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An extension of Fourier analysis for the *n*-torus in the magnetic field and its application to spectral analysis of the magnetic Laplacian

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We solved the Schrödinger equation for a particle in a uniform magnetic field in the *n*-dimensional torus. We obtained a complete set of solutions for a broad class of problems; the torus $T^n=\mathbf{R}^n/\Lambda$ is defined as a quotient of the Euclidean space $\mathbf{R}^n$ by an arbitrary *n*-dimensional lattice Λ. The lattice is not necessary either cubic or rectangular. The magnetic field is also arbitrary. However, we restrict ourselves within potential-free problems; the Schrödinger operator is assumed to be the Laplace operator defined with the covariant derivative. We defined an algebra that characterizes the symmetry of the Laplacian and named it the magnetic algebra. We proved that the space of functions on which the Laplacian acts is an irreducible representation space of the magnetic algebra. In this sense the magnetic algebra completely characterizes the quantum mechanics in the magnetic torus. We developed a new method for Fourier analysis for the magnetic torus and used it to solve the eigenvalue problem of the Laplacian. All the eigenfunctions are given in explicit forms. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1616203]

I. INTRODUCTION

In this paper we solve the Schrödinger equation for a particle in a uniform magnetic field in an *n*-dimensional torus. The problem looks plain at first sight but actually it turns out to be a hard problem, which has not been solved before. Hence we begin this paper by a quick explanation of the problem. After that we will describe our strategy to solve it. Subsequently we will briefly mention studies by other people and describe our motivation of this study. At the end of the Introduction we will give guides for quick access to main results of this paper.

An *n*-dimensional torus, or *n*-torus, is defined as $T^n=\mathbf{R}^n/\mathbf{Z}^n$. In the coordinate a point $(t^1,\ldots,t^j+1,\ldots,t^n)$ is identified with $(t^1,\ldots,t^j,\ldots,t^n)$ for each $j=1,\ldots,n$. The eigenvalue problem of the ordinary Laplacian in the torus is the equation

$$-\Delta f=-\sum_{j=1}^{n}\left(\frac{\partial}{\partial t^j}\right)^2 f=\varepsilon f \tag{1.1}$$

with the periodic boundary condition

$$f(t^1,\ldots,t^j+1,\ldots,t^n)=f(t^1,\ldots,t^j,\ldots,t^n). \tag{1.2}$$

The eigenvalue problem can be immediately solved by Fourier expansion. A plane-wave function

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$$\chi_k(t^1,\ldots,t^n)=e^{2\pi i\Sigma_{j=1}^n k_j t^j} \tag{1.3}$$

with quantized momenta $k_j\in\mathbf{Z}$ is a solution. The whole set of eigenfunctions $\{\chi_k\,|\,(k_1,\ldots,k_n)\in\mathbf{Z}^n\}$ constitutes a complete orthonormal set of the space of periodic functions over the torus. This is a well-known result.

In this paper we would like to solve an eigenvalue problem of the magnetic Laplacian; the magnetic Laplacian is defined by replacing the partial derivative in the ordinary Laplacian by a covariant derivative as

$$\Delta f=\sum_{j,l=1}^n g^{jl}\left(\frac{\partial}{\partial t^j}-2\pi i A_j\right)\left(\frac{\partial}{\partial t^l}-2\pi i A_l\right)f. \tag{1.4}$$

Here A_l is a component of the $U(1)$ gauge field

$$A_l=\frac{1}{2}\sum_{j=1}^n \phi_{jl}\,t^j+\alpha_l \tag{1.5}$$

with integers $\{\phi_{jl}=-\phi_{lj}\}$ and real numbers $\{\alpha_j\}$. The gauge field $A=\Sigma_{l=1}^n A_l\,\mathrm{d}t^l$ generates a uniform magnetic field $B=\mathrm{d}A=(1/2)\Sigma_{j,l=1}^n\phi_{jl}\,\mathrm{d}t^j\wedge\mathrm{d}t^l$. Moreover, we would like to consider a general oblique torus $T^n=\mathbf{R}^n/\Lambda$; Λ is an n-dimensional lattice. Edges of the unit cell of the lattice do not necessarily cross at a right angle and they do not necessarily have a same length. Hence we introduce a metric g^{jl} in the definition of the magnetic Laplacian (1.4) to take inclined and stretched or shortened unit cells into account. The eigenvalue problem of (1.4) is accompanied by the condition

$$f(t^1,\ldots,t^j+1,\ldots,t^n)=e^{\pi i\Sigma_{k=1}^n\phi_{jk}t^k}f(t^1,\ldots,t^j,\ldots,t^n), \tag{1.6}$$

which we call a twisted periodic condition. Thus the plain problem (1.1) with (1.2) is generalized to the magnetic problem (1.4) with (1.6). At first glance it looks rather straightforward to generalize the problem in this way but it is actually highly nontrivial and difficult to generalize the solution.

Let us see where the difficulty lies. In the case of the ordinary Laplacian, the plane-wave solution (1.3) is a simultaneous eigenfunction of the momentum operators

$$p_j=-i\frac{\partial}{\partial t^j}\quad(j=1,\ldots,n) \tag{1.7}$$

as $p_j\chi_k=2\pi k_j\chi_k$. The Laplacian can be expressed in terms of the momentum operators as $-\Delta=\Sigma_{j=1}^n(p_j)^2$ and, of course, it commutes with the momentum operators. Thus integers $(k_1,\ldots,k_n)$ are good quantum numbers. Then the whole set of simultaneous eigenfunctions $\{\chi_k\}$ forms the complete solutions of the Laplacian problem. This is the way how Fourier analysis works. However, when we turn to the magnetic Laplacian, we may seek for a simultaneous eigenfunction of magnetic momentum operators

$$p_j=-i\left(\frac{\partial}{\partial t^j}-2\pi i A_j\right)\quad(j=1,\ldots,n). \tag{1.8}$$

But such a simultaneous eigenfunction does not exist because magnetic momenta do not commute with each other and instead exhibit commutators

$$[\,p_j,p_l\,]=2\pi i\,\phi_{jl}\,. \tag{1.9}$$

The magnetic Laplacian can be still expressed in terms of the magnetic momentum operators as $-\Delta=\Sigma_{j=1}^{n}(p_j)^2$ but it does not commute with p_j. Hence, the strategy of ordinary Fourier analysis does not work well for the magnetic Laplacian.

To solve the problem we developed a new method, which we call Fourier analysis for the magnetic torus. This is a main subject of this paper. Let us describe our strategy: First, we will find a group of operators that commute with magnetic momentum operators. We call the group a magnetic translation group. Second, we enlarge a family of operators to define an algebra, which includes magnetic momenta and magnetic translations as its elements. We call the algebra a magnetic algebra and construct its representations. Third, we show that the space of twisted periodic functions over the torus is actually an irreducible representation space of the magnetic algebra. By diagonalizing a maximal commutative subalgebra of the magnetic algebra we obtain a complete orthonormal set of twisted periodic functions. This set of orthogonal functions provides a kind of unitary transformation as the set of plane-wave functions provides the Fourier transformation which bridges between the momentum space and the real space. We note that it is easy to diagonalize the Laplacian in the momentum space. Finally, we get a whole set of eigenfunctions in the real space by applying the unitary transformation. In this procedure the third step is the hardest part and is actually accomplished by lengthy cumbersome calculations. However, the strategy is clear.

We would like to briefly review studies by other people on spectral analysis in magnetic field. Brown[1] first examined the symmetry structure of the Schrödinger equation for an electron in a lattice in a uniform magnetic field and found that the symmetry is described by a noncommutative discrete translation group. At almost the same time Zak[2] also found the same symmetry structure and named the group a magnetic translation group (MTG). Zak[3] immediately built a representation theory of the MTG in the three-dimensional lattice. From the viewpoint of functional analysis, Avron, Herbst, and Simon have been studying spectral problems of the Schrödinger operators in a magnetic field in a series of papers.[4–6] Dubrovin and Novikov[7,8] studied the spectrum of the Pauli operator in a two-dimensional lattice with a periodic magnetic field and intensively analyzed the gap structure above the ground state. Florek[9,10] constructed tensor product representations of the MTG to analyze a three-particle system in a lattice in a magnetic field. Kuwabara[11,12] has been studying quantum-classical correspondence from the viewpoint of spectral geometry. For example, he[11] proved that if the whole set of level spacings of the quantum spectrum is not dense in $\mathbf{R}$, every trajectory of the corresponding classical particle is a closed orbit. Arai[13] found a quantum plane and quantum group structure in the quantum system in a singular magnetic field. Thus we can see that quantum mechanics in a magnetic field has been an active research area. However, we do not find a literature in which the quantum mechanics in an n-torus is solved.

Our study on quantum mechanics in magnetic fields originates from studies of extra-dimension models of the space–time. In extra-dimension models the space–time is assumed to be a base space of a fiber bundle with a compact fiber or a noncompact fiber. The history of extra-dimension models is rather old, but an interest in these models is recently renewed as Arkani-Hamed, Dimopoulos, and Dvali[14] pointed out that the extra-dimension model may solve the hierarchy problem of high energy physics. Inspired with extra-dimension models we[15] built a model which has a circle S^1 as a fiber over an any-dimensional space–times $\mathbf{R}^{D-1}$. Then we found that a twisted boundary condition in the S^1-direction causes spontaneous breaking of the translational symmetry. Based on this observation, we[16] proposed a new mechanism of supersymmetry breaking. Next we[17,18] built a model which has a two-dimensional sphere S^2 as a fiber over the four-dimensional space–times $\mathbf{R}^4$. We solved dynamics in the sphere in a magnetic monopole background and then found that the monopole induces spontaneous breaking of the rotational symmetry and the CP symmetry. We also built a model which has an n-dimensional torus as a fiber and tried to analyze dynamics in the torus in a background magnetic field. However, its analysis was not a straightforward task. Then we studied the symmetry structure of quantum mechanics in the torus in the magnetic field. We[19] constructed the MTG in the n-torus and classified irreducible representations of the MTG.

Armed with these tools we are now ready to solve the spectral problem in the n-torus T^n

$=\mathbf{R}^n/\Lambda$. We decide to solve the problem exhaustively; in our treatment the dimensions of the torus is taken to be arbitrary, lengths and angles of edges of the unit cell of Λ are arbitrary, and an arbitrary constant magnetic field is applied to the torus. Thus we aim to solve the widest class of quantum mechanics in the n-torus in uniform magnetic fields.

For busy readers here we give guides for quick access to main results. In Sec. II we provide a geometric setting to define the problem. The problem to be solved is the eigenvalue problem of the magnetic Laplacian (2.13) with the twist condition (2.5). In Sec. III we find a family of operators that commute with the covariant derivative. Actually they are composition of ordinary displacements and gauge transformations as shown at (3.2). These displacement vectors form a restricted family of vectors as shown at (3.8). These displacement operators generate the magnetic translation group (MTG), which is noncommutative as shown at (3.12). Along (3.16)–(3.28) we construct irreducible representations of the MTG. In Sec. IV we introduce a coordinate system, which will be revealed to be useful later. In Sec. V we define the magnetic algebra by adding differential operators (5.2)–(5.4) and multiplicative operators (5.9) to the MTG. Then we construct and classify irreducible representations of the magnetic algebra. Section VI is devoted to calculation of simultaneous eigenfunctions (6.8) of a maximal commutative subalgebra of the magnetic algebra. Then we obtain a complete orthonormal set of functions over the magnetic torus, which provide an extension of Fourier analysis for the magnetic torus. This is one of the main products of this paper. In Sec. VII by applying this method we solve the original problem, the eigenvalue problem of the magnetic Laplacian. There we obtain a whole set of eigenfunctions (7.11) and eigenvalues (7.12). These are the main results of this paper.

II. GAUGE FIELD IN THE TORUS

Let $\mathbf{t}=(t^1,\ldots,t^n)$ denote a coordinate of an n-dimensional torus $T^n=\mathbf{R}^n/\mathbf{Z}^n$. Namely, a point $(t^1,\ldots,t^j+1,\ldots,t^n)$ is identified with $(t^1,\ldots,t^j,\ldots,t^n)$ in T^n. A uniform magnetic field is generated by the gauge field

$$A=\sum_{k=1}^{n} A_k\,\mathrm{d}t^k=\frac{1}{2}\sum_{j,k=1}^{n}\phi_{jk}\,t^j\,\mathrm{d}t^k+\sum_{k=1}^{n}\alpha_k\,\mathrm{d}t^k. \tag{2.1}$$

Here $\{\phi_{jk}=-\phi_{kj}\}$ and $\{\alpha_j\}$ are real constants. Then the magnetic field is

$$B=\mathrm{d}A=\frac{1}{2}\sum_{j,k=1}^{n}\phi_{jk}\,\mathrm{d}t^j\wedge\mathrm{d}t^k. \tag{2.2}$$

Therefore, the number ϕ_{jk} represents magnetic flux which penetrates the (t^j,t^k)-face of the torus. We call the array of numbers (ϕ_{jk}) a magnetic flux matrix.

