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A New Algebraic Approach to Stabilization for Boundary Control Systems of Parabolic Type

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Abstract. We study the stabilization problem of linear parabolic boundary control systems. While the control system is described by a pair of standard linear differential operators $(\mathcal{L}, \tau)$, the corresponding semigroup generator generally admits *no* Riesz basis of eigenvectors. In the sense that very little information on the fractional powers of this generator is needed, our approach has enough generality as a prototype to be used for other types of parabolic systems. We propose in this paper a new algebraic approach to the stabilization, which gives - to the best of the author's knowledge - the simplest framework of the problem. The control system with the scheme of boundary observation/boundary feedback is turned into the differential equations with no boundary input in usual and standard L^2-spaces in a readable manner.

1. Introduction

We consider in this paper the stabilization problem for a class of linear boundary control systems of parabolic type by means of feedback control. Now the problem has a history of two decades (see the literature, e.g., [1, 3, 5, 11 – 14, 16]), and looks somewhat matured. But, there still remain unresolved difficulties and an interest in new viewpoints and frameworks of the problem. The problem is most interesting in the scheme of a finite number of boundary observation and boundary control. Several approaches to the problem have been developed to cope with this scheme. An analytic approach based on an *integral transform* of the state variable is found in [11, 12]. This approach - via the fractional powers of an elliptic operator (see [4, 7]) - is effective in the problems with the Robin boundaries. When enough fractional structure is not known, an algebraic approach is developed in [13] for the problems with the mixed boundaries. When the controlled plant admits no Riesz basis, the algebraic method for stabilization has been further developed in our latest paper [14]. It is also pointed out the limit of the above analytic approach, by showing that it encounters an essential difficulty in well-posedness of control systems with the Dirichlet boundaries. At this point the superiority of the algebraic approach is evident. Another attempt to control systems with unbounded observations and controls is to study differential equations in spaces equipped with weaker topologies than usual. Along this line, the abstract setting of "regular linear systems" (RLS) is introduced in [2, 16] to cope with these unboundedness. In RLS, the original unboundedness is regarded as boundedness in spaces with weaker topologies. In [15], spaces with weak topologies, i.e., the spaces of linear forms (distributions) are introduced in studying optimal control problems. Differential equations are interpreted as the weaker ones in these extended spaces. But this setting cannot solve the above difficulty in studying the feedback stabilization problem with the Dirichlet boundaries. Anyhow the original unboundedness does not dissapear and remains implicitly in these settings.

The purpose of this paper is to establish a stabilization result as in [11 - 14] by introducing an alternative new algebraic method: A specific feature of the paper is that the method gives - to the best of the author's knowledge - the simplest framework among the literature. The method is also readable, since the argument always stays with the *usual and standard* L^2-spaces and is based on differential equations with *no* boundary input.

This point seems important, since the proposed method owns some property in common with finite-dimensional control systems (and thus readable). The coefficient operator of the controlled plant consists of a standard elliptic differential operator $\mathcal{L}$ of 2nd order in a bounded domain $\Omega\,(\subset \mathbb{R}^m)$ and the associated boundary operator of mixed nature, denoted by τ: The operator τ consists partially of the Dirichlet type and partially of the Neumann type. In standard cases, the domain of the operator L, which is derived from the pair $(\mathcal{L},\, \tau)$, is often characterized as $\mathcal{D}(L) = \{u \in H^2(\Omega);\ \tau u = 0 \text{ on } \partial\Omega\}$. In our problem, L is obtained as the closure in $L^2(\Omega)$ of a closable operator $\hat{L}$. Thus $\mathcal{D}(L)$ is unclearer than in the standard cases. For example, we do not know exactly if $(\lambda - L)^{-1} f \in \mathcal{D}(L)$ with $\lambda \in \rho(L)$ and $f \in L^2(\Omega)$ would be an $H^2(\Omega)$-function. To achieve stabilization, another differential equation describing a dynamic compensator is introduced. Since a Riesz basis corresponding to $(\mathcal{L},\, \tau)$ is not generally expected, the compensator of general type is employed in an arbitrary separable Hilbert space. Our new strategy is to carry out first the stabilization argument on the algebraically transformed control system which includes only the distributed feedback terms. These feedback terms reflect the original boundary feedback. As in [11 - 14], the compensator is then reduced to a finite-dimensional one with the stability property unchanged.

Our boundary control system is described by the following system of linear differential equations (see the figure below):

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } \ \mathbb{R}^1_+ \times \Omega, \\ \tau u = \displaystyle\sum_{k=1}^{M} \langle v,\, \rho_k \rangle_{\mathbb{R}^\ell}\, h_k \quad \text{on } \ \mathbb{R}^1_+ \times \Gamma, \\ \dfrac{dv}{dt} + B_1 v = \displaystyle\sum_{k=1}^{N} \langle u,\, w_k \rangle_{\Gamma}\, \xi_k \quad \text{in } \ \mathbb{R}^1_+, \\ u(0,\cdot) = u_0(\cdot) \quad \text{in } \ \Omega, \qquad v(0) = v_0. \end{cases} \tag{1.1}$$

Eqn. (1.1) reveals the control scheme which is finally obtained from (1.4). In (1.1), let us observe how the observation/control scheme is constructed : The controlled plant Σ_p with state $u = u(t,\cdot)$ is characterized by the pair of linear differential operators $(\mathcal{L}, \tau)$ in a bounded domain Ω of $\mathbb{R}^m$ with the boundary Γ which consists of a finite number of smooth components of $(m-1)$-dimension. The *compensator* Σ_c with state $v = v(t)$ is described by the differential equation in $\mathbb{R}^\ell$, the dimension ℓ being suitably chosen.

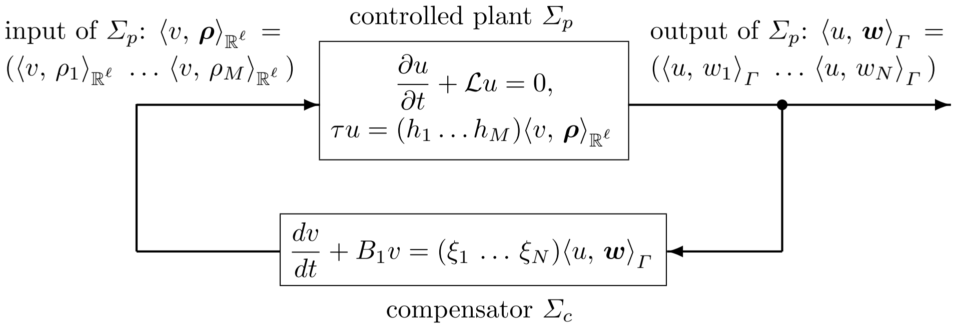
Throughout the paper, the inner products in $L^2(\Omega)$ and $L^2(\Gamma)$ are denoted by $\langle \cdot, \cdot \rangle_\Omega$ and $\langle \cdot, \cdot \rangle_\Gamma$, respectively. Let w_k be in $L^2(\Gamma)$, $1 \leqslant k \leqslant N$. Then the output (observation) of Σ_p is denoted as

$$\langle u, w_k \rangle_\Gamma, \quad 1 \leqslant k \leqslant N, \tag{1.2}$$

which enters Σ_c as the input through the actuators ξ_k. The output of Σ_c is denoted as

$$\langle v, \rho_k \rangle_{\mathbb{R}^\ell}, \quad 1 \leqslant k \leqslant M,$$

which enters Σ_p as the input through the actuators h_k on Γ. Thus (1.1) forms a closed loop system with state $(u(t, \cdot), v(t)) \in L^2(\Omega) \times \mathbb{R}^\ell$. These relationships are shown in the following figure:



We employ a typical but general differential operator for the controlled plant Σ_p. Let us define the pair $(\mathcal{L}, \tau)$ as follows:

$$\begin{aligned} \mathcal{L}u &= -\sum_{i,j=1}^m \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^m b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \\ \tau u &= \alpha(\xi)u + (1-\alpha(\xi))\frac{\partial u}{\partial \nu}, \end{aligned} \tag{1.3}$$

where $a_{ij}(x) = a_{ji}(x)$ for $1 \leqslant i, j \leqslant m$, $x \in \overline{\Omega}$; for some positive δ

$$\sum_{i,j=1}^m a_{ij}(x)\xi_i\xi_j \geqslant \delta|\xi|^2, \quad \forall \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m, \quad \forall x \in \overline{\Omega};$$

and

$$0 \leqslant \alpha(\xi) \leqslant 1 \quad \text{with} \quad \alpha(\xi) \not\equiv 1, \qquad \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^m a_{ij}(\xi)\nu_i(\xi) \left.\frac{\partial u}{\partial x_j}\right|_\Gamma,$$

$\boldsymbol{\nu}(\xi) = (\nu_1(\xi), \dots, \nu_m(\xi))$ being the unit outer normal at $\xi \in \Gamma$. As for the regularity of the coefficients, it is enough to assume that $a_{ij}(\cdot)$, $b_i(\cdot)$, $c(\cdot)$, and $\alpha(\cdot)$ belong to $C^2(\overline{\Omega})$, $C^2(\overline{\Omega})$, $C^\omega(\overline{\Omega})$, and $C^{2+\omega}(\Gamma)$, respectively, where ω, $0 < \omega < 1$ will denote a constant depending on each function. As for the actuators, we assume that h_k belong to $C^{2+\omega}(\Gamma)$, $1 \leqslant k \leqslant M$.

