x)^{1 - \chi} x \right].$$

From what we have analyzed so far, we must explore our solution in conditions which satisfy  $\phi(x) = 0$ . We define  $\phi(x)$  as

$$\phi(x) = (1 - \alpha) \left( x + \frac{\alpha}{1 - \alpha} \right) - \left( \frac{\alpha}{\rho} \right) E (1 - x)^{1 - \chi} x$$
$$= (1 - \alpha) \phi(x).$$

Here, we newly define a function  $\varphi(x)$  as

$$\varphi(x) = \left(x + \frac{\alpha}{1 - \alpha}\right) - (1 - \alpha)^{1 - \chi} G^{\chi - 1} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(1 - \chi)}{1 - \alpha}} (1 - x)^{1 - \chi} x.$$

At this stage, we again define B as

$$B = (1-\alpha)^{1-\chi} G^{\chi-1} \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha(1-\chi)}{1-\alpha}}.$$

In this situation,  $\varphi(x)$  can be expressed as

$$\varphi(x) = x + \frac{\alpha}{1 - \alpha} - B(1 - x)^{1 - \chi} x = x[1 - B(1 - x)^{1 - \chi}] + \frac{\alpha}{1 - \alpha}$$

From what we have analyzed, we finally obtain  $\phi(x) = (1 - \alpha)\phi(x)$ . This shows that solutions satisfying  $\phi(x) = 0$  are those in the steady states which we are exploring.

#### Appendix C

Let us consider the function  $\varphi(x)$ , defined as

$$\varphi(x) = x[1 - B(1 - x)^{1 - \chi}] + \frac{\alpha}{1 - \alpha}$$

First, we focus on the contents in the bracket above, here defined as  $f(x) = 1 - B(1-x)^{1-\chi}$ .

We can easily obtain the following information with regard to  $f(x) : f'(x) = B(1-\chi)(1-x)^{-\chi} > 0$ ,  $f(0) = 1 - B \ge 0$  and f(1) = 1.

Based on the above information, we can say that if  $B \leq 1$ ,  $f \geq 0$ , and that therefore, since  $\varphi(x) > 0$ , there is no steady state. So, we assume B > 1.<sup>11)</sup>

Differentiating  $\varphi(x)$  with regard to *x*, we obtain the following:

$$\varphi'(x) = B(1-x)^{-\chi} [(2-\chi)x - 1] + 1$$
  
=  $B(1-x)^{-\chi} [(2-\chi)x - 1 + \frac{1}{B}(1-x)^{\chi}].$ 

Focusing on the contents in the second bracket in the above relationships, we investigate how  $\varphi'(x)$  behaves.

<sup>11)</sup> Here, we should consider this relationship from an economic viewpoint. *B* was defined as follows:  $B = \left(\frac{w^2}{G}\right)^{1-\chi}$ . Then, since  $\chi < 0$ , the following relationship, that is,  $w^* > G$ , must hold. In the context of this article,  $w^*$  indicates a real wage rate at a steady state, which shows output in exchange for a unit of working time. On the other hand, *G* indicates total wage income taxes levied when working hours equal  $H^*$  and the tax rates are  $\tau^*$ . B > 1 shows that the former is larger than the latter.



Figure 2  $x^*$  exists uniquely

We define  $\eta_1(x) = \left(\frac{1}{B}\right)(1-x)^{\chi}$ , which has properties such as  $\eta'_1(x) = -\left(\frac{1}{B}\right)\chi(1-x)^{\chi-1} > 0$ ,  $\eta_1(0) = 1/B > 0$  and  $\eta_1(1) = \infty$ .

On the other hand, we define  $\eta_2(x) = 1 - (2 - \chi)x$ , which is monotonously decreasing with regard to x. Additionally,  $\eta_2(0) = 1$  and  $\eta_2(1) = \chi - 1 < 0$ .

Depicting the above relationships, we obtain the following:

When G is given,  $x^*$  is naturally determined satisfying  $1 - (2 - \chi)x^* = 1/B(1 - x^*)^{-\chi}$ .

From the above figure, it becomes obvious that  $\varphi(x)$  monotonously decreases when  $0 < x \le x^*$ , and increases when  $x^* < x < 1$ .

Additionally,  $\varphi(x)$  has properties such as  $\varphi(0) = \alpha/(1-\alpha) > 0$  and  $\varphi(1) = 1 + \alpha/(1-\alpha) > 0$ .

## Appendix D

From (d) in *Corollary 1* the following can hold:

$$B = \frac{x + \frac{\alpha}{1 - \alpha}}{x(1 - x)^{1 - \chi}} = \frac{x + A}{x(1 - x)^{\Omega}}.$$

Here, we denote  $\Omega = 1 - \chi$ ,  $A = \frac{\alpha}{1-\alpha}$ , which respectively lead to  $\Omega > 1$  and  $0 < A < \infty$ . Then we express the above mathematical formula as  $\xi(x)$ . It is obvious that  $\xi(0) \to \infty$  and  $\xi(1) \to \infty$ . Differentiating  $\xi(x)$  with regard to x, we obtain

$$\xi'(x) = -\frac{(1-x)^{-1-\Omega}(A - Ax - x(A + x)\Omega)}{x^2}.$$

Therefore, we obtain x satisfying  $\xi'(x) = 0$  as  $\frac{-A(1+\Omega) + \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{2\Omega}$ . Then, substituting the above x into  $\xi(x)$ , we can obtain the minimum point in  $\xi(x)$  as

$$\begin{split} \Psi(A,\Omega) &= 2^{-1+\Omega} \left( \frac{A+2\Omega+A\Omega-\sqrt{A}\sqrt{4\Omega+A(1+\Omega)^2}}{\Omega} \right)^{-\Omega} \\ &\times \left( 2+A+A\Omega+\sqrt{A}\sqrt{4\Omega+A(1+\Omega)^2} \right). \end{split}$$