Let us introduce a complex scalar field f in the torus. The scalar field couples to the gauge field via the covariant derivative

$$\mathrm{D}f=\mathrm{d}f-2\pi iAf. \tag{2.3}$$

We put the coefficient $2\pi i$ in front of A for later convenience. Topology of the torus imposes a boundary condition on the scalar field. The gauge field itself is not a periodic function on $\mathbf{R}^n$ but it changes its form as

$$A(t^1,\ldots,t^j+1,\ldots,t^n)=A(t^1,\ldots,t^j,\ldots,t^n)+\frac{1}{2}\sum_{k=1}^{n}\phi_{jk}\,\mathrm{d}t^k. \tag{2.4}$$

Therefore, if we make the gauge transformation

$$f(t^1,\ldots,t^j+1,\ldots,t^n)=e^{\pi i\Sigma_{k=1}^n\phi_{jk}t^k}f(t^1,\ldots,t^j,\ldots,t^n), \tag{2.5}$$

the covariant derivative (2.3) remains covariant as

$$\mathrm{D}f(t^1,\ldots,t^j+1,\ldots,t^n)=e^{\pi i\Sigma_{k=1}^n\phi_{jk}t^k}\,\mathrm{D}f(t^1,\ldots,t^j,\ldots,t^n). \tag{2.6}$$

We call the condition (2.5) a twisted periodic condition. There are two ways to bring a point $(t^1,\ldots,t^j+1,\ldots,t^k+1,\ldots,t^n)$ to $(t^1,\ldots,t^j,\ldots,t^k,\ldots,t^n)$. The first way is

$$\begin{aligned}f(t^1,\ldots,t^j+1,\ldots,t^k+1,\ldots,t^n)&=e^{\pi i\{\phi_{jk}+\Sigma_{l=1}^n\phi_{jl}t^l\}}f(t^1,\ldots,t^j,\ldots,t^k+1,\ldots,t^n)\\&=e^{\pi i\{\phi_{jk}+\Sigma_{l=1}^n\phi_{jl}t^l+\Sigma_{l=1}^n\phi_{kl}t^l\}}f(t^1,\ldots,t^j,\ldots,t^k,\ldots,t^n).\end{aligned} \tag{2.7}$$

The other way is

$$\begin{aligned}f(t^1,\ldots,t^j+1,\ldots,t^k+1,\ldots,t^n)&=e^{\pi i\{\phi_{kj}+\Sigma_{l=1}^n\phi_{kl}t^l\}}f(t^1,\ldots,t^j+1,\ldots,t^k,\ldots,t^n)\\&=e^{\pi i\{\phi_{kj}+\Sigma_{l=1}^n\phi_{kl}t^l+\Sigma_{l=1}^n\phi_{jl}t^l\}}f(t^1,\ldots,t^j,\ldots,t^k,\ldots,t^n).\end{aligned} \tag{2.8}$$

To make these two expressions coincide we need to have

$$e^{\pi i\phi_{jk}}=e^{\pi i\phi_{kj}},$$

namely,

$$e^{\pi i(\phi_{jk}-\phi_{kj})}=e^{2\pi i\phi_{jk}}=1. \tag{2.9}$$

Therefore, compatibility of the periodic conditions (2.7) and (2.8) demands that ϕ_{jk} is an integer. Hence, the magnetic flux through each face of the torus is quantized. We call the torus where the magnetic field has been introduced a magnetic torus.

Since two displacements $t^j\mapsto t^j+1$ and $t^k\mapsto t^k+1$ are commutative, we can write the twisted periodic condition (2.5) in a more general form

$$f(\mathbf{t}+\mathbf{m})=e^{\pi i\Sigma_{j,k=1}^n\phi_{jk}m^jt^k}f(\mathbf{t}) \tag{2.10}$$

with an arbitrary $\mathbf{m}=(m^1,\ldots,m^n)\in\mathbf{Z}^n$. An inner product of two twisted periodic functions $f(\mathbf{t})$ and $g(\mathbf{t})$ is defined by

$$\langle f|g\rangle=\int_0^1\mathrm{d}t^1\cdots\int_0^1\mathrm{d}t^n\,f^*(\mathbf{t})g(\mathbf{t}). \tag{2.11}$$

Equipped with this inner product the space of twisted periodic functions becomes a Hilbert space.

To define the Laplacian we need to introduce a metric into the torus. Let Λ be an n-dimensional lattice in the Euclidean space $\mathbf{R}^n$. We equip the torus T^n with a Riemannian structure by identifying T^n with the quotient space $\mathbf{R}^n/\Lambda$. Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ be a set of vectors that generates the lattice Λ. Their inner products is denoted by

$$g_{jk}=\langle\mathbf{u}_j,\mathbf{u}_k\rangle \tag{2.12}$$

and its inverse is denoted by g^{jk}. Then the magnetic Laplacian is defined as

$$\Delta f = \sum_{j,k=1}^{n} g^{jk} \left(\frac{\partial}{\partial t^j} - 2\pi i A_j \right) \left(\frac{\partial}{\partial t^k} - 2\pi i A_k \right) f. \tag{2.13}$$

It is also referred to as the Bochner Laplacian in the literature. The purpose of this paper is to solve the eigenvalue problem of the magnetic Laplacian accompanied by the twisted periodic condition (2.5).

III. MAGNETIC TRANSLATION GROUP

Our goal is to find a complete set of eigenvalues and eigenfunctions of the magnetic Laplacian (2.13) as announced above. A royal road to solving an eigenvalue problem is to detect symmetry. In this section we determine a group of operators that commute with the Laplacian and construct irreducible representations of the group.

The vector space $\mathbf{R}^n$ acts on the torus as isometries. However, the gauge field restricts the admissible class of vectors as seen below. An arbitrary vector $\mathbf{v} \in \mathbf{R}^n$ displaces the gauge field (2.1) as

$$A(\mathbf{t}) \mapsto A(\mathbf{t}-\mathbf{v}) = A(\mathbf{t}) - d\left(\frac{1}{2} \sum_{j,k=1}^{n} \phi_{jk} v^j t^k \right). \tag{3.1}$$

If we perform a gauge transformation of the scalar field simultaneously with the displacement

$$f(\mathbf{t}) \mapsto f'(\mathbf{t}) = (U(\mathbf{v}) f)(\mathbf{t}) = e^{\pi i \Sigma_{j,k=1}^{n} \phi_{jk} v^j t^k} f(\mathbf{t}-\mathbf{v}), \tag{3.2}$$

then the covariant derivative is changed covariantly

$$\mathrm{D}f(\mathbf{t}) \mapsto \mathrm{D}f'(\mathbf{t}) = e^{\pi i \Sigma_{j,k=1}^{n} \phi_{jk} v^j t^k} (\mathrm{D}f)(\mathbf{t}-\mathbf{v}). \tag{3.3}$$

In other words, the transformation $U(\mathbf{v})$ commutes with the covariant derivative as

$$(\mathrm{D}U(\mathbf{v}) f)(\mathbf{t}) = (U(\mathbf{v}) \mathrm{D} f)(\mathbf{t}). \tag{3.4}$$

Hence it commutes with the magnetic Laplacian, which is defined in terms of the covariant derivative. The operator $U(\mathbf{v})$ is unitary with respect to the inner product (2.11).

The displaced function (3.2) also must satisfy the twisted periodic condition. If the original function f satisfies the condition (2.10), the displaced function changes its form as

$$f'(\mathbf{t}+\mathbf{m}) = e^{2\pi i \Sigma_{j,k=1}^{n} \phi_{jk} v^j m^k} e^{\pi i \Sigma_{j,k=1}^{n} \phi_{jk} m^j t^k} f'(\mathbf{t}) \tag{3.5}$$

for $\mathbf{m} \in \mathbf{Z}^n$. Thus the displaced function satisfies the condition (2.10) if and only if

$$\sum_{j,k=1}^{n} \phi_{jk} v^j m^k \tag{3.6}$$

is an integer for an arbitrary $\mathbf{m} \in \mathbf{Z}^n$. In other words,

$$\sum_{j=1}^{n} \phi_{jk} v^j \quad (k = 1, \ldots, n) \tag{3.7}$$

must be an integer. We call such a restricted vector $\mathbf{v}$ a magnetic shift. The set of magnetic shifts forms an Abelian group

$$\Omega^n = \{\mathbf{v} \in \mathbf{R}^n \mid \phi\mathbf{v} \in \mathbf{Z}^n\}. \tag{3.8}$$

There is a sequence of Abelian subgroups $\mathbf{Z}^n \subset \Omega^n \subset \mathbf{R}^n$. In particular, an integer vector $\mathbf{m} \in \mathbf{Z}^n$ induces a displacement

$$(U(\mathbf{m})f)(\mathbf{t}) = e^{\pi i \Sigma_{j,k=1}^n \phi_{jk} m^j t^k} f(\mathbf{t}-\mathbf{m}).$$

However, owing to the twisted periodic condition (2.10), this is reduced to the identity transformation

$$(U(\mathbf{m})f)(\mathbf{t}) = f(\mathbf{t}). \tag{3.9}$$

Thus we conclude that the group G of effective transformations is generated by

$$\{U(\mathbf{v}) \mid \mathbf{v} \in \Omega^n/\mathbf{Z}^n\}. \tag{3.10}$$

We call the group G the magnetic translation group. A product of transformations is

$$U(\mathbf{v})U(\mathbf{w}) = e^{\pi i \Sigma_{j,k=1}^n \phi_{jk} v^j w^k} U(\mathbf{v}+\mathbf{w}). \tag{3.11}$$

Their commutator is

$$U(\mathbf{v})U(\mathbf{w})U(-\mathbf{v})U(-\mathbf{w}) = e^{2\pi i \Sigma_{j,k=1}^n \phi_{jk} v^j w^k}. \tag{3.12}$$

We can say that the MTG is a central extension of the Abelian group $\Omega^n/\mathbf{Z}^n$ by $U(1)$.

We can express the MTG in a standard form. Let $L(n,\mathbf{Z})$ denote a group of the n-dimensional matrices $\{S\}$ of integers such that $\det S = \pm 1$. The matrix $S \in L(n,\mathbf{Z})$ acts on $\mathbf{t} \in \mathbf{R}^n$ by $\mathbf{t} \mapsto S\mathbf{t}$ and this action induces an automorphism of the torus $T^n = \mathbf{R}^n/\mathbf{Z}^n$. It also induces a transformation of the magnetic flux matrix as

$$\phi_{jk} \mapsto \phi'_{jk} = \sum_{l,p=1}^n \phi_{lp} S^l{}_j S^p{}_k. \tag{3.13}$$

The Frobenius lemma[20] tells that for any integral antisymmetric matrix there exists a transformation to bring it into a standard form

$$(\phi_{jk}) = \begin{pmatrix} 0 & q_1 & & & & & & & & \\ -q_1 & 0 & & & & & & & & \\ & & 0 & q_2 & & & & & & \\ & & -q_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & q_m & & & \\ & & & & & -q_m & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{pmatrix}, \tag{3.14}$$

where $\{q_i\}$ are positive integers and they constitute a sequence $q_1|q_2|\cdots|q_m$, which implies that q_i divides q_{i+1}. For example, we may have a sequence $3|6|12|48$. Of course, $2m \leqslant n$. The vector

subspace of the zero eigenvalue of the matrix ϕ has dimensions $n-2m$ and it is called null directions. In the following we suppose that the flux matrix is in the standard form (3.14).

Now we can write the magnetic shifts (3.8) in a more explicit form. Let $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ be the standard basis of $\mathbf{R}^n$ in the $(t^1,\ldots,t^n)$-coordinate. Then any magnetic shift is uniquely expressed as

$$\mathbf{v}=\sum_{j=1}^{m}\left(\frac{s_{2j-1}}{q_j}\mathbf{e}_{2j-1}+\frac{s_{2j}}{q_j}\mathbf{e}_{2j}\right)+\sum_{k=1}^{n-2m}\theta_{2m+k}\mathbf{e}_{2m+k} \tag{3.15}$$

with integers $\{s_1,\ldots,s_{2m}\}$ and real numbers $\{\theta_{2m+1},\ldots,\theta_n\}$. Namely, the magnetic shifts are generated by $\{(1/q_j)\mathbf{e}_{2j-1},(1/q_j)\mathbf{e}_{2j}\,|\,j=1,\ldots,m\}$ with integral coefficients and $\{\mathbf{e}_{2m+k}\,|\,k=1,\ldots,n-2m\}$ with real coefficients. Hence, if the flux matrix ϕ has null directions $n-2m>0$, the MTG has a continuous component. Otherwise, the MTG is a completely discrete group.

Here we summarize our discussion; the MTG is generated by the unitary operators

$$U_j=U\left(\frac{1}{q_j}\mathbf{e}_{2j-1}\right),\quad V_j=U\left(\frac{1}{q_j}\mathbf{e}_{2j}\right)\quad(j=1,\ldots,m) \tag{3.16}$$

and

$$W_k(\theta)=U(\theta\mathbf{e}_{2m+k})\quad(k=1,\ldots,n-2m). \tag{3.17}$$

According to (3.9), (3.11), (3.12), and (3.14), these generators satisfy the following relations:

$$(U_j)^{q_j}=(V_j)^{q_j}=1, \tag{3.18}$$

$$U_jV_jU_j^{-1}V_j^{-1}=e^{2\pi i/q_j}, \tag{3.19}$$

$$W_k(1)=1, \tag{3.20}$$

$$W_k(\theta)W_k(\theta')=W_k(\theta+\theta'), \tag{3.21}$$

and other trivial commutators.

To solve the eigenvalue problem of the magnetic Laplacian we need to prepare the whole set of irreducible representations of the MTG. Let $\{|r_1,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle\}$ be elements of a representation space that are labeled by

$$r_j\in\mathbf{Z}/\mathbf{Z}_{q_j}\quad(j=1,\ldots,m), \tag{3.22}$$

$$d_k\in\mathbf{Z}\quad(k=1,\ldots,n-2m). \tag{3.23}$$

Then the generators (3.16) and (3.17) are represented by

$$U_j|r_1,\ldots,r_j,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle=|r_1,\ldots,r_j+1,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle, \tag{3.24}$$

$$V_j|r_1,\ldots,r_j,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle=e^{-2\pi ir_j/q_j}|r_1,\ldots,r_j,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle, \tag{3.25}$$

$$W_k(\theta)|r_1,\ldots,r_j,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle=e^{-2\pi id_k\theta}|r_1,\ldots,r_j,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle. \tag{3.26}$$

Thus

$$\mathcal{H}_d = \bigoplus_{r_1=0}^{q_1} \cdots \bigoplus_{r_m=0}^{q_m} \mathbf{C}|r_1,\ldots,r_m;d_1,\ldots,d_{n-2m}\rangle \tag{3.27}$$

provides an irreducible representation space of the MTG. Its dimension is

$$\dim \mathcal{H}_d = q_1 \times q_2 \times \cdots \times q_m . \tag{3.28}$$

The labels r_j and d_k in (3.22) and (3.23) will become good quantum numbers for the Laplacian (2.13). The dimension (3.28) will give the degree of degeneracy of each eigenvalue.

