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Tomita, Yoshihiro

Yuan, Xi

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EFFECTIVE PROPERTIES OF COSSERAT COMPOSITES WITH PERIODIC MICROSTRUCTURE

Xi Yuan*, Yoshihiro Tomita**
*Gradate School of Science & Technology, Kobe University, Kobe 657-8501, Japan
**Faculty of Engineering, Kobe University, Kobe 657-8501, Japan

1.Introduction

In the last three decades, numbers of homogenization theories have been developed in order to predict the macroscopic behavior of heterogeneous materials from their microstructure. However, most of these theories are developed in the field of the classical continuum mechanics and give results independent of the scale of the microstructure. Despite the conspicuous development of the generalized continuum or nonlocal theories recently, works concerning the homogenization analysis of heterogeneous nonlocal continuum, e.g. [1,2], is rather rare and most characters of such kind of heterogeneous medium are not clear yet.

The aim of this work is to develop a homogenization method for heterogeneous Cosserat materials. Such materials [3-5] are characterized by rotational degrees of freedom $\boldsymbol{\varphi}$, which are independent of the translational motion $\mathbf{u}$. They belong to nonlocal continuum and have found much application, e.g. [6-8], now. We [9,10] once developed a homogenization method for this type of material by asymptotic homogenization approach. In this paper, a homogenization method is derived after the discussion of properties of the Cosserat composite with periodic microstructure on the micromechanics sense. Then some scale-dependent behaviors of porous media are discussed.

Throughout the paper, a bold character denotes a tensor. Dot means first order contraction and **:** the double contraction. $\otimes$ stands for tensor product and ∇ is the Nabla operator.

2.Micropolar composite with periodic microstructure

2.1. Basic field equations

A composite Ω with a periodic microstructure can be defined by the smallest repeatable element, i.e., representative volume element (RVE) or unit cell Y (see Fig.1). Here we suppose that the constituents of the RVE are micropolar materials and their spatial distribution and mechanical properties are given. The static elastic problem with no body force and body couple is considered in this paper. The equations governing the equilibrium stress and deformation fields in such a linear isotropic micropolar elastic composite material are then [5]

$$\boldsymbol{\varepsilon} = \nabla \otimes \mathbf{u} - \mathbf{e} \bullet \boldsymbol{\varphi}, \qquad \boldsymbol{\kappa} = \nabla \otimes \boldsymbol{\varphi}, \tag{1a}$$

$$\nabla \bullet \mathbf{t} = \mathbf{0}, \qquad \nabla \bullet \boldsymbol{\mu} + \mathbf{e} : \mathbf{t} = \mathbf{0}. \tag{1b}$$

$$\begin{matrix} \mathbf{t} = \mathbf{D}^{u} : \boldsymbol{\varepsilon} \\ \boldsymbol{\mu} = \mathbf{D}^{\phi} : \boldsymbol{\kappa} \end{matrix} \quad \text{or} \quad \begin{matrix} \mathbf{t} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\, \mathbf{I}_2 + \mu\, \boldsymbol{\varepsilon} + \mu_c\, \boldsymbol{\varepsilon}^T, \\ \boldsymbol{\mu} = \alpha \mathrm{tr}(\boldsymbol{\kappa})\, \mathbf{I}_2 + \gamma\, \boldsymbol{\kappa} + \beta\, \boldsymbol{\kappa}^T, \end{matrix} \tag{1c}$$

where $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ are strain and curvature strain tensors, $\mathbf{u}$ and $\boldsymbol{\varphi}$ are displacement and rotation vectors, $\mathbf{t}$ and $\boldsymbol{\mu}$ are stress and couple stress tensors. Besides, $\mathbf{I}_2$ denotes unit tensors of the second order. The elastic moduli λ and $\mu+\mu_c=2C$ are readily recognized as Lame coefficients, while $\mu-\mu_c=2G$ denotes the Cosserat shear modulus which couples the skew-symmetric stress-strain components. Moreover, an intrinsic length scale l_c and a coupling factor N can be introduced as

$$l_c^2 = \gamma / 2C, \qquad N^2 = G/(C+G). \tag{2}$$

With N=0, equation (1c) gives the same stress-strain relation as the classical elastic theory.

