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On multidimensional inverse scattering for Stark Hamiltonians

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Based on the Enss-Weder ["The geometrical approach to multidimensional inverse scattering," J. Math. Phys. **36**, 3902–3921 (1995)] time-dependent method, we study one of multidimensional inverse scattering problems for Stark Hamiltonians. We first show that when the space dimension is greater than or equal to 2, the high velocity limit of the scattering operator determines uniquely the potential such as $|x|^{-\gamma}$ with $\gamma > 1/2$ which is short range under the Stark effect. This is an improvement of previous results obtained by Nicoleau ["Inverse scattering for Stark Hamiltonians with short-range potentials," Asymptotic Anal. **35**, 349–359 (2003)] and Weder ["Multidimensional inverse scattering in an electric field," J. Funct. Anal. **139**, 441–465 (1996)]. Moreover, we prove that for a given long-range part of the potential under the Stark effect, the high velocity limit of the Dollard-type modified scattering operator determines uniquely the short-range part of the potential.
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I. INTRODUCTION

In this paper, we consider one of the inverse scattering problems for quantum systems in a constant electric field $\mathcal{E} \in \boldsymbol{R}^n$. Throughout this paper, we assume that $n \geqslant 2$. For brevity's sake, we suppose that $\mathcal{E} = e_1 = (1, 0, \ldots, 0)$. The free Stark Hamiltonian under consideration is given by

$$H_0 = -\frac{1}{2}\Delta - x_1, \tag{1.1}$$

acting on $L^2(\boldsymbol{R}^n)$, where x_1 is the first component of $x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n$. It is well known that H_0 is essentially self-adjoint on $\mathcal{S}(\boldsymbol{R}^n)$. The self-adjoint realization of H_0 is also denoted by H_0. We here assume that the potential V is the multiplication operator by $V(x)$, and that $V(x)$ is represented as a sum of parts of *very short range, short range, and long range* under the Stark effect:

$$V(x) = V^{\mathrm{vs}}(x) + V^{\mathrm{s}}(x) + V^{\mathrm{l}}(x), \tag{1.2}$$

where $V^{\mathrm{vs}} \in \mathcal{V}^{\mathrm{vs}}$, $V^{\mathrm{s}} \in \mathcal{V}^{\mathrm{s}}$, and $V^{\mathrm{l}} \in \mathcal{V}^{\mathrm{l}}$. We here assume that $\mathcal{V}^{\mathrm{vs}}$ is the class of real-valued potentials $V^{\mathrm{vs}}(x)$ satisfying that $V^{\mathrm{vs}}(x) = V_1^{\mathrm{vs}}(x) + V_2^{\mathrm{vs}}(x)$ with $V_2^{\mathrm{vs}}(x)$ bounded, $V_1^{\mathrm{vs}}(x)$ $-\Delta/2$-bounded with relative bound less than 1, and $V_1^{\mathrm{vs}}(x)x_1$ $-\Delta/2$-bounded, and that

$$\int_0^\infty \|V^{\mathrm{vs}}(x)(-\Delta+1)^{-1}F(|x| \geqslant R)\|_{\mathcal{B}(L^2)}\mathrm{d}R < \infty, \tag{1.3}$$

where $F(|x| \geqslant R)$ is the characteristic function of $\{x \in \boldsymbol{R}^n \,|\, |x| \geqslant R\}$. Since $V^{\mathrm{vs}}(x)$ is a multiplication operator, the condition (1.3) is equivalent to

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$$\int_0^\infty \|F(|x| \geqslant R)V^{\mathrm{vs}}(x)(-\Delta+1)^{-1}\|_{\mathcal{B}(L^2)}\mathrm{d}R < \infty, \tag{1.4}$$

as is well known (see, e.g., Ref. 13). Moreover, $\mathcal{V}^{\mathrm{s}}$ is the class of real-valued potentials $V^{\mathrm{s}}(x)$ satisfying that $V^{\mathrm{s}}(x) \in C^1(\boldsymbol{R}^n)$ and that

$$|V^{\mathrm{s}}(x)| \leqslant C\langle x\rangle^{-\gamma}, \tag{1.5}$$

$$|\partial^\beta V^{\mathrm{s}}(x)| \leqslant C_\beta \langle x\rangle^{-1-\alpha}, \quad |\beta| = 1, \tag{1.6}$$

with some $1/2 < \alpha \leqslant \gamma \leqslant 1$, where $\langle x\rangle = \sqrt{1+|x|^2}$. Finally, $\mathcal{V}^{\mathrm{l}}$ is the class of real-valued potentials $V^{\mathrm{l}}(x)$ satisfying that $V^{\mathrm{l}}(x) \in C^2(\boldsymbol{R}^n)$ and that

$$|\partial^\beta V^{\mathrm{l}}(x)| \leqslant C_\beta \langle x\rangle^{-\gamma_D-\mu|\beta|}, \quad |\beta| \leqslant 2,$$

with $0 < \gamma_D \leqslant 1/2$ and $1-\gamma_D < \mu \leqslant 1$.

We first consider the case where $V^{\mathrm{l}}=0$. It is known that for $V \in \mathcal{V}^{\mathrm{vs}}+\mathcal{V}^{\mathrm{s}}$, $H=H_0+V$ is self-adjoint, and the wave operators

$$W^\pm = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-itH_0} \tag{1.7}$$

exist (as for two-body direct scattering under the Stark effect, see, e.g., Refs. 3, 8, 6, 10, 12, 14, 15, 17, and 18; as for N-body direct scattering under the Stark effect, see, e.g., Refs. 2 and 9). Then the scattering operator $S=S(V)$ is defined by

$$S = (W^+)^* W^-. \tag{1.8}$$

Then one of the main results of this paper is represented as follows.

