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How precisely can we reduce the three-flavor neutrino oscillation to the two-flavor one only from $\delta m_{12}^2/\delta m_{13}^2 \lesssim 1/15$?

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We derive a reduction formula that expresses the survival rate for the three-flavor neutrino oscillation using the two-flavor one to next-to-leading order when there is one resonance due to the matter effect. We numerically find that the next-to-leading reduction formula is extremely accurate and the improvement is relevant for the precision test of solar neutrino oscillation and the indirect measurement of CP violation in the leptonic sector. We also derive a reduction formula, which is slightly different from that previously obtained, in the case when there are two resonances. We numerically verify that this reduction formula is quite accurate and is valid for a wider parameter region than are those previously obtained.

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I. INTRODUCTION

The physics of neutrino oscillation is currently under very active investigation, since it leads to physics beyond the standard model. Two-flavor neutrino oscillation, however, is adopted in most analyses of the data, although everyone knows that there are three active neutrino flavors. Two-flavor oscillation is easy to investigate in comparison with three-flavor oscillation because there are only two parameters: a mass-squared difference and a mixing angle. In addition, some exact solutions of the oscillation probability are known in the two-flavor oscillation scheme even in the presence of a matter effect [1–8]. To make the analysis realistic, however, we need to work in the full three-flavor system, where we have six parameters, two mass-squared differences, three mixing angles, and one CP phase. The analysis using three-flavor oscillation is particularly important when we compare the outputs from different experiments, sensitive to different mass-squared differences and mixing angles. The importance of the analysis in the full three-flavor context for terrestrial neutrino oscillation experiments has been discussed [9,10].

The simplest way to investigate the three-flavor oscillation is by relying on a numerical calculation. There is no serious technical difference between the two-flavor oscillation and the three-flavor one in numerical calculations.¹ The parameter space, however, becomes very large, six dimensional, and is difficult to exhaust. What is worse, we cannot easily understand the physical consequences from the numerical results intuitively, even if the parameter space is exhausted.

A more elegant way to investigate the three-flavor oscillation is to reduce the three-flavor oscillation to an effective two-flavor one using the hierarchy between the two mass differences or the smallness of some mixing angle. For example, the following reduction formula for the survival probability of the electron neutrino $S_{3\nu}$ is found in the case that one mass difference is much larger than the other one and the matter effect, $\Delta_{12}, A(x) \ll \Delta_{13}$ [11],

$$S_{3\nu} = S_{2\nu,eff} \cos^4 \theta_{13} + \sin^4 \theta_{13}. \quad (1)$$

Here, θ_{13} is the mixing angle defined in Sec. II A, while $S_{2\nu,eff}$ is the survival rate calculated in the effective two-flavor scheme with effective matter effect $A_{eff}(x) \equiv \cos^2 \theta_{13} A(x)$. We show the definitions of each quantity in detail in Sec. II. This relation is often used in the analysis of the solar neutrino [12–19]. Although it is exact in the limit $\Delta_{12}/\Delta_{13}, A(x)/\Delta_{13} \rightarrow 0$, the actual hierarchy is not so good. Taking the large mixing angle Mikheyev-Smirnov-Wolfenstein (LMA-MSW) solution for the solar neutrino problem and the allowed range of Δ_{13} from the analysis of the atmospheric neutrino oscillation, we find that the hierarchies are rather mild, $\Delta_{12}/\Delta_{13} \lesssim 1/15$, $A/\Delta_{13} \lesssim 1/10$.² This indicates that the above relation possibly has an error around several percent or more.

Such an error is not so important in just verifying the existence of the neutrino masses. Current interest in neutrino physics is, however, not only in verifying the finite masses but also in determining the precise values of the parameters. The allowed region of the mass differences and the mixing angles may be affected due to error in the formula used in the analysis. This error would be more serious in attempts to observe the CP violation in the leptonic sector using the sizes of the unitarity triangle, since it requires a more precise determination of the survival rate [20–23].

²The atmospheric and reactor data also indicate that the hierarchy of the mass differences is not very large [10].

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¹There is, however, a trick only for the two-flavor neutrino oscillation. The equation of motion of the two-flavor neutrino oscillation, which is a coupled first order differential equation, can be converted to a single *first order* differential equation [8]. This makes the numerical calculation quite easy.

