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(Citation)
Physical Review Letters,84(7):1447-1450

(Issue Date)
2000-02

(Resource Type)
journal article

(Version)
Version of Record

(URL)
https://hdl.handle.net/20.500.14094/90000095

Two-Dimensional Evolution Equation of Finite-Amplitude Internal Gravity Waves in a Uniformly Stratified Fluid

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(Received 31 August 1999)

We derive a fully nonlinear evolution equation that can describe the two-dimensional motion of finite-amplitude long internal waves in a uniformly stratified three-dimensional fluid of finite depth. The derived equation is the two-dimensional counterpart of the evolution equation obtained by Grimshaw and Yi [J. Fluid Mech. **229**, 603 (1991)]. In the small-amplitude limit, our equation is reduced to the celebrated Kadomtsev-Petviashvili equation.

PACS numbers: 47.55.Hd, 47.20.Ky, 47.35.+i

Nonlinear internal waves in a stratified fluid of finite depth have been under intense investigation for recent decades. Various phenomena, such as modulation and stability of wave trains [1], interaction of internal solitary waves [2], and formation of these solitary waves by resonant flow over topography [3,4], etc., have been investigated extensively from both the analytical and experimental points of view. For the analytical description of these nonlinear waves, various kinds of approximated nonlinear evolution equations have been derived, depending on the physical situations [1,3,5–7]. One of the most well-known and fundamental equations, for instance, is the so-called Korteweg–de Vries (KdV) equation [3,5]. This equation is based on an asymptotic multiscale expansion technique which uses the assumptions of a long wavelength and a small wave amplitude. Here the word "long" means that the horizontal wavelength is large compared to the depth of the fluid. If the weak dispersion effects in the transverse direction are also taken into account, the Kadomtsev-Petviashvili (KP) equation [6] is obtained as the two-dimensional counterpart of the KdV equation. Both evolution equations contain the leading-order terms representing time evolution, nonlinearity, and wave dispersion, and it has been found that these equations can predict reasonably well the time evolution of long nonlinear waves of small, but not infinitesimal, amplitude.

However, in the anomalous, but important and basic, case of a uniformly and weakly stratified fluid where the so-called Boussinesq approximation is valid, the nonlinear term does not appear in an evolution equation derived by the above conventional weakly nonlinear analysis. In this case, the wave amplitude must be scaled with unity to derive an evolution equation that can describe the nonlinear effects of long internal waves. Using a novel approach pioneered by Warn [8], Grimshaw and Yi (GY) [9] derived a fully nonlinear evolution equation that can predict the one-dimensional motion of finite-amplitude long internal waves in a uniformly and weakly stratified two-dimensional fluid. Various properties of these finite-amplitude long waves, such as one-dimensional stability [10] and overtaking collision of solitary waves [11], and time evolution of topographically excited finite-amplitude waves [9,12], etc., have been investigated so far on the basis of this equation. In the latter study, comparison between solutions of the GY equation and those of the Euler equations was also made and the results reveal excellent agreement.

The purpose of this Letter is to derive the two-dimensional counterpart of the GY equation. The method of derivation follows that of GY; however, two-dimensional effects are incorporated by allowing for the weak dependence of the wave pattern on the transverse direction. In addition to being valid for waves of finite amplitude, the obtained equation has the following merits: (i) It is expressed as a single equation with respect to the wave amplitude. (ii) It has the analytical solutions of both solitary and periodic waves. These merits indicate that, apart from the computational efficiency in numerical calculation, this equation may be amenable to analytical investigations of various two-dimensional phenomena of finite-amplitude internal waves, such as transverse stability and oblique interaction of both solitary and periodic internal waves.

In the other type of stratification, Choi and Camassa [13] derived fully nonlinear model equations that govern the evolution of finite-amplitude long internal waves in two-layer fluid systems (although they derived one-dimensional equations, their extension to two dimensions is straightforward). Their equations are fully two dimensional and not limited to the description of unidirectional or weakly two-dimensional waves. However, they possess neither of the above-mentioned merits (i) and (ii).