IV. DIAGONALIZATION OF THE MAGNETIC FIELD STRENGTH

To solve the eigenvalue problem of the magnetic Laplacian we need to take the metric (2.12) into account. Remember that the basis $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ generates the lattice Λ in the Euclidean space $\mathbf{R}^n$ and that the torus is isometric to $\mathbf{R}^n/\Lambda$. Let $\mathbf{x}=(x^1,\ldots,x^n)$ be an orthonormal coordinate of $\mathbf{R}^n$. It is related to the normalized coordinate $\mathbf{t}=(t^1,\ldots,t^n)$ via

$$\mathbf{x} = t^1\mathbf{u}_1 + \cdots + t^n\mathbf{u}_n = U\mathbf{t}. \tag{4.1}$$

In these coordinates the magnetic field (2.2) is expressed as

$$B = \frac{1}{2}\sum_{j,k=1}^{n} B_{jk}\,\mathrm{d}x^j\wedge dx^k = \frac{1}{2}\sum_{j,k,l,p=1}^{n} B_{jk}\,u^j{}_l\,u^k{}_p\,\mathrm{d}t^l\wedge\mathrm{d}t^p = \frac{1}{2}\sum_{l,p=1}^{n}\phi_{lp}\,\mathrm{d}t^l\wedge\mathrm{d}t^p. \tag{4.2}$$

The number B_{jk} is areal density of magnetic flux which penetrates the (x^j,x^k)-plane. If we perform a coordinate transformation

$$\mathbf{x} = R\mathbf{y} \tag{4.3}$$

by an orthogonal transformation $R\in O(n,\mathbf{R})$, the components of the field strength is transformed as

$$B = \frac{1}{2}\sum_{j,k=1}^{n} B_{jk}\,\mathrm{d}x^j\wedge\mathrm{d}x^k = \frac{1}{2}\sum_{j,k,l,p=1}^{n} B_{jk}\,R^j{}_l\,R^k{}_p\,\mathrm{d}y^l\wedge\mathrm{d}y^p. \tag{4.4}$$

By a suitable orthogonal transformation the field strength matrix (B_{jk}) can be brought into a standard form

$$\nu = {}^tRB\,R = \begin{pmatrix} 0 & \nu_1 & & & & & & & \\ -\nu_1 & 0 & & & & & & & \\ & & 0 & \nu_2 & & & & & \\ & & -\nu_2 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & \nu_m & & \\ & & & & & -\nu_m & 0 & & \\ & & & & & & & 0 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 0 \end{pmatrix}, \tag{4.5}$$

where $\{\nu_j\}$ are positive real numbers. The $(t^1,\ldots,t^n)$-coordinate system block-diagonalizes the magnetic flux in the form of (3.14) while $(y^1,\ldots,y^n)$ block-diagonalizes the magnetic field strength in the form of (4.5). The transformations (4.1) and (4.3) are combined into

$$\mathbf{y}={}^tR\mathbf{x}={}^tRU\mathbf{t}=L\mathbf{t}. \tag{4.6}$$

Then we obtain relations among the matrices

$$\phi={}^tUBU={}^tUR\,\nu\,{}^tRU={}^tL\,\nu L. \tag{4.7}$$

The phase factor in (3.2) is rewritten as

$$\begin{aligned}\sum_{j,k=1}^{n} v^j\phi_{jk}t^k&=\sum_{j,k,l,p=1}^{n} v^jL^l{}_j\nu_{lp}L^p{}_kt^k\\&=\sum_{j,l,p=1}^{n} v^jL^l{}_j\nu_{lp}y^p\\&=\sum_{j=1}^{n}\sum_{l=1}^{m} v^j(L^{2l-1}_j\nu_ly^{2l}-L^{2l}_j\nu_ly^{2l-1}).\end{aligned} \tag{4.8}$$

The gauge field (2.1) is expressed in the y-coordinate as

$$A=\frac{1}{2}\sum_{j=1}^{m}\nu_j(y^{2j-1}\,\mathrm{d}y^{2j}-y^{2j}\,\mathrm{d}y^{2j-1})+\sum_{j,k=1}^{n}\alpha_j(L^{-1})^j{}_k\,\mathrm{d}y^k. \tag{4.9}$$

We set

$$\beta_k=\sum_{j=1}^{n}\alpha_j(L^{-1})^j{}_k \tag{4.10}$$

for later use.

Actually, we can choose a transformation matrix L that has zeros in this pattern

$$L=\begin{pmatrix} L^{2i-1}_{2p-1} & L^{2i-1}_{2q} & L^{2i-1}_{2m+r}\\ L^{2j}_{2p-1} & L^{2j}_{2q} & L^{2j}_{2m+r}\\ L^{2m+k}_{2p-1} & L^{2m+k}_{2q} & L^{2m+k}_{2m+r}\end{pmatrix}=\begin{pmatrix} * & 0 & 0\\ * & * & 0\\ * & * & *\end{pmatrix} \tag{4.11}$$

with $i,j,p,q=1,\ldots,m$ and $k,r=1,\ldots,n-2m$. The inverse matrix L^{-1} also has the same pattern of zeros. This distribution of zeros is proved in Appendix A. Then we can rewrite the magnetic shift operators (3.16) and (3.17) with a help of (4.8) in the y-coordinate as

$$U_jf(y^i)=e^{\pi i\Sigma_{l=1}^{m}(1/q_j)(L^{2l-1}_{2j-1}\nu_ly^{2l}-L^{2l}_{2j-1}\nu_ly^{2l-1})}f(y^i-L^i{}_{2j-1}(1/q_j)), \tag{4.12}$$

$$V_jf(y^i)=e^{-\pi i\Sigma_{l=1}^{m}(1/q_j)L^{2l}_{2j}\nu_ly^{2l-1}}f(y^i-L^i{}_{2j}(1/q_j)), \tag{4.13}$$

$$W_k(\theta)f(y^i)=f(y^i-L^i{}_{2m+k}\theta) \tag{4.14}$$

for $j=1,\ldots,m$ and $k=1,\ldots,n-2m$. From (4.11) and (4.7) we get a formula

$$\begin{aligned}\nu_i L^{2i-1}_{2j-1} &= \sum_{l,p=1}^{m} \nu_l L^{2l-1}_{2j-1} L^{2l}_{2p} (L^{-1})^{2p}_{2i} \\ &= \sum_{l,p=1}^{m} (L^{2l-1}_{2j-1} \nu_l L^{2l}_{2p} - L^{2l}_{2j-1} \nu_l L^{2l-1}_{2p})(L^{-1})^{2p}_{2i} \\ &= \sum_{p=1}^{m} q_j \delta_{jp} (L^{-1})^{2p}_{2i} = q_j (L^{-1})^{2j}_{2i}, \end{aligned} \tag{4.15}$$

which will be repeatedly used later.

V. MAGNETIC ALGEBRA

In this section we introduce new operators which act on twisted periodic functions. These new operators and the operators in the MTG generate an algebra, which we call a magnetic algebra. We construct and classify its irreducible representations. In the next section we will prove that the space of twisted periodic functions is actually an irreducible representation of the magnetic algebra. In this sense, the magnetic algebra completely characterizes the quantum mechanics in the magnetic torus.

Now we introduce a family of Hermite operators. Expanding the covariant derivative (2.3) in terms of the y-coordinate

$$Df = i \sum_{l=1}^{n} P_l f \, \mathrm{d}y^l, \tag{5.1}$$

we define differential operators

$$P_{2j-1} = -i\left(\frac{\partial}{\partial y^{2j-1}} + \pi i \nu_j y^{2j} - 2\pi i \beta_{2j-1} \right), \tag{5.2}$$

$$P_{2j} = -i\left(\frac{\partial}{\partial y^{2j}} - \pi i \nu_j y^{2j-1} - 2\pi i \beta_{2j} \right) \quad (j = 1, \ldots, m), \tag{5.3}$$

$$P_{2m+k} = -i\left(\frac{\partial}{\partial y^{2m+k}} - 2\pi i \beta_{2m+k} \right) \quad (k = 1, \ldots, n-2m). \tag{5.4}$$

These are Hermitian with respect to the inner product (2.11). Since $(y^1, \ldots, y^n)$ is an orthonormal coordinate, the Laplacian (2.13) becomes

$$-\Delta f = \sum_{i=1}^{n} (P_i)^2 f. \tag{5.5}$$

Nontrivial commutators among P's are

$$[P_{2j-1}, P_{2j}] = 2\pi i \nu_j \quad (j = 1, \ldots, m). \tag{5.6}$$

The other commutators vanish. We call the operators $\{P_{2j-1}, P_{2j}\}$ transverse momenta while we call the operators $\{P_{2m+k}\}$ longitudinal momenta. Since the covariant derivative commutes with the magnetic shifts as seen at (3.4), the momentum operators $\{P_i\}$ commute with the shift operators $\{U_j, V_j, W_k\}$, which are defined at (3.16) and (3.17).

Next we introduce another family of unitary operators. For this purpose we need an observation; in the t-coordinate that expresses the magnetic flux matrix in the standard form (3.14), t^{2m+k} $(k=1,\ldots,n-2m)$ are genuine cyclic coordinates. That is to say, the twisted periodic function (2.5) is periodic with respect to these coordinates as

$$f(t^1,\ldots,t^{2m+k}+1,\ldots,t^n)=f(t^1,\ldots,t^{2m+k},\ldots,t^n) \quad (k=1,\ldots,n-2m) \tag{5.7}$$

and the magnetic shift (3.2) is reduced to an ordinary continuous shift

$$(W_k(\theta)f)(t^1,\ldots,t^{2m+k},\ldots,t^n)=f(t^1,\ldots,t^{2m+k}-\theta,\ldots,t^n). \tag{5.8}$$

Then we define an operator T^k for each $k=1,\ldots,n-2m$ which acts on f by multiplication

$$(T^k f)(\mathbf{t})=e^{2\pi i\, t^{2m+k}} f(\mathbf{t})=e^{2\pi i \Sigma_{i=1}^{n}(L^{-1})_i^{2m+k} y^i} f(\mathbf{t}). \tag{5.9}$$

Here we used the inverse L^{-1} of the coordinate transformation (4.6). The operators $\{T^k\}$ are unitary operators with respect to the inner product (2.11). They satisfy

$$W_k(\theta)T^k=e^{-2\pi i\theta}T^k W_k(\theta), \tag{5.10}$$

$$[P_i,T^k]=2\pi(L^{-1})_i^{2m+k}T^k \quad (i=1,\ldots,n;\ k=1,\ldots,n-2m) \tag{5.11}$$

and commute with the other generators of the MTG.

Combining all the operators introduced above we define an algebra $\mathcal{A}$ with the generators $\{P_i,U_j,V_j,W_k(\theta),T^k \mid i=1,\ldots,n;\ j=1,\ldots,m;\ k=1,\ldots,n-2m;\ \theta\in\mathbf{R}\}$ and with the relations (3.18)–(3.21), (5.6), (5.10), (5.11) and other trivial commutators. We call the algebra $\mathcal{A}$ a magnetic algebra. In the following we will construct all the irreducible representations of the algebra $\mathcal{A}$ and classify their unitary equivalence classes.

A subset of generators $\{P_{2j-1},P_{2m+l},V_j,W_l(\theta) \mid j=1,\ldots,m;\ l=1,\ldots,n-2m;\ \theta\in\mathbf{R}\}$ generates a maximal Abelian subalgebra of $\mathcal{A}$. Hence these generators are simultaneously diagonalizable. Their simultaneous eigenstate $|k,r,d\rangle$ is labeled by $r_j\in\mathbf{Z}/\mathbf{Z}_{q_j}$ of (3.22) and $d_l\in\mathbf{Z}$ of (3.23) with new labels

$$k_{2j-1},k_{2m+l}\in\mathbf{R}. \tag{5.12}$$

The generators $\{P_{2j-1},P_{2m+l}\}$ act on these states as

$$P_{2j-1}|k,r,d\rangle=2\pi k_{2j-1}|k,r,d\rangle, \tag{5.13}$$

$$P_{2m+l}|k,r,d\rangle=2\pi k_{2m+l}|k,r,d\rangle, \tag{5.14}$$

and $\{V_j,W_l(\theta)\}$ act as (3.25), (3.26), respectively. The coefficient 2π was set for later convenience. Other generators $\{P_{2j},T^l \mid j=1,\ldots,m;\ l=1,\ldots,n-2m\}$ act on the states as

$$\langle\psi|P_{2j}|k,r,d\rangle=i\nu_j\frac{\partial}{\partial k_{2j-1}}\langle\psi|k,r,d\rangle, \tag{5.15}$$

$$T^l|k_i,r,d_k\rangle=e^{2\pi i\{\Sigma_{j=1}^{m}(k_{2j-1}+\Delta^l k_{2j-1})\Delta^l k_{2j}/\nu_j+\Sigma_{p=1}^{n-2m}Z^{lp}d_p\}}|k_i+\Delta^l k_i,r,d_k+\delta_k{}^l\rangle, \tag{5.16}$$

with $\Delta^l k_i=(L^{-1})_i^{2m+l}$. We will determine the $(n-2m)\times(n-2m)$ matrix Z^{lp} later. Here $|\psi\rangle$ represents an arbitrary state. The rests $\{U_j \mid j=1,\ldots,m\}$ act as (3.24). Therefore, in an irreducible representation space the eigenvalues k_{2m+i} are linked to d_l via

$$k_{2m+i}=\sum_{l=1}^{n-2m} d_l(L^{-1})^{2m+l}_{2m+i}-\beta_{2m+i}=\sum_{l=1}^{n-2m}(d_l-\alpha_{2m+l})(L^{-1})^{2m+l}_{2m+i}\quad(i=1,\ldots,n-2m). \tag{5.17}$$

Here the real number β_{2m+i} coincides with the one that appeared in (5.4). We also used the relations (4.10) and (4.11).

