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PFAFFIAN OF LAURICELLA'S HYPERGEOMETRIC SYSTEM F_A

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ABSTRACT. We give a Pfaffian system of differential equations associated with Lauricella's hypergeometric series $F_A(a,b,c;x)$ of m-variables. This system is integrable of rank 2^m. To express the connection form of this system, we make use of the intersection form of twisted cohomology groups with respect to integrals representing solutions of this system.

1. INTRODUCTION

Lauricella's hypergeometric series $F_A(a,b,c;x)$ of m-variables $x=(x_1,\dots,x_m)$ with parameters a, $b=(b_1,\dots,b_m)$ and $c=(c_1,\dots,c_m)$ is defined as

$$F_A(a,b,c;x)=\sum_{n\in\mathbb{N}^m}\frac{(a,\sum_{i=1}^m n_i)\prod_{i=1}^m(b_i,n_i)}{\prod_{i=1}^m(c_i,n_i)\prod_{i=1}^m(1,n_i)}\prod_{i=1}^m x_i^{n_i},$$

where $\mathbb{N}=\{0,1,2,\dots\}$, $n=(n_1,\dots,n_m)$, $c_1,\dots,c_m\notin-\mathbb{N}=\{0,-1,-2,\dots,\}$, and $(c_i,n_i)=c_i(c_i+1)\cdots(c_i+n_i-1)=\Gamma(c_i+n_i)/\Gamma(c_i)$. It is known that Lauricella's hypergeometric system $F_A(a,b,c)$ of differential equations satisfied by $F_A(a,b,c;x)$ is of rank 2^m with the singular locus

$$\big\{x\in\mathbb{C}^m \mid \prod_{i=1}^m x_i\prod_{v\in\mathbb{Z}_2^m}(1-v\,{}^tx)=0\big\},$$

where $v=(v_1,\dots,v_m)$ and $v_i\in\mathbb{Z}_2=\{0,1\}\subset\mathbb{N}$. In this paper, we give a Pfaffian system of $F_A(a,b,c)$ under the non-integral conditions (2.2) for linear combinations of parameters a, b and c. The connection form of the Pfaffian system is expressed in terms of logarithmic 1-forms of defining equations of the singular locus, see Corollary 4.3. When the number of variables is two, this system is called Appell's F_2, of which Pfaffian system is studied by several authors; refer to [K] and the references therein.

To express the connection form of this system, we study linear transformations $\mathcal{R}_0^i$ and $\mathcal{R}_v$ representing local behaviors of the connection form around the components $S_0^i=\{x\in\mathbb{C}^m \mid x_i=0\}$ and $S_v=\{x\in\mathbb{C}^m \mid 1-v\,{}^tx=0\}$ of the singular locus. They can be regarded as linear transformations of the twisted cohomology groups with respect to integrals representing solutions of this system. We show that they have two eigenvalues for generic parameters. It is a key property for characterizing $\mathcal{R}_0^i$ and $\mathcal{R}_v$ that eigenspaces of each of them are orthogonal to each other with respect to the intersection form of the twisted cohomology groups. By using the intersection form, we express $\mathcal{R}_0^i$ and $\mathcal{R}_v$ without choosing a basis of the twisted cohomology group, see Lemma 4.4 and Theorem 4.1. Their representation matrices in Corollary 4.2 imply the Pfaffian system of Lauricella's F_A.

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The monodromy representation of this system is studied in [MY]. Its circuit transformations are expressed in terms of the intersection form of twisted homology groups, which are dual to the twisted cohomology groups.

Refer to [M2] for the study of a Pfaffian and the monodromy representation of Lauricella's system F_D in terms of the intersection form of twisted (co)homology groups.

2. Lauricella's hypergeometric system F_A

In this section, we collect some facts about Lauricella's hypergeometric system F_A of differential equations, for which we refer to [AK], [L], [MY] and [Y]. Lauricella's hypergeometric series $F_A(a,b,c;x)$ converges in the domain

$$\mathbb{D} = \Big\{ x \in \mathbb{C}^m \Big| \sum_{i=1}^m |x_i| < 1 \Big\}$$

and admits the integral representation

$$\Big[\prod_{i=1}^m \frac{\Gamma(c_i)}{\Gamma(b_i)\Gamma(c_i-b_i)} \Big] \int_{(0,1)^m} u(a,b,c;t,x) \frac{dT}{t_1 \cdots t_m}, \tag{2.1}$$

where $dT = dt_1 \wedge \cdots \wedge dt_m$,

$$u(t,x) = u(a,b,c;t,x) = \Big[\prod_{i=1}^m t_i^{b_i}(1-t_i)^{c_i-b_i-1} \Big] (1-\sum_{i=1}^m t_i x_i)^{-a},$$

and parameters b and c satisfy $\mathrm{Re}(c_i) > \mathrm{Re}(b_i) > 0$ $(i=1,\dots,m)$.

Differential operators

$$x_i(1-x_i)\partial_i^2 - x_i \sum_{1\le j\le m}^{j\ne i} x_j \partial_i \partial_j + [c_i-(a+b_i+1)x_i]\partial_i - b_i \sum_{1\le j\le m}^{j\ne i} x_j\partial_j - ab_i$$

for $i=1,\dots,m$ annihilate the series $F_A(a,b,c;x)$, where $\partial_i = \dfrac{\partial}{\partial x_i}$. We define Lauricella's hypergeometric system $F_A(a,b,c)$ by differential equations corresponding to these operators.

We define the local solution space $Sol(U)$ of the system $F_A(a,b,c)$ on a domain U in $\mathbb{C}^m$ by the $\mathbb{C}$-vector space

$$\{F(x) \in \mathcal{O}(U) \mid P(x,\partial)\cdot F(x) = 0 \text{ for any } P(x,\partial) \in F_A(a,b,c)\},$$

where $\mathcal{O}(U)$ is the $\mathbb{C}$-algebra of single valued holomorphic functions on U. The rank of $F_A(a,b,c)$ is defined by $\sup_U \dim(Sol(U))$. It is known that the rank of $F_A(a,b,c)$ is 2^m and 2^m functions $F_A(a,b,c;x)$ and $\partial_I F_A(a,b,c;x)$ are linearly independent, where $I=\{i_1,\dots,i_r\}$ runs over the non-empty subsets of $\{1,\dots,m\}$ and $\partial_I = \partial_{i_1}\cdots\partial_{i_r}$.

If the rank of $F_A(a,b,c)$ is greater than $\dim(Sol(U_x))$ for any neighborhood U_x of $x \in \mathbb{C}^m$ then x is called a singular point of $F_A(a,b,c)$. The singular locus of $F_A(a,b,c)$ is defined as the set of such points. It is also shown in [MY] that the singular locus is

$$(\bigcup_{v\in\mathbb{Z}_2^m} S_v) \bigcup (\bigcup_{i=1}^m S_0^i),$$

where

$$S_v = \{x \in \mathbb{C}^m \mid v\,{}^t x = \textstyle\sum_{i=1}^m v_i x_i = 1\}, \quad v \in \mathbb{Z}_2^m,$$
$$S_0^i = \{x \in \mathbb{C}^m \mid x_i = 0\}, \qquad i = 1, \dots, m,$$

and we regard $S_{(0,\dots,0)}$ as the empty set. We set

$$\begin{aligned} X &= \mathbb{C}^m - \Big[\big(\bigcup_{v \in \mathbb{Z}_2^m} S_v\big) \bigcup \big(\bigcup_{i=1}^m S_0^i\big)\Big], \\ S &= (\mathbb{P}^1)^m - X = \big(\bigcup_{v \in \mathbb{Z}_2^m} S_v\big) \bigcup \big(\bigcup_{i=1}^m S_0^i\big) \bigcup \big(\bigcup_{i=1}^m S_\infty^i\big), \end{aligned}$$

where $S_\infty^i = \{x \in (\mathbb{P}^1)^m \mid x_i = \infty\}$.