Let us now proceed to derive the evolution equation. We consider the three-dimensional flow of an incompressible and inviscid stratified fluid of finite depth h ($-\infty < x^* < \infty$, $-\infty < y^* < \infty$, $0 < z^* < h$; $x^* - y^* - z^*$ is the Cartesian coordinate system). To simplify the discussion, we assume that the stratification is uniform so that the buoyancy frequency squared defined by $N^2(z^*) = -(g/\rho)(d\rho/dz^*)$ takes a constant value N_0^2, where g and $\rho(z^*)$ are the gravitational acceleration and the undisturbed density profile, respectively. Moreover, we apply the Boussinesq approximation under which the Boussinesq parameter defined by $\beta = N_0^2 h/g$ is zero. Notice, however, that the effects of small deviations from uniformly stratified Boussinesq fluid can be incorporated straightforwardly in the same manner as the case of two-dimensional flow (or one-dimensional wave motion) [9].

Concentrating on finite-amplitude long internal waves that move in the negative x^* direction with weak dependence on the y^* direction, it is convenient to adopt a reference frame moving with speed $-cN_0h$ in the x^* direction where cN_0h is the limiting speed of infinitely long waves [see (20) below]. Then the time evolution is expected to be slow in this reference frame and the dimensionless equations governing the wave motion are written in the form

$$u_x + w_z + \varepsilon^2 v_y = 0, \tag{1}$$

$$J_1(u,\phi) + p_x = O(\varepsilon^2), \tag{2}$$

$$J_1(v,\phi) + p_y = O(\varepsilon^2), \tag{3}$$

$$J_1(q + \overline{u}/c^2, \phi) - (\zeta + \overline{u}/c)_x + \varepsilon^2 F_1 = O(\varepsilon^4), \tag{4}$$

$$J_1(\zeta + \overline{u}/c, \phi) + \varepsilon^2 F_2 = 0, \tag{5}$$

where

$$q = u_z - \varepsilon^2 w_x, \tag{6}$$

$$\phi = cz + \overline{u}, \tag{7}$$

$$\overline{u} = \int_0^z u\, dz', \tag{8}$$

$$F_1 = u_{zt} + \left[J_2\left(u, \int_0^z v\, dz'\right)\right]_z, \tag{9}$$

$$F_2 = \zeta_t + J_2\left(\zeta - z, \int_0^z v\, dz'\right), \tag{10}$$

with boundary conditions at the rigid upper and lower plane surfaces

$$\zeta = 0 \quad \text{at } z = 0 \text{ and } 1, \tag{11}$$

and those far upstream

$$u, v, w, \zeta \to 0 \quad \text{as } x \to -\infty, \tag{12}$$

Here (u, v, w) are the velocity components in the (x, y, z) directions relative to the base flow $(c, 0, 0)$, p is the dynamic pressure, and ζ is the vertical particle displacement. These dimensionless quantities, which are scaled to be $O(1)$, are related to the corresponding dimensional ones, with asterisks, by the relations $x = x^*/l$, $y = \varepsilon y^*/l$, $z = z^*/h$, $t = \varepsilon^3 N_0 t^*$, $(u, v, w) = (u^*, \varepsilon^{-1} v^*, \varepsilon^{-1} w^*)/N_0 h$, $p = p^*/\rho_0 N_0^2 h^2$, and $\zeta = \zeta^*/h$, where l is a characteristic scale of horizontal length, ρ_0 is a typical value of density, and $\varepsilon = h/l$ which, according to the long-wave assumption, is a small dimensionless parameter. The above scalings are determined to assure the balance among time evolution, wave dispersion, and transverse effects. $J_1(a,b) = a_x b_z - a_z b_x$ and $J_2(a,b) = a_y b_z - a_z b_y$ stand for the Jacobian operators. The subscripts x, y, z, and t denote partial differentiations. These notations will be used throughout the paper.