The matrix Z^{lp} is determined by the condition $T^{l'}T^{l}=T^{l}T^{l'}$. The action of $T^{l'}T^{l}$ gives

$$\begin{aligned}T^{l'}T^{l}|k_i,r,d_k\rangle=&e^{2\pi i\{\Sigma_{j=1}^{m}(k_{2j-1}+\Delta^{l}k_{2j-1})\Delta^{l}k_{2j}/\nu_j+\Sigma_{p=1}^{n-2m}Z^{lp}d_p\}}\\&\times e^{2\pi i\{\Sigma_{j=1}^{m}(k_{2j-1}+\Delta^{l}k_{2j-1}+\Delta^{l'}k_{2j-1})\Delta^{l'}k_{2j}/\nu_j+\Sigma_{p=1}^{n-2m}Z^{l'p}(d_p+\delta_p{}^{l})\}}\\&\times|k_i+\Delta^{l}k_i+\Delta^{l'}k_i,r,d_k+\delta_k{}^{l}+\delta_k{}^{l'}\rangle\end{aligned} \tag{5.18}$$

while the action of $T^{l}T^{l'}$ gives

$$\begin{aligned}T^{l}T^{l'}|k_i,r,d_k\rangle=&e^{2\pi i\{\Sigma_{j=1}^{m}(k_{2j-1}+\Delta^{l'}k_{2j-1})\Delta^{l'}k_{2j}/\nu_j+\Sigma_{p=1}^{n-2m}Z^{l'p}d_p\}}\\&\times e^{2\pi i\{\Sigma_{j=1}^{m}(k_{2j-1}+\Delta^{l'}k_{2j-1}+\Delta^{l}k_{2j-1})\Delta^{l}k_{2j}/\nu_j+\Sigma_{p=1}^{n-2m}Z^{lp}(d_p+\delta_p{}^{l'})\}}\\&\times|k_i+\Delta^{l'}k_i+\Delta^{l}k_i,r,d_k+\delta_k{}^{l'}+\delta_k{}^{l}\rangle.\end{aligned} \tag{5.19}$$

To give $T^{l'}T^{l}=T^{l}T^{l'}$ the matrix $Z^{ll'}$ must satisfy

$$\sum_{j=1}^{m}(\Delta^{l}k_{2j-1}\Delta^{l'}k_{2j}/\nu_j)+Z^{l'l}=\sum_{j=1}^{m}(\Delta^{l'}k_{2j-1}\Delta^{l}k_{2j}/\nu_j)+Z^{ll'}. \tag{5.20}$$

A general solution of the above equation is

$$Z^{ll'}=\sum_{j=1}^{m}(\Delta^{l}k_{2j-1}\Delta^{l'}k_{2j}/\nu_j)+S^{ll'}. \tag{5.21}$$

Here we leave an arbitrary symmetric matrix $S^{ll'}=S^{l'l}$ yet undetermined. Actually any choice of $S^{ll'}$ results in an equivalent representation, and therefore we take $S^{ll'}=0$.

A restricted set of vectors $\{|k,r,d\rangle\}$ that are labeled by the mutually independent parameters

$$k_{2j-1}\in\mathbf{R},\quad r_j\in\mathbf{Z}/\mathbf{Z}_{q_j},\quad d_l\in\mathbf{Z}\quad(j=1,\ldots,m;\ l=1,\ldots,n-2m) \tag{5.22}$$

spans a Hilbert space $\mathcal{H}_\alpha$ for each fixed value of $(\alpha_{2m+1},\alpha_{2m+2},\ldots,\alpha_n)$. Thus we conclude that a unitary equivalence class of irreducible representations of the algebra $\mathcal{A}$ has one-to-one correspondence with the parameter $(\alpha_{2m+1},\alpha_{2m+2},\ldots,\alpha_n)\in\mathbf{R}^{n-2m}/\mathbf{Z}^{n-2m}$.

VI. FOURIER ANALYSIS FOR THE MAGNETIC TORUS

Now let us turn to the space of twisted periodic functions. It is a representation space of the magnetic algebra. We will calculate the whole family of eigenfunctions of the maximal Abelian subalgebra of the magnetic algebra. These eigenfunctions $\chi_{k,r,d}(t^1,\ldots,t^n)=\langle t|k,r,d\rangle$ satisfy

$$P_{2j-1}\chi_{k,r,d}=-i\left(\frac{\partial}{\partial y^{2j-1}}+\pi i\nu_j y^{2j}-2\pi i\beta_{2j-1}\right)\chi_{k,r,d}=2\pi k_{2j-1}\chi_{k,r,d}, \tag{6.1}$$

$$P_{2j}\chi_{k,r,d}=-i\left(\frac{\partial}{\partial y^{2j}}-\pi i\nu_j y^{2j-1}-2\pi i\beta_{2j}\right)\chi_{k,r,d}=i\nu_j\frac{\partial}{\partial k_{2j-1}}\chi_{k,r,d}, \tag{6.2}$$

$$P_{2m+l}\,\chi_{k,r,d}=-i\left(\frac{\partial}{\partial y^{2m+l}}-2\pi i\beta_{2m+l}\right)\chi_{k,r,d}=2\pi\left(\sum_{i=1}^{n-2m} d_i(L^{-1})^{2m+i}_{2m+l}-\beta_{2m+l}\right)\chi_{k,r,d}, \tag{6.3}$$

$$U_j\,\chi_{k,r,d}=e^{\pi i t^{2j}}\chi_{k,r,d}\left(t^{2j-1}-\frac{1}{q_j}\right)=\chi_{k,(r+1),d}, \tag{6.4}$$

$$V_j\,\chi_{k,r,d}=e^{-\pi i t^{2j-1}}\chi_{k,r,d}\left(t^{2j}-\frac{1}{q_j}\right)=e^{-2\pi i r_j/q_j}\chi_{k,r,d}, \tag{6.5}$$

$$W_l(\theta)\,\chi_{k,r,d}=\chi_{k,r,d}(t^{2m+l}-\theta)=e^{-2\pi i d_l\theta}\chi_{k,r,d}, \tag{6.6}$$

$$T^l\,\chi_{k,r,d}=e^{2\pi i\, t^{2m+l}}\chi_{k,r,d}=e^{2\pi i\{\sum_{j=1}^{m}(k_{2j-1}+\Delta k_{2j-1})\Delta k_{2j}/\nu_j+\sum_{p=1}^{n-2m}Z^{lp}d_p\}}\chi_{k+\Delta k,r,(d+1)}, \tag{6.7}$$

for $j=1,\ldots,m$; $l=1,\ldots,n-2m$, and $\theta\in\mathbf{R}$. Here $(r+1)$ is an abbreviation of $(r_i+\delta_{ij})$ and $(d+1)$ is an abbreviation of $(d_p+\delta_p{}^l)$. And $\Delta k_i=(L^{-1})^{2m+l}_{i}$ as given in (5.16). Then $\chi_{k,r,d}$ is a simultaneous eigenfunction of $\{P_{2j-1},P_{2m+l},V_j,W_l(\theta)\}$. In the rest of this section we will solve the set of equations (6.1)–(6.7) to get the solutions

$$\begin{aligned}\chi_{k,r,d}(y^1,\ldots,y^n)=c\,\exp&\left[2\pi i\left\{-\sum_{j=1}^{m}\sum_{l=1}^{n-2m} d_l(L^{-1})^{2m+l}_{2j-1}\beta_{2j}/\nu_j-\frac{1}{2}\sum_{j=1}^{m}\nu_j y^{2j-1}y^{2j}\right\}\right]\\
&\times\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}\exp\left[2\pi i\left\{\frac{1}{2}\sum_{i,j,l=1}^{m}\nu_l L^{2l-1}_{2i-1}L^{2l}_{2j-1}(q_i\sigma_i+r_i)(q_j\sigma_j\right.\right.\\
&+r_j)/(q_iq_j)+\sum_{j=1}^{m}\beta_{2j-1}\left\{y^{2j-1}-\sum_{i=1}^{m}L^{2j-1}_{2i-1}(q_i\sigma_i+r_i)/q_i\right\}\\
&+\sum_{j,l=1}^{m}(q_j\sigma_j+r_j-\gamma_j)(L^{-1})^{2j}_{2l}\left\{y^{2l}-\sum_{i=1}^{m}L^{2l}_{2i-1}(q_i\sigma_i+r_i)/q_i\right\}\\
&+\sum_{k,l=1}^{n-2m}d_k(L^{-1})^{2m+k}_{2m+l}\left\{y^{2m+l}-\sum_{i=1}^{m}L^{2m+l}_{2i-1}(q_i\sigma_i+r_i)/q_i\right\}\\
&\left.\left.+\sum_{j=1}^{m}k_{2j-1}\left\{y^{2j-1}+\beta_{2j}/\nu_j-\sum_{i=1}^{m}L^{2j-1}_{2i-1}(q_i\sigma_i+r_i-\gamma_i)/q_i\right\}\right\}\right],\end{aligned} \tag{6.8}$$

where c is a common normalization constant. The coefficients γ_j are given later at (6.13). The fact that the eigenfunctions are uniquely determined up to the common coefficient c implies that the space of twisted periodic function is an irreducible representation space of the magnetic algebra. Consequently, the eigenfunctions (6.8) constitute a complete orthonormal set of the space of twisted periodic functions over the torus. This is one of the main results of this paper. Hence an arbitrary twisted periodic function over the torus can be expanded as

$$f(y^1,\ldots,y^n)=\sum_{k,r,d}\lambda_{k,r,d}\,\chi_{k,r,d}(y^1,\ldots,y^n) \tag{6.9}$$

with unique coefficients $\lambda_{k,r,d}$. Therefore, the complete set $\{\chi_{k,r,d}\}$ provides a new basis for Fourier analysis in the magnetic torus.

In the rest of this section we give detailed lengthy calculations to prove the above statements. The reader may skip them to the next section, where we calculate solutions of the eigenvalue problem of the magnetic Laplacian using the main result (6.8). First, a simultaneous solution of (6.1) and (6.3) is

$$\chi_{k,r,d}(y^1,y^2,\ldots,y^n)=e^{2\pi i\Sigma_{j=1}^{m}\{(k_{2j-1}+\beta_{2j-1})y^{2j-1}-(1/2)\nu_j y^{2j-1}y^{2j}\}}e^{2\pi i\Sigma_{j,l=1}^{n-2m}d_j(L^{-1})_{2m+l}^{2m+j}y^{2m+l}}\times\phi_{k,r,d}(y^2,y^4,\ldots,y^{2m}), \tag{6.10}$$

where $\phi_{k,r,d}(y^2,y^4,\ldots,y^{2m})$ is an arbitrary function to be specified later.

Next let us turn to the other equation (6.5). Using (4.13) we can rewrite (6.5) as

$$e^{-\pi i\Sigma_{l=1}^{m}(1/q_j)L^{2l}_{\ 2j}\nu_l y^{2l-1}}\chi_{k,r,d}(y^i-L^{i}_{\ 2j}/q_j)=e^{-2\pi i r_j/q_j}\chi_{k,r,d}(y^i). \tag{6.11}$$

As discussed at (4.11) we have taken the matrix L such that $L^{2l-1}_{2j}=0$. Therefore, when (6.10) is substituted, the LHS of (6.11) becomes

$$\begin{aligned}&e^{-\pi i\Sigma_{l=1}^{m}(1/q_j)L^{2l}_{\ 2j}\nu_l y^{2l-1}}\chi_{k,r,d}(y^i-L^{i}_{\ 2j}/q_j)\\&\quad=e^{2\pi i\Sigma_{l=1}^{m}\{(k_{2l-1}+\beta_{2l-1})y^{2l-1}-(1/2)\nu_l y^{2l-1}y^{2l}\}}e^{2\pi i\Sigma_{p,l=1}^{n-2m}d_p(L^{-1})_{2m+l}^{2m+p}y^{2m+l}}\\&\qquad\times e^{-2\pi i\Sigma_{p,l=1}^{n-2m}d_p(L^{-1})_{2m+l}^{2m+p}L^{2m+l}_{\ 2j}/q_j}\phi_{k,r,d}(y^{2i}-L^{2i}_{\ 2j}/q_j).\end{aligned} \tag{6.12}$$

Hence, if we set

$$\gamma_j=\sum_{p,l=1}^{n-2m}d_p(L^{-1})_{2m+l}^{2m+p}L^{2m+l}_{\ 2j}, \tag{6.13}$$

(6.11) implies that

$$e^{-2\pi i\gamma_j/q_j}\phi_{k,r,d}(y^{2i}-L^{2i}_{\ 2j}/q_j)=e^{-2\pi i r_j/q_j}\phi_{k,r,d}(y^{2i}). \tag{6.14}$$