Our task is to determine the parameters in the feedback control scheme of (1.1) for the stabilization. More precisely it is stated as follows:

Given a set of h_k and w_k, determine suitable feedback parameters, that is, the dimension ℓ, the matrix B_1, the vectors ξ_k, and ρ_k, so that the state $u(t,\cdot)$ as well as $v(t)$ in (1.1) *decays exponentially as $t \to \infty$ for every initial state u_0 and v_0.*

Since our main purpose is to establish a new and simple algebraic framework for the boundary control system, let us review the existing approach briefly according to [13, 14]. The basic system of differential equations is described by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ \tau u = \displaystyle\sum_{k=1}^{M} \langle v,\, \rho_k \rangle_H \, h_k \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\ \dfrac{dv}{dt} + Bv = \displaystyle\sum_{k=1}^{N} \langle u,\, w_k \rangle_\Gamma \, \xi_k + \sum_{k=1}^{M'} \langle v,\, \rho_k \rangle_H \, \zeta_k \quad \text{in } \mathbb{R}^1_+ \times H, \\ u(0,\cdot) = u_0(\cdot) \in L^2(\Omega), \qquad v(0) = v_0 \in H. \end{cases} \tag{1.4}$$

In (1.4) the differential equation with state $v(t)$ characterizes the compensator Σ_c in a separable Hilbert space H, which is finally reduced to the one in $\mathbb{R}^\ell$; B denotes a linear closed operator in H with dense domain; and $M < M'$. The ξ_k, ρ_k, and ζ_k as well as B are the parameters specified later. The stabilization problem is first studied for the system (1.4) with state $(u(t,\cdot), v(t))$, and then reduced to the one for (1.1) with state $(u(t,\cdot), v_1(t))$. Roughly speaking, the matrix B_1 in (1.1) is derived from the B, ρ_k, and ζ_k in (1.4); the ξ_k in (1.1) from the ξ_k in (1.4); and the ρ_k in (1.1) from the ρ_k, $1 \leqslant k \leqslant M$ in (1.4). We will see later in Section 3 that (1.4) is obtained algebraically by (3.22) – (3.24).

The role of the compensator Σ_c is that the state $v(t)$ approximates the state $u(t)$ as $t \to \infty$ in an appropriate topology. To see this briefly, set $Lu = \mathcal{L}u$ for u with the boundary condition $\tau u = 0$ (the precise definition of L is stated in Section 2). Let $X \in \mathcal{L}(L^2(\Omega); H)$ be the unique solution to the operator equation: $XL - BX = C$, where the operator C is defined by $C = -\sum_{k=1}^{N} \langle \cdot,\, w_k \rangle_\Gamma \, \xi_k$. In [14], Σ_c is designed so that $\| Xu(t) - v(t) \|_H \to 0$ as $t \to \infty$ with a particular property of X (Proposition 3.3). In this scheme, we note that $u(t)$ does not belong to $\mathcal{D}(L)$, while $v(t)$ belongs to $\mathcal{D}(B)$. In the case where L admits a Riesz basis, the so called identity compensator is employed in a more constructive manner: In [13] we set $H = L^2(\Omega)$, $X = 1$, and thus $B = L - C$, $\mathcal{D}(B) = \mathcal{D}(L)$. Given a large constant $c > 0$, let $\varphi_k \in H^2(\Omega)$, $1 \leqslant k \leqslant M$, denote the unique solutions to the boundary value problems: $(c + \mathcal{L})\varphi_k = 0$ in Ω, $\tau\varphi_k = h_k$ on Γ. The solutions φ_k are denoted by $\varphi_k = N_{-c}h_k$ (see (2.12), Section 2). As long as c is large enough, the operator $S_{-c}u = u - \sum_{k=1}^{M} \langle u,\, \rho_k\rangle_\Omega\, N_{-c}h_k$ determines a bounded bijection from $L^2(\Omega)$ onto itself. The compensator Σ_c is then designed in two different manners, so that one of the following estimates holds:

(i) $\|u(t) - S_{-c}^{-1}v(t)\| \to 0, \quad t \to \infty,$

(ii) $\|S_{-c}u(t) - v(t)\| \to 0, \quad t \to \infty.$

Although the above (i) and (ii) mean the same, the basic systems of differential equations satisfy different boundary conditions in each scheme. In (i), the control system with state $(u(t),\, \tilde{v}(t))$ is first studied, where $\tilde{v}(t) = S_{-c}^{-1}v(t)$ satisfies the feedback boundary condition: $\tau\tilde{v}(t) = \sum_{k=1}^{M} \langle \tilde{v}(t),\, \rho_k\rangle_\Omega\, h_k$. In this scheme, both $u(t)$ and $\tilde{v}(t)$ do not belong to $\mathcal{D}(L)$. When the stabilization of $(u(t), \tilde{v}(t))$ is achieved, Σ_c is finally transformed into another equation with state $v(t) = S_{-c}\tilde{v}(t)$. In (ii), the control system has state $(\tilde{u}(t), v(t))$, where $v(t)$ belongs to $\mathcal{D}(L)$, but $\tilde{u}(t) = S_{-c}u(t)$ does not. Instead, $\tilde{u}(t) - v(t)$ satisfies the feedback boundary condition: $\tau(\tilde{u} - v) = -\sum_{k=1}^{M} \langle \tilde{u} - v,\, \rho_k\rangle_\Omega\, h_k$. Thus the study of systems of differential equations with more complicated boundary conditions is required.

On the other hand, our approach in this paper is much simpler: It is, subsequently to the above approaches, the *fourth* algebraic approach, and gives - to the best of the author's knowledge - the simplest and clearest framework among the literature. Our idea is to introduce a distributed feedback law, regardless of the complexity of the boundary condition. Of course this feedback law reflects the original boundary feedback law. Then we study the system of differential equations with state $(q(t), v(t)) \in L^2(\Omega) \times H$ such that $(q(t),\, v(t))$ belongs to $\mathcal{D}(L) \times \mathcal{D}(B)$. Thus the standard argument of the semigroup theory is applied to the system in a more readable manner without complicated arguments on boundary inputs. At the same time it is not necessary to extend the differential equation for $u(t)$ to the more abstract equation in a space of linear forms including $L^2(\Omega)$. Thus the equation for $u(t)$ always stays in $L^2(\Omega)$ in our approach. The compensator Σ_c is constructed so that $\|Xq(t) - v(t)\|_H \to 0$ as $t \to \infty$: It turns out that this estimate approximately creates the desirable output $\langle v(t), \rho_k \rangle_H$ of Σ_c.

The basic regularity problem as well as some preliminary results are discussed within the framework of both the L^2- and the classical theories in Section 2. Based on the well known observability and controllability conditions, the main result on stabilization is stated and proven in Section 3, where the new system of differential equations with no boundary input is introduced in the framework of the L^2-theory.

2. Preliminary results

We begin the section by characterizing the coefficient operators L and then B which appeared in (1.4). Set

$$\hat{L}u = \mathcal{L}u, \quad \mathcal{D}(\hat{L}) = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega});\ \mathcal{L}u \in L^2(\Omega),\ \tau u = 0\}. \tag{2.1}$$

The closure of $\hat{L}$ in $L^2(\Omega)$ is denoted by L. More precisely the domain $\mathcal{D}(L)$ consists of functions $u \in L^2(\Omega)$ with the property that there is a sequence $\{u_n\} \subset \mathcal{D}(\hat{L})$ such that u_n converges to u and $\hat{L}u_n$ converges in $L^2(\Omega)$ as $n \to \infty$. Then Lu is defined as the limit of $\hat{L}u_n$. It is well known (see [8]) that L has a compact resolvent $(\lambda - L)^{-1}$; that the spectrum $\sigma(L)$ lies in the complement $(\overline{\Sigma} - b)^c$ of some sector $\overline{\Sigma} - b$, where $\overline{\Sigma} = \{\lambda \in \mathbb{C};\ \theta_0 \leqslant |\arg \lambda| \leqslant \pi\}$, $0 < \theta_0 < \pi/2$, $b \in \mathbb{R}^1$; and that the estimates

$$\begin{aligned} &\|(\lambda - L)^{-1}\| \leqslant \frac{\text{const}}{1 + |\lambda|}, \quad \text{and} \\ &\|(\lambda - L)^{-1}\|_{\mathcal{L}(L^2(\Omega);\, H^1(\Omega))} \leqslant \frac{\text{const}}{1 + |\lambda|^{1/2}}, \qquad \lambda \in \overline{\Sigma} - b \end{aligned} \tag{2.2}$$

hold, where the norm $\|\cdot\|$ denotes the $L^2(\Omega)$- or the $\mathcal{L}(L^2(\Omega);\ L^2(\Omega))$-norm. The latter estimate is derived from the relation (2.5) below. There is a set of *generalized* eigenpairs $\{\lambda_i,\ \varphi_{ij}\}$ such that (see [6])

(i) $\sigma(L) = \{\lambda_1, \lambda_2, \dots, \lambda_i, \dots\}$, $\quad \mathrm{Re}\,\lambda_1 \leqslant \mathrm{Re}\,\lambda_2 \leqslant \cdots \leqslant \mathrm{Re}\,\lambda_i \leqslant \cdots \to \infty$; and

(ii) $L\varphi_{ij} = \lambda_i \varphi_{ij} + \sum_{k<j} \alpha^i_{jk} \varphi_{ik}$, $\quad i \geqslant 1,\ 1 \leqslant j \leqslant m_i\ (< \infty)$.