We define a partial order and a total order on $\mathbb{Z}_2^m$.

Definition 2.1. *For $v = (v_1, \dots, v_m), w = (w_1, \dots, w_m) \in \mathbb{Z}_2^m$,*

(i) $v \succeq w$ if and only if $w_i = 1 \Rightarrow v_i = 1$;
(ii) $v \succ w$ if and only if $v \succeq w$ and $v \neq w$;
(iii) $v > w$ if and only if $v \neq w$ and they satisfy one of
  (1) $|v| > |w|$,
  (2) $|v| = |w|$ and $v_i < w_i$, where i is the minimum index satisfying $v_i \neq w_i$.

Here

$$|v| = \sum_{i=1}^m v_i.$$

It is easy to see that

$$v \succ w \Rightarrow v > w,$$
$$(0, \dots, 0) \prec e_i \prec e_i + e_j \prec e_i + e_j + e_k \prec \cdots \prec (1, \dots, 1),$$
$$(0, \dots, 0) < e_1 < e_2 < \cdots < e_m < e_1 + e_2 < e_1 + e_3 < \cdots < (1, \dots, 1),$$

where e_i is the i-th unit row vector, and i, j, k are mutually different. Note that the cardinality of the set $\{w \in \mathbb{Z}_2^m \mid v \succeq w\}$ for a fixed $v \in \mathbb{Z}_2^m$ is $2^{|v|}$. By the bijection

$$\mathbb{Z}_2^m \ni v \mapsto I_v = \{i \in \{1, \dots, m\} \mid v_i = 1\} \in 2^{\{1,\dots,m\}}$$

between $\mathbb{Z}_2^m$ and the power set $2^{\{1,\dots,m\}}$ of $\{1, \dots, m\}$, the partial order $\succeq$ on $\mathbb{Z}_2^m$ corresponds to the partial order $\supset$ on $2^{\{1,\dots,m\}}$.

We set

$$\begin{aligned} (\beta_{0,1}, \dots, \beta_{0,m}) &= (b_1, \dots, b_m), \\ (\beta_{1,1}, \dots, \beta_{1,m}) &= (c_1 - 1 - b_1, \dots, c_m - 1 - b_m), \\ \gamma_v &= a - v\,{}^t c + |v| \quad (v \in \mathbb{Z}_2^m). \end{aligned}$$

We regard theses parameters as indeterminates. Throughout this paper, we assume that

$$\beta_{0,i},\ \beta_{1,i},\ \gamma_v \notin \mathbb{Z} \tag{2.2}$$

for any $i \in \{1, \dots, m\}$ and $v \in \mathbb{Z}_2^m$, when we assign complex values to them.

3. Twisted cohomology groups

In this section, we regard vector spaces as defined over the rational function field $\mathbb{C}(\alpha) = \mathbb{C}(a, b_1, \dots, b_m, c_1, \dots, c_m)$ when we do not specify a field. We denote the vector space of rational k-forms on $\mathbb{C}^m$ with poles only along S by $\Omega^k_X(*S)$. Note that $\Omega^0_X(*S)$ admits the structure of an algebra over $\mathbb{C}(\alpha)$. We set

$$\begin{aligned}
\widetilde{X} &= \Big\{(t,x) \in \mathbb{C}^m \times X \Big| \big(1-\sum_{i=1}^m t_i x_i\big) \prod_{i=1}^m t_i(1-t_i) \neq 0\Big\} \subset (\mathbb{P}^1)^{2m},\\
\widetilde{S} &= (\mathbb{P}^1)^{2m} - \widetilde{X}.
\end{aligned}$$

We define projections

$$\mathrm{pr}_T : \widetilde{X} \ni (t,x) \mapsto t \in \mathbb{C}^m, \quad \mathrm{pr}_X : \widetilde{X} \ni (t,x) \mapsto x \in X.$$

Note that

$$\mathrm{pr}_T(\mathrm{pr}_X^{-1}(x)) = \{t \in \mathbb{C}^m \mid (1-\sum_{i=1}^m t_i x_i) \prod_{i=1}^m t_i(1-t_i) \neq 0\} = \mathbb{C}^m_x$$

for any fixed $x \in X$.

Let $\Omega^k_{\widetilde{X}}(*\widetilde{S})$ be the vector space of rational k-forms on $\widetilde{X}$ with poles only along $\widetilde{S}$ and $\Omega^{p,q}_{\widetilde{X}}(*\widetilde{S})$ be the subspace of $\Omega^{p+q}_{\widetilde{X}}(*\widetilde{S})$ consisting elements which are p-forms with respect to the variables $t_1, \dots, t_m$.

We set

$$\begin{aligned}
\omega_T &= \sum_{i=1}^m \omega_{T_i} dt_i \in \Omega^{1,0}_{\widetilde{X}}(*\widetilde{S}), \quad \omega_{T_i} = \frac{\beta_{0,i}}{t_i} + \frac{\beta_{1,i}}{t_i - 1} + \frac{a x_i}{1 - t\,{}^t x},\\
\omega_X &= \sum_{i=1}^m \omega_{X_i} dx_i \in \Omega^{0,1}_{\widetilde{X}}(*\widetilde{S}), \quad \omega_{X_i} = \frac{a t_i}{1 - t\,{}^t x},\\
\omega &= \omega_T + \omega_X \in \Omega^1_{\widetilde{X}}(*\widetilde{S}).
\end{aligned}$$

We define a twisted exterior derivation on $\widetilde{X}$ by

$$\nabla_T = d_T + \omega_T \wedge,$$

where d_T is the exterior derivation with respect to the variable t, i.e.,

$$d_T f(t,x) = \sum_{i=1}^m \frac{\partial f}{\partial t_i}(t,x) dt_i.$$

We define an $\Omega^0_X(*S)$-module by

$$\mathcal{H}^m(\nabla_T) = \Omega^{m,0}_{\widetilde{X}}(*\widetilde{S}) \big/ \nabla_T(\Omega^{m-1,0}_{\widetilde{X}}(*\widetilde{S})).$$

It admits the structure of a vector bundle over X. We define two sets $\{\varphi_v\}_{v \in \mathbb{Z}_2^m}$ and $\{\psi_v\}_{v \in \mathbb{Z}_2^m}$ of 2^m elements of $\Omega^{m,0}_{\widetilde{X}}(*\widetilde{S})$ as

$$\varphi_v = \frac{dT}{\prod_{i=1}^m (t_i - v_i)}, \quad \psi_v = \frac{(1 - v\,{}^t x) dT}{(1 - t\,{}^t x) \prod_{1 \le i \le m} (t_i - v_i)} \tag{3.1}$$

where $v = (v_1, \dots, v_m) \in \mathbb{Z}_2^m$. To express ψ_v as a linear combination of φ_v's, we give some Lemmas.