If we introduce another coordinate system $(z^1,z^2,\ldots,z^m)$ which is related to $(y^2,y^4,\ldots,y^{2m})$ via

$$y^{2i}=\sum_{j=1}^{m}L^{2i}_{\ 2j}(z^j/q_j), \tag{6.15}$$

then (6.14) is rewritten as

$$\phi_{k,r,d}(z^1,\ldots,z^j-1,\ldots,z^m)=e^{2\pi i(\gamma_j-r_j)/q_j}\phi_{k,r,d}(z^1,\ldots,z^j,\ldots,z^m). \tag{6.16}$$

Moreover, if we set

$$\psi_{k,r,d}(z^1,\ldots,z^m)=e^{2\pi i\Sigma_{j=1}^{m}(\gamma_j-r_j)z^j/q_j}\phi_{k,r,d}(z^1,\ldots,z^m), \tag{6.17}$$

then (6.16) implies that

$$\psi_{k,r,d}(z^1,\ldots,z^j-1,\ldots,z^m)=\psi_{k,r,d}(z^1,\ldots,z^j,\ldots,z^m). \tag{6.18}$$

Hence $\psi_{k,r,d}$ is a periodic function with the period 1 and can be expanded in a Fourier series

$$\psi_{k,r,d}(z^1,\ldots,z^j,\ldots,z^m)=\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}c_{k,r,d,\sigma}e^{2\pi i\Sigma_{j=1}^{m}\sigma_j z^j}. \tag{6.19}$$

Note that the inverse transformation of (6.15) is given by

$$(z^j/q_j)=\sum_{i=1}^{m}(L^{-1})^{2j}_{2i}y^{2i}. \tag{6.20}$$

Combining the above equations we can write down the eigenfunction (6.10) in a more specific form as

$$\chi_{k,r,d}(y^1,y^2,\ldots,y^n)=e^{2\pi i\Sigma_{j=1}^{m}\{(k_{2j-1}+\beta_{2j-1})y^{2j-1}-(1/2)\nu_j y^{2j-1}y^{2j}\}}e^{2\pi i\Sigma_{j,l=1}^{n-2m}d_j(L^{-1})^{2m+j}_{2m+l}y^{2m+l}}$$
$$\times\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}c_{k,r,d,\sigma}\,e^{2\pi i\Sigma_{j,l=1}^{m}(q_j\sigma_j+r_j-\gamma_j)(L^{-1})^{2j}_{2l}y^{2l}}. \tag{6.21}$$

Moreover, referring to (4.11) and (4.14), we can see that (6.21) satisfies (6.6). Thus we have seen that $\chi_{k,r,d}$ is a simultaneous eigenfunction of $\{P_{2j-1},P_{2m+l},V_j,W_l(\theta)\}$ as announced above.

The remaining task is to solve (6.2), (6.4), and (6.7). Let us begin with (6.2). The left-hand side (LHS) of (6.2) is

$$-i\left(\frac{\partial}{\partial y^{2j}}-\pi i\nu_j y^{2j-1}-2\pi i\beta_{2j}\right)\chi_{k,r,d}$$
$$=\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}2\pi\left(-\frac{1}{2}\nu_j y^{2j-1}+\sum_{i=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{2j}-\frac{1}{2}\nu_j y^{2j-1}-\beta_{2j}\right)$$
$$\times e^{2\pi i\Sigma_{i=1}^{m}\{(k_{2i-1}+\beta_{2i-1})y^{2i-1}-(1/2)\nu_i y^{2i-1}y^{2i}\}}e^{2\pi i\Sigma_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{2m+l}y^{2m+l}}c_{k,r,d,\sigma}$$
$$\times e^{2\pi i\Sigma_{i,l=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{2l}y^{2l}}.$$

On the other hand, the right-hand side (RHS) of (6.2) is

$$i\nu_j\frac{\partial}{\partial k_{2j-1}}\chi_{k,r,d}=i\nu_j\,e^{2\pi i\Sigma_{i=1}^{m}\{(k_{2i-1}+\beta_{2i-1})y^{2i-1}-(1/2)\nu_i y^{2i-1}y^{2i}\}}e^{2\pi i\Sigma_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{2m+l}y^{2m+l}}$$
$$\times\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}\left(\frac{\partial c_{k,r,d,\sigma}}{\partial k_{2j-1}}+2\pi i y^{2j-1}c_{k,r,d,\sigma}\right)$$
$$\times e^{2\pi i\Sigma_{i,l=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{2l}y^{2l}}.$$

Therefore, we have an equation

$$i\nu_j\frac{\partial c_{k,r,d,\sigma}}{\partial k_{2j-1}}=2\pi\left(\sum_{i=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{2j}-\beta_{2j}\right)c_{k,r,d,\sigma} \tag{6.22}$$

and get its solution

$$c_{k,r,d,\sigma}=c_{0,r,d,\sigma}\,e^{-2\pi i\{\Sigma_{i,j=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{2j}k_{2j-1}/\nu_j-\Sigma_{j=1}^{m}\beta_{2j}k_{2j-1}/\nu_j\}}. \tag{6.23}$$

Thus (6.21) becomes

$$\chi_{k,r,d}(y^1,y^2,\ldots,y^n)=e^{2\pi i\Sigma_{j=1}^{m}\{(k_{2j-1}+\beta_{2j-1})y^{2j-1}+k_{2j-1}\beta_{2j}/\nu_j-(1/2)\nu_j y^{2j-1}y^{2j}\}}$$
$$\times e^{2\pi i\Sigma_{j,l=1}^{n-2m}d_j(L^{-1})^{2m+j}_{\ 2m+l}y^{2m+l}}$$
$$\times\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}c_{0,r,d,\sigma}\,e^{2\pi i\Sigma_{j,l=1}^{m}(q_j\sigma_j+r_j-\gamma_j)(L^{-1})^{2j}_{\ 2l}(y^{2l}-k_{2l-1}/\nu_l)}. \tag{6.24}$$

Next we turn to (6.4). With the aid of (4.12) and (4.15) the LHS of (6.4) becomes

$$e^{\pi i\Sigma_{l=1}^{m}(1/q_j)(L^{2l-1}_{\ 2j-1}\nu_l y^{2l}-L^{2l}_{\ 2j-1}\nu_l y^{2l-1})}\chi_{k,r,d}(y^i-L^{i}_{\ 2j-1}/q_j)$$
$$=e^{2\pi i\Sigma_{i=1}^{m}\{-\beta_{2i-1}L^{2i-1}_{\ 2j-1}/q_j-(1/2)(L^{-1})^{2j}_{\ 2i}L^{2i}_{\ 2j-1}/q_j\}}\,e^{-2\pi i\Sigma_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{\ 2m+l}L^{2m+l}_{\ 2j-1}/q_j}$$
$$\times e^{2\pi i\Sigma_{i=1}^{m}\{(k_{2i-1}+\beta_{2i-1})y^{2i-1}+k_{2i-1}\beta_{2i}/\nu_i-(1/2)\nu_i y^{2i-1}y^{2i}\}}e^{2\pi i\Sigma_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{\ 2m+l}y^{2m+l}}$$
$$\times\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}c_{0,r,d,\sigma}\,e^{-2\pi i\Sigma_{i,l=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{\ 2l}L^{2l}_{\ 2j-1}/q_j}$$
$$\times e^{2\pi i\Sigma_{i,l=1}^{m}(q_i\sigma_i+r_i+\delta_{ij}-\gamma_i)(L^{-1})^{2i}_{\ 2l}(y^{2l}-k_{2l-1}/\nu_l)}. \tag{6.25}$$

To make this coincide with the RHS of (6.4) we have a recursive equation

$$e^{2\pi i\Sigma_{i=1}^{m}\{-\beta_{2i-1}L^{2i-1}_{\ 2j-1}/q_j-(1/2)(L^{-1})^{2j}_{\ 2i}L^{2i}_{\ 2j-1}/q_j\}}\,e^{-2\pi i\Sigma_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{\ 2m+l}L^{2m+l}_{\ 2j-1}/q_j}$$
$$\times e^{-2\pi i\Sigma_{i,l=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{\ 2l}L^{2l}_{\ 2j-1}/q_j}\,c_{0,r,d,\sigma}=c_{0,(r+1),d,\sigma}. \tag{6.26}$$

Here $(r+1)$ means $(r_i+\delta_{ij})$. If we define an $m\times m$ matrix

$$Y_{ij}=\sum_{l=1}^{m}(L^{-1})^{2i}_{\ 2l}L^{2l}_{\ 2j-1}/q_j=\sum_{l=1}^{m}\nu_l L^{2l-1}_{\ 2i-1}L^{2l}_{\ 2j-1}/(q_iq_j), \tag{6.27}$$

it is symmetric as

$$Y_{ij}-Y_{ji}=\sum_{l=1}^{m}(L^{2l-1}_{\ 2i-1}\nu_l L^{2l}_{\ 2j-1}-L^{2l}_{\ 2i-1}\nu_l L^{2l-1}_{\ 2j-1})/(q_iq_j)=\phi_{2i-1,2j-1}/(q_iq_j)=0 \tag{6.28}$$

by virtue of (4.7). Then the solution of (6.26) is

$$c_{0,r,d,\sigma}=c_{0,0,d,\sigma}\,e^{-2\pi i\Sigma_{i,j=1}^{m}\beta_{2i-1}L^{2i-1}_{\ 2j-1}r_j/q_j}e^{-2\pi i\Sigma_{j=1}^{m}\Sigma_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{\ 2m+l}L^{2m+l}_{\ 2j-1}r_j/q_j}$$
$$\times e^{-2\pi i\Sigma_{i,j,l=1}^{m}(q_i\sigma_i+(1/2)r_i-\gamma_i)(L^{-1})^{2i}_{\ 2l}L^{2l}_{\ 2j-1}r_j/q_j}. \tag{6.29}$$

Therefore, (6.24) becomes

$$\chi_{k,r,d}(y^1,y^2,\ldots,y^n)=\exp\Bigg[2\pi i\Bigg\{\sum_{j=1}^{m}\Big\{(k_{2j-1}+\beta_{2j-1})y^{2j-1}+k_{2j-1}\beta_{2j}/\nu_j-\frac{1}{2}\nu_j y^{2j-1}y^{2j}\Big\}$$
$$+\sum_{i,l=1}^{n-2m} d_i(L^{-1})^{2m+i}_{2m+l}\Big(y^{2m+l}-\sum_{j=1}^{m}L^{2m+l}_{2j-1}r_j/q_j\Big)$$
$$-\sum_{i,j=1}^{m}\beta_{2i-1}L^{2i-1}_{2j-1}r_j/q_j+\frac{1}{2}\sum_{i,j=1}^{m}Y_{ij}r_ir_j\Bigg\}\Bigg]$$
$$\times\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}c_{0,0,d,\sigma}\exp\Bigg[2\pi i\sum_{j,l=1}^{m}(q_j\sigma_j+r_j-\gamma_j)(L^{-1})^{2j}_{2l}$$
$$\times\Big(y^{2l}-k_{2l-1}/\nu_l-\sum_{i=1}^{m}L^{2l}_{2i-1}r_i/q_i\Big)\Bigg]. \tag{6.30}$$

Since $(U_j)^{q_j}=1$ by (3.18), the substitution $r_j\mapsto r_j+q_j$ must leave $\chi_{k,r,d}$ invariant. This substitution gives

$$\chi_{k,(r+q),d}(\mathbf{y})=\exp\Bigg[2\pi i\Bigg\{\sum_{i=1}^{m}\Big\{(k_{2i-1}+\beta_{2i-1})y^{2i-1}+k_{2i-1}\beta_{2i}/\nu_i-\frac{1}{2}\nu_i y^{2i-1}y^{2i}\Big\}$$
$$+\sum_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{2m+l}\Big(y^{2m+l}-\sum_{p=1}^{m}L^{2m+l}_{2p-1}r_p/q_p\Big)-\sum_{i,l=1}^{m}\beta_{2i-1}L^{2i-1}_{2l-1}r_l/q_l$$
$$+\frac{1}{2}\sum_{i,p=1}^{m}Y_{ip}r_ir_p-\sum_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{2m+l}L^{2m+l}_{2j-1}-\sum_{i=1}^{m}\beta_{2i-1}L^{2i-1}_{2j-1}+\sum_{i=1}^{m}Y_{ij}r_iq_j$$
$$+\frac{1}{2}Y_{jj}q_jq_j\Big)\Bigg\}\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}c_{0,0,d,(\sigma-1)}\exp\Bigg[2\pi i\Bigg\{-\sum_{i=1}^{m}(q_i\sigma_i+r_i-\gamma_i)Y_{ij}q_j$$
$$+\sum_{i,l=1}^{m}(q_i\sigma_i+r_i-\gamma_i)(L^{-1})^{2i}_{2l}\Big(y^{2l}-k_{2l-1}/\nu_l-\sum_{p=1}^{m}L^{2l}_{2p-1}r_p/q_p\Big)\Bigg\}\Bigg]. \tag{6.31}$$

Here $(\sigma-1)$ is an abbreviation of $(\sigma_i-\delta_{ij})$. We used (4.15). Then we have another recursive equation

$$c_{0,0,d,\sigma}=c_{0,0,d,(\sigma-1)}\exp\Bigg[2\pi i\Bigg\{-\sum_{i,l=1}^{n-2m}d_i(L^{-1})^{2m+i}_{2m+l}L^{2m+l}_{2j-1}-\sum_{i=1}^{m}\beta_{2i-1}L^{2i-1}_{2j-1}$$
$$+\sum_{i=1}^{m}Y_{ij}r_iq_j+\frac{1}{2}Y_{jj}q_jq_j-\sum_{i=1}^{m}(q_i\sigma_i+r_i-\gamma_i)Y_{ij}q_j\Bigg\}\Bigg]. \tag{6.32}$$