To make a reduction of these governing equations, we follow Warn [8] and GY [9] to replace the vertical coordinate z with ϕ [see (28) below for the validity of this transformation]. Using a formula $J_1(a,\phi) = \phi_z(\partial a/\partial x)_\phi$, where $(\partial/\partial x)_\phi$ denotes the derivative with respect to x keeping ϕ, y, and t constant, then (5) can be integrated along a contour line of ϕ in the x-ϕ plane to give

$$\zeta + \frac{\overline{u}}{c} = \varepsilon^2\left(- \int_{\overline{x}}^x \left[\frac{F_2}{\phi_z}\right]_{\phi=C} dx' + F_3\right), \tag{13}$$

where

$$F_3 = \varepsilon^{-2}(\zeta + \overline{u}/c)|_{x=\overline{x}}. \tag{14}$$

Here $\int[\,]_{\phi=C}\, dx'$ indicates an integration with respect to x keeping ϕ, y, and t constant. $\overline{x} = \overline{x}(x, y, \phi, t)$ denotes the x coordinate of a position where we first encounter the upper plane ($z = 1$) moving along a contour line of ϕ in the negative x direction from a spatial point (x, y, ϕ) at a given time t. If the contour line does not encounter the upper plane at all, $\overline{x} = -\infty$. Note that $\overline{x}$ does not always take the value of $-\infty$ because the flow is three dimensional. Therefore $a|_{x=\overline{x}}$ denotes the value of a at $(x,z) = (\overline{x}, 1)$ or at $x = -\infty$ (this notation will be used hereafter). It also should be noted here that F_3 defined by (14) is $O(1)$, since $\zeta + \overline{u}/c$ is everywhere $O(\varepsilon^2)$ from (12) and (13).

Equation (4) may also be integrated along the contour line of ϕ. Then, on using (13), we find the following closed boundary-value problem for the vertical particle displacement ζ subject to the boundary conditions (11) and (12):

$$\zeta_{zz} + \frac{\zeta}{c^2} + \varepsilon^2 G[\zeta] = O(\varepsilon^4), \tag{15}$$

where

$$G = \left(\frac{\partial^2}{\partial z^2} + \frac{1}{c^2}\right)\int_{\overline{x}}^x \left[\frac{F_2}{\phi_z}\right]_{\phi=C} dx' + \frac{w_x}{c} + F_4 - \int_{\overline{x}}^x \left[\frac{1}{c\phi_z}\left\{F_1 + \frac{\partial}{\partial x'}\int_{\overline{x}}^{x'}\left[\frac{F_2}{\phi_z}\right]_{\phi=C} dx''\right\}\right]_{\phi=C} dx' \tag{16}$$

and

$$F_4 = \int_{\overline{x}}^x \left[\frac{1}{c\phi_z}\frac{\partial F_3}{\partial x'}\right]_{\phi=C} dx' - \left(\frac{\partial^2}{\partial z^2} + \frac{1}{c^2}\right)F_3 + \frac{1}{c\varepsilon^2}\left(q + \frac{\overline{u}}{c^2}\right)\Bigg|_{x=\overline{x}}. \tag{17}$$

Here $q + \overline{u}/c^2$ is $O(\varepsilon^2)$ from (4) and (12), since $\zeta + \overline{u}/c$ is $O(\varepsilon^2)$. We already know that F_3 is $O(1)$ so that F_4 defined by (17) is $O(1)$.

Now we expand ζ in an asymptotic series

$$\zeta = \zeta_0 + \varepsilon^2\zeta_1 + O(\varepsilon^4). \tag{18}$$

Substituting (18) into the governing equation (15) subject to the boundary conditions (11), we find that

$$\zeta_0 = A(x, y, t)W(z), \tag{19}$$

where $W(z)$ is the nth modal function given by

$$W(z) = \sin(z/c_n), \qquad c = c_n = 1/n\pi, \tag{20}$$

with $n = 1, 2, \ldots$. Then $A(x, y, t)$ represents the wave amplitude which will be governed by the evolution equation we are seeking. Using (1)–(3), (12), (13), and (19), the other physical quantities, such as u, v, and w, are also expressed with respect to A as

$$u = -c_n A W_z + O(\varepsilon^2), \tag{21}$$

$$v = -c_n \int_{\overline{x}}^{x} \left[\frac{W_z - A/c_n^2}{1 - AW_z} A_y\right]_{\phi=C} dx' + v|_{x=\overline{x}} + O(\varepsilon^2), \tag{22}$$

$$w = c_n A_x W + O(\varepsilon^2). \tag{23}$$

At the next order in ε^2, an inhomogeneous boundary-value problem for ζ_1 is constructed from (15) subject to the boundary conditions (11). This problem has a solution if the following compatibility condition is satisfied.