Lemma 3.1. *We have*

$$\psi_v = \frac{1 - v\,{}^t x}{1 - t\,{}^t x}\varphi_v = \varphi_v + \sum_{j=1}^{m} \frac{x_j dT}{(1 - t\,{}^t x)\prod_{1\le i\le m}^{i\neq j}(t_i - v_i)}.$$

Proof. A straightforward calculation implies this lemma. □

Lemma 3.2. *We have*

$$\frac{a x_j dT}{(1 - t\,{}^t x)\prod_{1\le i\le m}^{i\neq j}(t_i - v_i)} = \begin{cases} -\beta_{0,j}\varphi_v - \beta_{1,j}\varphi_{\sigma_j\cdot v} & \text{if} \quad v_j = 0, \\ -\beta_{0,j}\varphi_{\sigma_j\cdot v} - \beta_{1,j}\varphi_v & \text{if} \quad v_j = 1, \end{cases}$$

as elements of $\mathcal{H}^m(\nabla_T)$*, where*

$$\sigma_j : \mathbb{Z}_2^m \ni v \mapsto \sigma_j \cdot v \in \mathbb{Z}_2^m, \quad \sigma_j \cdot v \equiv v + e_j \bmod 2.$$

Proof. Put

$$\check{\varphi}_v^j = (-1)^{j-1}\frac{dt_1 \wedge \cdots \wedge dt_{j-1} \wedge dt_{j+1} \wedge \cdots \wedge dt_m}{\prod_{1\le i\le m}^{i\neq j}(t_i - v_i)} \in \Omega_{\widetilde{X}}^{m-1,0}(*\widetilde{S})$$

for $1 \le j \le m$ and $v \in \mathbb{Z}_2^m$. Since

$$dt_i \wedge (\frac{\partial}{\partial t_i}\check{\varphi}_v^j) = 0 \quad (1 \le i \le m),$$
$$dt_i \wedge (\omega_{T_i}\check{\varphi}_v^j) = 0 \quad (1 \le i \le m, i \neq j),$$

we have

$$\begin{aligned} &\nabla_T(\check{\varphi}_v^j) = \omega_{T_j} dt_j \wedge \check{\varphi}_v^j \\ = &\frac{\beta_{0,j} dT}{t_j \prod\limits_{1\le i\le m}^{i\neq j}(t_i - v_i)} + \frac{\beta_{1,j} dT}{(t_j - 1)\prod\limits_{1\le i\le m}^{i\neq j}(t_i - v_i)} + \frac{a x_j dT}{(1 - t\,{}^t x)\prod\limits_{1\le i\le m}^{i\neq j}(t_i - v_i)}. \end{aligned}$$

If $v_j = 0$ then the first and second terms of the last line are $\beta_{0,j}\varphi_v$ and $\beta_{1,j}\varphi_{\sigma_j\cdot v}$, respectively; if $v_j = 1$ then they are $\beta_{0,j}\varphi_{\sigma_j\cdot v}$ and $\beta_{1,j}\varphi_v$. Note that $\nabla_T(\check{\varphi}_v^j) = 0$ as an element of $\mathcal{H}^m(\nabla_T)$. □

Proposition 3.1. *For any* $v \in \mathbb{Z}_2^m$*, the form*

$$a\psi_v = a\frac{1 - v\,{}^t x}{1 - t\,{}^t x}\varphi_v \in \Omega_{\widetilde{X}}^{m,0}(*\widetilde{S})$$

is equal to

$$\Big[a - \sum_{j=1}^{m}\beta_{v_j,j}\Big]\varphi_v - \sum_{j=1}^{m}\beta_{1-v_j,j}\varphi_{\sigma_j\cdot v}$$

as an element of $\mathcal{H}^m(\nabla_T)$.

Proof. Rewrite the right hand side of the identity in Lemma 3.1 by Lemma 3.2. Then we have

$$a\psi_v = a\varphi_v - \sum_{1\le j\le m}^{v_j=0}(\beta_{0,j}\varphi_v + \beta_{1,j}\varphi_{\sigma_j\cdot v}) - \sum_{1\le j\le m}^{v_j=1}(\beta_{0,j}\varphi_{\sigma_j\cdot v} + \beta_{1,j}\varphi_v).$$

Note that

$$\sum_{1\le j\le m}^{v_j=0}\beta_{0,j}\varphi_v + \sum_{1\le j\le m}^{v_j=1}\beta_{1,j}\varphi_v = \Big(\sum_{j=1}^{m}\beta_{v_j,j}\Big)\varphi_v,$$

and that

$$\beta_{1-v_j,j}\varphi_{\sigma_j\cdot v} = \begin{cases} \beta_{1,j}\varphi_{\sigma_j\cdot v} & \text{if} \quad v_j = 0, \\ \beta_{0,j}\varphi_{\sigma_j\cdot v} & \text{if} \quad v_j = 1, \end{cases}$$

for $1 \le j \le m$. □

We consider the structure of the fiber of $\mathcal{H}^m(\nabla_T)$ at x. Let $\Omega^p_{\mathbb{C}^m_x}(*x)$ be the pull-back of $\Omega^{p,0}_{\widetilde{X}}(*\widetilde{S})$ under the map $\imath_x : \mathbb{C}^m_x \to \widetilde{X}$ for a fixed $x \in X$. Each fiber of $\mathcal{H}^m(\nabla_T)$ at x is isomorphic to the rational twisted cohomology group

$$H^m(\Omega^\bullet_{\mathbb{C}^m_x}(*x), \nabla_T) = \Omega^m_{\mathbb{C}^m_x}(*x)/\nabla_T(\Omega^{m-1}_{\mathbb{C}^m_x}(*x))$$

on $\mathbb{C}^m_x$ with respect to ∇_T induced from the map $\imath_x$. We denote the pull-back of φ_v under the map $\imath_x$ by $\varphi_{x,v}$.

Fact 3.1 ([AK]). (i) *The space $H^m(\Omega^\bullet_{\mathbb{C}^m_x}(*x), \nabla_T)$ is 2^m-dimensional and it is spanned by the classes of $\varphi_{x,v}$ for any $v \in \mathbb{Z}_2^m$.*

(ii) *There is a canonical isomorphism $\jmath_x$ from $H^m(\Omega^\bullet_{\mathbb{C}^m_x}(*x), \nabla_T)$ to*

$$H^m(\mathcal{E}^\bullet_c(x), \nabla_T) = \ker(\nabla_T : \mathcal{E}^m_c(x) \to \mathcal{E}^{m+1}_c(x))/\nabla_T(\mathcal{E}^{m-1}_c(x)),$$

where $\mathcal{E}^k_c(x)$ is the vector space of smooth k-forms with compact support in $\mathbb{C}^m_x$.

By Fact 3.1, we have the following.

Proposition 3.2. *The $\Omega^0_X(*S)$-module $\mathcal{H}^m(\nabla_T)$ is of rank 2^m. The classes of φ_v $(v \in \mathbb{Z}_2^m)$ in $\Omega^{m,0}_{\widetilde{X}}(*\widetilde{S})$ form a frame of the vector bundle $\mathcal{H}^m(\nabla_T)$ over X.*

Set

$$\mathcal{H}^m(\nabla_T^\vee) = \Omega^{m,0}_{\widetilde{X}}(*\widetilde{S})/\nabla_T^\vee(\Omega^{m-1,0}_{\widetilde{X}}(*\widetilde{S})),$$

where $\nabla_T^\vee = d_T - \omega_T\wedge$. This $\Omega^0_X(*S)$-module can be regarded as vector bundles over X. The classes of φ_v $(v \in \mathbb{Z}_2^m)$ also form a frame of this vector bundle. Each fiber of $\mathcal{H}^m(\nabla_T^\vee)$ at x is the rational twisted cohomology group $H^m(\Omega^\bullet_{\mathbb{C}^m_x}(*x), \nabla_T^\vee)$ on $\mathbb{C}^m_x$ defined by the coboundary $\nabla_T^\vee$ instead of ∇_T. We define the intersection form between $H^m(\Omega^\bullet_{\mathbb{C}^m_x}(*x), \nabla_T)$ and $H^m(\Omega^\bullet_{\mathbb{C}^m_x}(*x), \nabla_T^\vee)$ by

$$\mathcal{I}(\varphi_x, \varphi'_x) = \int_{\mathbb{C}^m_x} \jmath_x(\varphi_x) \wedge \varphi'_x \in \mathbb{C}(\alpha),$$

where $\varphi_x, \varphi'_x \in \Omega^m_{\mathbb{C}^m_x}(*x)$, and $\jmath_x$ is given in Fact 3.1. This integral converges since $\jmath_x(\varphi_x)$ is a smooth m-form on $\mathbb{C}^m_x$ with compact support. It is bilinear over $\mathbb{C}(\alpha)$.