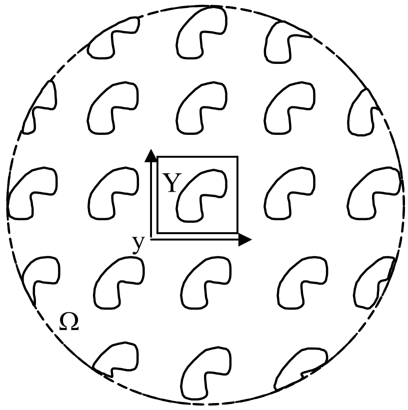


FIG.1
Composite with periodic microstructure and unit cell

Assume that the microscopic scale (the size of unit cell Y) is small enough compared to the macroscopic scale (the size of Ω) that the heterogeneities can be 'smeared-out' and Ω can be homogenized to behaves as a homogeneous body. Then imposing some boundary displacement, force, or couple force will result in a homogeneous strain **E** and a homogeneous curvature **K** throughout Y. These homogeneous strain and curvature generate a homogeneous stress **T** and a homogeneous couple stress **M** and the homogenized (or effective) constitutive relations express the relations between **E**, **K** and **T**, **M**.

Let **y** denote the position of a point in the unit cell. The local strain field $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ can be split into the overall strain **E**, **K** and a perturbation terms $\boldsymbol{\varepsilon}*$ and $\boldsymbol{\kappa}*$, which account for the presence of heterogeneities. Then it obtains that:

$$\boldsymbol{\varepsilon} = \mathbf{E} + \boldsymbol{\varepsilon}^*, \qquad \boldsymbol{\kappa} = \mathbf{K} + \boldsymbol{\kappa}^* \tag{3}$$

Correspondingly, the displacement and microrotation field **u** and $\boldsymbol{\varphi}$ can be split into **U**, $\boldsymbol{\Phi}$ and periodic fields **u*** and $\boldsymbol{\varphi}^*$ respectively, where

$$\begin{matrix} \mathbf{K} = \nabla \otimes \boldsymbol{\Phi}, & \boldsymbol{\kappa}^* = \nabla \otimes \boldsymbol{\varphi}^* \\ \mathbf{E} = \nabla \otimes \mathbf{U} - \boldsymbol{\Phi}, & \boldsymbol{\varepsilon}^* = \nabla \otimes \mathbf{u}^* - \boldsymbol{\varphi}^* \end{matrix} \tag{4}$$

Here all the deformation fields, $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\mathbf{t}$, $\boldsymbol{\mu}$, $\mathbf{u}^*$ and $\boldsymbol{\varphi}^*$, conforming themselves to the periodic arrangement of the cells, are supposed periodic at the microscopic scale, which means

$$\mathbf{g}^*(\mathbf{y}) = \mathbf{g}^*(\mathbf{y} + \mathbf{d}), \tag{5}$$

where **g** denotes an arbitrary deformation parameter, **d** is the constant vector that determines the period of the structure.

2.2. Properties of overall stress and strain tensors

The definitions of overall strain and curvature strain tensors have been given in Eq. (4). In addition, we define the overall stress and couple stress as

$$\mathbf{T} = <\mathbf{t}>, \qquad \mathbf{M} = <\boldsymbol{\mu}>, \tag{6}$$

where <**g**> denotes the average of any variable **g** over the volume of unit cell Y, i.e.

$$<\mathbf{g}> = \frac{1}{|Y|}\int_Y \mathbf{g}\, dY. \tag{7}$$

One of the most important properties of the above-defined overall strain and stress tensors is:

Lemma 1: Let $\mathbf{t}$ and $\boldsymbol{\mu}$ be self-equilibrated stress and couple stress field on Y which fulfill (1b) and $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ the compatible strain and curvature fields given by (4) and satisfy (1a), then if boundary condition (5) is satisfied, the following relation holds:

$$<\mathbf{t}:\boldsymbol{\varepsilon}+\boldsymbol{\mu}:\boldsymbol{\kappa}>=\mathbf{T}:\mathbf{E}+\mathbf{M}:\mathbf{K} \tag{8}$$

In another word, the average of the microscopic internal work is precisely the macroscopic work of effective stress <**t**>, <**μ**>. This conclusion plays a fundamental role in the discussion of effective properties of micropolar composite materials.