$$\int_0^1 G[\zeta_0]W(z)\, dz = 0. \tag{24}$$

This condition leads to the desired evolution equation for $A(x, y, t)$. In order to simplify calculation of the condition (24), we introduce the variable ξ expressed as

$$\xi = z - A(x, y, t)W(z). \tag{25}$$

We then obtain $\phi = c_n\xi + O(\varepsilon^2)$ from (7) and (21) so that the transformation from z to ϕ is, to the leading order, equivalent to that from z to ξ. The key idea to calculate the compatibility condition (24) is to use ξ as an independent variable in place of ϕ. Then calculation is remarkably simplified since $\overline{x} = -\infty$ everywhere in the flow field and we find from (17) and (22) that, on using the boundary conditions (12),

$$F_4 = 0, \tag{26}$$

$$v = -c_n \int_{-\infty}^{x} [z_\xi^{-2} z_{\xi A} A_y]_{\xi=C}\, dx' + O(\varepsilon^2). \tag{27}$$

Here z is represented by a function of A and ξ from (25), while z_ξ and $z_{\xi A}$ are given by (34) and (36) below. It should be noted that, for $z = z(\xi, A)$ to exist, $\xi_z \neq 0$, or equivalently,

$$|A| < c_n, \tag{28}$$

must be satisfied. The condition (28) assures us that the approximate density gradient in the z direction, with an error of $O(\varepsilon^2)$, is negative everywhere in the flow field, or that the flow is statically stable everywhere.

Now substituting (16), with (9), (10), (19)–(21), (23), and (25)–(27), into the compatibility condition (24), and using formulas $\partial/\partial y = (\partial/\partial y)_\xi - A_y W \partial/\partial\xi$ and $\partial/\partial z = z_\xi^{-1}\partial/\partial\xi$, we obtain, upon integration with respect to x and ξ by parts, the following desired evolution equation:

$$-\int_{-\infty}^{x} K(A, A')A'_t\, dx' + \frac{c_n^3}{2} A_{xx} + \int_{-\infty}^{x}\int_{-\infty}^{x'} (c_n I_1 A''_{yy} + I_2 A''^2_y + I_3 A'_y A''_y)\, dx''\, dx' = 0, \tag{29}$$

where

$$K(A, A') = \int_0^1 \{z_A z'_A(1 + z'_\xi) - (z - z')z_A z'_{\xi A}\}\, d\xi, \tag{30}$$

$$I_1 = I_1(A, A', A'') = c_n^2 \int_0^1 z_\xi^{-2} z_{\xi A} z'_\xi z''^{-2}_\xi z''_{\xi A}\, d\xi, \tag{31}$$

$$I_2 = I_2(A, A', A'') = -c_n^3 \int_0^1 z_\xi^{-2} z_{\xi A} z'_\xi z''^{-2}_\xi (2 z''^{-1}_\xi z''^2_{\xi A} - z''_{\xi AA})\, d\xi, \tag{32}$$

$$I_3 = I_3(A, A', A'') = c_n^3 \int_0^1 z_{\xi A}(z_\xi^{-2} + z'^{-2}_\xi) z'_{\xi A} z''^{-2}_\xi z''_{\xi A}\, d\xi, \tag{33}$$

and $z' = z(\xi, A')$, $z'' = z(\xi, A'')$, $A' = A(x', y, t)$, $A'' = A(x'', y, t)$, while z_ξ, z_A, $z_{\xi A}$, and $z_{\xi AA}$ are given by

$$z_\xi = \frac{\partial z}{\partial \xi} = \frac{1}{1 - AW_z}, \tag{34}$$

$$z_A = \frac{\partial z}{\partial A} = \frac{W}{1 - AW_z}, \tag{35}$$

$$z_{\xi A} = \frac{\partial^2 z}{\partial\xi\,\partial A} = \frac{W_z - A/c_n^2}{(1 - AW_z)^3}, \tag{36}$$

$$z_{\xi AA} = \frac{\partial^3 z}{\partial\xi\,\partial A^2} = \frac{3(W_z - A/c_n^2)^2}{(1 - AW_z)^5} - \frac{1 + W^2 - AW_z}{c_n^2(1 - AW_z)^4}. \tag{37}$$