Remember that $q_iq_jY_{ij}=\Sigma_{l=1}^{m}\nu_lL^{2l-1}_{2i-1}L^{2l}_{2j-1}=q_jq_iY_{ji}$ is symmetric. The solution of (6.32) is

$$c_{0,0,d,\sigma}=c_{0,0,d,0}\exp\Bigg[2\pi i\Bigg\{-\sum_{i,l=1}^{n-2m}\sum_{j=1}^{m}d_i(L^{-1})^{2m+i}_{2m+l}L^{2m+l}_{2j-1}\sigma_j-\sum_{i,j=1}^{m}\beta_{2i-1}L^{2i-1}_{2j-1}\sigma_j$$
$$+\sum_{i,j=1}^{m}Y_{ij}r_iq_j\sigma_j-\sum_{i,j=1}^{m}((1/2)q_i\sigma_i+r_i-\gamma_i)Y_{ij}q_j\sigma_j\Bigg\}\Bigg]. \tag{6.33}$$

Substituting it into (6.30) and using (4.15), we get

$$\chi_{k,r,d}(\mathbf{y}) = c_{0,0,d,0}\, e^{-\pi i \Sigma_{j=1}^{m} \nu_j y^{2j-1} y^{2j}} \sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty} \exp\Bigg[2\pi i \Bigg\{ \frac{1}{2} \sum_{i,j=1}^{m} Y_{ij}(q_i\sigma_i + r_i)(q_j\sigma_j + r_j)$$
$$+ \sum_{j=1}^{m} \beta_{2j-1} \Bigg\{ y^{2j-1} - \sum_{i=1}^{m} L^{2j-1}_{2i-1}(q_i\sigma_i + r_i)/q_i \Bigg\}$$
$$+ \sum_{j,l=1}^{m} (q_j\sigma_j + r_j - \gamma_j)(L^{-1})^{2j}_{2l} \Bigg\{ y^{2l} - \sum_{i=1}^{m} L^{2l}_{2i-1}(q_i\sigma_i + r_i)/q_i \Bigg\}$$
$$+ \sum_{i,l=1}^{n-2m} d_i (L^{-1})^{2m+i}_{2m+l} \Bigg\{ y^{2m+l} - \sum_{j=1}^{m} L^{2m+l}_{2j-1}(q_j\sigma_j + r_j)/q_j \Bigg\}$$
$$+ \sum_{j=1}^{m} k_{2j-1} \Bigg\{ y^{2j-1} + \beta_{2j}/\nu_j - \sum_{l=1}^{m} L^{2j-1}_{2l-1}(q_l\sigma_l + r_l - \gamma_l)/q_l \Bigg\} \Bigg\} \Bigg]. \tag{6.34}$$

Finally, we are going to solve (6.7). Its LHS becomes

$$e^{2\pi i\, t^{2m+l}} \chi_{k,r,d}$$
$$= c_{0,0,d,0}\, e^{-\pi i \Sigma_{j=1}^{m} \nu_j y^{2j-1} y^{2j}} \sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty} \exp\Bigg[2\pi i \Bigg\{ \frac{1}{2} \sum_{i,j=1}^{m} Y_{ij}(q_i\sigma_i + r_i)(q_j\sigma_j + r_j)$$
$$+ \sum_{j=1}^{m} \beta_{2j-1} \Bigg\{ y^{2j-1} - \sum_{i=1}^{m} L^{2j-1}_{2i-1}(q_i\sigma_i + r_i)/q_i \Bigg\}$$
$$+ \sum_{j,p=1}^{m} \Bigg(q_j\sigma_j + r_j - \gamma_j + \sum_{i=1}^{m} (L^{-1})^{2m+l}_{2i} L^{2i}_{2j} \Bigg) (L^{-1})^{2j}_{2p} \Bigg\{ y^{2p} - \sum_{i=1}^{m} L^{2p}_{2i-1}(q_i\sigma_i + r_i)/q_i \Bigg\}$$
$$+ \sum_{p,i=1}^{m} (L^{-1})^{2m+l}_{2p} L^{2p}_{2i-1}(q_i\sigma_i + r_i)/q_i$$
$$+ \sum_{i,p=1}^{n-2m} (d_i + \delta^l_i)(L^{-1})^{2m+i}_{2m+p} \Bigg\{ y^{2m+p} - \sum_{j=1}^{m} L^{2m+p}_{2j-1}(q_j\sigma_j + r_j)/q_j \Bigg\}$$
$$+ \sum_{p=1}^{n-2m} \sum_{j=1}^{m} (L^{-1})^{2m+l}_{2m+p} L^{2m+p}_{2j-1}(q_j\sigma_j + r_j)/q_j$$
$$+ \sum_{j=1}^{m} (k_{2j-1} + (L^{-1})^{2m+l}_{2j-1}) \Bigg\{ y^{2j-1} + \beta_{2j}/\nu_j - \sum_{p=1}^{m} L^{2j-1}_{2p-1}(q_p\sigma_p + r_p - \gamma_p)/q_p \Bigg\}$$
$$- \sum_{j=1}^{m} (L^{-1})^{2m+l}_{2j-1} \Bigg\{ \beta_{2j}/\nu_j - \sum_{p=1}^{m} L^{2j-1}_{2p-1}(q_p\sigma_p + r_p - \gamma_p)/q_p \Bigg\} \Bigg\} \Bigg]. \tag{6.35}$$

We set $\Delta^l k_i = (L^{-1})^{2m+l}_{i}$ as before. The change $d_p \mapsto d_p + \delta_p{}^l$ causes a change of γ_j as

$$\begin{aligned}\Delta^l\gamma_j &= \sum_{i=1}^{n-2m} (L^{-1})^{2m+l}_{2m+i} L^{2m+i}_{2j} \\ &= \sum_{i=1}^{n} (L^{-1})^{2m+l}_{i} L^{i}_{2j} - \sum_{i=1}^{m} (L^{-1})^{2m+l}_{2i-1} L^{2i-1}_{2j} - \sum_{i=1}^{m} (L^{-1})^{2m+l}_{2i} L^{2i}_{2j} \\ &= 0 - 0 - \sum_{i=1}^{m} (L^{-1})^{2m+l}_{2i} L^{2i}_{2j} \end{aligned} \tag{6.36}$$

via (6.13) with (4.11). Moreover, we can see that

$$\begin{aligned}\sum_{p=1}^{m} L^{2j-1}_{2p-1}\Delta^l\gamma_p/q_p &= -\sum_{p,i=1}^{m} L^{2j-1}_{2p-1}(L^{-1})^{2m+l}_{2i} L^{2i}_{2p}/q_p \\ &= -\sum_{p,i=1}^{m} (L^{-1})^{2p}_{2j}(L^{-1})^{2m+l}_{2i} L^{2i}_{2p}/\nu_j \\ &= -\sum_{i=1}^{m} (L^{-1})^{2m+l}_{2i}\delta^{2i}_{2j}/\nu_j \\ &= -(L^{-1})^{2m+l}_{2j}/\nu_j \\ &= -\Delta^l k_{2j}/\nu_j . \end{aligned} \tag{6.37}$$

Therefore (6.35) becomes

$$\begin{aligned} & e^{2\pi i\, t^{2m+l}}\chi_{k,r,d} \\ &= c_{0,0,d,0}\, e^{-\pi i\sum_{j=1}^{m}\nu_j y^{2j-1}y^{2j}} \exp\Bigg[2\pi i \Bigg\{ \sum_{j=1}^{m} (k_{2j-1}+\Delta k_{2j-1})\Delta k_{2j}/\nu_j \\ &\quad - \sum_{j=1}^{m} (L^{-1})^{2m+l}_{2j-1}\Bigg(\beta_{2j}/\nu_j + \sum_{p=1}^{m} L^{2j-1}_{2p-1}\gamma_p/q_p \Bigg)\Bigg\}\Bigg] \\ &\quad \times \sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty} \exp\Bigg[2\pi i \Bigg\{ \frac{1}{2}\sum_{i,j=1}^{m} Y_{ij}(q_i\sigma_i+r_i)(q_j\sigma_j+r_j) \\ &\quad + \sum_{j=1}^{m} \beta_{2j-1}\Bigg\{ y^{2j-1} - \sum_{i=1}^{m} L^{2j-1}_{2i-1}(q_i\sigma_i+r_i)/q_i \Bigg\} \\ &\quad + \sum_{j,p=1}^{m} (q_j\sigma_j+r_j-\gamma_j-\Delta\gamma_j)(L^{-1})^{2j}_{2p}\Bigg\{ y^{2p} - \sum_{i=1}^{m} L^{2p}_{2i-1}(q_i\sigma_i+r_i)/q_i \Bigg\} \\ &\quad + \sum_{i,p=1}^{n-2m} (d_i+\delta^l_i)(L^{-1})^{2m+i}_{2m+p}\Bigg\{ y^{2m+p} - \sum_{j=1}^{m} L^{2m+p}_{2j-1}(q_j\sigma_j+r_j)/q_j \Bigg\} \\ &\quad + \sum_{j=1}^{m} (k_{2j-1}+\Delta k_{2j-1})\Bigg\{ y^{2j-1} + \beta_{2j}/\nu_j - \sum_{p=1}^{m} L^{2j-1}_{2p-1}(q_p\sigma_p+r_p-\gamma_p-\Delta\gamma_p)/q_p \Bigg\}\Bigg\}\Bigg]. \end{aligned} \tag{6.38}$$

In the course of calculation we used the fact $\Sigma_{p=1}^{n}(L^{-1})^{2m+l}_{p} L^{p}_{2i-1}=0$. With the aid of (4.15), (6.36) and the definition (5.21) we can deduce that

$$\sum_{j,p=1}^{m} (L^{-1})^{2m+l}_{2j-1} L^{2j-1}_{2p-1} \gamma_p / q_p = -\sum_{i=1}^{n-2m} Z^{li} d_i . \tag{6.39}$$

Thus we reach

$$\begin{aligned} e^{2\pi i\, t^{2m+l}} \chi_{k,r,d} = {} & e^{-2\pi i \Sigma_{j=1}^{m} (L^{-1})^{2m+l}_{2j-1} \beta_{2j}/\nu_j} c_{0,0,d,0} c^{-1}_{0,0,(d+1),0} \\ & \times e^{2\pi i \{\Sigma_{j=1}^{m} (k_{2j-1}+\Delta k_{2j-1}) \Delta k_{2j}/\nu_j + \Sigma_{i=1}^{n-2m} Z^{li} d_i\}} \chi_{k+\Delta k,r,(d+1)} . \end{aligned} \tag{6.40}$$

To satisfy (6.7) we meet another recursive equation

$$c_{0,0,(d+1),0} = e^{-2\pi i \Sigma_{j=1}^{m} (L^{-1})^{2m+l}_{2j-1} \beta_{2j}/\nu_j} c_{0,0,d,0} \tag{6.41}$$

and we get the solution

$$c_{0,0,d,0} = e^{-2\pi i \Sigma_{j=1}^{m} \Sigma_{l=1}^{n-2m} d_l (L^{-1})^{2m+l}_{2j-1} \beta_{2j}/\nu_j} c_{0,0,0,0} . \tag{6.42}$$

Substituting it into (6.34) we reach the final result

$$\begin{aligned} \chi_{k,r,d}(\mathbf{y}) = c_{0,0,0,0} \exp\Bigg[2\pi i \Bigg\{ & -\sum_{j=1}^{m} \sum_{l=1}^{n-2m} d_l (L^{-1})^{2m+l}_{2j-1} \beta_{2j}/\nu_j - \frac{1}{2} \sum_{j=1}^{m} \nu_j y^{2j-1} y^{2j} \Bigg\} \Bigg] \\ & \times \sum_{\sigma_1, \sigma_2, \ldots, \sigma_m = -\infty}^{\infty} \exp\Bigg[2\pi i \Bigg\{ \frac{1}{2} \sum_{i,j=1}^{m} Y_{ij} (q_i \sigma_i + r_i)(q_j \sigma_j + r_j) \\ & + \sum_{j=1}^{m} \beta_{2j-1} \Bigg\{ y^{2j-1} - \sum_{i=1}^{m} L^{2j-1}_{2i-1} (q_i \sigma_i + r_i)/q_i \Bigg\} \\ & + \sum_{j,l=1}^{m} (q_j \sigma_j + r_j - \gamma_j)(L^{-1})^{2j}_{2l} \Bigg\{ y^{2l} - \sum_{i=1}^{m} L^{2l}_{2i-1} (q_i \sigma_i + r_i)/q_i \Bigg\} \\ & + \sum_{k,l=1}^{n-2m} d_k (L^{-1})^{2m+k}_{2m+l} \Bigg\{ y^{2m+l} - \sum_{i=1}^{m} L^{2m+l}_{2i-1} (q_i \sigma_i + r_i)/q_i \Bigg\} \\ & + \sum_{j=1}^{m} k_{2j-1} \Bigg\{ y^{2j-1} + \beta_{2j}/\nu_j - \sum_{i=1}^{m} L^{2j-1}_{2i-1} (q_i \sigma_i + r_i - \gamma_i)/q_i \Bigg\} \Bigg\} \Bigg] . \end{aligned} \tag{6.43}$$

This is the result (6.8) announced previously. The eigenfunction $\chi_{k,r,d}$ is determined up to a unique normalization constant $c_{0,0,0,0}$. Thus we conclude that the space of twisted periodic functions over the torus is an irreducible representation space of the magnetic algebra $\mathcal{A}$. Finally, we have proved that the set of functions $\{\chi_{k,r,d}\}$ is a complete orthonormal set in the space of twisted periodic functions over the torus as announced at (6.9).