To prove (8), we firstly consider the internal work of perturbation strains $\boldsymbol{\varepsilon}^*$ and $\boldsymbol{\kappa}^*$. After the introduction of (4), and considering the equilibrium equation (2), it can be obtained that:

$$\begin{aligned}
&<\mathbf{t}:\boldsymbol{\varepsilon}^*+\boldsymbol{\mu}:\boldsymbol{\kappa}^*>\\
&=<\mathbf{t}:\nabla\otimes\mathbf{u}^*>+<\boldsymbol{\mu}:\nabla\otimes\boldsymbol{\varphi}^*-\mathbf{e}:\mathbf{t}.\boldsymbol{\varphi}^*>\\
&=<\nabla.(\mathbf{t}.\mathbf{u}^*)>-<\nabla.\mathbf{t}.\mathbf{u}^*>+<\nabla.(\boldsymbol{\mu}.\boldsymbol{\varphi}^*)>-<(\nabla.\boldsymbol{\mu}+\mathbf{e}:\mathbf{t}).\boldsymbol{\varphi}^*>\\
&=<\nabla.(\mathbf{t}.\mathbf{u}^*)>+<\nabla.(\boldsymbol{\mu}.\boldsymbol{\varphi}^*)>
\end{aligned} \tag{9}$$

It then can be obtained from Gauss's theorem and the periodicity of $\mathbf{t}$, $\boldsymbol{\mu}$ and $\mathbf{u}^*,\boldsymbol{\varphi}^*$ that the above equation gives zero value. Then Eq. (8) can be easily obtained after the introduction of Eq. (4) into Eq. (9).

2.3 Solution of perturbed deformation field and effective macroscopic properties

The local stress and strain fields induced at the microscopic unit cell Y by overall strain $\mathbf{E}$, $\mathbf{K}$ or overall stress $\mathbf{T}$, $\mathbf{M}$ can be obtained by solving the following equilibrium and periodic boundary conditions:

$$\begin{gathered}
\nabla.\mathbf{t}(\mathbf{y})=\mathbf{0},\qquad \nabla.\boldsymbol{\mu}(\mathbf{y})+\mathbf{e}:\mathbf{t}(\mathbf{y})=\mathbf{0}\\
\mathbf{t}(\mathbf{y})=\mathbf{D}^{u}(\mathbf{y}):(\boldsymbol{\varepsilon}^*(\mathbf{y})+\mathbf{E}),\qquad \boldsymbol{\mu}(\mathbf{y})=\mathbf{D}^{\phi}(\mathbf{y}):(\boldsymbol{\kappa}^*(\mathbf{y})+\mathbf{K})\\
\forall\,\mathbf{y}\in \mathrm{Y},\qquad \mathbf{u}^*,\boldsymbol{\kappa}^*,\mathbf{t},\boldsymbol{\mu}\quad \text{are periodic}
\end{gathered} \tag{10}$$

Once the problem (10) is solved, the macroscopic stress <**t**> and couple stress <**μ**> can then be computed and the effective stiffness $\mathbf{D}$ of the composite is determined through the relation

$$\mathbf{D}:\begin{bmatrix}\mathbf{E}\\ \boldsymbol{\Sigma}\end{bmatrix}=<\begin{bmatrix}\mathbf{t}\\ \boldsymbol{\mu}\end{bmatrix}>=<\mathbf{D}_{mic}:\begin{bmatrix}\boldsymbol{\varepsilon}^*+\mathbf{E}\\ \boldsymbol{\kappa}^*+\mathbf{K}\end{bmatrix}>, \tag{11}$$

where $\mathbf{D}_{mic}=\begin{bmatrix}\mathbf{D}^{u} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}^{\phi}\end{bmatrix}$.

It can be proved that $\mathbf{D}$ share the same symmetry as $\mathbf{D}_{mic}$, namely $\mathbf{D}_{ijmn}=\mathbf{D}_{mnij}$ [10].