In our general boundary condition, the elliptic theory for L owes much to the fundamental solution $U(t, x, y)$, as introduced later in this section. In the specific case where $\alpha(\xi) \equiv 1$ or $\alpha(\xi) < 1$ on Γ, however, the elliptic theory for L is standard, and much deeper results are well known (see [6] for details). In such cases, $\mathcal{D}(L)$ is simply characterized by $\{u \in H^2(\Omega);\ \tau u = 0 \text{ on } \Gamma\}$, so that $(\lambda - L)^{-1} f$ for $f \in L^2(\Omega)$ is an $H^2(\Omega)$-function. As mentioned in Section 1, these facts seem unclear in our case: We do not know exactly if $(\lambda - L)^{-1} f$ would belong to $H^2(\Omega)$ for any $f \in L^2(\Omega)$. To discuss the detailed classical regularity, we need the associated Green function $G(x, y)$ and the C^α-theory.

Let the pair $(\mathcal{L}^*, \tau^*)$ be the formal adjoint of $(\mathcal{L}, \tau)$:

$$\begin{aligned} \mathcal{L}^* \varphi &= - \sum_{i,j=1}^m \frac{\partial}{\partial x_i}\Big(a_{ij}(x) \frac{\partial \varphi}{\partial x_j}\Big) - \operatorname{div}(\boldsymbol{b}(x)\varphi) + c(x)\varphi, \\ \tau^* \varphi &= \alpha(\xi)\varphi + (1 - \alpha(\xi))\Big(\frac{\partial \varphi}{\partial \nu} + (\boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi))\varphi\Big), \end{aligned} \tag{2.3}$$

where $\boldsymbol{b}(x) = (b_1(x), \dots, b_m(x))$. The pair $(\mathcal{L}^*, \tau^*)$ defines the operator $\hat{L}^*$ just as in (2.1). Then the adjoint of L, denoted by L^*, is obtained as the closure of $\hat{L}^*$ in $L^2(\Omega)$. There is a set of generalized eigenpairs $\{\overline{\lambda_i}, \ \psi_{ij}\}$ such that

(i) $\sigma(L^*) = \{\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_i}, \dots\}$; and
(ii) $L^* \psi_{ij} = \overline{\lambda_i} \psi_{ij} + \sum_{k<j} \beta^i_{jk} \psi_{ik}, \quad i \geqslant 1, \ 1 \leqslant j \leqslant m_i \, (< \infty)$.

As for the genuine solutions to the boundary value problem associated with $(\mathcal{L}, \tau)$, we note the following classical result: If f is in $C^\omega(\overline{\Omega})$ and $-c$ is a real number in $\rho(L)$, then the boundary value problem

$$(c + \mathcal{L})u = f \quad \text{in } \Omega, \qquad \tau u = 0 \quad \text{on } \Gamma \tag{2.4}$$

admits a unique solution $u \in \mathcal{D}(\hat{L})$ (see [8, Theorem 19.2]). In other words, $u = L_c^{-1} f$ *is a genuine solution in* $\mathcal{D}(\hat{L})$ *as long as* f *is Hölder continuous and* $-c \in \rho(L)$ *is a real number.* This result is proven by the standard expression of the solution u by the Green's function $G(x, y)$ (see (2.7) below). The corresponding result also holds for L^* (see [8, Theorem 19.2*]).

Setting $\Gamma_1 = \{\xi \in \Gamma; \ \alpha(\xi) = 1\} \neq \varnothing$, let $H^1_\alpha(\Omega)$ be the space defined by

$$H^1_\alpha(\Omega) = \Big\{ u \in H^1(\Omega); \ u = 0 \ \text{ on } \ \Gamma_1, \quad \Big(\frac{\alpha(\xi)}{1 - \alpha(\xi)}\Big)^{1/2} u \in L^2(\Gamma \setminus \Gamma_1) \Big\}.$$

The sesqui-linear form associated with the pair $(\mathcal{L}, \tau)$ is defined by

$$\begin{aligned} B(u, \psi) = &\Big\langle \frac{\alpha(\xi)}{1 - \alpha(\xi)} u, \psi \Big\rangle_{\Gamma \setminus \Gamma_1} \\ &+ \sum_{i,j=1}^m \Big\langle a_{ij}(x) \frac{\partial u}{\partial x_j}, \frac{\partial \psi}{\partial x_i} \Big\rangle_\Omega + \sum_{i=1}^m \Big\langle b_i(x) \frac{\partial u}{\partial x_i}, \psi \Big\rangle_\Omega + \langle c(x) u, \psi \rangle_\Omega \, . \end{aligned}$$

It is clear that $\langle Lu,\, \psi \rangle_\Omega = B(u,\psi)$ for $u \in \mathcal{D}(L)$ and $\psi \in H^1_\alpha(\Omega)$. Thus, when $c > 0$ is chosen large enough, we see that

$$\mathrm{Re}\, \langle L_c u,\, u\rangle_\Omega \geqslant \mathrm{const}\ \|u\|^2_{H^1(\Omega)}, \quad \text{and thus} \quad \|L_c u\| \geqslant \mathrm{const}\ \|u\|_{H^1(\Omega)}, \quad u \in \mathcal{D}(L),$$

where $L_c = L + c$. Similarly we obtain the estimate

$$\mathrm{Re}\, \langle L^*_c u,\, u\rangle_\Omega \geqslant \mathrm{const}\ \|u\|^2_{H^1(\Omega)}, \quad \text{and thus} \quad \|L^*_c u\| \geqslant \mathrm{const}\ \|u\|_{H^1(\Omega)}, \quad u \in \mathcal{D}(L^*).$$

The operator L with $\boldsymbol{b}(x)$ being set $\mathbf{0}$ is denoted by L^0. The operator L^0 is self-adjoint. Choosing a $c > 0$ again large enough, if necessary, both L_c and L^0_c are m-accretive. Recall that $\mathcal{D}({L^0_c}^{1/2}) = H^1_\alpha(\Omega)$ (see [13]). Thus we see – via a generalization of the Heinz inequality in [9] – that

$$\mathcal{D}(L_c^{\omega/2}) = \mathcal{D}({L^0_c}^{\omega/2}) \subset H^\omega(\Omega), \quad 0 \leqslant \omega \leqslant 1.$$

Due to the first part of (2.2), $-L$ is the infinitesimal generator of an analytic semigroup e^{-tL}, $t > 0$. The following is not directly connected to our stabilization study, but is necessary to obtain the regularity of the state of the control system. It is also interesting in the sense that it connnects the modern theory with the classical one: It is well known (see [8]) that there is a unique fundamental solution $U(t,x,y)$, $t > 0$, $x,\ y \in \overline{\Omega}$ such that

(i) $\left(\dfrac{\partial}{\partial t} + \mathcal{L}_x\right) U(t,x,y) = 0, \quad \tau_\xi U(t,\xi,y) = 0,$

where the subindex x to $\mathcal{L}$, for example, means to apply $\mathcal{L}$ to $U(t,x,y)$ as a function of x, and the subsequent subindices τ_ξ, etc. will be self-explanatory;

(ii) $\left(\dfrac{\partial}{\partial t} + \mathcal{L}^*_y\right) U(t,x,y) = 0, \quad \tau^*_\xi U(t,x,\xi) = 0$; and

(iii) $e^{-tL}u = \displaystyle\int_\Omega U(t,x,y)u(y)\,dy, \quad u \in L^2(\Omega),$

$\|e^{-tL}\| \leqslant e^{-Ct}$, $t \geqslant 0$, where $C = \inf_{x \in \overline{\Omega}} c(x)$ †.

If $u(t,x)$ is a genuine solution to the initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}u = f(t,x) \quad \text{in } \mathbb{R}^1_+ \times \Omega, \qquad \tau u = g(t,\xi) \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{2.5}$$

†Generally speaking, the estimate: $\|e^{-tL}\| \leqslant M e^{-C't}$ with $M \geqslant 1$ is derived from the first part of (2.2), where $C' \leqslant \inf \mathrm{Re}\ \sigma(L)$. The fact $M = 1$ is not essential in our arguments.

then $u(t,x)$ is expressed as

$$\begin{aligned}
&u(t,x)\\
&=\int_\Omega U(t,x,y)u_0(y)\,dy+\int_0^t ds\int_\Omega U(t-s,x,y)f(s,y)\,dy\\
&\quad+\int_0^t ds\int_\Gamma\Big\{(1-\boldsymbol{b}(\xi)\cdot\boldsymbol{\nu}(\xi))U(t-s,x,\xi)-\frac{\partial}{\partial\nu_\xi}U(t-s,x,\xi)\Big\}g(s,\xi)\,d\Gamma.
\end{aligned}\tag{2.6}$$

If $u_0(x)$, $f(t,x)$, and $g(t,\xi)$ satisfy some regularity conditions, the right-hand side of (2.6) gives a unique genuine solution. For example, the following conditions are sufficient enough: The function u_0 is in $L^2(\Omega)$; $f(t,x)$ is uniformly Hölder continuous on $[0,T]\times\overline{\Omega}$ for $\forall T>0$; and $g_t(t,\xi)$, $g_{\xi_i}(t,\xi)$, and $g_{\xi_i\xi_j}(t,\xi)$ are uniformly Hölder continuous on $[0,T]\times\Gamma$ for $\forall T>0$, $1\leqslant i,\ j\leqslant m$ (see [8] for weaker sufficient conditions).