VII. EIGENFUNCTIONS OF THE MAGNETIC LAPLACIAN

In this section we will write down explicitly solutions of the eigenvalue problem of the magnetic Laplacian (2.13) or (5.5), which is expressed in the y-coordinate as

$$\begin{aligned} \Delta f = {} & \sum_{j=1}^{m} \left[\left(\frac{\partial}{\partial y^{2j-1}} + \pi i \nu_j y^{2j} - 2\pi i \beta_{2j-1} \right)^2 f + \left(\frac{\partial}{\partial y^{2j}} - \pi i \nu_j y^{2j-1} - 2\pi i \beta_{2j} \right)^2 f \right] \\ & + \sum_{k=1}^{n-2m} \left(\frac{\partial}{\partial y^{2m+k}} - 2\pi i \beta_{2m+k} \right)^2 f . \end{aligned} \tag{7.1}$$

The function f must satisfy the twisted periodic condition (2.5). We will obtain solutions using the Fourier analysis that is developed in the preceding section.

Here we would like to describe the outline of our method. As mentioned in the preceding section, the Laplacian commutes with the magnetic shift operators, $\{U_j, V_j, W_k\}$. Hence the labels $(r,d)=(r_1,r_2,\ldots,r_m,d_1,d_2,\ldots,d_{n-2m})$, which are defined in (3.22)–(3.26), are good quantum numbers. Moreover, the Laplacian commutes with the longitudinal momentum operators $\{P_{2m+k}\}$. Hence the corresponding momentum eigenvalues $\{k_{2m+k}\}$ are also good quantum numbers and are related to the labels d_l via (5.17). On the other hand, the Laplacian does not commute with the transverse momentum operators $\{P_{2j-1}, P_{2j}\}$. Hence the transverse momentum eigenvalues $\{k_{2j-1}\}$ do not remain good quantum numbers. The Laplacian admits a new set of good quantum numbers $(n_1,n_2,\ldots,n_m)$, which will be introduced later. It will be revealed that eigenfunctions of the Laplacian are actually matrix elements of a unitary transformation,

$$\psi_{n,r,d}(k_1,k_3,\ldots,k_{2m-1})=\langle k,r,d|n,r,d\rangle, \tag{7.2}$$

which relates the quantum numbers n's to k's. In the k-space it is rather easy to get eigenfunctions by the standard method of a harmonic oscillator. On the other hand, the set of eigenfunctions of the momenta and magnetic shifts,

$$\chi_{k,r,d}(y^1,y^2,\ldots,y^n)=\langle y|k,r,d\rangle, \tag{7.3}$$

plays a role a unitary transformation which bridges between the momentum space and the real space like the usual Fourier transformation. Hence the Laplacian eigenfunctions are transformed into the y-coordinate representations by

$$\psi_{n,r,d}(y^1,y^2,\ldots,y^n)=\langle y|n,r,d\rangle=\int_{-\infty}^{\infty} \mathrm{d}k_1\,\mathrm{d}k_3\cdots\mathrm{d}k_{2m-1}\langle y|k,r,d\rangle\,\langle k,r,d|n,r,d\rangle. \tag{7.4}$$

This will give the desired result.

Now let us carry out the program outlined above. We define creation and annihilation operators associated with the transverse momenta as

$$a_j^\dagger=\frac{1}{\sqrt{4\pi\nu_j}}(P_{2j-1}-iP_{2j}),\quad a_j=\frac{1}{\sqrt{4\pi\nu_j}}(P_{2j-1}+iP_{2j})\quad (j=1,\ldots,m). \tag{7.5}$$

It is easily verified that $[a_j,a_k^\dagger]=\delta_{jk}$. Then the Laplacian (5.5) becomes

$$-\Delta=\sum_{j=1}^{m} 4\pi\nu_j\left(a_j^\dagger a_j+\frac{1}{2}\right)+\sum_{k=1}^{n-2m}(P_{2m+k})^2. \tag{7.6}$$

The eigenstate $|\Omega\rangle$ for the lowest eigenvalue satisfies

$$0=\langle k|a_j|\Omega\rangle=\frac{1}{\sqrt{4\pi\nu_j}}\langle k|(P_{2j-1}+iP_{2j})|\Omega\rangle=\frac{1}{\sqrt{4\pi\nu_j}}\left(2\pi k_{2j-1}+\nu_j\frac{\partial}{\partial k_{2j-1}}\right)\langle k|\Omega\rangle. \tag{7.7}$$

Here we used (5.13) and (5.15). The solution is

$$\langle k|\Omega\rangle=e^{-\pi\Sigma_{j=1}^{m}(k_{2j-1})^2/\nu_j}. \tag{7.8}$$

States for higher eigenvalues are generated by creation operators as

$$\begin{aligned}\langle k|n\rangle &= \frac{1}{\sqrt{n_1!\cdots n_m!}}\langle k|(a_1^\dagger)^{n_1}\cdots(a_m^\dagger)^{n_m}|\Omega\rangle \\ &= \frac{1}{\sqrt{n_1!\cdots n_m!}}\prod_{j=1}^{m}\left[\frac{1}{\sqrt{4\pi\nu_j}}\left(2\pi k_{2j-1}-\nu_j\frac{\partial}{\partial k_{2j-1}}\right)\right]^{n_j}\langle k|\Omega\rangle \\ &= \frac{1}{\sqrt{n_1!\cdots n_m!}}e^{\pi\Sigma_{j=1}^{m}(k_{2j-1})^2/\nu_j}\prod_{j=1}^{m}\left[\frac{1}{\sqrt{4\pi\nu_j}}\left(-\nu_j\frac{\partial}{\partial k_{2j-1}}\right)\right]^{n_j}e^{-2\pi\Sigma_{j=1}^{m}(k_{2j-1})^2/\nu_j} \quad (7.9)\end{aligned}$$

for $n_1,n_2,\ldots,n_m=0,1,2,\ldots$. We are suppressing other labels (r,d). In Appendix B we prove that

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}k\,e^{2\pi ikz}\cdot e^{\pi k^2/\nu}\left(-\nu\frac{\partial}{\partial k}\right)^n e^{-2\pi k^2/\nu} &= \sqrt{\nu}\,e^{\pi\nu z^2}\left(-i\frac{\partial}{\partial z}\right)^n e^{-2\pi\nu z^2} \\ &= \sqrt{\nu}(i\sqrt{2\pi\nu})^n e^{-\pi\nu z^2}H_n(z\sqrt{2\pi\nu}). \quad (7.10)\end{aligned}$$

In the second line $H_n(\xi)$ is the nth Hermite polynomial. Substituting (6.43) and (7.9) into (7.4) and applying (7.10) we obtain

$$\begin{aligned}\langle y|n,r,d\rangle &= \int_{-\infty}^{\infty}\mathrm{d}k_1\,\mathrm{d}k_3\cdots\mathrm{d}k_{2m-1}\,\langle y|k,r,d\rangle\langle k,r,d|n,r,d\rangle \\ &= c\,\exp\Bigg[2\pi i\Bigg\{-\sum_{j=1}^{m}\sum_{l=1}^{n-2m}d_l(L^{-1})^{2m+l}_{2j-1}\beta_{2j}/\nu_j \\ &\quad -\frac{1}{2}\sum_{j=1}^{m}\nu_j y^{2j-1}y^{2j}\Bigg\}\Bigg]_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}\exp\Bigg[2\pi i\Bigg\{\frac{1}{2}\sum_{i,j=1}^{m}Y_{ij}(q_i\sigma_i+r_i)(q_j\sigma_j+r_j) \\ &\quad +\sum_{j=1}^{m}\beta_{2j-1}\Bigg\{y^{2j-1}-\sum_{i=1}^{m}L^{2j-1}_{2i-1}(q_i\sigma_i+r_i)/q_i\Bigg\} \\ &\quad +\sum_{j,l=1}^{m}(q_j\sigma_j+r_j-\gamma_j)(L^{-1})^{2j}_{2l}\Bigg\{y^{2l}-\sum_{i=1}^{m}L^{2l}_{2i-1}(q_i\sigma_i+r_i)/q_i\Bigg\} \\ &\quad +\sum_{k,l=1}^{n-2m}d_k(L^{-1})^{2m+k}_{2m+l}\Bigg\{y^{2m+l}-\sum_{i=1}^{m}L^{2m+l}_{2i-1}(q_i\sigma_i+r_i)/q_i\Bigg\}\Bigg\}\Bigg] \\ &\quad \times\prod_{j=1}^{m}\Bigg[\sqrt{\frac{\nu_j}{n_j!}}\left(\frac{i}{\sqrt{2}}\right)^{n_j}e^{-\pi\nu_j\{y^{2j-1}+\beta_{2j}/\nu_j-\Sigma_{i=1}^{m}L^{2j-1}_{2i-1}(q_i\sigma_i+r_i-\gamma_i)/q_i\}^2} \\ &\quad \times H_{n_j}\Bigg(\sqrt{2\pi\nu_j}\Bigg\{y^{2j-1}+\beta_{2j}/\nu_j-\sum_{i=1}^{m}L^{2j-1}_{2i-1}(q_i\sigma_i+r_i-\gamma_i)/q_i\Bigg\}\Bigg)\Bigg]. \quad (7.11)\end{aligned}$$

This is the main result of this paper. Using (5.17), we can calculate eigenvalues of the Laplacian (7.6) as

$$-\frac{1}{2}\Delta\psi_{n,r,d}=\Bigg[\sum_{j=1}^{m}2\pi\nu_j\left(n_j+\frac{1}{2}\right)+\frac{1}{2}\sum_{k=1}^{n-2m}\Bigg\{\sum_{l=1}^{n-2m}2\pi(d_l-\alpha_{2m+l})(L^{-1})^{2m+l}_{2m+k}\Bigg\}^2\Bigg]\psi_{n,r,d}. \quad (7.12)$$

Eigenvalues depend on quantum numbers $n_1,n_2,\ldots,n_m=0,1,2,\ldots$ and $d_1,d_2,\ldots,d_{n-2m}=0,\pm1,\pm2,\ldots$ but not on $r=(r_1,r_2,\ldots,r_m)\in\mathbf{Z}_{q_1}\times\mathbf{Z}_{q_2}\times\cdots\times\mathbf{Z}_{q_m}$. Thus each eigenvalue is degenerated by $q_1q_2\cdots q_m$ folds as predicted in (3.28). If a ratio $\nu_i/\nu_j(i\neq j)$ is rational, degeneracy happens more. On the other hand, if $\alpha_{2m+l}=0$, the eigenvalue for $-d_l$ coincides with the one for d_l. If $\alpha_{2m+l}=1/2$, the eigenvalue for $(-d_l+1)$ coincides with the one for d_l. Moreover, for specific values of $\{(L^{-1})^{2m+l}_{2m+k}\}$ we may meet more multifold degeneracy.

Let us discuss physical meanings of the eigenvalue (7.12). It is energy of an electrically charged particle moving in the magnetic field in the torus. In (7.12) we set the coefficient $-1/2$ in front of Δ to adjust the equation to the conventional Schrödinger equation. In the context of classical mechanics, the particle exhibits a cyclic motion with the frequency ν_j in the (y^{2j-1},y^{2j})-plane for each $j=1,2,\ldots,m$. And it exhibits a uniform straight motion along the y^{2m+k}-axis. The whole motion is a superposition of those cyclic and straight motions. When we turn to quantum mechanics, energy of the cyclic motion is quantized and results in the so-called Landau level $2\pi\nu_j(n_j+1/2)$. On the other hand, the longitudinal momentum P_{2m+k} associated with the straight motion is quantized to be $2\pi k_{2m+k}=2\pi\Sigma_{l=1}^{n-2m}(d_l-\alpha_{2m+l})(L^{-1})^{2m+l}_{2m+k}=2\pi\Sigma_{l=1}^{n-2m}d_l(L^{-1})^{2m+l}_{2m+k}-\beta_{2m+k}$ with integers $(d_1,d_2,\ldots,d_{n-2m})$ as explained in (5.17). Along the course of the straight motion the particle flies around the torus and picks up the so-called Aharonov–Bohm effect. Then the momentum is shifted by the Aharonov–Bohm parameters $(\beta_{2m+1},\ldots,\beta_n)$. Accordingly, the kinetic energy of the straight motion is also quantized. The total energy is then given as (7.12).

Moreover, let us examine meanings of other Aharonov–Bohm parameters $(\beta_1,\ldots,\beta_{2m})$. These do not affect the energy (7.12) and hence they have a geometric significance rather than a physical significance. To understand their meaning we rewrite the eigenfunction (6.43) as

$$\begin{aligned}\chi_{k,r,d}(\mathbf{y})=c\exp\Bigg[2\pi i\Bigg\{&\sum_{l=1}^{n-2m}\sum_{j=1}^{m}d_l\{(L^{-1})^{2m+l}_{2j}\beta_{2j-1}-(L^{-1})^{2m+l}_{2j-1}\beta_{2j}\}/\nu_j-\frac{1}{2}\sum_{j=1}^{m}\nu_jy^{2j-1}y^{2j}\\&+\sum_{j=1}^{m}\beta_{2j-1}y^{2j-1}\Bigg\}\Bigg]\sum_{\sigma_1,\sigma_2,\ldots,\sigma_m=-\infty}^{\infty}\exp\Bigg[2\pi i\Bigg\{\frac{1}{2}\sum_{i,j=1}^{m}Y_{ij}(q_i\sigma_i+r_i)(q_j\sigma_j+r_j)\\&+\sum_{j,l=1}^{m}(q_j\sigma_j+r_j-\gamma_j)(L^{-1})^{2j}_{2l}\Bigg\{y^{2l}-\beta_{2l-1}/\nu_l-\sum_{i=1}^{m}L^{2l}_{2i-1}(q_i\sigma_i+r_i)/q_i\Bigg\}\\&+\sum_{k,l=1}^{n-2m}d_k(L^{-1})^{2m+k}_{2m+l}\Bigg\{y^{2m+l}-\sum_{i=1}^{m}L^{2m+l}_{2i-1}(q_i\sigma_i+r_i)/q_i\Bigg\}\\&+\sum_{j=1}^{m}k_{2j-1}\Bigg\{y^{2j-1}+\beta_{2j}/\nu_j-\sum_{i=1}^{m}L^{2j-1}_{2i-1}(q_i\sigma_i+r_i-\gamma_i)/q_i\Bigg\}\Bigg\}\Bigg].\end{aligned}\tag{7.13}$$

To get the above expression we used

$$\sum_{i=1}^{m}L^{2j-1}_{2i-1}\gamma_i/q_i=-\sum_{l=1}^{n-2m}d_l(L^{-1})^{2m+l}_{2j}/\nu_j,\tag{7.14}$$

which is easily derived from (6.36). Thus we can see that the parameters $(\beta_1,\ldots,\beta_{2m})$ induce a displacement

$$(y^{2j-1},y^{2j})\to(y^{2j-1}+\beta_{2j}/\nu_j,y^{2j}-\beta_{2j-1}/\nu_j)\tag{7.15}$$

of the profile $|\chi_{k,r,d}(\mathbf{y})|$. This is the geometric significance of the transverse Aharonov–Bohm parameters.