Here the primes and double primes appended to A on the far right side of (30)–(33) are omitted and their expressions should follow those of the corresponding principal variables z (for instance, $z''_{\xi A} = \partial^2 z''/\partial\xi\,\partial A''$). Equation (29) is the two-dimensional counterpart of the GY equation describing the behavior of finite-amplitude long internal waves of the nth mode in a uniformly stratified three-dimensional Boussinesq fluid. The third term on the left-hand side of (29) is a newly derived term in this Letter and represents the effects of weak wave dispersion in the transverse (y) direction. When the y dependence of the flow is absent, this term disappears and (29) corresponds with the GY equation [9]. Here we give the following important property of the integral kernels I_1, I_2, and I_3, included in the newly derived term:

$$\int_{-\infty}^{x}\int_{-\infty}^{x'}(c_n I_1 A''_{xx} + I_2 A''^2_x + I_3 A'_x A''_x)\,dx''\,dx' = c_n A/2\,, \tag{38}$$

where the primes and double primes appended to x in the integrand are omitted and their expressions should follow those of the corresponding principal variables A (for instance, $A'_x = \partial A'/\partial x'$).

In (29), nonlinearity derives from transience and transverse dispersion only. However, the nonlinear terms in the form of A^m $(m = 2, 3, \ldots)$ show up if one allows for small deviations from uniformly stratified Boussinesq fluid. Assuming that a deviation of the stratification profile from the uniform case and the Boussinesq parameter β are both scaled with ε^2, we may put the buoyancy frequency $N(z)$ in the form $N^2(z) = N_0^2[1 + \beta M(z)]$, and $\beta = \sigma\varepsilon^2$ $[\sigma = O(1)]$. Then the term represented by

$$b(A) = \sigma c_n\left\{\int_0^1 A W z_A M(\xi)\,d\xi + \frac{c_n}{6}A^2[1 - (-1)^n]\right\} \tag{39}$$

is added on the left-hand side of (29) [9].

Using (38) and $\int_{-\infty}^{x} K(A, A')A'_x\,dx' = A$, we find that the evolution equation (29), with an additive term of $+b(A)$, has both solitary and periodic cnoidal wave solutions. If, for instance, we put $b(A) = 3\sigma A^2$, then the solitary wave solution is

$$A = a\,\mathrm{sech}^2(kx + ly - \omega t)\,, \tag{40}$$

$$-\frac{\omega}{k} - \frac{c_n l^2}{2k^2} = 2\sigma a = 2c_n^3 k^2. \tag{41}$$

Notice that, in the case of a uniformly stratified Boussinesq fluid $(\sigma = 0)$, no solitary wave solution exists and (29) has only a sinusoidal wave train solution.

Next we consider a small-amplitude limit $A \to 0$ for which it can be shown that

$$I_1(A, A', A'') = \frac{1}{2} - \frac{1}{16c_n^2} \times (9A^2 + 4AA' - 24AA'' - 4A'^2 + 4A'A'' + 9A''^2) + \ldots\,, \tag{42}$$

$$I_2(A, A', A'') = \frac{1}{8c_n}(12A - 2A' - 9A'') + \ldots\,, \tag{43}$$

$$I_3(A, A', A'') = \frac{1}{4c_n}(A + A' - 2A'') + \ldots\,. \tag{44}$$

The kernel K can be expressed by $K(A, A') = 1 - (3A^2 - 8AA' + 3A'^2)/4c_n^2 + \ldots$, so that substituting these equations into (29) (with an additive term of $+3\sigma A^2$), we obtain, correct to $O(A^2)$,

$$-A_t + 6\sigma A A_x + \frac{c_n^3}{2}A_{xxx} + \frac{c_n}{2}\int_{-\infty}^{x} A'_{yy}\,dx' = 0\,, \tag{45}$$

which is nothing but the KP equation [6].

In conclusion, we have derived the fully nonlinear evolution equation (29) that can predict the two-dimensional motion of finite-amplitude long internal waves in a uniformly and weakly stratified three-dimensional fluid. It should be noted here that the flow past obstacles or topography [14] is also describable by incorporating a forcing term into the evolution equation [9]. In fact, we are now examining various two-dimensional phenomena of finite-amplitude internal waves on the basis of this equation. These matters will be reported elsewhere.

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