As for the solution $u(x)$ to (2.4) with $f\in C^\omega(\overline{\Omega})$, we have the expression (see [8])

$$\begin{aligned}
u(x)&=\int_\Omega G(x,y)f(y)\,dy,\quad\text{where}\\
G(x,y)&=\int_0^\infty e^{-ct}U(t,x,y)\,dt,\quad (x,y)\in\overline{\Omega}\times\overline{\Omega},\ x\neq y.
\end{aligned}\tag{2.7}$$

Let P_{λ_i} be the projection operator corresponding to the eigenvalue λ_i of L. Generally speaking, P_{λ_i} is not an orthogonal projector. Then the adjoint $P^*_{\lambda_i}$ is the projector corresponding to the eigenvalue $\overline{\lambda_i}$ of L^*. Setting $P_{\lambda_i}u=\sum_{j=1}^{m_i}u_{ij}\varphi_{ij}$, we have the relationship:

$$\begin{pmatrix}u_{i1}\\ \vdots\\ u_{im_i}\end{pmatrix}=\Pi_{\lambda_i}^{-1}\begin{pmatrix}\langle u,\ \psi_{i1}\rangle_\Omega\\ \vdots\\ \langle u,\ \psi_{im_i}\rangle_\Omega\end{pmatrix},\tag{2.8}$$

where the non-singular matrix Π_{λ_i} is defined by

$$\Pi_{\lambda_i}=\left(\langle\varphi_{ij},\ \psi_{il}\rangle_\Omega\ ;\ \begin{matrix}j\ \rightarrow\ 1,\dots,m_i\\ l\ \downarrow\ 1,\dots,m_i\end{matrix}\right).$$

Throughout the paper it is assumed that Re $\lambda_1\leqslant 0$. Thus some $e^{-tL}u_0$ does not converge to 0 as $t\to\infty$. Let K be the integer such that

$$\text{Re }\lambda_K\leqslant 0<\text{Re }\lambda_{K+1},\tag{2.9}$$

and set $P_K=P_{\lambda_1}+\cdots+P_{\lambda_K}$. The restriction of L onto the subspace $P_KL^2(\Omega)$ is, according to the basis $\{\varphi_{ij};\ 1\leqslant i\leqslant K,\ 1\leqslant j\leqslant m_i\}$, equivalent to the upper triangular matrix Λ, the diagonal elements of which are $\underbrace{\lambda_1,\dots,\lambda_1}_{m_1},\ \underbrace{\lambda_2,\dots,\lambda_2}_{m_2},\dots,\ \underbrace{\lambda_K,\dots\lambda_K}_{m_K}$.

Let us define the operator B. Let H be a separable Hilbert space equipped with the inner product: $\langle \cdot, \cdot \rangle_H$, and choose an orthonormal basis for H. We relabel the basis as

$$\{\eta_{ij}^{\pm};\ i \geqslant 1,\ 1 \leqslant j \leqslant n_i\,(< \infty)\}.$$

Every vector $v \,\epsilon\, H$ is expressed in terms of $\{\eta_{ij}^{\pm}\}$ as

$$v = \sum_{i,j} v_{ij}^{+}\eta_{ij}^{+} + \sum_{i,j} v_{ij}^{-}\eta_{ij}^{-}, \quad v_{ij}^{\pm} = \langle v,\, \eta_{ij}^{\pm}\rangle_H\,.$$

Let $\{\mu_i\}$ be a sequence of increasing positive numbers: $0 < \mu_1 < \mu_2 < \cdots \to \infty$, and define B as

$$\begin{gathered} Bv = \sum_{i,j} \mu_i\omega^{+} v_{ij}^{+}\eta_{ij}^{+} + \sum_{i,j} \mu_i\omega^{-} v_{ij}^{-}\eta_{ij}^{-}, \quad \text{where} \\ \omega^{\pm} = a \pm \sqrt{-1}\,\sqrt{1-a^2}, \quad 0 < a < 1. \end{gathered} \tag{2.10}$$

It is easily seen that B is a closed operator with dense domain $\mathcal{D}(B) = \Big\{v \,\epsilon\, H;\ \sum_{i,j} |v_{ij}^{\pm}\mu_i|^2 < \infty\Big\}$. In addition,

(i) $\sigma(B) = \{\mu_i\omega^{\pm};\ i \geqslant 1\}$; and

(ii) $(\mu_i\omega^{\pm} - B)\eta_{ij}^{\pm} = 0, \quad i \geqslant 1,\ \ 1 \leqslant j \leqslant n_i.$

Thus we see that $-B$ is the infinitesimal generator of an analytic semigroup e^{-tB}, $t > 0$, which is expressed by

$$e^{-tB}v = \sum_{i,j} e^{-\mu_i\omega^{+}t} v_{ij}^{+}\eta_{ij}^{+} + \sum_{i,j} e^{-\mu_i\omega^{-}t} v_{ij}^{-}\eta_{ij}^{-}.$$

The semigroup e^{-tB} satisfies the decay estimate

$$\|e^{-tB}\|_H \leqslant e^{-a\mu_1 t}, \quad t \geqslant 0. \tag{2.11}$$

For functions $h \,\epsilon\, C^{2+\omega}(\mathit{\Gamma})$, let R be a non-unique operator of prolongation such that

$$Rh \,\epsilon\, C^{2+\omega}(\overline{\mathit{\Omega}}), \quad Rh|_{\mathit{\Gamma}} = \frac{\partial}{\partial\nu} Rh\bigg|_{\mathit{\Gamma}} = h.$$

Then it is clear that $\tau Rh = h$ on $\mathit{\Gamma}$. If $-c \,\epsilon\, \rho(L)$ is a real number and h belongs to $C^{2+\omega}(\mathit{\Gamma})$, the boundary value problem

$$(c + \mathcal{L})u = 0 \quad \text{in } \mathit{\Omega}, \quad \tau u = h \quad \text{on } \mathit{\Gamma}$$

admits a unique solution $u \,\epsilon\, C^2(\mathit{\Omega}) \cap C^1(\overline{\mathit{\Omega}})$ (see [8]). In view of (2.4), the solution is expressed by $u = Rh - L_c^{-1}(c + \mathcal{L})Rh$. The function

$$N_\lambda h = Rh - (\lambda - L)^{-1}(\lambda - \mathcal{L})Rh \tag{2.12}$$

is analytic in $\lambda \,\epsilon\, \rho(L)$, and coincides with the above genuine solution when $\lambda = -c$ [‡]. For our actuators h_k, we thus define $N_{-c}h_k$ when c is a real number: $c > -\mathrm{Re}\,\lambda_1$, such that $N_{-c}h_k \,\epsilon\, C^2(\Omega) \cap C^1(\overline{\Omega})$ and $\mathcal{L}N_{-c}h_k \,\epsilon\, L^2(\Omega)$.

We rewrite here the system of differential equations (1.4), which is fundamental in our stabilization study:

$$\begin{cases} \dfrac{du}{dt} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ \tau u = \displaystyle\sum_{k=1}^{M} \langle v,\, \rho_k \rangle_H \, h_k \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\ \dfrac{dv}{dt} + Bv = \displaystyle\sum_{k=1}^{N} \langle u,\, w_k \rangle_\Gamma \, \xi_k + \sum_{k=1}^{M'} \langle v,\, \rho_k \rangle_H \, \zeta_k \quad \text{in } \mathbb{R}^1_+ \times H, \\ u(0,\cdot) = u_0(\cdot) \,\epsilon\, L^2(\Omega), \qquad v(0) = v_0 \,\epsilon\, H. \end{cases} \tag{2.13}$$

Here $\partial u/\partial t$ is replaced by du/dt, which means the differentiation of u in the topology of $L^2(\Omega)$. In (2.13) the equation for v means the compensator Σ_c which is finally reduced to a finite-dimensional equation. The output of Σ_c is a set of linear functionals $\langle v,\, \rho_k \rangle_H$, $1 \leqslant k \leqslant M'$, a part of which enters the plant Σ_p as the input through the h_k on Γ. In the stabilization procedure in Section 3, the vectors ρ_k are chosen as linear combinations of a finite number of $\eta^{\pm}_{ij}$. Thus, we assume that the ρ_k belong to $\mathcal{D}(B^*)$. We show that the problem (2.13) is well posed in $L^2(\Omega) \times H$. Actually we have the following result, where the classical regularity result is guaranteed by the general theory for eqn. (2.5) - via the fundamental solution $U(t,x,y)$ - such as the conditions stated below (2.6).

Theorem 2.1 [14]. *The problem* (2.13) *is well posed in* $L^2(\Omega) \times H$*, and the solution* $u(t,\cdot)$ *is in* $C^2(\Omega) \cap C^1(\overline{\Omega})$*, and* $\mathcal{L}u(t,\cdot) \,\epsilon\, L^2(\Omega)$*,* $t > 0$*. The semigroup generated by* (2.13) *is analytic in* $t > 0$.

3. Main result

To begin with, we first interpret the functions h_k and w_k on Γ in the framework of the control theory for systems in the finite-dimensional subspace $P_K L^2(\Omega)$. As for the h_k, set

[‡]More is true. In fact, N_λ belongs to $\mathcal{L}(H^{3/2}(\Gamma);\, H^2(\Omega))$ in the case where $\alpha(\xi) \equiv 1$ and to $\mathcal{L}(H^{1/2}(\Gamma);\, H^2(\Omega))$ in the case where $0 \leqslant \alpha(\xi) < 1$. See, e.g., J. L. Lions and E. Magenes, "Non-Homogeneous Boundary Value Problems and Applications," vol. I, Springer-Verlag, New York, 1972.