VIII. CONCLUSION

Here we summarize our discussions. As well-known, a $U(1)$ gauge field replaces the partial derivative by the covariant derivative and generates a magnetic field. As a natural extension of the eigenvalue problem of the ordinary Laplacian in the n-torus, we formulated the eigenvalue problem of the magnetic Laplacian. The ordinary Laplacian admits continuous Abelian symmetry and therefore the usual Fourier analysis is applicable. However, the magnetic Laplacian does not admit continuous Abelian symmetry and therefore the usual Fourier analysis is not applicable to it. Hence, we developed an alternative method, which became an extension of the usual Fourier analysis. We identified symmetry structure of the magnetic Laplacian and defined the magnetic translation group (MTG), which is discrete and non-Abelian in general. Moreover, we defined the magnetic algebra by extending the MTG. We proved that the space of functions on which the magnetic Laplacian acts is an irreducible representation space of the magnetic algebra. By diagonalizing the maximal Abelian subalgebra of the magnetic algebra we obtained a complete orthonormal set of functions $\{\chi_{k,r,d}(\mathbf{y})\}$ over the magnetic torus; those functions are labeled by a set of good quantum numbers (k,r,d). It was rather easy to diagonalize the magnetic Laplacian in the $\mathbf{k}$-space representation. Applying a unitary transformation by $\{\chi_{k,r,d}(\mathbf{y})\}$ to the eigenstate of the magnetic Laplacian, we finally obtained the eigenfunction in the $\mathbf{y}$-space representation. The eigenvalues of the magnetic Laplacian were naturally interpreted as sums of energies of cyclic motions in the transverse directions to the magnetic field and energies of linear motions in the longitudinal direction to it.

New results of this paper are the definition and representations of the magnetic algebra, the proof of irreducibility of the space of twisted periodic functions as a representation space of the magnetic algebra, the complete orthogonal set of functions (6.8) which provides a basis of the extended Fourier analysis, the eigenfunctions (7.11) of the magnetic Laplacian in explicit forms, and the eigenvalues (7.12).

Before closing this paper we would like to discuss briefly possible directions for further development. We treated only the Laplace operator in this paper but for application to physics it is more desirable to treat the Schrödinger operator

$$H = -\tfrac{1}{2}\Delta + V, \tag{8.1}$$

which has a potential energy term V. The potential V is a periodic function; in the t-coordinate it satisfies $V(t^1,\ldots,t^i+1,\ldots,t^n)=V(t^1,\ldots,t^i,\ldots,t^n)$ for each i. It acts on the twisted periodic function $f(\mathbf{t})$ by multiplication. To take the potential term into account we may introduce new operators X^k by

$$(X^j f)(\mathbf{t}) = e^{2\pi i\, t^j} f(\mathbf{t}) \quad (j=1,2,\ldots,2m), \tag{8.2}$$

which belong to the same family of operators T^k of (5.9). Then any periodic potential operator can be expanded as

$$\begin{aligned} V(t^1,\ldots,t^n) &= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_n=-\infty}^{\infty} c_\sigma\, e^{2\pi i(\sigma_1 t^1+\cdots+\sigma_n t^n)} \\ &= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_n=-\infty}^{\infty} c_\sigma\, (X^1)^{\sigma_1}\cdots(X^{2m})^{\sigma_{2m}}(T^1)^{\sigma_{2m+1}}\cdots(T^{n-2m})^{\sigma_n}. \end{aligned} \tag{8.3}$$

We can easily calculate commutators of X's with other operators to get an algebra which is an extension of the magnetic algebra. The resulted algebra is isomorphic to the so-called noncommutative torus[21] although we do not yet examine these relation thoroughly.

Another direction for future development is to solve an eigenvalue problem of the Dirac operator in the n-torus in the background magnetic field. We also construct supersymmetric field theory, which have both scalar and spinor fields as its constituents to pursue a new mechanism of supersymmetry breaking.

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APPENDIX A: DISTRIBUTION OF ZEROS

As shown in (4.11) we prove existence of a transformation matrix L that has zeros in the pattern

$$L=\begin{pmatrix} L^{2i-1}_{2p-1} & L^{2i-1}_{2q} & L^{2i-1}_{2m+r} \\ L^{2j}_{2p-1} & L^{2j}_{2q} & L^{2j}_{2m+r} \\ L^{2m+k}_{2p-1} & L^{2m+k}_{2q} & L^{2m+k}_{2m+r} \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \tag{A1}$$

with $i,j,p,q=1,\ldots,m$ and $k,r=1,\ldots,n-2m$.

In $\mathbf{R}^n$ we have an antisymmetric bilinear form B. We say that a vector $\mathbf{u}$ is longitudinal with respect to B if it satisfies

$$B(\mathbf{u},\mathbf{v})=0 \tag{A2}$$

for arbitrary $\mathbf{v}\in\mathbf{R}^n$. We call

$$M^0=\{\mathbf{u}\in\mathbf{R}^n \mid \forall\mathbf{v}\in\mathbf{R}^n,\ B(\mathbf{u},\mathbf{v})=0\} \tag{A3}$$

a longitudinal vector subspace. Let $(t^1,t^2,\ldots,t^n)$ be the coordinate system that expresses B in the standard form

$$B=\frac{1}{2}\sum_{i,j=1}^{n}\phi_{ij}\,\mathrm{d}t^i\wedge\mathrm{d}t^j=\sum_{j=1}^{m}q_j\,\mathrm{d}t^{2j-1}\wedge\mathrm{d}t^{2j} \tag{A4}$$

as in (3.14). Let $(y^1,y^2,\ldots,y^n)$ be another coordinate system that is related to $(t^1,t^2,\ldots,t^n)$ by a linear transformation $y^i=\Sigma_{j=1}^{n}L^i{}_j\,t^j$. The matrix L is not yet specified. Basis vectors generated by these coordinates are related as

$$\frac{\partial}{\partial t^j}=\sum_{i=1}^{n}\frac{\partial y^i}{\partial t^j}\frac{\partial}{\partial y^i}=\sum_{i=1}^{n}L^i{}_j\frac{\partial}{\partial y^i}, \tag{A5}$$

$$\frac{\partial}{\partial y^j}=\sum_{i=1}^{n}\frac{\partial t^i}{\partial y^j}\frac{\partial}{\partial t^i}=\sum_{i=1}^{n}(L^{-1})^i{}_j\frac{\partial}{\partial t^i}. \tag{A6}$$

Define vector subspaces M^- and M^+ of $\mathbf{R}^n$ as

$$M^-=\mathbf{R}\frac{\partial}{\partial t^1}\oplus\mathbf{R}\frac{\partial}{\partial t^3}\oplus\cdots\oplus\mathbf{R}\frac{\partial}{\partial t^{2m-1}}, \tag{A7}$$

$$M^+ = \mathbf{R}\frac{\partial}{\partial t^2} \oplus \mathbf{R}\frac{\partial}{\partial t^4} \oplus \cdots \oplus \mathbf{R}\frac{\partial}{\partial t^{2m}}. \tag{A8}$$

Then $M^- \oplus M^+ \oplus M^0 = \mathbf{R}^n$. Now let us remember that $\mathbf{R}^n$ is equipped with inner product structure. Then the two-form $B: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ can be regarded as an antisymmetric operator $\hat{B}: \mathbf{R}^n \to \mathbf{R}^n$. As a square of a linear operator $\hat{B}^2$ is well-defined and becomes a symmetric operator and therefore is diagonalizable by an orthogonal transformation. $\hat{B}^2$ has nonpositive eigenvalues. The eigenspace W^0 associated with the zero eigenvalue of $\hat{B}^2$ coincides with M^0. Of course,

$$\left\{ \frac{\partial}{\partial t^{2m+1}}, \frac{\partial}{\partial t^{2m+2}}, \ldots, \frac{\partial}{\partial t^n} \right\} \tag{A9}$$

is a basis of $M^0 = W^0$. We take an orthonormal basis

$$\left\{ \frac{\partial}{\partial y^{2m+1}}, \frac{\partial}{\partial y^{2m+2}}, \ldots, \frac{\partial}{\partial y^n} \right\} \tag{A10}$$

of W^0. This implies that

$$L^{2i-1}_{2m+r} = L^{2j}_{2m+r} = (L^{-1})^{2i-1}_{2m+r} = (L^{-1})^{2j}_{2m+r} = 0 \tag{A11}$$

in (A5) and (A6).

Let us step into a difficult part of the proof. Each eigenspace associated with each nonzero eigenvalue of $\hat{B}^2$ can be decomposed into two-dimensional subspaces such that each two-dimensional subspace W_j is irreducible with respect to the action of $\hat{B}$. Thus we get an orthogonal decomposition

$$\mathbf{R}^n = W_1 \perp W_2 \perp \cdots \perp W_m \perp W^0. \tag{A12}$$

Next we define vector subspaces

$$W_j^+ = W_j \cap (M^+ \oplus M^0), \tag{A13}$$

$$W_j^- = W_j \cap (M^+ \oplus M^0)^\perp, \tag{A14}$$

then we can show that both W_j^+ and W_j^- have one dimension. First, note that $\dim W_j^+ + \dim W_j^- = \dim W_j = 2$. If $\dim W_j^+ = 2$, W_j^+ coincides with W_j itself. Since the two-form B is degenerated on $(M^+ \oplus M^0)$, it must be degenerated also on $W_j = W_j^+ \subset (M^+ \oplus M^0)$. This contradicts the fact that W_j is irreducible with respect to $\hat{B}$. On the other hand, if $\dim W_j^- = 2$, W_j^- coincides with W_j itself. Then we can take an arbitrary one-dimensional subspace $M_{-1} \subset W_j = W_j^- \subset (M^+ \oplus M^0)^\perp$. Since W_j is irreducible with respect to $\hat{B}$, B is degenerated on $M_{-1} \oplus M^+ \oplus M^0$. This contradicts the fact that $M^+ \oplus M^0$ is a maximal degenerated subspace of B. Hence we conclude that $\dim W_j^+ = \dim W_j^- = 1$.

We take a normalized vector $\partial/\partial y^{2j}$ of W_j^+. And we take another normalized vector $\partial/\partial y^{2j-1}$ of W_j^- such that $\nu_j = B(\partial/\partial y^{2j-1}, \partial/\partial y^{2j}) > 0$ for each $j = 1, \ldots, m$. Then we obtain a complete orthonormal basis $\{\partial/\partial y^i \mid i = 1,2,\ldots,n\}$ that expresses B in the standard form $B = \Sigma_{j=1}^m \nu_j \, dy^{2j-1} \wedge dy^{2j}$.

Since $\partial/\partial y^{2q} \in W_q^+ \subset M^+ \oplus M^0$, we can say that

$$(L^{-1})^{2i-1}_{2q} = 0 \tag{A15}$$

in (A6). By an elementary argument of linear algebra we can say that

$$L_{2q}^{2i-1}=0. \tag{A16}$$

The proof is over.

APPENDIX B: FOURIER TRANSFORMATION OF THE HERMITE POLYNOMIALS

In our convention the Hermite polynomial is defined as

$$H_n(\xi)=(-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}. \tag{B1}$$

The formula (7.10) can be deduced by a partial integration and a change of variables as

$$\begin{aligned}\int_{-\infty}^{\infty} dk\, e^{2\pi ikz}\cdot e^{\pi k^2/\nu}\left(-\frac{\partial}{\partial k}\right)^n e^{-2\pi k^2/\nu} &= \int_{-\infty}^{\infty} dk\, e^{-2\pi k^2/\nu}\left(\frac{\partial}{\partial k}\right)^n e^{2\pi ikz}\cdot e^{\pi k^2/\nu}\\ &= e^{\pi\nu z^2}\int_{-\infty}^{\infty} dk\, e^{-2\pi k^2/\nu}\left(\frac{\partial}{\partial k}\right)^n e^{\pi(k+i\nu z)^2/\nu}\\ &= e^{\pi\nu z^2}\int_{-\infty}^{\infty} dk\, e^{-2\pi k^2/\nu}\left(-\frac{i}{\nu}\frac{\partial}{\partial z}\right)^n e^{\pi(k+i\nu z)^2/\nu}\\ &= e^{\pi\nu z^2}\left(-\frac{i}{\nu}\frac{\partial}{\partial z}\right)^n \int_{-\infty}^{\infty} dk\, e^{-\pi(k-i\nu z)^2/\nu-2\pi\nu z^2}\\ &= e^{\pi\nu z^2}\left(-\frac{i}{\nu}\frac{\partial}{\partial z}\right)^n \sqrt{\nu}\, e^{-2\pi\nu z^2}.\end{aligned} \tag{B2}$$

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