For $w = (w_1, \dots, w_m) \in \mathbb{Z}_2^m$ with $|w| = r$, we have a sequence of $w^{(r)}, w^{(r-1)}, \dots, w^{(1)} \in \mathbb{Z}_2^m$ such that $|w^{(j)}| = j$ and

$$w = w^{(r)} \succ w^{(r-1)} \succ w^{(r-2)} \succ \cdots \succ w^{(1)} \succ (0, \dots, 0).$$

Let $\mathfrak{S}_w$ be the set of such sequences $(w, w^{(r-1)}, \dots, w^{(1)})$ for given $w \in \mathbb{Z}_2^m$. Note that its cardinality is $r!$. We put

$$A_w = \sum_{(w,w^{(r-1)},\dots,w^{(1)}) \in \mathfrak{S}_w} \frac{1}{\prod_{j=1}^r \gamma_{w^{(j)}}}.$$

For example,

$$\begin{aligned}
A_{(1,1)} &= \frac{1}{\gamma_{(1,1)}} \times \Big(\frac{1}{\gamma_{(1,0)}} + \frac{1}{\gamma_{(0,1)}}\Big) = \frac{1}{a-c_1-c_2+2} \times \Big(\frac{1}{a-c_1+1} + \frac{1}{a-c_2+1}\Big),\\
A_{(1,1,1)} &= \frac{1}{\gamma_{(1,1,1)}\gamma_{(1,1,0)}}\Big(\frac{1}{\gamma_{(1,0,0)}} + \frac{1}{\gamma_{(0,1,0)}}\Big) + \frac{1}{\gamma_{(1,1,1)}\gamma_{(1,0,1)}}\Big(\frac{1}{\gamma_{(1,0,0)}} + \frac{1}{\gamma_{(0,0,1)}}\Big)\\
&\quad + \frac{1}{\gamma_{(1,1,1)}\gamma_{(0,1,1)}}\Big(\frac{1}{\gamma_{(0,1,0)}} + \frac{1}{\gamma_{(0,0,1)}}\Big)\\
&= \frac{1}{a-c_1-c_2-c_3+3} \times \Big[\frac{1}{a-c_1-c_2+2}(\frac{1}{a-c_1+1}+\frac{1}{a-c_2+1})\\
&\quad +\frac{1}{a-c_1-c_3+2}(\frac{1}{a-c_1+1}+\frac{1}{a-c_3+1}) +\frac{1}{a-c_2-c_3+2}(\frac{1}{a-c_2+1}+\frac{1}{a-c_3+1})\Big].
\end{aligned}$$

Proposition 3.3. *We have*

$$\begin{aligned}
\mathcal{I}(\varphi_{x,v}, \varphi_{x,v'}) &= (2\pi\sqrt{-1})^m \left[\sum_{w\in\mathbb{Z}_2^m} A_w \prod_{1\le i\le m}^{w_i=0} \frac{\delta(v_i, v_i')}{\beta_{v_i,i}}\right],\\
\mathcal{I}(\varphi_{x,v}, \psi_{x,v'}) &= (2\pi\sqrt{-1})^m \begin{cases} \dfrac{1}{\Pi\beta_v}, & \text{if } v=v',\\ 0, & \text{otherwise},\end{cases}\\
\mathcal{I}(\psi_{x,v}, \psi_{x,v'}) &= (2\pi\sqrt{-1})^m \begin{cases} \dfrac{a-\Sigma\beta_v}{a\Pi\beta_v}, & \text{if } v=v',\\ \dfrac{-1}{a\prod_{1\le i\le m}^{v_i=v_i'}\beta_{v_i,i}}, & \text{if } \#(v\cap v') = m-1,\\ 0, & \text{otherwise},\end{cases}
\end{aligned}$$

where $v=(v_1,\dots,v_m)$, $v=(v_1',\dots,v_m')\in\mathbb{Z}_2^m$, δ *denotes Kronecker's symbol,*

$$\Sigma\beta_v = \sum_{i=1}^m \beta_{v_i,i}, \quad \Pi\beta_v = \prod_{i=1}^m \beta_{v_i,i},$$

and we regard

$$\prod_{1\le i\le m}^{w_i=0} \frac{\delta(v_i, v_i')}{\beta_{v_i,i}} = 1$$

for $w=(1,\dots,1)$. *The matrix*

$$C = \frac{1}{(2\pi\sqrt{-1})^m}\mathcal{I}(\varphi_{x,v}, \varphi_{x,v'})_{v,v'\in\mathbb{Z}_2^m}$$

satisfies

$$\det(C) = \frac{a^{2^m}}{\big(\prod_{w\in\mathbb{Z}_2^m}\gamma_w\big)\big(\prod_{i=1}^m(\beta_{0,i}\beta_{1,i})^{2^{m-1}}\big)},$$

where we array $v, v'\in\mathbb{Z}_2^m$ *by the total order in Definition 2.1. When we assign the parameters to complex values under the assumption (2.2), each intersection number is well-defined and* $\det(C)\neq 0$*; the matrix* C *is invertible.*

Proof. Since the pole divisor of $\psi_{x,v'}$ is normal crossing in $\mathbb{C}^m$, we can evaluate the intersection numbers $\mathcal{I}(\varphi_{x,v},\psi_{x,v'})$ and $\mathcal{I}(\psi_{x,v},\psi_{x,v'})$ by using results in [M1]. To compute $\mathcal{I}(\varphi_{x,v},\varphi_{x,v'})$, we blow up the space $(\mathbb{P}^1)^m$ to $\widetilde{\mathbb{P}}^m$ so that the pole divisor of $\omega_{x,T}=\imath_x^*(\omega_T)$ in $\widetilde{\mathbb{P}}^m$ becomes normal crossing. Then the pole divisor of $\omega_{x,T}$ in $\widetilde{\mathbb{P}}^m$ consists of $\ell_{0,i}$, $\ell_{1,i}$ $(1\le i\le m)$ and ℓ_w $(w\in\mathbb{Z}_2^m)$ with correspondence

$$\begin{array}{rclrcl}\ell_{0,i} & \leftrightarrow & \{t\in\mathbb{C}^m \mid t_i=0\}, & \ell_{1,i} & \leftrightarrow & \{t\in\mathbb{C}^m \mid t_i=1\},\\ \ell_{(0,\dots,0)} & \leftrightarrow & \{t\in\mathbb{C}^m \mid t\,{}^t x=1\}, & \ell_w & \leftrightarrow & \infty(\mathbb{P}^r_w)\times\mathbb{P}^{m-r}_{\sigma\cdot w}\quad (|w|=r>0),\end{array}$$

via natural bi-rational maps, where $\mathbb{P}^r_w$ is the projective space of $\mathbb{C}^r$ coordinated by t_i with $w_i=1$, $\infty(\mathbb{P}^r_w)$ is the hyperplane at infinity in $\mathbb{P}^r_w$ and $\sigma\cdot w=(1,\dots,1)-w$. The residues of $\omega_{x,T}$ in $\widetilde{\mathbb{P}}^m$ along these components are

$$\beta_{0,i},\quad \beta_{1,i},\quad -\gamma_{(0,\dots,0)},\quad \gamma_w\ ((0,\dots,0)\neq w\in\mathbb{Z}_2^m),$$

respectively. We consider conditions that m components of the pole divisor of $\varphi_{x,v}$ in $\widetilde{\mathbb{P}}^m$ intersect only at a point. Here note that $\ell_{(0,\dots,0)}$ is not its component. For fixed r $(0\le r\le m)$, let $w^{(1)},\dots,w^{(r)}$ be elements of $\mathbb{Z}_2^m$ satisfying

$$(0,\dots,0)\prec w^{(1)}\prec w^{(2)}\prec\cdots\prec w^{(r)}=w=(w_1,\dots,w_m),\quad |w|=r,$$

and let $I_{\sigma\cdot w}$ be a subset of $\{1,\dots,m\}$ given by $I_{\sigma\cdot w}=\{i\in\{1,\dots,m\}\mid w_i=0\}$. Then m components $\ell_{w^{(1)}},\dots,\ \ell_{w^{(r)}}$, $\ell_{v_i,i}$ $(i\in I_{\sigma\cdot w})$ intersect only at a point. We can show that the converse holds true. Figure 1 indicates the pole divisor of $\varphi_{x,(1,1)}$ in $\widetilde{\mathbb{P}}^2$. In this case, there are five intersection points of two components; one can easily check the above fact. This fact