$P_{\lambda_i} N_{-c} h_k = \sum_{j=1}^{m_i} \zeta_{ij}^k \varphi_{ij}$. Then, by (2.8)

$$\begin{pmatrix} \zeta_{i1}^k \\ \vdots \\ \zeta_{im_i}^k \end{pmatrix} = \Pi_{\lambda_i}^{-1} \begin{pmatrix} \langle N_{-c} h_k,\, \psi_{i1} \rangle_\Omega \\ \vdots \\ \langle N_{-c} h_k,\, \psi_{im_i} \rangle_\Omega \end{pmatrix}.$$

Green's formula implies that

$$\begin{aligned} &\langle \mathcal{L}_c N_{-c} h_k,\, \psi_{ij} \rangle_\Omega - \langle N_{-c} h_k,\, L_c^* \psi_{ij} \rangle_\Omega \\ &= -\left\langle \frac{\partial N_{-c} h_k}{\partial \nu},\, \psi_{ij} \right\rangle_\Gamma + \left\langle N_{-c} h_k,\, \frac{\partial \psi_{ij}}{\partial \nu} \right\rangle_\Gamma + \langle (\boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi)) N_{-c} h_k,\, \psi_{ij} \rangle_\Gamma \\ &= -\left\langle h_k,\, \big(1 - \boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi)\big) \psi_{ij} - \frac{\partial \psi_{ij}}{\partial \nu} \right\rangle_\Gamma = -\langle h_k,\, \sigma \psi_{ij} \rangle_\Gamma\,, \end{aligned}$$

where σ denotes the boundary operator defined by

$$\sigma \psi_{ij} = \big(1 - \boldsymbol{b}(\xi) \cdot \boldsymbol{\nu}(\xi)\big) \psi_{ij} - \frac{\partial \psi_{ij}}{\partial \nu}.$$

Thus there is a non-singular $m_i \times m_i$ matrix R_i such that

$$\begin{pmatrix} \zeta_{i1}^k \\ \vdots \\ \zeta_{im_i}^k \end{pmatrix} = R_i \begin{pmatrix} \langle h_k,\, \sigma \psi_{i1} \rangle_\Gamma \\ \vdots \\ \langle h_k,\, \sigma \psi_{im_i} \rangle_\Gamma \end{pmatrix}, \quad 1 \leqslant i \leqslant K.$$

The above relation is rewritten as

$$P_K N_{-c} h_k = \sum_{i=1}^{K} \sum_{j=1}^{m_i} \zeta_{ij}^k \varphi_{ij} \quad \Longleftrightarrow \quad \begin{pmatrix} \zeta_{11}^k \\ \vdots \\ \zeta_{ij}^k \\ \vdots \\ \zeta_{Km_K}^k \end{pmatrix} = \mathrm{diag}\,(R_1 \dots R_K) \begin{pmatrix} \langle h_k,\, \sigma \psi_{11} \rangle_\Gamma \\ \vdots \\ \langle h_k,\, \sigma \psi_{ij} \rangle_\Gamma \\ \vdots \\ \langle h_k,\, \sigma \psi_{Km_K} \rangle_\Gamma \end{pmatrix}.$$

Setting $S = m_1 + \dots + m_K$, define the $S \times M$ matrices Z and $\hat{H}$ as

$$\begin{aligned} &Z = \left(\zeta_{ij}^k\,;\, \begin{matrix} k & \to 1, \dots, M \\ (i,j) & \downarrow (1,1), \dots, (K, m_K) \end{matrix} \right), \quad \text{and} \\ &\hat{H} = \begin{pmatrix} H_1 \\ H_2 \\ \vdots \\ H_K \end{pmatrix}, \quad \text{where} \quad H_i = \left(\langle h_k,\, \sigma \psi_{ij} \rangle_\Gamma\,;\, \begin{matrix} k \to 1, \dots, M \\ j \ \downarrow \ 1, \dots, m_i \end{matrix} \right), \end{aligned} \tag{3.1}$$

respectively. Then, $Z = R\hat{H}$, where $R = \mathrm{diag}\,(R_1 \dots R_K)$. It is clear that the controllability condition for the pair $(\varLambda, Z)$:

$$\mathrm{rank}\,(Z \ \varLambda Z \ \dots \varLambda^{S-1} Z) = S$$

is equivalent to the controllability condition for the pair $(R^{-1}\varLambda R, \hat{H})$. As for the w_k, we define the $N \times m_i$ matrices W_i by

$$W_i = \left(\langle w_k, \, \varphi_{ij} \rangle_\Gamma \; ; \begin{array}{c} k \ \downarrow \ 1, \dots, N \\ j \rightarrow 1, \dots, m_i \end{array} \right), \quad 1 \leqslant i \leqslant K. \tag{3.2}$$

Our stabilization procedure is based on the control system (2.13), which is well posed in $L^2(\varOmega) \times H$ according to Theorem 2.1. Assuming the well known finite-dimansional observability and controllability conditions on the w_k and the h_k, respectively, we first achieve the stabilization of (2.13) and then reduce it to (1.1), where the matrix B_1 is determined by the parameters: μ_i, $\omega^\pm$, ρ_k, and ζ_k: They all are what we can manipulate. In order to study the stabilization, we assume that

$$\begin{gathered} \mu_i \leqslant \mathrm{const}\, i^\gamma, \quad i \geqslant 1, \quad \text{for some } \gamma;\ 0 < \gamma < 2, \\ \mathrm{Re}\ \lambda_{K+1} < a\mu_1, \quad \text{and} \quad \sigma(L) \cap \sigma(B) = \varnothing. \end{gathered} \tag{3.3}$$

The above conditions are fulfilled by adjusting the parameters $\omega^\pm$ and μ_1. The vectors $\xi_k \in H$ are expressed as $\xi_k = \sum_{i,j} \xi_{ij}^k \eta_{ij}^+ + \sum_{i,j} \overline{\xi_{ij}^k} \eta_{ij}^-$. Then we define the $n_i \times N$ matrices $\varXi_i$ by

$$\varXi_i = \left(\xi_{ij}^k \, ; \begin{array}{c} k \rightarrow 1, \dots, N \\ j \ \downarrow \ 1, \, \dots, n_i \end{array} \right), \quad i \geqslant 1. \tag{3.4}$$

Our aim is to construct the feedback control system (1.1) and to derive an exponential decay of solutions $(u(t,\cdot), v(t))$ to (1.1) with the prescribed decay rate $r < \mathrm{Re}\ \lambda_{K+1}$. The following is our main result, in which the second and the third steps of the proof mainly reflect the new idea among others:

Theorem 3.1. (i) *Let r be an arbitrary positive number smaller than* Re λ_{K+1}. *Suppose that $(R^{-1}\varLambda R, \hat{H})$ is a controllable pair. Suppose further that*

$$\begin{aligned} &\mathrm{rank}\ W_i = m_i, \quad 1 \leqslant i \leqslant K, \quad \textit{and} \\ &\mathrm{rank}\ \varXi_i = N, \quad i \geqslant 1. \end{aligned} \tag{3.5}$$

Then for any r_1; $r < r_1 < \mathrm{Re}\, \lambda_{K+1}$, there exist vectors $\zeta_k \in H$ and $\rho_k \in \mathcal{D}(B^*)$ which ensure the decay estimate

$$\|u(t,\cdot)\| + \|v(t)\|_H \leqslant \text{const } e^{-r_1 t}(\|u_0\| + \|v_0\|_H), \quad t \geqslant 0 \tag{3.6}$$

for every solution $(u(t,\cdot), v(t))$ to (2.13).

(ii) *Eqn.* (1.1) *is derived from* (2.13) *by suitably choosing an integer $l < \infty$, and it is well posed in $L^2(\Omega) \times \mathbb{R}^l$, where the solution $u(t,\cdot)$ is in $C^2(\Omega) \cap C^1(\overline{\Omega})$, and $\mathcal{L}u \in L^2(\Omega)$, $t > 0$. Every solution (u, v) to* (1.1) *satisfies the decay estimate*

$$\|u(t,\cdot)\| + |v(t)|_l \leqslant \text{const } e^{-rt}(\|u_0\| + |v_0|_l), \quad t \geqslant 0. \tag{3.7}$$

Proof of Theorem 3.1.