Theorem 1.1: *Let $V_1, V_2 \in \mathcal{V}^{\mathrm{vs}}+\mathcal{V}^{\mathrm{s}}$. If $S(V_1)=S(V_2)$, then $V_1=V_2$.*

It is well known that V^{vs} is short range and V^{s} is long range in the absence of the external electric field, and that both V^{vs} and V^{s} are short range in the presence of a constant electric field. Thus this theorem implies that a certain potential that may be long range in the absence of the electric field can be determined by the scattering operator S under the Stark effect. This theorem was first proven by Weder[16] under the conditions $\gamma > 3/4$ and $n \geqslant 2$. Later Nicoleau[11] proved this theorem for real-valued $V \in C^\infty(\boldsymbol{R}^n)$ satisfying

$$|\partial^\beta V(x)| \leqslant C_\beta \langle x\rangle^{-\gamma-|\beta|},$$

with some $\gamma > 1/2$, under the condition $n \geqslant 3$. Our result improves their result. The key for improving the result of Weder[16] in Ref. 11 is to introduce the Dollard-type modifier $\mathrm{e}^{-i\int_0^t V^{\mathrm{s}}(p_\perp s + e_1 s^2/2)\mathrm{d}s}$ due to White,[17] where $p=-i\nabla=(p_1,p_\perp)$. The assumption $n-1 \geqslant 2$ is needed for his method. We will here use the Graf-type (or Zorbas-type) modifier $\mathrm{e}^{-i\int_0^t V^{\mathrm{s}}(vs+e_1 s^2/2)\mathrm{d}s}$ (see Refs. 6 and 19) with a certain constant vector $v \in \boldsymbol{R}^n$ instead of $\mathrm{e}^{-i\int_0^t V^{\mathrm{s}}(p_\perp s+e_1 s^2/2)\mathrm{d}s}$ in order to deal with the case where $n \geqslant 2$ and to relax the smoothness condition on potentials supposed by Nicoleau.[11] Our proof as well as theirs are based on the Enss-Weder time-dependent method.[5]

We next consider the case where $V^{\mathrm{l}} \neq 0$. It is known that for $V \in \mathcal{V}^{\mathrm{vs}}+\mathcal{V}^{\mathrm{s}}+\mathcal{V}^{\mathrm{l}}$, $H=H_0+V$ is self-adjoint, and the Dollard-type modified wave operators due to Jensen-Yajima[10]

$$W_D^\pm = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\mathrm{e}^{-itH_0}M_D(t), \quad M_D(t) = \mathrm{e}^{-i\int_0^t V^{\mathrm{l}}(ps+e_1 s^2/2)\mathrm{d}s} \tag{1.9}$$

as well as the Graf-type modified wave operators

$$W_G^{\pm} = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH}e^{-itH_0}M_G(t), \quad M_G(t) = e^{-i\int_0^t V^{l}(e_1 s^2/2)ds}, \tag{1.10}$$

and the Dollard-type modified wave operators due to White[17]

$$W_W^{\pm} = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH}e^{-itH_0}M_W(t), \quad M_W(t) = e^{-i\int_0^t V^{l}(p_\perp s + e_1 s^2/2)ds} \tag{1.11}$$

exist by virtue of the condition $\gamma_D+\mu>1$ (see Refs. 1 and 17). Then the Dollard-type modified scattering operator $S_D=S_D(V^{l};V^{vs}+V^{s})$ is defined by

$$S_D = (W_D^{+})^{*}W_D^{-}. \tag{1.12}$$

The reason why $W_D^{\pm}$ are used for defining the modified scattering operator in place of $W_G^{\pm}$ and $W_W^{\pm}$ is that $W_D^{\pm}$ match the Enss-Weder time-dependent method as seen below. Then we obtain the following new result.

Theorem 1.2: *Let $V^{l}\in\mathcal{V}^{l}$ be given. Let $V_1,V_2\in\mathcal{V}^{vs}+\mathcal{V}^{s}$. If $S_D(V^{l};V_1)=S_D(V^{l};V_2)$, then $V_1=V_2$. Moreover, any one of the Dollard-type modified scattering operators S_D determines uniquely the total potential $V(x)$.*

The plan of this paper is as follows: In Sec. II, we consider the case where $V^{l}=0$. In Sec. III, we consider the general case. The main purpose of these two sections is deriving the reconstruction formula (see Theorems 2.1 and 3.1) for potentials in the class $\mathcal{V}^{vs}+\mathcal{V}^{s}+\mathcal{V}^{l}$, which is fairly suitable for studying the scattering problems under the Stark effect. We note that Theorems 2.1 and 3.1 imply that the high velocity limit of the (modified) scattering operator determines uniquely the short-range part of the potential.

II. SHORT-RANGE CASE

In this section, we consider the case where $V^{l}=0$. The main purpose of this section is showing the following reconstruction formula, which yields the proof of Theorem 1.1.

Theorem 2.1: *Let $\hat{v}\in\boldsymbol{R}^n$ be given such that $|\hat{v}|=1$ and $|\hat{v}\cdot e_1|<1$. Set $v=|v|\hat{v}$. Let $\eta>0$ be given, and Φ_0, $\Psi_0\in L^2(\boldsymbol{R}^n)$ be such that $\hat{\Phi}_0$, $\hat{\Psi}_0\in C_0^\infty(\boldsymbol{R}^n)$ with $\text{supp}\,\hat{\Phi}_0$, $\text{supp}\,\hat{\Psi}_0\subset\{\xi\in\boldsymbol{R}^n \,|\, |\xi|<\eta\}$. Here $\hat{\Phi}_0$ and $\hat{\Psi}_0$ stand for the Fourier transforms of Φ_0 and Ψ_0, respectively. Set $\Phi_v=e^{ivx}\Phi_0$ and $\Psi_v=e^{ivx}\Psi_0$. Then*

$$\lim_{|v|\to\infty}|v|(i[S,p_j]\Phi_v,\Psi_v) = \int_{-\infty}^{\infty}[(V^{vs}(x+\hat{v}\tau)p_j\Phi_0,\Psi_0) - (V^{vs}(x+\hat{v}\tau)\Phi_0,p_j\Psi_0) + i((\partial_j V^{s})(x+\hat{v}\tau)\Phi_0,\Psi_0)]d\tau \tag{2.1}$$

holds. Here $p_j=-i\partial_j$.

We will make preparations for the proof of Theorem 2.1. Throughout this paper, we need the following proposition due to Enss[4] (see Proposition 2.10 in Ref. 4).

Proposition 2.1: *For any $f\in C_0^\infty(\boldsymbol{R}^n)$ with $\text{supp}\, f\subset\{x\in\boldsymbol{R}^n \,|\, |x|<\eta\}$ for some $\eta>0$, and any $l\in\boldsymbol{N}$, there exists a constant C_l dependent on f only such that*

$$\|F(x\in\mathcal{M}')e^{it\Delta/2}f(p-v)F(x\in\mathcal{M})\| \leq C_l(1+r+|t|)^{-l} \tag{2.2}$$

for $v\in\boldsymbol{R}^n$, $t\in\boldsymbol{R}$, and measurable sets $\mathcal{M}$, $\mathcal{M}'$ with the property that $r=\text{dist}(\mathcal{M}',\mathcal{M}+vt)-\eta|t|\geq 0$. Here $F(x\in\mathcal{M})$ stands for the characteristic function of $\mathcal{M}$.

The following lemma has been proven by Weder.[16]

Lemma 2.1: *Let v and Φ_v be as in Theorem 2.1. Then*

$$\int_{-\infty}^{\infty}\|V^{vs}(x)e^{-itH_0}\Phi_v\|dt = O(|v|^{-1}) \tag{2.3}$$

holds as $|v|\to\infty$ for $V^{vs}\in\mathcal{V}^{vs}$.