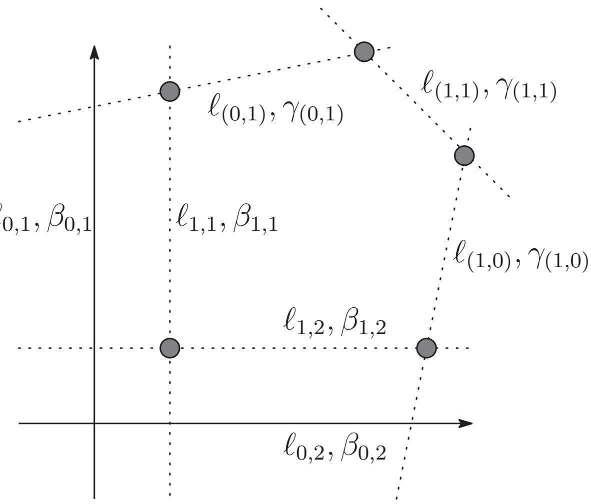

FIGURE 1. Pole divisor of $\varphi_{x,v}$ in $\widetilde{\mathbb{P}}^2$

together with results in [M1] enables us to compute $\mathcal{I}(\varphi_{x,v},\varphi_{x,v'})$. Note that

$$A_w\prod_{1\le i\le m}^{w_i=0}\frac{\delta(v_i,v_i')}{\beta_{v_i,i}}=\begin{cases}A_{(1,\dots,1)} & \text{if}\quad w=(1,\dots,1),\\ \displaystyle\prod_{i=1}^m\frac{\delta(v_i,v_i')}{\beta_{v_i,i}} & \text{if}\quad w=(0,\dots,0),\end{cases}$$

$A_{(1,\dots,1)}$ is the contribution for intersection points in $\ell_{(1,\dots,1)}$ and $\prod_{i=1}^{m}\dfrac{\delta(v_i,v_i')}{\beta_{v_i,i}}$ is that in $\mathbb{C}^m$.

It is easy to see that

$$\det\Big(\frac{1}{(2\pi\sqrt{-1})^m}\mathcal{I}(\varphi_{x,v},\psi_{x,v'})_{v,v'\in\mathbb{Z}_2^m}\Big)=\frac{1}{\prod_{i=1}^m(\beta_{0,i}\beta_{1,i})^{2^{m-1}}}.$$

By following the method in Appendix of [MY], we have

$$\det\Big(\frac{1}{(2\pi\sqrt{-1})^m}\mathcal{I}(\psi_{x,v},\psi_{x,v'})_{v,v'\in\mathbb{Z}_2^m}\Big)=\frac{\prod_{w\in\mathbb{Z}_2^m}\gamma_w}{a^{2^m}\prod_{i=1}^m(\beta_{0,i}\beta_{1,i})^{2^{m-1}}}.$$

These imply the value of $\det(C)$. □

By this fact, we can regard the intersection form $\mathcal{I}$ as that between $\mathcal{H}^m(\nabla_T)$ and $\mathcal{H}^m(\nabla_T^\vee)$. It is bilinear over $\Omega_X^0(*S)$ and the intersection matrix C is defined by the frame $\{\varphi_v\}_{v\in\mathbb{Z}_2^m}$. Let $\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ (resp. $\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T^\vee)$) be the linear span of φ_v ($v\in\mathbb{Z}_2^m$) over the field $\mathbb{C}(\alpha)$ contained in $\mathcal{H}^m(\nabla_T)$ (resp. $\mathcal{H}^m(\nabla_T^\vee)$). We have

$$\psi_v\in\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$$

for any $v\in\mathbb{Z}_2^m$ by Proposition 3.1.

4. Connections

We introduce operators

$$\nabla_k=\partial_k+\frac{at_k}{1-t\,{}^tx},\quad (k=1,\dots,m),$$

then we have

$$\partial_k\int_{\mathrm{reg}(0,1)^m}u(t,x)\varphi=\int_{\mathrm{reg}(0,1)^m}u(t,x)(\nabla_k\varphi),\tag{4.1}$$

where $\mathrm{reg}(0,1)^m$ is the regularization of the domain $(0,1)^m$ of integration defined in [AK]. Thanks to the regularization, the integral converges whenever we assign complex values to parameters under the condition (2.2), and the order of the integration and the operator ∂_k can be changed. We set

$$\nabla_X=\sum_{i=1}^m dx_i\wedge\nabla_i=d_X+\omega_X\wedge,$$

where d_X is the exterior derivation with respect to x:

$$d_Xf=\sum_{i=1}^m(\partial_i f)dx_i,\quad f\in\Omega^0_{\widetilde{X}}(*\widetilde{S}).$$

It is easy to see that

$$\nabla_T\circ\nabla_X+\nabla_X\circ\nabla_T=0.\tag{4.2}$$

We set

$$\begin{aligned}\mathcal{H}^{m,1}(\nabla_T)&=\Omega^{m,1}_{\widetilde{X}}(*\widetilde{S})/\nabla_T(\Omega^{m-1,1}_{\widetilde{X}}(*\widetilde{S})),\\ \mathcal{H}^{m,1}(\nabla_T^\vee)&=\Omega^{m,1}_{\widetilde{X}}(*\widetilde{S})/\nabla_T^\vee(\Omega^{m-1,1}_{\widetilde{X}}(*\widetilde{S})).\end{aligned}$$

Proposition 4.1. *There is a natural map $\nabla_X:\mathcal{H}^m(\nabla_T)\to\mathcal{H}^{m,1}(\nabla_T)$ induced from the derivation ∇_X.*

Proof. We have only to show that if $\psi \in \nabla_T(\Omega^{m-1,0}_{\widetilde{X}}(*\widetilde{S}))$ then

$$\nabla_X(\psi) \in \nabla_T(\Omega^{m-1,1}_{\widetilde{X}}(*\widetilde{S})).$$

For any $\psi \in \nabla_T(\Omega^{m-1,0}_{\widetilde{X}}(*\widetilde{S}))$, there exists $f \in \Omega^{m-1,0}_{\widetilde{X}}(*\widetilde{S})$ such that $\nabla_T(f) = \psi$. By (4.2), we have

$$\nabla_X(\psi) = \nabla_X \circ \nabla_T(f) = -\nabla_T \circ \nabla_X(f) = \nabla_T(-\nabla_X(f)),$$

which belongs to $\nabla_T(\Omega^{m-1,1}_{\widetilde{X}}(*\widetilde{S}))$. $\square$

By this proposition, we can regard the map ∇_X as a connection of the vector bundle $\mathcal{H}^m(\nabla_T)$ over X. It is characterized as follows.