First Step (*operator equation*). Before introducing the coupled control system (3.23) below, we first consider the operator equation

$$XL - BX = C \quad \text{on } \mathcal{D}(L), \quad \text{where} \quad C = -\sum_{k=1}^{N} \langle \,\cdot\,,\, w_k \rangle_\Gamma \,\xi_k, \quad \xi_k \in H. \tag{3.8}$$

Here the domain $\mathcal{D}(C)$ is given by $\cup_{s>1/2} H^s(\Omega)$. Recall that $\sigma(L) \cap \sigma(B) = \varnothing$ in (3.3). Then our first result is the following:

Proposition 3.2 [11, 14]. *The operator equation* (3.8) *admits a unique operator solution $X \in \mathcal{L}(L^2(\Omega); H)$. The solution X is expressed as*

$$\begin{gathered} Xu = \sum_{i,j} \sum_{k=1}^{N} f_k(\mu_i \omega^+; u) \xi_{ij}^k \eta_{ij}^+ + \sum_{i,j} \sum_{k=1}^{N} f_k(\mu_i \omega^-; u) \overline{\xi_{ij}^k} \eta_{ij}^-, \quad u \in L^2(\Omega), \\ \text{where} \quad f_k(\lambda; u) = \langle (\lambda - L)^{-1} u,\, w_k \rangle_\Gamma, \quad 1 \leqslant k \leqslant N. \end{gathered} \tag{3.9}$$

Remark. In [14], a somewhat stronger condition: $\sum_{i,j} |\xi_{ij}^k \mu_i^{1/4+\epsilon}|^2 < \infty$, where $\epsilon > 0$, is assumed. This condition is removed in our theorem, although it is essential so that the range of X is contained in $\mathcal{D}(B)$.

Proposition 3.3 [11, 14]. *Under the assumptions* (3.3) *and* (3.5) *on the Ξ_i, we have the inclusion relation:*

$$P_K^* L^2(\Omega) \subset \overline{X^* H}. \tag{3.10}$$

In (3.10) *the overline on the right-hand side means the closure in $L^2(\Omega)$, and the left-hand side is a finite-dimensional subspace spanned by ψ_{ij}, $1 \leqslant i \leqslant K$, $1 \leqslant j \leqslant m_i$.*

For u and $z_k \in L^2(\Omega)$, $1 \leqslant k \leqslant M$, set $\langle u, \boldsymbol{z}\rangle_\Omega = {}^t(\langle u, z_1\rangle_\Omega \dots \langle u, z_M\rangle_\Omega)$, where ${}^t(\dots)$ denotes the transpose of vectors. Similar expressions appearing later will be self-explanatory. Throughout the theorem we may assume with no loss of generality that 0 belongs to $\rho(L)$, and set $N = N_0$ in (2.12). The following lemma is easily examined by direct computaitons. It is closely connected to the second and the third steps:

Lemma 3.4. *The function $\boldsymbol{G}(\cdot) \in \mathcal{L}((L^2(\Omega))^M)$ defined by*

$$\boldsymbol{y} = \boldsymbol{G}(\boldsymbol{z}) = \Big(1 + \overline{\Big(\langle N h_{\underset{k\to}{k}}, \boldsymbol{z}\rangle_\Omega\Big)}\Big)^{-1} \boldsymbol{z}, \quad \boldsymbol{z} \in (L^2(\Omega))^M \tag{3.11}$$

admits the inverse $\boldsymbol{G}^{-1}(\cdot) \in \mathcal{L}((L^2(\Omega))^M)$ as long as $\det\Big(1 + \overline{\Big(\langle N h_{\underset{k\to}{k}}, \boldsymbol{z}\rangle_\Omega\Big)}\Big) \neq 0$. The inverse $\boldsymbol{G}^{-1}$ is given by

$$\boldsymbol{z} = \boldsymbol{G}^{-1}(\boldsymbol{y}) = \Big(1 - \overline{\Big(\langle N h_{\underset{k\to}{k}}, \boldsymbol{y}\rangle_\Omega\Big)}\Big)^{-1} \boldsymbol{y}, \quad \textit{and}$$

$$1 - \overline{\Big(\langle N h_{\underset{k\to}{k}}, \boldsymbol{y}\rangle_\Omega\Big)} = \Big(1 + \overline{\Big(\langle N h_{\underset{k\to}{k}}, \boldsymbol{z}\rangle_\Omega\Big)}\Big)^{-1},$$

where $\Big(\langle N h_{\underset{k\to}{k}}, \boldsymbol{z}\rangle_\Omega\Big)$ denotes the $M \times M$ matrix defined by

$$\Big(\langle N h_{\underset{k\to}{k}}, \boldsymbol{z}\rangle_\Omega\Big) = \Big(\langle N h_k, \boldsymbol{z}\rangle_\Omega\ ;\ k \to 1, \dots, M\Big) = \Big(\langle N h_k, z_j\rangle_\Omega\ ;\ \begin{matrix} k \to 1, \dots, M \\ j \downarrow 1, \dots, M \end{matrix}\Big).$$

Given a set of $y_k \in L^2(\Omega)$, $1 \leqslant k \leqslant M$, let $\hat{F}$ be the operator defined by

$$\hat{F}u = \mathcal{L}u, \quad u \in \mathcal{D}(\hat{F}) = \{u \in C^2(\Omega) \cap C^1(\overline{\Omega});\ \mathcal{L}u \in L^2(\Omega),\ \ \tau_f u = 0 \ \text{ on } \Gamma\}, \tag{3.12}$$

where τ_f denotes the boundary operator defined by

$$\tau_f u = \tau u - \sum_{k=1}^M \langle u, y_k\rangle_\Omega\, h_k.$$

We consider the differential equation: $du/dt + \hat{F}u = 0$. This means the boundary feedback control system described by

$$\begin{gathered} \frac{du}{dt} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ \tau u = \sum_{k=1}^M \langle u, y_k\rangle_\Omega\, h_k \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \qquad u(0,\cdot) = u_0 \quad \text{in } \Omega. \end{gathered} \tag{3.13}$$

A specific feature of the operator $\hat{F}$ is stated in the following proposition, where the last statement of (ii) looks merely a standard perturbation result as long as $\alpha(\xi) \equiv 1$ or $0 \leqslant \alpha(\xi) < 1$ on Γ. A more careful consideration is required in our general case.

Proposition 3.5 [14]. (i) *The operator $\hat{F}$ admits the closure F in $L^2(\Omega)$. The closure F is densely defined, and generates an analytic semigroup e^{-tF}, $t > 0$. For $\lambda \in \rho(L)$, let T_λ be the operator on $L^2(\Omega)$ which is defined by $z = T_\lambda u = u - (N_\lambda h_1 \dots N_\lambda h_M) \langle u, \boldsymbol{y} \rangle_\Omega$. Then there exists an $a \in \mathbb{R}^1$ such that T_λ, $\lambda \in \overline{\Sigma} - a$ is a bounded bijection from $L^2(\Omega)$ onto itself; that $\overline{\Sigma} - a$ is contained in $\rho(F)$; and that*

$$(\lambda - F)^{-1} = T_\lambda^{-1}(\lambda - L)^{-1}, \quad \textit{and} \quad \|(\lambda - F)^{-1}\| \leqslant \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \overline{\Sigma} - a.$$

(ii) *Suppose that $(R^{-1}\Lambda R, \hat{H})$ or (Λ, Z) is a controllable pair. Then there exists a set of $y_k \in P_K^* L^2(\Omega)$, $1 \leqslant k \leqslant M$, such that the following estimate holds* [§] :

$$\|e^{-tF}\| \leqslant \text{const } e^{-r_2 t}, \quad t \geqslant 0, \quad r_1 < r_2 < \text{Re } \lambda_{K+1}. \tag{3.14}$$

Consider the case where the above y_k are replaced by $\tilde{y}_k \in L^2(\Omega)$ in $\hat{F}$. If the perturbation $\sum_{k=1}^M \|\tilde{y}_k - y_k\|$ is small enough, the estimate (3.14) *is changed into a little altered one:*

$$\|e^{-tF}\| \leqslant \text{const } e^{-r_1 t}, \quad t \geqslant 0. \tag{3.14$'$}$$

Choose a set of $y_k = y_k^0 \in P_K^* L^2(\Omega)$, $1 \leqslant k \leqslant M$, stated in Proposition 3.5, (ii). We may assume with no loss of generality that

$$\det\left(1 - \overline{\left(\langle N h_k, \boldsymbol{y}_0 \rangle_\Omega\right)}_{k\to}\right) \neq 0, \quad \boldsymbol{y}_0 = {}^t(y_1^0 \dots y_M^0).$$

If not, we may adjust the y_k^0 slightly within $P_K^* L^2(\Omega)$ so that the estimate (3.14) is correct. By Lemma 3.4, we find a unique $\boldsymbol{z}_0 = {}^t(z_1^0 \dots z_M^0) \in (P_K^* L^2(\Omega))^M$ such that $\boldsymbol{z}_0 = \boldsymbol{G}^{-1}(\boldsymbol{y}_0)$. By Proposition 3.3, we find suitable sequences of functions $X^*\rho_k$ which are arbitrarily close to z_k^0 in the $L^2(\Omega)$-topology, $1 \leqslant k \leqslant M$. Thus,

$$\|\boldsymbol{G}(X^*\boldsymbol{\rho}) - \boldsymbol{y}_0\| \to 0, \quad \text{and} \quad \det\left(1 + \overline{\left(\langle N h_k, X^*\boldsymbol{\rho} \rangle_\Omega\right)}_{k\to}\right) \neq 0. \tag{3.15}$$

[§] When y_k belong to $P_K^* L^2(\Omega)$, we have the expression:

$$e^{-tF} = T_{-c}^{-1} \cdot \exp\left(-t\left(L - \textstyle\sum_{k=1}^M \langle \cdot, L_c^* y_k \rangle_\Omega N_{-c} h_k\right)\right) \cdot T_{-c}, \quad t \geqslant 0.$$