Proof: We will sketch the proof for consistency. By virtue of the Avron-Herbst formula[3]

$$\mathrm{e}^{-itH_0} = \mathrm{e}^{-it^3/6}\mathrm{e}^{itx_1}\mathrm{e}^{-ip_1t^2/2}\mathrm{e}^{it\Delta/2} \tag{2.4}$$

and

$$\mathrm{e}^{-ivx}\mathrm{e}^{it\Delta/2}\mathrm{e}^{ivx} = \mathrm{e}^{-iv^2t/2}\mathrm{e}^{-ipvt}\mathrm{e}^{it\Delta/2}, \tag{2.5}$$

one has

$$\|V^{\mathrm{vs}}(x)\mathrm{e}^{-itH_0}\Phi_v\| = \|V^{\mathrm{vs}}(x+vt+e_1t^2/2)\mathrm{e}^{it\Delta/2}\Phi_0\|.$$

Take $f\in C_0^\infty(\boldsymbol{R}^n)$ such that $f\hat{\Phi}_0=\hat{\Phi}_0$ and $\mathrm{supp}\, f\subset\{\xi\in\boldsymbol{R}^n\,|\,|\xi|<\eta\}$. Then one has

$$\|V^{\mathrm{vs}}(x+vt+e_1t^2/2)\mathrm{e}^{it\Delta/2}\Phi_0\| = \|V^{\mathrm{vs}}(x+vt+e_1t^2/2)(-\Delta+1)^{-1}\mathrm{e}^{it\Delta/2}f(p)(-\Delta+1)\Phi_0\| \leqslant I_1+I_2+I_3,$$

with

$$I_1 = \|V^{\mathrm{vs}}(x+vt+e_1t^2/2)(-\Delta+1)^{-1}\|_{\mathcal{B}(L^2)}\|F(|x|\geqslant 3\lambda|v||t|)\mathrm{e}^{it\Delta/2}f(p)F(|x|\leqslant\lambda|v||t|)\|_{\mathcal{B}(L^2)} \times\|(-\Delta+1)\Phi_0\|,$$

$$I_2 = \|V^{\mathrm{vs}}(x+vt+e_1t^2/2)(-\Delta+1)^{-1}\|_{\mathcal{B}(L^2)}\|F(|x|\geqslant 3\lambda|v||t|)\mathrm{e}^{it\Delta/2}f(p)F(|x|>\lambda|v||t|)\langle x\rangle^{-2}\|_{\mathcal{B}(L^2)} \times\|\langle x\rangle^2(-\Delta+1)\Phi_0\|,$$

$$I_3 = \|V^{\mathrm{vs}}(x+vt+e_1t^2/2)(-\Delta+1)^{-1}F(|x|<3\lambda|v||t|)\|_{\mathcal{B}(L^2)}\|\mathrm{e}^{it\Delta/2}f(p)(-\Delta+1)\Phi_0\|,$$

where $\lambda>0$, which is independent of $|v|$, will be determined below. Since

$$\|V^{\mathrm{vs}}(x+vt+e_1t^2/2)(-\Delta+1)^{-1}\|_{\mathcal{B}(L^2)} = \|V^{\mathrm{vs}}(x)(-\Delta+1)^{-1}\|_{\mathcal{B}(L^2)},$$

we have

$$I_1+I_2\leqslant C(1+|v||t|)^{-2},$$

for $\lambda|v|\geqslant\eta$. Here we used Proposition 2.1 for estimating I_1 under the condition $\lambda|v|\geqslant\eta$. As for I_3, we note that

$$\|V^{\mathrm{vs}}(x+vt+e_1t^2/2)(-\Delta+1)^{-1}F(|x|<3\lambda|v||t|)\|_{\mathcal{B}(L^2)} = \|V^{\mathrm{vs}}(x)(-\Delta+1)^{-1}F(|x-vt-e_1t^2/2|<3\lambda|v||t|)\|_{\mathcal{B}(L^2)}.$$

We put $\delta=|\hat{v}e_1|<1$. Then we have

$$\begin{aligned}|vt+e_1t^2/2|^2 &= |v|^2t^2+t^4/4+t^3ve_1\geqslant|v|^2|t|^2+|t|^4/4-\delta|v||t|^3\\ &= |t|^2(|t|^2-4\delta|v||t|+4\delta^2|v|^2)/4+(1-\delta^2)|v|^2|t|^2\geqslant(1-\delta^2)|v|^2|t|^2.\end{aligned} \tag{2.6}$$

If one takes λ as $4\lambda=(1-\delta^2)^{1/2}>0$, one has

$$F(|x-vt-e_1t^2/2|<3\lambda|v||t|) = F(|x-vt-e_1t^2/2|<3\lambda|v||t|)F(|x|\geqslant\lambda|v||t|).$$

Then we have

$$I_3\leqslant\|V^{\mathrm{vs}}(x)(-\Delta+1)^{-1}F(|x|\geqslant\lambda|v||t|)\|_{\mathcal{B}(L^2)}\|(-\Delta+1)\Phi_0\|.$$

Therefore we obtain

$$\int_{-\infty}^{\infty} (I_1 + I_2 + I_3)\mathrm{d}t = O(|v|^{-1})$$

by assumption, which implies the lemma. □

The following lemma is the key in this paper.

Lemma 2.2: *Let v and Φ_v be as in Theorem 2.1. Then*

$$\int_{-\infty}^{\infty} \|(V^s(x) - V^s(vt + e_1 t^2/2))\mathrm{e}^{-itH_0}\Phi_v\|\mathrm{d}t = \begin{cases} O(|v|^{-\alpha}) & \text{if } \alpha < 1 \\ O(|v|^{-(1-\epsilon)}) & \text{if } \alpha = 1 \end{cases} \tag{2.7}$$

holds with any $0 < \epsilon < 1$ as $|v| \to \infty$ for $V^s \in \mathcal{V}^s$.