Proposition 4.2. *Let $v = (v_1, \dots, v_m)$ be an element of $\mathbb{Z}_2^m$. If $v_k = 0$ then*

$$\nabla_k(\varphi_v) = \frac{1}{x_k}(-\beta_{0,k}\varphi_v - \beta_{1,k}\varphi_{\sigma_k \cdot v});$$

if $v_k = 1$ then

$$\nabla_k(\varphi_v) = \frac{1}{x_k}(-\beta_{0,k}\varphi_{\sigma_k \cdot v} - \beta_{1,k}\varphi_v) + \frac{1}{1 - v\,{}^t x}\Big[\big(a - \sum_{j=1}^{m} \beta_{v_j,j}\big)\varphi_v + \sum_{j=1}^{m} \beta_{1-v_j,j}\varphi_{\sigma_j \cdot v}\Big].$$

Proof. Since $\partial_k \cdot \varphi_v = 0$, we have

$$\nabla_k(\varphi_v) = \omega_k \cdot \varphi_v = \frac{a t_k}{1 - t\,{}^t x} \cdot \frac{dT}{\prod_{i=1}^{m}(t_i - v_i)}.$$

If $v_i = 0$ then

$$\nabla_k(\varphi_v) = \frac{1}{x_k} \cdot \frac{a x_x dT}{(1 - t\,{}^t x)\prod_{1 \le i \le m}^{i \ne k}(t_i - v_i)} = \frac{-\beta_{0,k}\varphi_v - \beta_{1,k}\varphi_{\sigma_k \cdot v}}{x_k}$$

by Lemma 3.2. If $v_i = 1$ then

$$\begin{aligned}
\nabla_k(\varphi_v) &= \frac{a(t_k - 1) + a}{1 - t\,{}^t x} \cdot \frac{dT}{\prod_{i=1}^{m}(t_i - v_i)} = \frac{a\, dT}{(1 - t\,{}^t x)\prod_{1 \le i \le m}^{i \ne k}(t_i - v_i)} + \frac{a\, dT}{(1 - t\,{}^t x)\prod_{i=1}^{m}(t_i - v_i)} \\
&= \frac{-\beta_{0,k}\varphi_{\sigma_k \cdot v} - \beta_{1,k}\varphi_v}{x_k} + \frac{a\psi_v}{1 - v\,{}^t x}
\end{aligned}$$

by Lemma 3.2. Rewrite the last term by Proposition 3.1. $\square$

Corollary 4.1. *For any $v = (v_1, \dots, v_m) \in \mathbb{Z}_2^m$, we have*

$$\Big(\prod_{1 \le i \le m}^{v_i = 1} x_i \nabla_i\Big) \cdot \varphi_{(0,\dots,0)} = \sum_{w \preceq v} \Big[\prod_{1 \le i \le m}^{v_i = 1} (-\beta_{w_i,i})\Big]\varphi_w.$$

Proof. Use the induction on $|v|$ and Proposition 4.2. $\square$

We give some examples:

$$\begin{aligned}
(x_1\nabla_1)\cdot\varphi_{(0,0,0)} &= -\beta_{0,1}\varphi_{(0,0,0)}-\beta_{1,1}\varphi_{(1,0,0)},\\
(x_1x_2\nabla_1\nabla_2)\cdot\varphi_{(0,0,0)} &= \beta_{0,1}\beta_{0,2}\varphi_{(0,0,0)}+\beta_{1,1}\beta_{0,2}\varphi_{(1,0,0)}+\beta_{0,1}\beta_{1,2}\varphi_{(0,1,0)}+\beta_{1,1}\beta_{1,2}\varphi_{(1,1,0)},\\
(x_1x_2x_3\nabla_1\nabla_2\nabla_3)\cdot\varphi_{(0,0,0)} &= -\beta_{0,1}\beta_{0,2}\beta_{0,3}\varphi_{(0,0,0)}-\beta_{1,1}\beta_{0,2}\beta_{0,3}\varphi_{(1,0,0)}\\
&\quad -\beta_{0,1}\beta_{1,2}\beta_{0,3}\varphi_{(0,1,0)}-\beta_{0,1}\beta_{0,2}\beta_{1,3}\varphi_{(0,0,1)}\\
&\quad -\beta_{1,1}\beta_{1,2}\beta_{0,3}\varphi_{(1,1,0)}-\beta_{1,1}\beta_{0,2}\beta_{1,3}\varphi_{(1,0,1)}\\
&\quad -\beta_{0,1}\beta_{1,2}\beta_{1,3}\varphi_{(0,1,1)}-\beta_{1,1}\beta_{1,2}\beta_{1,3}\varphi_{(1,1,1)}.
\end{aligned}$$

To express ∇_X restricted to $\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ by the intersection form $\mathcal{I}$, we give some lemmas and a proposition.

Lemma 4.1. *Let φ be an element of $\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ and φ' be that of $\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla^\vee_T)$. Then we have*

$$\mathcal{I}(\nabla_i\varphi,\varphi')+\mathcal{I}(\varphi,\nabla^\vee_i\varphi')=0,$$

where $1\le i\le m$ and $\nabla^\vee_i=\partial_i-\dfrac{at_i}{1-t\,{}^tx}$.

Proof. It is clear by Proposition 3.3 that

$$\partial_i\mathcal{I}(\varphi,\varphi')=0$$

for $1\le i\le m$. For any compact set K in $\mathbb{C}^m_x$, we have

$$\begin{aligned}
&\partial_i\int_K\varphi\wedge\varphi'=\int_K\partial_i\varphi\wedge\varphi'+\int_K\varphi\wedge\partial_i\varphi'\\
=&\int_K\partial_i(u(t,x)\cdot\varphi)\wedge\frac{\varphi'}{u(t,x)}+\int_K u(t,x)\cdot\varphi\wedge\partial_i\Big(\frac{\varphi'}{u(t,x)}\Big)=\int_K(\nabla_i\varphi)\wedge\varphi'+\int_K\varphi\wedge(\nabla^\vee_i\varphi').
\end{aligned}$$

We can show that the commutativity of $\jmath_x$ and $\nabla^\vee_i$ by following results in [M2]. These imply this lemma. □

We define maps

$$\begin{aligned}
\mathcal{R}_k &: \mathcal{H}^m(\nabla_T)\ni\varphi\mapsto \operatorname*{Res}_{x_k=0}(\nabla_X(\varphi)) \in\mathcal{H}^m(\nabla_T),\\
\mathcal{R}_{k,v} &: \mathcal{H}^m(\nabla_T)\ni\varphi\mapsto \operatorname*{Res}_{x_k=S_v\cap L_k}(\nabla_X(\varphi)) \in\mathcal{H}^m(\nabla_T),
\end{aligned}$$

where $\operatorname*{Res}_{x_k=0}(\eta)$ and $\operatorname*{Res}_{x_k=S_v\cap L_k}(\eta)$ are the residues of $\eta\in\Omega^{m,1}_{\widetilde{X}}(*\widetilde{S})$ with respect to the variable x_k at 0 and at the intersection point $S_v\cap L_k$ of S_v and the line L_k in X fixing the variables $x_1,\dots,x_{k-1},x_{k+1},\dots,x_m$.

Proposition 4.3 (Orthogonal Principle)**.** (i) *For $\varphi\in\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ and $\varphi'\in\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla^\vee_T)$, we have*

$$\mathcal{I}(\mathcal{R}_k(\varphi),\varphi')+\mathcal{I}(\varphi,\mathcal{R}^\vee_k(\varphi'))=0,\quad \mathcal{I}(\mathcal{R}_{k,v}(\varphi),\varphi')+\mathcal{I}(\varphi,\mathcal{R}^\vee_{k,v}(\varphi'))=0,$$

where $\mathcal{R}^\vee_k$ and $\mathcal{R}^\vee_{k,v}$ are naturally defined by $\nabla^\vee_X=\sum_{i=1}^m dx_i\nabla^\vee_i$ and the residue.

(ii) *Let φ and φ' be eigenvectors of $\mathcal{R}_k$ and $\mathcal{R}^\vee_k$ (resp. $\mathcal{R}_{k,v}$ and $\mathcal{R}^\vee_{k,v}$) with eigenvalues μ and μ', respectively. If $\mu+\mu'\neq 0$ then $\mathcal{I}(\varphi,\varphi')=0$.*

Proof. (i) We have only to see coefficients of $1/x_k$ and $1/(1-v\,{}^t x)$ of the identity in Lemma 4.1.