For the convenience of finite element calculation, we transfer the boundary condition problem (10) into its variational form:

$$\int_Y\{\boldsymbol{\varepsilon}(\mathbf{V}):\mathbf{D}_{mic}:\boldsymbol{\delta\varepsilon}(\mathbf{v})+\boldsymbol{\varepsilon}(\mathbf{v}^*):\mathbf{D}_{mic}:\boldsymbol{\delta}\,\boldsymbol{\varepsilon}(\mathbf{v})\}dY=\mathbf{0} \tag{12}$$

where $\mathbf{v}=\{\mathbf{u},\boldsymbol{\phi}\}$, $\mathbf{V}=\{\mathbf{U},\boldsymbol{\Phi}\}$, $\boldsymbol{\varepsilon}(\mathbf{v})=\{\nabla\otimes\mathbf{u}-\mathbf{e}:\boldsymbol{\varphi},\nabla\otimes\boldsymbol{\varphi}\}$. The solution of Eq. (12) yields:

$$\int_Y\mathbf{D}_{mic}:\boldsymbol{\varepsilon}(\boldsymbol{\chi}):\delta\,\boldsymbol{\varepsilon}(\mathbf{v})dY=\int_Y\mathbf{D}_{mic}:\boldsymbol{\delta\varepsilon}(\mathbf{v})dY \tag{13}$$

$$\mathbf{v}^*=\{\mathbf{U},\boldsymbol{\Phi}\}+\boldsymbol{\chi}:\boldsymbol{\varepsilon}(\mathbf{U},\boldsymbol{\Phi}) \tag{14}$$

When the overall (macroscopic) deformation field $\mathbf{U}$, $\boldsymbol{\Phi}$ is given, the microscopic deformation field $\mathbf{v}^*$ can then obtained from Eqs. (13), (14).

3. Analysis of scale dependent deformation behaviors of porous materials

The above homogenization method is applied to investigate the elastic plane strain problem of porous materials. The shapes of the voids inside the matrix are supposed ellipse and the geometry of the unit cell is given in Fig. 2. As shown in the figure, the unit cell has its height and width of 2L and contains an elliptic void of aspect ratio a/b at its center. The geometry of the voids is also represented by equivalent radius R, where R^2=ab.

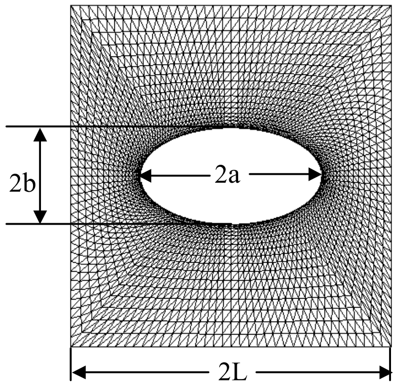


FIG. 2
Geometry and typical finite element mesh of unit cell

Consistent with definition of the material constants of isotropic elastic micropolar media given in Eqs. (1c) and (2), here we define the characteristic length of the homogenized material as

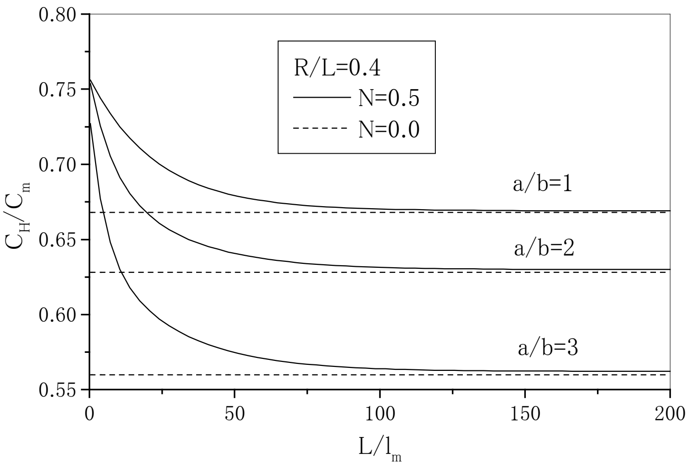
$$l_H = \sqrt{\frac{D_{3131}}{2(D_{1212}+D_{1221})}}\,, \qquad (15)$$

and the homogenized value of the Cosserat shear moduli C and G as:

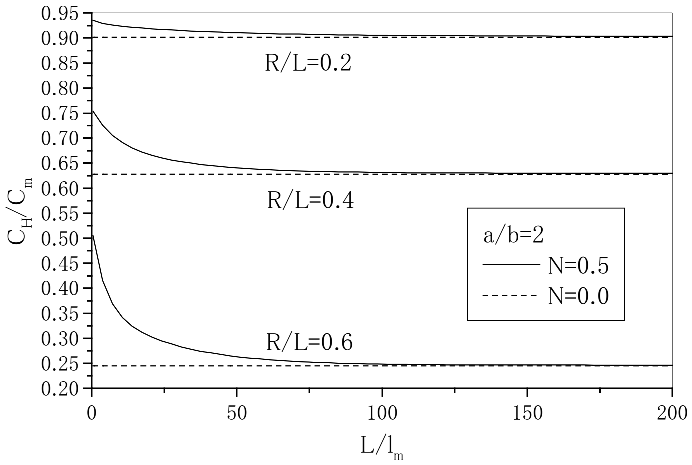
$$C_H = D_{1212}+D_{1221}, \quad G_H = D_{1212}-D_{1221} \qquad (16)$$

The unit cell problem Eq. (13) is solved by finite element method. Triangle element is used and typical finite element mesh of the unit cell is also given in Fig.2. To emphasize the difference of current method with classical homogenization approach, we focus our attention on the analysis of some size effect phenomenon, i.e., the dependence of the above-defined elastic constants on the size of unit cell L. The influence of the geometry of unit cell, relative void radius R/L and aspect ratio a/b, are also analyzed. In the following part, subscript m and H denote matrix and homogenized values, respectively.

The value of homogenized Cosserat elastic modulus C_H is shown in Fig.3. It indicates that with the increase of the size of unit cell L, C_H decreases and converges to some constants, while the classical results (when coupling factor N=0) show no such size dependence. Furthermore, the values of above constants are dependent on the geometry of the unit cell. When aspect ratio a/b or relative radius R/L of the internal void becomes larger, the constants become smaller. Furthermore, larger value of a/b and R/L also give rise to quicker change of C_H. In other words, it shows more obvious size dependence in this case.



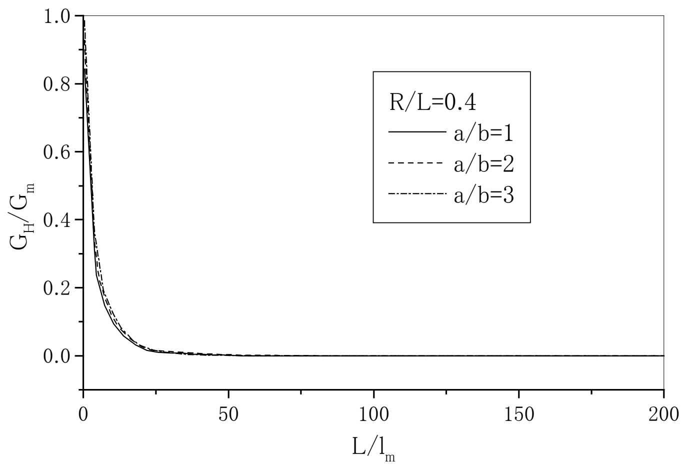
(a)



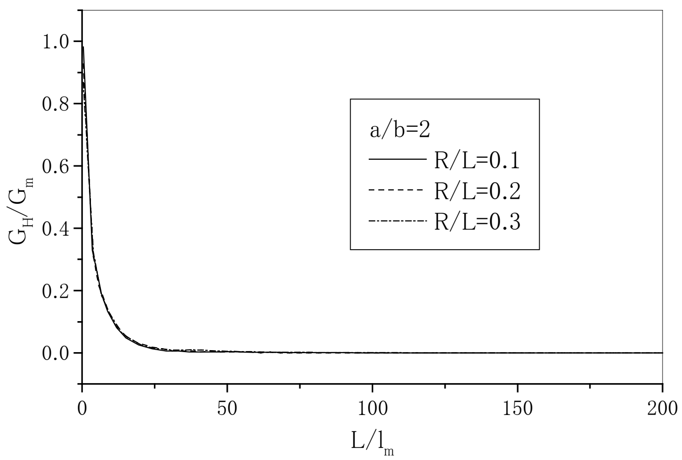
(b)