Proof: As in the proof of Lemma 2.1, take $f \in C_0^\infty(\boldsymbol{R}^n)$ such that $f\hat{\Phi}_0 = \hat{\Phi}_0$ and $\mathrm{supp}\, f \subset \{\xi \in \boldsymbol{R}^n \,|\, |\xi| < \eta\}$. Then one has

$$\|(V^s(x) - V^s(vt + e_1 t^2/2))\mathrm{e}^{-itH_0}\Phi_v\| = \|(V^s(x + vt + e_1 t^2/2) - V^s(vt + e_1 t^2/2))\mathrm{e}^{it\Delta/2} f(p)\Phi_0\|$$

by virtue of Eqs. (2.4) and (2.5). This can be estimated as

$$\|(V^s(x) - V^s(vt + e_1 t^2/2))\mathrm{e}^{-itH_0}\Phi_v\| \leqslant I_1 + I_2 + I_3,$$

with

$$I_1 = 2 \sup_{y \in \boldsymbol{R}^n} |V^s(y)| \|F(|x| \geqslant 3|v|^\rho |t|)\mathrm{e}^{it\Delta/2} f(p) F(|x| \leqslant |v|^\rho |t|)\|_{\mathcal{B}(L^2)} \|\Phi_0\|,$$

$$I_2 = 2 \sup_{y \in \boldsymbol{R}^n} |V^s(y)| \|F(|x| \geqslant 3|v|^\rho |t|)\mathrm{e}^{it\Delta/2} f(p) F(|x| > |v|^\rho |t|)\langle x\rangle^{-2}\|_{\mathcal{B}(L^2)} \|\langle x\rangle^2 \Phi_0\|,$$

$$I_3 = \|(V^s(x + vt + e_1 t^2/2) - V^s(vt + e_1 t^2/2))F(|x| < 3|v|^\rho |t|)\|_{\mathcal{B}(L^2)} \|\mathrm{e}^{it\Delta/2} f(p)\Phi_0\|,$$

where $0 < \rho < 1$, which is independent of $|v|$, will be determined below. By using Proposition 2.1 for estimating I_1 under the condition $|v|^\rho \geqslant \eta$, we have

$$I_1 + I_2 \leqslant C(1 + |v|^\rho |t|)^{-2},$$

for $|v|^\rho \geqslant \eta$, which implies

$$\int_{-\infty}^{\infty} (I_1 + I_2)\mathrm{d}t = O(|v|^{-\rho}). \tag{2.8}$$

We set $\delta = |\hat{v} \cdot e_1| < 1$. If $|x| < 3|v|^\rho |t|$ and $3|v|^{\rho-1} \leqslant (1-\delta)/4$,

$$\begin{aligned} |x + vt + e_1 t^2/2|^2 &= |x + vt|^2 + t^4/4 + t^2(x + vt)e_1 > (1 - 3|v|^{\rho-1})^2 |v|^2 |t|^2 + |t|^4/4 - (\delta + 3|v|^{\rho-1})|v||t|^3 \\ &\geqslant ((3+\delta)/4)^2 |v|^2 |t|^2 + |t|^4/4 - (1 + 3\delta)|v||t|^3/4. \end{aligned}$$

Since $|t|^4/4 - (1+3\delta)|v||t|^3/4 + ((1+3\delta)/4)^2 |v|^2 |t|^2 \geqslant 0$, we have

$$|x + vt + e_1 t^2/2|^2 \geqslant \{((3+\delta)/4)^2 - ((1+3\delta)/4)^2\}|v|^2|t|^2 = (1-\delta^2)|v|^2|t|^2/2.$$

We here set $c_1 = ((1-\delta^2)/2)^{1/2}$. Moreover, since $((3+\delta)/4)^2|v|^2|t|^2 - (1+3\delta)|v||t|^3/4 + ((1+3\delta)/(3+\delta))^2|t|^4/4 \geqslant 0$, we have

$$|x + vt + e_1 t^2/2|^2 \geqslant \{1 - ((1+3\delta)/(3+\delta))^2\}|t|^4/4 = 2(1-\delta^2)|t|^4/(3+\delta)^2.$$

We here set $c_2 = (2(1-\delta^2))^{1/2}/(3+\delta)$. Thus we obtain

$$|x+vt+e_1t^2/2| \geqslant \max\{c_1|v||t|,c_2|t|^2\}, \tag{2.9}$$

if $|x|<3|v|^{\rho}|t|$ and $3|v|^{\rho-1}\leqslant(1-\delta)/4$. Now we introduce $V_{v,t}(x)$ as

$$V_{v,t}(x)=V^s(x)g(x/(|v||t|)),$$

where $g\in C^\infty(\boldsymbol{R}^n)$ such that

$$g(x)=\begin{cases}1 & \text{if } |x|\geqslant c_1\\ 0 & \text{if } |x|\leqslant c_1/2.\end{cases}$$

Then I_3 is estimated as

$$I_3\leqslant \|(V_{v,t}(x+vt+e_1t^2/2)-V_{v,t}(vt+e_1t^2/2))F(|x|<3|v|^{\rho}|t|)\|_{\mathcal{B}(L^2)}\|\Phi_0\|$$

if $3|v|^{\rho-1}\leqslant(1-\delta)/4$. By using

$$V_{v,t}(x+vt+e_1t^2/2)-V_{v,t}(vt+e_1t^2/2)=\int_0^1(\nabla V_{v,t})(\theta x+vt+e_1t^2/2)x\mathrm{d}\theta,$$

we estimate I_3 as

$$I_3\leqslant 3\|\Phi_0\||v|^{\rho}|t|\int_0^1\|(\nabla V^s)(\theta x+vt+e_1t^2/2)F(|x|<3|v|^{\rho}|t|)\|_{\mathcal{B}(L^2)}\mathrm{d}\theta+C\|\Phi_0\||v|^{\rho-1}$$
$$\times\int_0^1\|V^s(\theta x+vt+e_1t^2/2)F(|x|<3|v|^{\rho}|t|)\|_{\mathcal{B}(L^2)}\mathrm{d}\theta.$$

By virtue of Eq. (2.9), we have

$$\int_{|t|\geqslant c_1|v|/c_2}I_3\mathrm{d}t\leqslant C'\left(|v|^{\rho}\int_{|t|\geqslant c_1|v|/c_2}|t|^{-1-2\alpha}\mathrm{d}t+|v|^{\rho-1}\int_{|t|\geqslant c_1|v|/c_2}|t|^{-2\gamma}\mathrm{d}t\right)\leqslant O(|v|^{\rho-2\alpha})+O(|v|^{\rho-2\gamma})$$
$$=O(|v|^{\rho-2\alpha})$$

because $|\theta x|\leqslant|x|$ for $0\leqslant\theta\leqslant1$. We here used that $1/2<\alpha\leqslant\gamma$. On the other hand,

$$\int_{|t|<c_1|v|/c_2}I_3\mathrm{d}t\leqslant C|v|^{\rho}\int_{|t|<c_1|v|/c_2}|t|(1+|v||t|)^{-1-\alpha}\mathrm{d}t+C|v|^{\rho-1}\int_{|t|<c_1|v|/c_2}(1+|v||t|)^{-\gamma}\mathrm{d}t$$
$$\leqslant 2C|v|^{\rho-1}\int_{|t|<c_1|v|/c_2}(1+|v||t|)^{-\alpha}\mathrm{d}t\leqslant 2C|v|^{\rho-2}\int_{|\tau|<c_1|v|^2/c_2}(1+|\tau|)^{-\alpha}\mathrm{d}\tau$$
$$\leqslant\begin{cases}O(|v|^{\rho-2\alpha}) & \text{if } \alpha<1\\ O(|v|^{\rho-2}\log|v|) & \text{if } \alpha=1\end{cases}$$

is obtained. Therefore we have

$$\int_{-\infty}^{\infty}I_3\mathrm{d}t\leqslant\begin{cases}O(|v|^{\rho-2\alpha}) & \text{if } \alpha<1\\ O(|v|^{\rho-2}\log|v|) & \text{if } \alpha=1.\end{cases} \tag{2.10}$$