Choose the above $\boldsymbol{\rho} = {}^t(\rho_1 \dots \rho_M) \in H^M$ such that the operator F with the parameters $\boldsymbol{y} = \boldsymbol{G}(X^*\boldsymbol{\rho}) \in (L^2(\Omega))^M$ guarantees the estimate (3.14′). Noting that the set $\{\eta_{ij}^{\pm}\}$ forms a complete orthonormal system for H, we can choose the ρ_k, which are expressed by a finite number of $\eta_{ij}^{\pm}$, say, $1 \leqslant i \leqslant n$. Based on the $\boldsymbol{\rho} \in (\mathcal{D}(B^*))^M$, we define the matrices Θ, G_1, and G_2 (and also G_3), respectively, as:

$$\begin{aligned} \Theta &= 1 + \overline{\Big(\underset{k\to}{\langle N h_k,} \, X^*\boldsymbol{\rho}\rangle_\Omega \Big)}, \\ G_1 &= \Big(\underset{k\to}{\langle N h_k,} \, \Theta^{-1} X^*\boldsymbol{\rho}\rangle_\Omega \Big) = \overline{\Theta}^{-1} \Big(\underset{k\to}{\langle N h_k,} \, X^*\boldsymbol{\rho}\rangle_\Omega \Big), \\ G_2 &= \Big(\underset{k\to}{\langle \xi_k,} \, \boldsymbol{\rho}\rangle_H \Big) = \Big(\langle \xi_k, \, \rho_j\rangle_H \, ; \begin{matrix} k \to 1, \dots, M \\ j \downarrow 1, \dots, M \end{matrix} \Big), \\ \text{and} \quad G_3 &= \Big(\underset{k\to}{\langle N h_k,} \, \boldsymbol{w}\rangle_\Gamma \Big) = \Big(\langle N h_k, \, w_j\rangle_\Gamma \, ; \begin{matrix} k \to 1, \dots, M \\ j \downarrow 1, \dots, M \end{matrix} \Big). \end{aligned} \tag{3.16}$$

On the analogy of the function $N_\lambda h$ (see (2.12)), we seek the solution to the boundary value problem:

$$(c + \mathcal{L})u = 0 \quad \text{in } \Omega, \qquad \tau_f u = g \quad \text{on } \Gamma, \tag{3.17}$$

where $g \in C^{2+\omega}(\Gamma)$ denotes the given function. If $c > 0$ is large enough such that $-c$ is in $\rho(L)$, then the boundary value problem admits a unique solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. The solution is denoted by $u = N_{-c}^f g$. Via the operator T_λ introduced in Proposition 3.5, (i), we actually obtain the expression of the solution as: $N_{-c}^f g = T_{-c}^{-1} N_{-c} g$.

Second Step (*differential equation with distributed feedback*). The purpose of this step is to derive the system of differential equations (3.23) with state $(q(t), v(t))$, which is most fundamental in our stabilization as well as well-posedness. Let $f_k(t)$, $1 \leqslant k \leqslant M$, be input functions, not specified at this point. Replacing $\langle u, y_k\rangle_\Omega$ by $f_k(t)$ in (3.13), let us consider the differential equation

$$\begin{gathered} \frac{du}{dt} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ \tau u = \sum_{k=1}^M f_k(t) h_k \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \qquad u(0, \cdot) = u_0 \quad \text{in } \Omega. \end{gathered} \tag{3.18}$$

Assuming for a moment that f_k are of class C^1 and setting

$$q(t) = u(t) - \sum_{k=1}^M f_k(t) N h_k,$$

we obtain the equation for $q(t)$:

$$\frac{dq}{dt} + Lq = -\sum_{k=1}^{M} f_k'(t) N h_k, \quad q(0) = q_0 = u_0 - \sum_{k=1}^{M} f_k(0) N h_k. \tag{3.19}$$

Our idea is to construct a dynamic compensator, based not on (3.18) but on (3.19) with state $q(t) \in \mathcal{D}(L)$. The system of differential equations in $L^2(\Omega) \times H$ is then described by

$$\begin{cases} \dfrac{dq}{dt} + Lq = -\displaystyle\sum_{k=1}^{M} f_k'(t) N h_k, \quad q(0) = q_0 \in L^2(\Omega), \\ \dfrac{dv}{dt} + Bv = -Cq - \displaystyle\sum_{k=1}^{M} f_k'(t) X N h_k, \quad v(0) = v_0 \in H. \end{cases} \tag{3.20}$$

Whatever $f_k(t)$ may be, Proposition 3.2 immediately implies that

$$\frac{d}{dt}(Xq - v) + B(Xq - v) = 0, \quad t > 0, \text{ or } \quad Xq(t) - v(t) = e^{-tB}(Xq_0 - v_0), \quad t \geqslant 0.$$

Creating this relation is the role of the compensator. Owing to the decay property of e^{-tB}, the above right-hand side goes to 0 exponentially as $t \to \infty$:

$$\|Xq(t) - v(t)\|_H \leqslant e^{-a\mu_1 t} \|Xq_0 - v_0\|_H, \quad t \geqslant 0. \tag{3.21}$$

Let $\boldsymbol{g}(q, v)$ be the vector-valued function defined by

$$\boldsymbol{g}(q, v) = \overline{\Theta}^{-1} \left(G_2 \langle q, \boldsymbol{w} \rangle_\Gamma - \langle v, B^* \boldsymbol{\rho} \rangle_H \right) = \begin{pmatrix} g_1(q, v) \\ \vdots \\ g_M(q, v) \end{pmatrix}, \tag{3.22}$$

where the matrices Θ and G_2 are defined in (3.16). Replacing $f_k'(t)$ by $g_k(q, v)$ in (3.20), we obtain the system of differential equations with state (q, v):

$$\begin{cases} \dfrac{dq}{dt} + Lq = -(Nh_1 \ \dots \ Nh_M) \boldsymbol{g}(q, v), \quad q(0) = q_0 \in L^2(\Omega), \\ \dfrac{dv}{dt} + Bv = -Cq - (XNh_1 \ \dots \ XNh_M) \boldsymbol{g}(q, v), \quad v(0) = v_0 \in H. \end{cases} \tag{3.23}$$

Eqn. (3.23) is clearly well posed in $L^2(\Omega) \times H$, and the decay estimate (3.21) holds. As in Theorem 2.1, $q(t)$ belongs to $\mathcal{D}(\hat{L})$ for each $t > 0$: This is proven by the classical theory for eqn. (2.5) and the property of the fundamental solution $U(t, x, y)$.

Third Step (*stabilization*). We begin with the well posed equation (3,23). In (3.23), set

$$u(t) = q(t) + \sum_{k=1}^{M} f_k(t) N h_k = q(t) + (Nh_1 \ \dots \ Nh_M) \boldsymbol{f}(t), \quad \text{where} \quad f_k(t) = \langle v(t), \rho_k \rangle_H \quad 1 \leqslant k \leqslant M. \tag{3.24}$$

Then $u(t)$ belongs to $C^2(\Omega) \cap C^1(\overline{\Omega})$ for each $t > 0$. In view of (3.21), we calculate as

$$
\begin{aligned}
|\boldsymbol{f}(t) - \langle q(t), X^* \boldsymbol{\rho} \rangle_\Omega| &= \left| \boldsymbol{f}(t) - \langle u(t), X^* \boldsymbol{\rho} \rangle_\Omega + \sum_{k=1}^M f_k(t) \langle N h_k, X^* \boldsymbol{\rho} \rangle_\Omega \right| \\
&= |\boldsymbol{f}(t) - \langle u(t), \Theta \boldsymbol{y} \rangle_\Omega + \overline{\Theta} G_1 \boldsymbol{f}(t)| \\
&= |(1 + \overline{\Theta} G_1) \boldsymbol{f}(t) - \overline{\Theta} \langle u(t), \boldsymbol{y} \rangle_\Omega| \\
&\leqslant \mathrm{const}\, e^{-a\mu_1 t} (\|q_0\| + \|v_0\|_H), \quad t \geqslant 0.
\end{aligned}
$$

Since $\overline{\Theta} = 1 + \overline{\Theta} G_1$, we find that

$$
|\boldsymbol{f}(t) - \langle u(t), \boldsymbol{y} \rangle_\Omega| \leqslant \mathrm{const}\, e^{-a\mu_1 t} (\|q_0\| + \|v_0\|_H), \quad t \geqslant 0. \tag{3.25}
$$

Recall that $\boldsymbol{\rho}$ belongs to $(\mathcal{D}(B^*))^M$. Then we similarly obtain the estimate

$$
|\boldsymbol{f}'(t) - \langle u_t(t), \boldsymbol{y} \rangle_\Omega| \leqslant \mathrm{const}\, e^{-a\mu_1 t} (\|q_0\| + \|v_0\|_H), \quad t > 0. \tag{3.26}
$$

Looking at the equation for v in (3.23), we calculate as

$$
\begin{aligned}
\langle v_t, \boldsymbol{\rho} \rangle_H + \langle Bv, \boldsymbol{\rho} \rangle_H &= \sum_{k=1}^N \langle q, w_k \rangle_\Gamma \langle \xi_k, \boldsymbol{\rho} \rangle_H \\
&\quad - \Big(\langle XNh_1, \boldsymbol{\rho} \rangle_H \ \ldots \ \langle XNh_M, \boldsymbol{\rho} \rangle_H \Big) \boldsymbol{g}(q, v), \\
\text{or} \quad \boldsymbol{f}'(t) + \langle v, B^* \boldsymbol{\rho} \rangle_H &= G_2 \langle q, \boldsymbol{w} \rangle_\Gamma - \Big(\underset{k\to}{\langle Nh_k, X^* \boldsymbol{\rho} \rangle_\Omega} \Big) \boldsymbol{g}(q, v),
\end{aligned}
$$

from which we find that

$$
\boldsymbol{f}'(t) = \langle v_t, \boldsymbol{\rho} \rangle_H = \boldsymbol{g}(q, v), \quad t > 0. \tag{3.27}
$$