Our aim is to investigate whether  $\Psi(A, \Omega) > 1$  holds under A > 0 and  $\Omega > 1$ . We need to explore necessary conditions by differentiating  $\Psi$  concerning A and  $\Omega$ . The following holds:

$$\begin{split} \frac{\partial \Psi}{\partial A} &= \frac{1}{\sqrt{A}} 2^{-1+\Omega} \left( \sqrt{A} (1+\Omega) + \sqrt{4\Omega + A(1+\Omega)^2} \right) \\ & \times \left( \frac{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{\Omega} \right)^{-\Omega}, \end{split}$$

and

$$\begin{split} \frac{\partial \Psi}{\partial \Omega} &= 2^{-1+\Omega} \left( \frac{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{\Omega} \right)^{-\Omega} \\ &\times \left( 2 + A + A\Omega + \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2} \right) \\ &\times \log \left[ \frac{2\Omega}{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}} \right]. \end{split}$$

At this stage, as a preparation for later analyses, first, we can confirm that  $(2\Omega + A(1 + \Omega))^2 - (\sqrt{A}\sqrt{4\Omega + A(1 + \Omega)^2})^2 = 4(1 + A)\Omega^2 > 0$ . This eventually implies that  $2\Omega + A(1 + \Omega) > \sqrt{A}\sqrt{4\Omega + A(1 + \Omega)^2}$ . And then  $\frac{\partial \Psi}{\partial A} > 0$  holds, regardless of  $\Omega$ . Second, let us proceed to analyse  $\frac{\partial \Psi}{\partial \Omega}$ . Obviously, the signs of  $\frac{\partial \Psi}{\partial \Omega}$  depend on the last item. In general, if  $\Gamma$  is larger than 1,  $\frac{\partial \Psi}{\partial \Omega} > 0$ . Additionally,

$$\Gamma = \frac{2\Omega}{A + 2\Omega + A\Omega - \sqrt{A}\sqrt{4\Omega + A(1 + \Omega)^2}}$$

Here, considering a reciprocal of  $\Gamma$ , the following holds:

$$\frac{1}{\Gamma} = 1 + \frac{A(1+\Omega) - \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{2\Omega}$$

At this stage, we can verify from  $\xi'(x) = 0$ 

$$0 < \frac{-A(1+\Omega) + \sqrt{A}\sqrt{4\Omega + A(1+\Omega)^2}}{2\Omega} < 1.$$

This eventually refers to  $\Gamma > 1$ . Based on the above discussions, we obtain the conclusion  $\frac{\partial \Psi}{\partial \Omega} > 0$ .

Let us consider a two-dimensional plane, which has a horizontal line depicting A, and a vertical one depicting  $\Psi$ . In this situation,  $\Psi$  increases as A does, while  $\Psi$  shifts upwards as  $\Omega$  increases. On the other hand,  $\Psi(0, \Omega) = 1$ , for  $\Omega > 1$ . This proves  $\Psi > 1$  and results in  $\xi(x) > 1$ .

#### Appendix E

The following shows how  $\tau^*$ ,  $c^*$ ,  $H^*$  and G are determined, respectively. Of course,  $\tau^*$ ,  $c^*$  and  $H^*$  are values corresponding to G.

First, since  $\tau^* = 1 - x^*$ ,

$$\tau^* = \frac{\chi(2-\alpha) - 2 + \sqrt{\alpha(\alpha\chi^2 - 4\chi + 4)}}{2(1-\alpha)(\chi - 1)}$$

Second, we explore the value of  $c^*$ .

Based on the previous definition (on page 8), we obtain:

$$q = (e^{\delta^*})^{\frac{\chi-1}{\chi}} = (c^*)^{\frac{\chi-1}{\chi}}.$$

Furthermore, considering  $(19^*)$ ,

$$q = \left[\frac{G^{\chi}}{(w^*)^{\chi-1}} \frac{x}{(1-x)^{\chi}}\right]^{\frac{\chi-1}{\chi}}$$

we obtain the following required relationship :

$$c^* = q^{\frac{\chi}{\chi-1}} = \frac{G^{\chi}(1-\tau^*)}{(w^*)^{\chi-1}(\tau^*)^{\chi}}.$$

Here, G is formulated as follows:

$$G = (1 - \alpha) \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1 - \alpha}} \left[\frac{(\tau^*)^{\chi}}{1 - (2 - \chi)(1 - \tau^*)}\right]^{-\frac{1}{1 - \chi}}$$

Utilizing this G, we eventually reach:

$$c^* = (1-\alpha) \left(\frac{\alpha}{\rho}\right)^{\frac{\alpha}{1-\alpha}} (1-\tau^*) \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)}\right]^{-\frac{\chi}{1-\chi}}.$$

Third, we explore  $H^*$ . From (4<sup>\*</sup>) we can express  $H^*$  as follows:

$$H^* = \left[\frac{c^*}{(1-\tau^*) w^*}\right]^{\frac{1}{\chi}} = (c^*)^{\frac{1}{\chi}} (1-\tau^*)^{-\frac{1}{\chi}} (w^*)^{-\frac{1}{\chi}}.$$

Thus, substituting  $c^*$  and  $w^* = (1 - \alpha)(\frac{\alpha}{\rho})^{\frac{\alpha}{1-\alpha}}$  into the above relationship, we obtain the following:

$$H^* = \left[\frac{\tau^*}{1-(2-\chi)(1-\tau^*)}\right]^{-\frac{1}{1-\chi}}$$

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