Noticing Eqs. (2.8) and (2.10), and taking ρ as

$$\rho=\begin{cases}\alpha & \text{if } \alpha<1\\ 1-\epsilon & \text{if } \alpha=1,\end{cases}$$

with $0<\epsilon<1$, we obtain the lemma. □

We now introduce auxiliary wave operators,

$$\Omega^{\pm}_{G,v} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} U_{G,v}(t), \quad U_{G,v}(t) = e^{-itH_0} e^{-i\int_0^t V^{s}(vs+e_1 s^2/2)ds}.$$

We here emphasize that the Graf-type modifier $e^{-i\int_0^t V^{s}(vs+e_1 s^2/2)ds}$ commutes with any operators. This fact will be used frequently. Then we have the following.

Lemma 2.3: *Let v and Φ_v be as in Theorem 2.1. Then*

$$\sup_{t\in\mathbf{R}} \|(e^{-itH}\Omega^{-}_{G,v} - U_{G,v}(t))\Phi_v\| = \begin{cases} O(|v|^{-\alpha}) & \text{if } \alpha<1 \\ O(|v|^{-(1-\epsilon)}) & \text{if } \alpha=1 \end{cases} \tag{2.11}$$

holds with any $0<\epsilon<1$ as $|v|\to\infty$.

Proof: Noting that $\|(e^{-itH}\Omega^{-}_{G,v} - U_{G,v}(t))\Phi_v\| = \|(\Omega^{-}_{G,v} - e^{itH}U_{G,v}(t))\Phi_v\|$,

$$\|(\Omega^{-}_{G,v} - e^{itH}U_{G,v}(t))\Phi_v\| \leq \int_{-\infty}^{t} \|V^{vs}(x)U_{G,v}(\tau)\Phi_v\| d\tau + \int_{-\infty}^{t} \|(V^{s}(x) - V^{s}(v\tau + e_1\tau^2/2))U_{G,v}(\tau)\Phi_v\| d\tau$$

$$\leq \int_{-\infty}^{\infty} \|V^{vs}(x)e^{-i\tau H_0}\Phi_v\| d\tau + \int_{-\infty}^{\infty} \|(V^{s}(x) - V^{s}(v\tau + e_1\tau^2/2))e^{-i\tau H_0}\Phi_v\| d\tau$$

yields the lemma by virtue of Lemmas 2.1 and 2.2. We here used the commutativity of $e^{-i\int_0^t V^{s}(vs+e_1 s^2/2)ds}$ mentioned above. □

Proof of Theorem 2.1: Since the proof is quite similar to the one of Theorem 2.4 in Ref. 16, we sketch it.

We first note that S is represented as

$$S = (W^+)^* W^- = I_{G,v}(\Omega^{+}_{G,v})^* \Omega^{-}_{G,v}, \quad I_{G,v} = e^{-i\int_{-\infty}^{\infty} V^{s}(v\theta + e_1\theta^2/2)d\theta}.$$

Since $[S,p_j]=[S-I_{G,v}, p_j - v_j]$, $(p_j - v_j)\Phi_v = (p_j\Phi_0)_v$, and

$$i(S - I_{G,v})\Phi_v = I_{G,v} i(\Omega^{+}_{G,v} - \Omega^{-}_{G,v})^* \Omega^{-}_{G,v}\Phi_v = I_{G,v}\int_{-\infty}^{\infty} U_{G,v}(t)^* V_t(x) e^{-itH}\Omega^{-}_{G,v}\Phi_v dt,$$

with $V_t(x) = V^{vs}(x) + V^{s}(x) - V^{s}(vt + e_1 t^2/2)$, we have

$$|v|(i[S,p_j]\Phi_v, \Psi_v) = I_{G,v}\{I(v) + R(v)\}$$

with

$$I(v) = |v|\int_{-\infty}^{\infty} [(V_t(x)U_{G,v}(t)(p_j\Phi_0)_v, U_{G,v}(t)\Psi_v) - (V_t(x)U_{G,v}(t)\Phi_v, U_{G,v}(t)(p_j\Psi_0)_v)]dt,$$

$$R(v) = |v|\int_{-\infty}^{\infty} [((e^{-itH}\Omega^{-}_{G,v} - U_{G,v}(t))(p_j\Phi_0)_v, V_t(x)U_{G,v}(t)\Psi_v) - ((e^{-itH}\Omega^{-}_{G,v} - U_{G,v}(t))\Phi_v, V_t(x)U_{G,v}(t)(p_j\Psi_0)_v)]dt.$$

By Lemmas 2.1, 2.2, and 2.3, one has

$$|R(v)| = \begin{cases} O(|v|^{1-2\alpha}) & \text{if } \alpha<1 \\ O(|v|^{-1+2\epsilon}) & \text{if } \alpha=1. \end{cases}$$

We here used the commutativity of $e^{-i\int_0^t V^{s}(vs+e_1 s^2/2)ds}$. Since $\alpha>1/2$ by assumption and one can take ε as $0<\varepsilon<1/2$,

$$\lim_{|v|\to\infty} R(v) = 0$$

holds. Using the commutativity of $e^{-i\int_0^t V^s(vs+e_1s^2/2)\,ds}$, the Avron-Herbst formula (2.4) and (2.5), $I(v)$ is rewritten as

$$I(v) = |v|\int_{-\infty}^{\infty} [(V_t(x+vt+e_1t^2/2)e^{it\Delta/2}p_j\Phi_0, e^{it\Delta/2}\Psi_0) - (V_t(x+vt+e_1t^2/2)e^{it\Delta/2}\Phi_0, e^{it\Delta/2}p_j\Psi_0)]dt.$$