Thus (3.23) is rewritten as

$$
\begin{cases}
\dfrac{dq}{dt} + Lq = -(Nh_1 \ \ldots \ Nh_M) \boldsymbol{f}'(t), \quad q(0) = q_0 \in L^2(\Omega), \\
\dfrac{dv}{dt} + Bv = -Cq - (XNh_1 \ \ldots \ XNh_M) \boldsymbol{f}'(t), \quad v(0) = v_0 \in H.
\end{cases} \tag{3.23$'$}
$$

Then $u(t)$ defined by (3.24) satisfies the differential equation:

$$
\begin{cases}
\dfrac{du}{dt} + \mathcal{L} u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\
\tau u = (h_1 \ \ldots \ h_M) \boldsymbol{f}(t) \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\
u(0, \cdot) = u_0 = q_0 + (Nh_1 \ \ldots \ Nh_M) \langle v_0, \boldsymbol{\rho} \rangle_H \quad \text{in } \Omega.
\end{cases} \tag{3.18$'$}
$$

The behavior of $u(t)$ on Γ is described as $\tau_f u = (h_1 \ \ldots \ h_M)(\boldsymbol{f}(t) - \langle u(t), \boldsymbol{y} \rangle_\Omega)$. Set

$$
p(t) = u(t) - \Big(N^f_{-c} h_1 \ \ldots \ N^f_{-c} h_M \Big) \boldsymbol{\varepsilon}(t), \quad \text{with} \quad \boldsymbol{\varepsilon}(t) = \boldsymbol{f}(t) - \langle u(t), \boldsymbol{y} \rangle_\Omega,
$$

where $N^f_{-c}h_i$ are introduced in (3.17). The function $p(t)$, $t > 0$, belongs to $\mathcal{D}(\hat{F})$ and satisfies the equation

$$\frac{dp}{dt} + Fp = \left(N^f_{-c}h_1 \; \dots \; N^f_{-c}h_M\right)(c\boldsymbol{\varepsilon}(t) - \boldsymbol{\varepsilon}'(t)),$$
$$p(0) = u_0 - \left(N^f_{-c}h_1 \; \dots \; N^f_{-c}h_M\right)\left(\langle v_0, \boldsymbol{\rho}\rangle_H - \langle u_0, \boldsymbol{y}\rangle_\Omega\right).$$

By (3.25) and (3.26), we already know that

$$|\boldsymbol{\varepsilon}(t)|, \; |\boldsymbol{\varepsilon}'(t)| \leqslant \text{const}\, e^{-a\mu_1 t}(\|u_0\| + \|v_0\|_H), \quad t > 0.$$

In view of the estimate (3.14′), we obtain the decay estimate

$$\|p(t)\|, \; \|u(t)\|, \text{ and } |\boldsymbol{f}(t)| \leqslant \text{const } e^{-r_1 t}(\|u_0\| + \|v_0\|_H), \quad t \geqslant 0.$$

This immediately gives the decay estimate for every solution $(q(t), v(t))$ to (3.23):

$$\|q(t)\| + \|v(t)\|_H \leqslant \text{const } e^{-r_1 t}(\|q_0\| + \|v_0\|_H), \quad t \geqslant 0. \tag{3.28}$$

Fourth Step (*reduction to a finite-dimensional compensator*). We go back to eqn. (3.23) with the decay estimate (3.28). Let P^H_n be the projection operator in H corresponding to the eigenvalues $\mu_i\omega^\pm$ of B, $1 \leqslant i \leqslant n$, that is, $P^H_n v = \sum_{i,j\,(i\leqslant n)}(v^+_{ij}\eta^+_{ij} + v^-_{ij}\eta^-_{ij})$ for $v = \sum_{i,j}(v^+_{ij}\eta^+_{ij} + v^-_{ij}\eta^-_{ij}) \in H$. Recall that the vector $\boldsymbol{\rho}$ is chosen in the subspace $(P^H_n H)^M$ (see the first step). In (3.23), set $v_1(t) = P^H_n v(t)$. Note that $\boldsymbol{g}(q, v) = \boldsymbol{g}(q, v_1)$. Applying P^H_n to the both sides of the equation for v, we obtain the system of differential equations

$$\begin{cases} \dfrac{dq}{dt} + Lq = -(Nh_1 \; \dots \; Nh_M)\boldsymbol{g}(q, v_1), \\ \dfrac{dv_1}{dt} + B_1 v_1 = -P^H_n Cq - (P^H_n XNh_1 \; \dots \; P^H_n XNh_M)\boldsymbol{g}(q, v_1), \\ q(0) = q_0 \in L^2(\Omega), \quad v_1(0) = P^H_n v_0 \in P^H_n H. \end{cases} \tag{3.29}$$

In (3.29), B_1 denotes the restriction of B onto the invariant subspace $P^H_n H$, i.e., $B_1 = B|_{P^H_n H}$. Just as in (3.23), eqn. (3.29) with state (q, v_1) is well posed in $L^2(\Omega) \times P^H_n H$. The semigroup generated by (3.29) is analytic in $t > 0$. Solution $q(t, \cdot) \in \mathcal{D}(L)$ actually belongs to $\mathcal{D}(\hat{L})$ for each $t > 0$. Every solution $(q(t), v_1(t))$ to (3.29) with initial value $(q_0, v_0) \in L^2(\Omega) \times P^H_n H$ is derived from the solution $(\tilde{q}(t), \tilde{v}(t))$ to (3.23) with the same initial value, and is expressed by $(q(t), v_1(t)) = (\tilde{q}(t), P^H_n \tilde{v}(t))$. Thus every solution $(q(t), v_1(t))$ to (3.29) satisfies the decay estimate

$$\|q(t)\| + \|v_1(t)\|_H \leqslant \text{const } e^{-r_1 t}(\|q_0\| + \|v_0\|_H), \quad t \geqslant 0. \tag{3.28′}$$

The equation for v_1 in (3.29) means the finite-dimensioanl compensator in the subspace $P_n^H H$. In (3.24) note that $\boldsymbol{f}(t) = \langle v(t),\, \boldsymbol{\rho} \rangle_H = \langle v_1(t),\, \boldsymbol{\rho} \rangle_H$ satisfies the relation

$$\boldsymbol{f}'(t) = \langle (v_1)_t,\, \boldsymbol{\rho} \rangle_H = \boldsymbol{g}(q, v_1), \quad t > 0.$$

Thus (3.29) is rewritten as

$$\begin{cases} \dfrac{dq}{dt} + Lq = -(Nh_1\, \dots\, Nh_M)\boldsymbol{f}'(t), \\ \dfrac{dv_1}{dt} + B_1 v_1 = -P_n^H C q - (P_n^H X N h_1\, \dots\, P_n^H X N h_M)\boldsymbol{f}'(t), \\ q(0) = q_0 \,\epsilon\, L^2(\Omega), \quad v_1(0) \,\epsilon\, P_n^H H. \end{cases} \tag{3.29$'$}$$

We rewrite (3.29) in terms of $(u(t), v_1(t))$, where $u(t)$ is defined by (3.24) with v replaced by v_1. In view of (3.29$'$), we easily obtain

$$\begin{cases} \dfrac{du}{dt} + \mathcal{L}u = 0 \quad \text{in } \mathbb{R}^1_+ \times \Omega, \\ \tau u = (h_1\, \dots\, h_M)\langle v_1,\, \boldsymbol{\rho} \rangle_H \quad \text{on } \mathbb{R}^1_+ \times \Gamma, \\ \dfrac{dv_1}{dt} + B_1 v_1 = -P_n^H C u + (P_n^H C N h_1\, \dots\, P_n^H C N h_M)\langle v_1,\, \boldsymbol{\rho} \rangle_H \\ \qquad\qquad - (P_n^H X N h_1\, \dots\, P_n^H X N h_M)\tilde{\boldsymbol{g}}(u, v_1) \quad \text{in } \mathbb{R}^1_+, \\ u(0, \cdot) = u_0 \,\epsilon\, L^2(\Omega), \quad v_1(0) = v_{10} \,\epsilon\, P_n^H H. \end{cases} \tag{3.30}$$

where $u_0 = q_0 + (Nh_1\, \dots\, Nh_M)\langle v_{10},\, \boldsymbol{\rho} \rangle_H$, and

$$\tilde{\boldsymbol{g}}(u, v_1) = \boldsymbol{g}(q, v_1) = \overline{\Theta}^{-1}\left(G_2 \langle u,\, \boldsymbol{w} \rangle_\Gamma - G_2 G_3 \langle v_1,\, \boldsymbol{\rho} \rangle_H - \langle v_1,\, B^* \boldsymbol{\rho} \rangle_H \right). \tag{3.31}$$

In the equation for v_1, we get together the operator B_1 and the terms which include the inner products $\langle v_1,\, \boldsymbol{\rho} \rangle_H$ and $\langle v_1,\, B^* \boldsymbol{\rho} \rangle_H$ in a lump. The resultant operator is denoted by the same symbol B_1 with no confusion. Then we finally obtain the desired control system (1.1). Q.E.D.

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