FIG.3

Dependence of homogenized modulus C_H on the relative size of unit cell L/l_m and (a) aspect ratio of void, (b) relative radius of void R/L



(a)



(b)

FIG.4

Dependence of homogenized modulus G_H on the relative size of unit cell L/l_m and (a) aspect ratio of void, (b) relative radius of void R/L

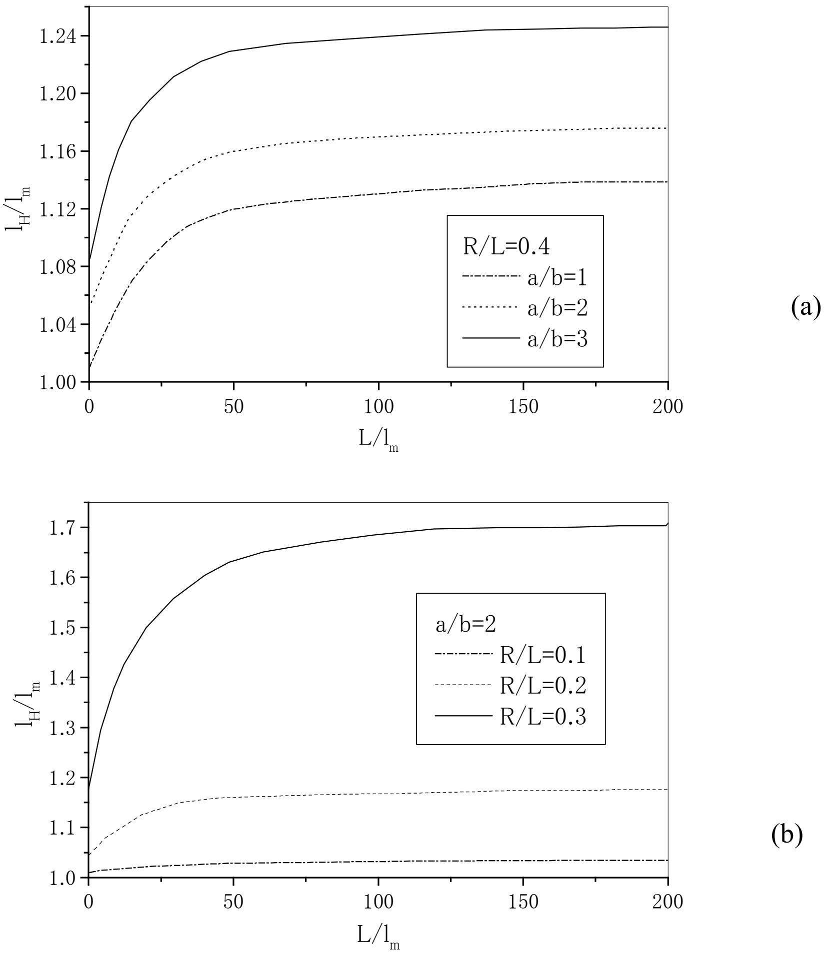


FIG.5

Dependence of homogenized characteristic length l_H on the relative size of unit cell L/l_m and (a) aspect ratio of void, (b) relative radius of void R/L

Fig.4 gives the value of homogenized Cosserat elastic modulus G_H. It shows that with the increase of the size of unit cell L, C_H decreases and converges to zero rapidly. Although it could be also seen the influence of the aspect ratio a/b and relative radius R/L of the internal void, it is quite little and almost ignorable.

Fig.5 shows the changes of the homogenized characteristic length l_H. It is a much interesting parameter because it greatly settles the nonlocal behavior of composite. The conclusion given in the figure is: larger the value of a/b or/and R/L, larger the value of l_H. That means it tends to show nonlocal properties. At the same time, if the size of unit cell is quite small, say at the same order of the characteristic length l_m, the characteristic length of the composite l_H is much smaller.

4. Conclusion:

Some properties of Cosserat composite with periodic microstructure are discussed. It is proved that the composite can be made equal to a homogenous Cosserat medium with the same value of strain energy. Additionally, a homogenization theorem is developed. The numerical implementation of the theorem is also presented and used to analyze some scale-dependent deformation behaviors of porous materials. The calculated results show that the homogenized elastic constants depend obviously on the size of unit cell and geometry of the internal void.

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