Since

$$(\{V^s(x+vt+e_1t^2/2) - V^s(vt+e_1t^2/2)\}e^{it\Delta/2}p_j\Phi_0, e^{it\Delta/2}\Psi_0) - (\{V^s(x+vt+e_1t^2/2) - V^s(vt+e_1t^2/2)\}e^{it\Delta/2}\Phi_0, e^{it\Delta/2}p_j\Psi_0) = i((\partial_j V^s)(x+vt+e_1t^2/2)e^{it\Delta/2}\Phi_0, e^{it\Delta/2}\Psi_0)$$

and $\partial_j V^s \in \mathcal{V}^{vs}$ by assumption, $I(v)$ is rewritten as

$$I(v) = |v|\int_{-\infty}^{\infty} [(V^{vs}(x+vt+e_1t^2/2)e^{it\Delta/2}p_j\Phi_0, e^{it\Delta/2}\Psi_0) - (V^{vs}(x+vt+e_1t^2/2)e^{it\Delta/2}\Phi_0, e^{it\Delta/2}p_j\Psi_0) + i((\partial_j V^s)(x+vt+e_1t^2/2)e^{it\Delta/2}\Phi_0, e^{it\Delta/2}\Psi_0)]dt = \int_{-\infty}^{\infty} l_v(\tau)d\tau$$

with

$$l_v(\tau) = (V^{vs}(x+\hat{v}\tau+e_1(\tau/|v|)^2/2)e^{i(\tau/|v|)\Delta/2}p_j\Phi_0, e^{i(\tau/|v|)\Delta/2}\Psi_0) - (V^{vs}(x+\hat{v}\tau + e_1(\tau/|v|)^2/2)e^{i(\tau/|v|)\Delta/2}\Phi_0, e^{i(\tau/|v|)\Delta/2}p_j\Psi_0) + i((\partial_j V^s)(x+\hat{v}\tau + e_1(\tau/|v|)^2/2)e^{i(\tau/|v|)\Delta/2}\Phi_0, e^{i(\tau/|v|)\Delta/2}\Psi_0).$$

Since

$$|l_v(\tau)| \leq C\{\|V^{vs}(x)(-\Delta+1)^{-1}F(|x| \geq \lambda|\tau|)\|_{\mathcal{B}(L^2)} + (1+|\tau|)^{-2} + (1+|\tau|)^{-1-\alpha}\}$$

is obtained as in the proof of Lemma 2.1, we see that

$$\lim_{|v|\to\infty} I(v) = \int_{-\infty}^{\infty} [(V^{vs}(x+\hat{v}\tau)p_j\Phi_0, \Psi_0) - (V^{vs}(x+\hat{v}\tau)\Phi_0, p_j\Psi_0) + i((\partial_j V^s)(x+\hat{v}\tau)\Phi_0, \Psi_0)]d\tau$$

by the Lebesgue dominated convergence theorem. Since

$$|vt+e_1t^2/2|^2 = |v|^2t^2 + t^4/4 + t^3ve_1 \geq |v|^2|t|^2 + |t|^4/4 - \delta|v||t|^3 = (|v|^2|t|^2 - \delta|v||t|^3 + \delta^2|t|^4/4) + (1-\delta^2)|t|^4/4 \geq (1-\delta^2)|t|^4/4,$$

as well as Eq. (2.6) holds, we see that $\lim_{|v|\to\infty} I_{G,v} = 1$ by the Lebesgue dominated convergence theorem because of $\gamma > 1/2$. These imply the theorem. □

By virtue of Theorem 2.1 and the Plancherel formula associated with the Radon transform (see Ref. 7), Theorem 1.1 can be shown in the same way as in the proof of Theorem 1.2 of Ref. 16 (see also Ref. 5). Thus we omit the proof of Theorem 1.1.

III. LONG-RANGE CASE

The main purpose of this section is showing the following reconstruction formula, which yields the proof of Theorem 1.2.

Theorem 3.1: *Let* $\hat{v} \in \boldsymbol{R}^n$ *be given such that* $|\hat{v}| = 1$ *and* $|\hat{v}\cdot e_1| < 1$. *Set* $v = |v|\hat{v}$. *Let* $\eta > 0$ *be given, and* $\Phi_0, \Psi_0 \in L^2(\boldsymbol{R}^n)$ *be such that* $\hat{\Phi}_0, \hat{\Psi}_0 \in C_0^\infty(\boldsymbol{R}^n)$ *with* supp $\hat{\Phi}_0$, supp $\hat{\Psi}_0 \subset \{\xi \in \boldsymbol{R}^n \mid |\xi| < \eta\}$. *Set* $\Phi_v = e^{ivx}\Phi_0$ *and* $\Psi_v = e^{ivx}\Psi_0$. *Then*

$$\lim_{|v|\to\infty}|v|(i[S_D,p_j]\Phi_v,\Psi_v)=\int_{-\infty}^{\infty}[(V^{\mathrm{vs}}(x+\hat{v}\tau)p_j\Phi_0,\Psi_0)-(V^{\mathrm{vs}}(x+\hat{v}\tau)\Phi_0,p_j\Psi_0)+i((\partial_jV^{\mathrm{s}})(x+\hat{v}\tau)\Phi_0,\Psi_0)+i((\partial_jV^{\mathrm{l}})(x+\hat{v}\tau)\Phi_0,\Psi_0)]\mathrm{d}\tau \tag{3.1}$$

holds.

We first need the following lemma.

Lemma 3.1: *Let v and Φ_v be as in Theorem 3.1. Then*

$$\sup_{t\in\boldsymbol{R}}\|\langle x\rangle^2 M_{D,v}(t)\Phi_0\|=O(1) \tag{3.2}$$

holds as $|v|\to\infty$, where $M_{D,v}(t)=\mathrm{e}^{-i\int_0^t V^{\mathrm{l}}(ps+vs+e_1s^2/2)\mathrm{d}s}$.

Proof: As in the proof of Lemma 2.1, take $f\in C_0^\infty(\boldsymbol{R}^n)$ such that $f\hat{\Phi}_0=\hat{\Phi}_0$ and $\mathrm{supp}\, f\subset\{\xi\in\boldsymbol{R}^n\,|\,|\xi|<\eta\}$. Since $x=i\partial_p$, one has

$$\begin{aligned}\||x|^2M_{D,v}(t)f(p)\langle x\rangle^{-2}\|_{\mathcal{B}(L^2)} &\le \|M_{D,v}(t)f(p)|x|^2\langle x\rangle^{-2}\|_{\mathcal{B}(L^2)}+2\left\|M_{D,v}(t)\left(\int_0^t s(\nabla V^{\mathrm{l}})(ps+vs+e_1s^2/2)\mathrm{d}s\right)f(p)x\langle x\rangle^{-2}\right\|_{\mathcal{B}(L^2)}+2\|M_{D,v}(t)(\nabla f)(p)x\langle x\rangle^{-2}\|_{\mathcal{B}(L^2)}\\ &\quad+2\left\|M_{D,v}(t)\left(\int_0^t s(\nabla V^{\mathrm{l}})(ps+vs+e_1s^2/2)ds\right)(\nabla f)(p)\langle x\rangle^{-2}\right\|_{\mathcal{B}(L^2)}\\ &\quad+\|M_{D,v}(t)(\Delta f)(p)\langle x\rangle^{-2}\|_{\mathcal{B}(L^2)}+\left\|M_{D,v}(t)\left(\int_0^t s(\nabla V^{\mathrm{l}})(ps+vs+e_1s^2/2)ds\right)^2 f(p)\langle x\rangle^{-2}\right\|_{\mathcal{B}(L^2)}\\ &\quad+\left\|M_{D,v}(t)\left(\int_0^t s^2(\Delta V^{\mathrm{l}})(ps+vs+e_1s^2/2)\mathrm{d}s\right)f(p)\langle x\rangle^{-2}\right\|_{\mathcal{B}(L^2)}.\end{aligned}$$