(ii) Note that

$$\mathcal{I}(\mathcal{R}_k(\varphi),\varphi')+\mathcal{I}(\varphi,\mathcal{R}_k^\vee(\varphi'))=\mathcal{I}(\mu\varphi,\varphi')+\mathcal{I}(\varphi,\mu'\varphi')=(\mu+\mu')\mathcal{I}(\varphi,\varphi').$$

By (i), we have $(\mu+\mu')\mathcal{I}(\varphi,\varphi')=0$. □

Lemma 4.2. (i) *Suppose that $c_k\neq 1$ when we assign a complex value to it. The eigenvalues of the map $\mathcal{R}_k$ are 0 and $-\beta_{0,k}-\beta_{1,k}=1-c_k$. The eigenspace W_k of the map $\mathcal{R}_k$ with eigenvalue 0 is 2^{m-1}-dimensional and expressed as*

$$W_k=\langle\varphi_v-\varphi_{\sigma_k\cdot v}\mid v\in\mathbb{Z}_2^m(0_k)\rangle,$$

which is the linear span of $\varphi_v-\varphi_{\sigma_k\cdot v}$ for elements v in

$$\mathbb{Z}_2^m(0_k)=\{v=(v_1,\dots,v_m)\in\mathbb{Z}_2^m\mid v_k=0\}.$$

The eigenspace of the map $\mathcal{R}_k$ with eigenvalue $1-c_k$ is 2^{m-1}-dimensional and

$$W_k^\perp=\langle\beta_{0,k}\varphi_v+\beta_{1,k}\varphi_{\sigma_k\cdot v}\mid v\in\mathbb{Z}_2^m(0_k)\rangle.$$

(ii) *Suppose that $\Sigma\beta_v-a\neq 0$ for a given $v\in\mathbb{Z}_2^m$ when we assign complex values to them. The eigenvalues of the map $\mathcal{R}_{k,v}$ are $\Sigma\beta_v-a$ and 0. The eigenspace W_v of the map $\mathcal{R}_{k,v}$ with eigenvalue $\Sigma\beta_v-a$ is spanned by ψ_v, and that with eigenvalue 0 is its orthogonal complement*

$$W_v^\perp=\{\varphi\in\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)\mid\mathcal{I}(\varphi,\psi_v)=0\},$$

which is spanned by φ_w for $w\neq v$.

Proof. (i) Let v be an element of $\mathbb{Z}_2^m(0_k)$. Proposition 4.2 implies that

$$\begin{aligned}\mathcal{R}_k(\varphi_v-\varphi_{\sigma_k\cdot v}) &= (-\beta_{0,k}\varphi_v-\beta_{1,k}\varphi_{\sigma_k\cdot v})-(-\beta_{0,k}\varphi_{\sigma_k\cdot(\sigma_k\cdot v)}-\beta_{1,k}\varphi_{\sigma_k\cdot v})=0,\\ \mathcal{R}_k(\beta_{0,k}\varphi_v+\beta_{1,k}\varphi_{\sigma_k\cdot v}) &= \beta_{0,k}(-\beta_{0,k}\varphi_v-\beta_{1,k}\varphi_{\sigma_k\cdot v})+\beta_{1,k}(-\beta_{0,k}\varphi_{\sigma_k\cdot(\sigma_k\cdot v)}-\beta_{1,k}\varphi_{\sigma_k\cdot v})\\ &= (-\beta_{0,k}-\beta_{1,k})(\beta_{0,k}\varphi_v+\beta_{1,k}\varphi_{\sigma_k\cdot v}).\end{aligned}$$

Thus $\varphi_v-\varphi_{\sigma_k\cdot v}$ is an eigenvector of $\mathcal{R}_k$ with eigenvalue 0, and $\beta_{0,k}\varphi_v+\beta_{1,k}\varphi_{\sigma_k\cdot v}$ is an eigenvector of $\mathcal{R}_k$ with eigenvalue $1-c_k$ for each $v\in\mathbb{Z}_2^m(0_k)$. Hence these eigenspaces are 2^{m-1}-dimensional.

(ii) Propositions 3.1 and 4.2 imply that

$$\begin{aligned}&\mathcal{R}_{k,v}(a\psi_v)=\mathcal{R}_{k,v}\Big[\big(a-\sum_{j=1}^m\beta_{v_j,j}\big)\varphi_v-\sum_{j=1}^m\beta_{1-v_j,j}\varphi_{\sigma_j\cdot v}\Big]\\ =\;& -\big(a-\sum_{j=1}^m\beta_{v_j,j}\big)\Big[\big(a-\sum_{j=1}^m\beta_{v_j,j}\big)\varphi_v-\sum_{j=1}^m\beta_{1-v_j,j}\varphi_{\sigma_j\cdot v}\Big]=(\Sigma\beta_v-a)(a\psi_v).\end{aligned}$$

Note that the image of $\mathcal{R}_{k,v}$ is spanned by ψ_v. Proposition 4.2 also implies that $\mathcal{R}_{k,v}\varphi_w=0$ for $w\neq v$. By Proposition 3.3, they are orthogonal to ψ_v with respect to the intersection form $\mathcal{I}$. □

Lemma 4.3. *Suppose that $c_k\neq 1$ when we assign a complex value to it. Then the projection $\mathrm{pr}_k:\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)\to W_k$ is expressed as*

$$\mathrm{pr}_k(\varphi)=\sum_{v\in\mathbb{Z}_2^m(0_k)}\frac{\beta_{1,k}\Pi\beta_v}{(2\pi\sqrt{-1})^m(\beta_{0,k}+\beta_{1,k})}\mathcal{I}\big(\varphi,(\psi_v-\psi_{\sigma_k\cdot v})\big)(\varphi_v-\varphi_{\sigma_k\cdot v}).$$

Proof. By Proposition 3.3, we have

$$\frac{\beta_{1,k}\Pi\beta_v}{(2\pi\sqrt{-1})^m(\beta_{0,k}+\beta_{1,k})}\mathcal{I}\big((\varphi_w-\varphi_{\sigma_k\cdot w}),(\psi_v-\psi_{\sigma_k\cdot v})\big)=\delta(v,w)$$

for $w\in\mathbb{Z}_2^m(0_k)$. Since

$$\mathcal{I}\big((\beta_{0,k}\varphi_v+\beta_{1,k}\varphi_{\sigma_k\cdot v}),(\psi_v-\psi_{\sigma_k\cdot v})\big)=0,$$

we have

$$\mathcal{I}(\varphi,(\psi_v-\psi_{\sigma_k\cdot v}))=0$$

for any element $\varphi\in W_k^\perp$. The restriction of the expression of pr_k to W_k is the identity, and that to $W_k^\perp$ is the zero map. □

Lemma 4.4. (i) *The map* $\mathcal{R}_k:\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)\to\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ *is expressed as*

$$\varphi\mapsto(1-c_k)\varphi+\sum_{v\in\mathbb{Z}_2^m(0_k)}\frac{\beta_{1,k}\Pi\beta_v}{(2\pi\sqrt{-1})^m}\mathcal{I}\big(\varphi,(\psi_v-\psi_{\sigma_k\cdot v})\big)(\varphi_v-\varphi_{\sigma_k\cdot v}).$$

(ii) *The map* $\mathcal{R}_{k,v}:\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)\to\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ *is expressed as*

$$\varphi\mapsto\frac{-a\Pi\beta_v}{(2\pi\sqrt{-1})^m}\mathcal{I}(\varphi,\psi_v)\psi_v.$$

Proof. (i) At first, we assume that $c_k\neq 1$ when we assign a complex value to it. The projection from $\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ to the eigenspace $W_k^\perp$ of $\mathcal{R}_k$ with eigenvalue $1-c_k$ is expressed as $\varphi\mapsto\varphi-\mathrm{pr}_k(\varphi)$. Thus we have

$$\mathcal{R}_k(\varphi)=(1-c_k)(\varphi-\mathrm{pr}_k(\varphi))=(1-c_k)\varphi+(\beta_{0,k}+\beta_{1,k})\mathrm{pr}_k(\varphi).$$