Set $\delta=|\hat{v}\cdot e_1|<1$. If $|\xi|\le\eta$ and $\eta/|v|\le(1-\delta)/4$, then

$$\begin{aligned}|\xi t+vt+e_1t^2/2|^2&=|\xi+v|^2t^2+t^4/4+t^3(\xi+v)e_1>(1-\eta/|v|)^2|v|^2|t|^2+|t|^4/4-(\delta+\eta/|v|)|v||t|^3\\&\ge((3+\delta)/4)^2|v|^2|t|^2+|t|^4/4-(1+3\delta)|v||t|^3/4.\end{aligned} \tag{3.3}$$

Since $((3+\delta)/4)^2|v|^2|t|^2-(1+3\delta)|v\|t|^3/4+((1+3\delta)/(3+\delta))^2|t|^4/4\ge0$, we have

$$|\xi t+vt+e_1t^2/2|^2\ge 2(1-\delta^2)|t|^4/(3+\delta)^2=(c_2|t|^2)^2. \tag{3.4}$$

This implies the lemma because $1-2(\gamma_D+\mu)<-1$ and $2-2(\gamma_D+2\mu)=2-4(\gamma_D+\mu)+2\gamma_D<-1$ by assumption. □

Lemma 3.2: *Let v* and *Φ_v be as in Theorem 3.1. Then*

$$\int_{-\infty}^{\infty}\|V^{\mathrm{vs}}(x)U_D(t)\Phi_v\|\mathrm{d}t=O(|v|^{-1}) \tag{3.5}$$

holds as $|v|\to\infty$ for $V^{\mathrm{vs}}\in\mathcal{V}^{\mathrm{vs}}$.

Proof: By virtue of the Avron-Herbst formula (2.4) and (2.5), one has

$$\|V^{\mathrm{vs}}(x)U_D(t)\Phi_v\| = \|V^{\mathrm{vs}}(x+vt+e_1t^2/2)\mathrm{e}^{it\Delta/2}M_{D,v}(t)\Phi_0\|.$$

Then the lemma can be proven in the same way as in the proof of Lemma 2.1, by virtue of Lemma 3.1. □

The following lemma can be also proven in the same way as in the proof of Lemma 2.2. Thus we omit the proof.

Lemma 3.3: *Let v and Φ_v be as in Theorem 3.1. Then*

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x)-V^{\mathrm{s}}(vt+e_1t^2/2))U_D(t)\Phi_v\|\mathrm{d}t = \begin{cases} O(|v|^{-\alpha}) & \text{if } \alpha<1 \\ O(|v|^{-(1-\epsilon)}) & \text{if } \alpha=1 \end{cases} \tag{3.6}$$

holds with any $0<\epsilon<1$ as $|v|\to\infty$ for $V^{\mathrm{s}}\in\mathcal{V}^{\mathrm{s}}$.

The following lemma is the key in this section.

Lemma 3.4: *Let v and Φ_v be as in Theorem 3.1. Then*

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{l}}(x)-V^{\mathrm{l}}(pt-e_1t^2/2))U_D(t)\Phi_v\|dt = O(|v|^{-\rho}) \tag{3.7}$$

holds with some $1/2<\rho<1$ as $|v|\to\infty$ for $V^{\mathrm{l}}\in\mathcal{V}^{\mathrm{l}}$.

Proof: Before showing the lemma, we note that

$$\mathrm{e}^{-itH_0}V^{\mathrm{l}}(pt+e_1t^2/2) = V^{\mathrm{l}}(pt-e_1t^2/2)\mathrm{e}^{-itH_0} \tag{3.8}$$

holds by virtue of the Avron-Herbst formula (2.4).

As in the proof of Lemma 2.1, take $f\in C_0^{\infty}(\boldsymbol{R}^n)$ such that $f\hat{\Phi}_0=\hat{\Phi}_0$ and $\mathrm{supp}\, f\subset\{\xi\in\boldsymbol{R}^n \,|\, |\xi|<\eta\}$. By virtue of the Avron-Herbst formula (2.4) and (2.5), one has

$$\|(V^{\mathrm{l}}(x)-V^{\mathrm{l}}(pt-e_1t^2/2))U_D(t)\Phi_v\| = \|(V^{\mathrm{l}}(x+vt+e_1t^2/2)-V^{\mathrm{l}}(pt+vt+e_1t^2/2))\mathrm{e}^{it\Delta/2}f(p)M_{D,v}(t)\Phi_0\|.$$

Set $\delta=|\hat{v}\cdot e_1|<1$. If $|\xi|\le\eta$ and $\eta/|v|\le(1-\delta)/4$, then

$$|\xi t+vt+e_1t^2/2|^2 \ge (1-\delta^2)|v|^2|t|^2/2 = (c_1|v||t|)^2$$

as well as Eq. (3.4) hold by virtue of Eq. (3.3). Thus we obtain

$$|\xi t+vt+e_1t^2/2| \ge (c_1|v||t|)^{\nu}\times(c_2|t|^2)^{1-\nu} = c_1^{\nu}c_2^{1-\nu}|v|^{\nu}|t|^{2-\nu}, \tag{3.9}$$

where $0<\nu<1$, which is independent of $|v|$, will be determined below. Now we introduce $V^{\mathrm{l}}_{v,t}(x)$ as

$$V^{\mathrm{l}}_{v,t}(x) = V^{\mathrm{l}}(x)g_{\nu}(x/(|v|^{\nu}|t|^{2-\nu})), \tag{3.10}$$

where $g_{\nu}\in C^{\infty}(\boldsymbol{R}^n)$ such that

$$g_{\nu}(x) = \begin{cases} 1, & |x|\ge c_1^{\nu}c_2^{1-\nu} \\ 0, & |x|\le c_1^{\nu}c_2^{1-\nu}/2. \end{cases}$$

If $|x|<3|v|^{\rho}|t|$ and $3|v|^{\rho-1}\le(1-\delta)/4$ for $0<\rho<1$, then we also obtain

$$|x+vt+e_1t^2/2| \ge c_1^{\nu}c_2^{1-\nu}|v|^{\nu}|t|^{2-\nu}$$

by virtue of Eq. (2.9). Then we have

$$\begin{aligned} &\|(V^{\mathrm{l}}(x)-V^{\mathrm{l}}(pt-e_1t^2/2))U_D(t)\Phi_v\| \\ &\quad = \|(V^{\mathrm{l}}(x+vt+e_1t^2/2)-V^{\mathrm{l}}(pt+vt+e_1t^2/2))\mathrm{e}^{it\Delta/2}f(p)M_{D,v}(t)\Phi_0\| \\ &\quad = \|(V^{\mathrm{l}}(x+vt+e_1t^2/2)-V^{\mathrm{l}}_{v,t}(pt+vt+e_1t^2/2))\mathrm{e}^{it\Delta/2}f(p)M_{D,v}(t)\Phi_0\| \end{aligned}$$

$$\leqslant \|(V^{l}(x+vt+e_1t^2/2)-V^{l}_{v,t}(x+vt+e_1t^2/2))e^{it\Delta/2}f(p)M_{D,v}(t)\Phi_0\|$$
$$+\|(V^{l}_{v,t}(x+vt+e_1t^2/2)-V^{l}_{v,t}(pt+vt+e_1t^2/2))e^{it\Delta/2}f(p)M_{D,v}(t)\Phi_0\|$$
$$\leqslant I_1+I_2+I_3,$$

with

$$I_1=\sup_{y\in\boldsymbol{R}^n}|V^{l}(y)|\|F(|x|\geqslant 3|v|^{\rho}|t|)e^{it\Delta/2}f(p)F(|x|\leqslant |v|^{\rho}|t|)\|_{\mathcal{B}(L^2)}\|M_{D,v}(t)\Phi_0\|,$$

$$I_2=\sup_{y\in\boldsymbol{R}^n}|V^{l}(y)|\|F(|x|\geqslant 3|v|^{\rho}|t|)e^{it\Delta/2}f(p)F(|x|>|v|^{\rho}|t|)\langle x\rangle^{-2}\|_{\mathcal{B}(L^2)}\|\langle x\rangle^{2}M_{D,v}(t)\Phi_0\|,$$

$$I_3=\|(V^{l}_{v,t}(x+pt+vt+e_1t^2/2)-V^{l}_{v,t}(pt+vt+e_1t^2/2))f(p)M_{D,v}(t)\Phi_0\|,$$

where $0<\rho<1$, which is independent of $|v|$, will be determined below. By virtue of Proposition 2.1 for estimating I_1 under the condition $|v|^{\rho}\geqslant\eta$ and Lemma 3.1, one has

$$I_1+I_2\leqslant C(1+|v|^{\rho}|t|)^{-2}$$

for $|v|^{\rho}\geqslant\eta$, which implies

$$\int_{-\infty}^{\infty}(I_1+I_2)dt=O(|v|^{-\rho}).$$

By the Baker-Campbell-Hausdorff formula,

$$V^{l}_{v,t}(x+pt+vt+e_1t^2/2)-V^{l}_{v,t}(pt+vt+e_1t^2/2)$$
$$=\int_0^1[(\nabla V_{v,t})(\theta x+pt+vt+e_1t^2/2)x+it(\Delta V_{v,t})(\theta x+pt+vt+e_1t^2/2)/2]d\theta,$$

we estimate I_3 as

$$I_3\leqslant C\{(|v|^{\nu}|t|^{2-\nu})^{-(\gamma_D+\mu)}+(|v|^{\nu}|t|^{2-\nu})^{-(\gamma_D+1)}+|v|^{-\nu(\gamma_D+2\mu)}|t|^{-(2-\nu)(\gamma_D+2\mu)+1}+|v|^{-\nu(\gamma_D+\mu+1)}$$
$$\times|t|^{-(2-\nu)(\gamma_D+\mu+1)+1}+|v|^{-\nu(\gamma_D+2)}|t|^{-(2-\nu)(\gamma_D+2)+1}\}.$$

On the other hand, one has

$$I_3\leqslant 2\sup_{y\in\boldsymbol{R}^n}|V^{l}(y)|\|\Phi_0\|.$$

Thus, by a straightforward computation, we obtain

$$\int_{-\infty}^{\infty}I_3dt=O(|v|^{-\nu/(2-\nu)})+O(|v|^{-\nu(\gamma_D+2\mu)/\{(2-\nu)(\gamma_D+2\mu)-1\}})+O(|v|^{-\nu(\gamma_D+\mu+1)/\{(2-\nu)(\gamma_D+\mu+1)-1\}})$$
$$+O(|v|^{-\nu(\gamma_D+2)/\{(2-\nu)(\gamma_D+2)-1\}})$$
$$=O(|v|^{-\nu/(2-\nu)}),$$

if $(2-\nu)(\gamma_D+2\mu)>2$. Since $0<\gamma_D\leqslant 1/2$ and $\gamma_D+\mu>1$, one has $\gamma_D+2\mu>2-\gamma_D\geqslant 3/2$. Take $1/2<\rho<1$ such that $\gamma_D+2\mu>\rho+1$, and make ν as $\nu/(2-\nu)=\rho$. Then $(2-\nu)(\gamma_D+2\mu)>2$ are satisfied by $\rho+1=2/(2-\nu)$, and

$$\int_{-\infty}^{\infty} \|(V^1(x) - V^1(pt - e_1 t^2/2)) U_D(t) \Phi_v\| \mathrm{d}t = O(|v|^{-\rho}) \tag{3.11}$$

holds. □

By virtue of (3.8), Lemmas 3.2, 3.3, and 3.4, the following lemma can be obtained as Lemma 2.3. Thus we omit the proof.

Lemma 3.5: *Let v and Φ_v be as in Theorem 3.1. Let ρ be as in Lemma 3.4. Then*

$$\sup_{t \in \boldsymbol{R}} \|(\mathrm{e}^{-itH} \Omega_D^- - U_D(t)) \Phi_v\| = O(|v|^{-\min\{\alpha,\rho\}}) \tag{3.12}$$

holds as $|v| \to \infty$.

Noting that $1 - 2\min\{\alpha, \rho\} < 0$ by assumption, Theorem 3.1 can be proven in the same way as in the proof of Theorem 2.1. Moreover, Theorem 1.2 can be shown in the same way as in the proof of Theorem 1.2 of Weder[16] (see also Ref. 5). Thus we omit the proofs.

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