Lemma 4.3 implies the expression. Note that this expression is valid even in the case $c_k=1$.
(ii) At first, we assume $\Sigma\beta_v-a\neq 0$ for a given $v\in\mathbb{Z}_2^m$ when we assign complex values to them. By Lemma 4.2 (ii), $\mathcal{R}_{k,v}$ is characterized as

$$\varphi\mapsto(\Sigma\beta_v-a)\mathcal{I}(\varphi,\psi_v)\mathcal{I}(\psi_v,\psi_v)^{-1}\psi_v.$$

By Proposition 3.3, we have

$$\mathcal{I}(\psi_v,\psi_v)=(2\pi\sqrt{-1})^m\frac{-(\Sigma\beta_v-a)}{a\Pi\beta_v},$$

which gives the expression. This expression is valid even in the case $\Sigma\beta_v-a=0$. □

Theorem 4.1. *Suppose that (2.2) when we assign complex values to the parameters. The restriction of* ∇_X *to the space* $\mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$ *is expressed as*

$$\begin{aligned}\varphi\ \mapsto\ &\sum_{k=1}^m(1-c_k)\frac{dx_k}{x_k}\wedge\varphi+\sum_{k=1}^m\sum_{v\in\mathbb{Z}_2^m(0_k)}\frac{\beta_{1,k}\Pi\beta_v}{(2\pi\sqrt{-1})^m}\mathcal{I}(\varphi,(\psi_v-\psi_{\sigma_k\cdot v}))\frac{dx_k}{x_k}\wedge(\varphi_v-\varphi_{\sigma_k\cdot v})\\&+\sum_{v\in\mathbb{Z}_2^m}\frac{-a\Pi\beta_v}{(2\pi\sqrt{-1})^m}\mathcal{I}(\varphi,\psi_v)\frac{d(1-v\,{}^tx)}{1-v\,{}^tx}\wedge\psi_v,\end{aligned}$$

where φ_v *and* ψ_v *are given in (3.1),* $\Pi\beta_v=\prod_{i=1}^m\beta_{v_i,i}$ *for* $v=(v_1,\dots,v_m)\in\mathbb{Z}_2^m$*, and we regard* $d(1-v\,{}^tx)$ *as* 0 *for* $v=(0,\dots,0)$.

Proof. By Proposition 4.2, we see that the connection ∇_X admits simple poles only along $S \subset (\mathbb{P}^1)^m$. Thus it is expressed as

$$\sum_{k=1}^{m}\Big(\frac{\mathcal{R}_k}{x_k} - \sum_{v\in\mathbb{Z}_2^m}\frac{\mathcal{R}_{k,v}}{1-v\ {}^t x}\Big)dx_k.$$

Use the expressions of $\mathcal{R}_k$ and $\mathcal{R}_{k,v}$ in Lemma 4.4. □

By using our frame $\{\varphi_v\}_{v\in\mathbb{Z}_2^m}$ of $\mathcal{H}^m(\nabla_T)$, we represent the connection ∇_X by matrices. We set a column vector Φ by arraying φ_v's by the total order in Definition 2.1:

$$\Phi = {}^t\big(\varphi_{(0,\dots,0)}, \varphi_{(1,0,\dots,0)}, \varphi_{(0,1,0,\dots,0)}, \dots, \varphi_{(1,\dots,1)}\big).$$

Let e_v $(v \in \mathbb{Z}_2^m)$ be the unit row vectors of size 2^m satisfying $\varphi_v = e_v\Phi$. Put

$$f_v = \frac{a-\Sigma\beta_v}{a}e_v - \sum_{j=1}^{m}\frac{\beta_{1-v_j,j}}{a}e_{\sigma_j\cdot v},$$

then we have

$$\psi_v = f_v\Phi$$

by Proposition 3.1.

Corollary 4.2. *Suppose that (2.2) when we assign complex values to the parameters. The map ∇_X is represented by the frame $\{\varphi_v\}_{v\in\mathbb{Z}_2^m}$ of $\mathcal{H}^m(\nabla_T)$ as*

$$\nabla_X\Phi = \Xi_\Phi \wedge \Phi,$$

$$\begin{aligned}\Xi_\Phi \;=\;& \sum_{k=1}^{m}(1-c_k)\mathrm{id}_{2^m}\frac{dx_k}{x_k} + \sum_{k=1}^{m}\sum_{v\in\mathbb{Z}_2^m(0_k)}(\beta_{1,k}\Pi\beta_v)C\ {}^t(f_v - f_{\sigma_k\cdot v})(e_v - e_{\sigma_k\cdot v})\frac{dx_k}{x_k}\\ &+\sum_{v\in\mathbb{Z}_2^m}(-a\Pi\beta_v)C\ {}^t f_v f_v\frac{d(1-v\ {}^t x)}{1-v\ {}^t x},\end{aligned}$$

where id_{2^m} is the unit matrix of size 2^m and the intersection matrix C is given in Proposition 3.3.

Proof. We identify a row vector $z = (\dots, z_v, \dots) \in \mathbb{C}(\alpha)^{2^m}$ with an element $\varphi = z\ \Phi \in \mathcal{H}^m_{\mathbb{C}(\alpha)}(\nabla_T)$. Then the intersection form is expressed as

$$\mathcal{I}(\varphi, \psi_v) = (2\pi\sqrt{-1})^m\ z\ C\ {}^t f_v.$$

Thus we have our representation Ξ_Φ of ∇_X by Theorem 4.1. □

We define a vector valued function $F(x) = {}^t(\dots, F_v(x), \dots)$ in $\mathbb{D}$ by

$$F_{(0,\dots,0)}(x) = \Big(\prod_{i=1}^{m}\frac{\Gamma(b_i)\Gamma(c_i-b_i)}{\Gamma(c_i)}\Big)F_A(a,b,c;x), \quad F_v(x) = \Big(\prod_{1\le i\le m}^{v_i=1} x_i\partial_i\Big)\cdot F_{(0,\dots,0)}(x),$$

where $F_v(x)$ $(v \in \mathbb{Z}_2^m)$ are arrayed by the total order in Definition 2.1.

Corollary 4.3 (Pfaffian system of $F_A(a,b,c)$)**.** *Suppose that (2.2) when we assign complex values to the parameters. The vector valued function $F(x)$ satisfies a Pfaffian system*

$$d_X F(x) = (P\ \Xi_{\Phi}\ P^{-1})F(x),$$

where Ξ_{Φ} is given in Corollary 4.2 and $P = (p_{vw})_{v,w\in\mathbb{Z}_2^m}$ is defined by

$$p_{vw} = \begin{cases} \displaystyle\prod_{1\le i\le m}^{v_i=1} (-\beta_{w_i,i}) & \text{if } v \succeq w, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the integral representation (2.1) of $F_A(a,b,c;x)$ and the equation (4.1), we have

$$F_v(x) = \int_{\mathrm{reg}(0,1)^m} u(t,x)\Big(\prod_{1\le i\le m}^{v_i=1} x_i\nabla_i\Big)\cdot \varphi_{(0,\dots,0)}.$$

Corollary 4.1 implies

$$F(x) = P\int_{\mathrm{reg}(0,1)^m} u(t,x)\Phi.$$

Since P is a lower triangular matrix with non-zero diagonal entries, it is invertible. Hence $F(x)$ satisfies the Pfaffian system. $\square$

Remark 4.1. *The (v,w)-entry of P^{-1} is*

$$\begin{cases} \displaystyle\prod_{1\le i\le m}^{v_i=1,w_i=0} \beta_{0,i} \Big/ \prod_{1\le i\le m}^{v_i=1}(-\beta_{1,i}) & \text{if } v\succeq w, \\ 0 & \text{otherwise.}\end{cases}$$

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