One of the aims of the present paper is to examine how precise the reduction formula is. We then propose a next-to-leading order reduction formula, which is surprisingly precise. We assume only the mass hierarchy $\Delta_{12}/\Delta_{13} \lesssim 1/15$ and do not impose any restriction on θ_{13} in order to keep our analysis as general as possible. While we average the survival rate with respect to the final time, we do not average it with respect to the initial time. The reduction formula obtained therefore can be used without averaging over the initial time.

Another example of the reduction formula is the following relation for two successive resonances [6]:

$$S_{3\nu} = P_L P_H \cos^2 \theta_{12} \cos^2 \theta_{13} + P_H (1 - P_L) \sin^2 \theta_{12} \cos^2 \theta_{13} + (1 - P_H) \sin^2 \theta_{13}. \quad (2)$$

Here, θ_{12} is the mixing angle defined in Sec. II A, and P_L and P_H are the jump probabilities for resonances with lower and higher number densities, respectively. This relation is used for the investigation into neutrino oscillation inside a supernova, which has two successive resonances due to the high matter density [25–29]. It is also used for investigation into hypothetical $O(\text{GeV})$ neutrinos from the sun produced by the annihilation of weakly interacting massive particles (WIMPs), which have two successive resonances due to the higher neutrino energy [30]. In Ref. [30], an improved reduction formula Eq. (55) is derived. The reduction formulas Eq. (2) and Eq. (55) are expressed by the jump probability, while the reduction formula Eq. (1) is expressed by the survival rate. We express the reduction formula by the survival rate also in this case and numerically study the difference between these expressions.

The present paper is organized as follows. In Sec. II A, we derive the next-to-leading order reduction formula in the case of one resonance with matter effect. The main result in this section is the next-to-leading reduction formula Eq. (24).

In Sec. II B, we examine the validity of the next-to-leading order reduction formula using numerical calculation. In Sec. III A, we derive the reduction formula in the case of two resonances. The reduction formula Eq. (54), which is the main result of this section, is expressed without the jump probability. In Sec. III B, we numerically verify the validity of the reduction formula obtained in Sec. III A, comparing to that expressed by the jump probability. In Sec. IV, we summarize the results obtained in the present paper.

II. REDUCTION FORMULA FOR ONE RESONANCE

We derive the next-to-leading order reduction formula in the case of one resonance in this section. While the formula is suitable for investigation of the solar neutrino problem, it would be applicable for other cases. We then verify its validity using numerical calculation.

A. Derivation of the next-to-leading order reduction formula

We derive the next-to-leading order reduction formula in the presence of one resonance. Neutrino propagation in matter for three flavors is governed by

$$i \frac{d\phi}{dt} = \left\{ V \begin{pmatrix} 0 & & \\ & \Delta_{12} & \\ & & \Delta_{13} \end{pmatrix} V^\dagger + \begin{pmatrix} A(t) & & \\ & 0 & \\ & & 0 \end{pmatrix} \right\} \phi \quad (3)$$

in the base of the weak eigenstate ϕ_α ($\alpha = e, \mu, \tau$). Here, $A(t) = \sqrt{2} G_F N_e(t)$ is the matter induced mass of the electron neutrino with $N_e(t)$ being the electron number density. The quantities $\Delta_{12} \equiv [\delta m_{12}^2 (= m_2^2 - m_1^2)]/2E$ and $\Delta_{13} \equiv [\delta m_{13}^2 (= m_3^2 - m_1^2)]/2E$ are given in terms of the neutrino masses m_i ($i = 1, 2, 3$) and the neutrino energy E . Throughout the present paper, we assume a hierarchical relation $0 < \Delta_{12} \ll \Delta_{13}$. The mixing matrix V is parametrized as

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (4)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \equiv V_{23}V_{13}V_{12}, \quad (5)$$

where s_{ij} and c_{ij} represent $\sin \theta_{ij}$ and $\cos \theta_{ij}$, respectively. Since we discuss only the survival rate of the electron neutrino, which does not depend on θ_{23} and δ , we set them to zero in the present paper.

For our purpose, it is convenient to work in the base $\xi \equiv V_{13}^\dagger V_{23}^\dagger \phi$, where the time evolution equation (3) is rewritten as

$$i\frac{d\xi}{dt} = \left\{ V_{12} \begin{pmatrix} 0 & & \\ & \Delta_{12} & \\ & & \Delta_{13} \end{pmatrix} V_{12}^\dagger + V_{13}^\dagger \begin{pmatrix} A(t) & & \\ & 0 & \\ & & 0 \end{pmatrix} V_{13} \right\} \xi \quad (6)$$

$$= \left\{ \begin{pmatrix} \Delta_{12}s_{12}^2 + Ac_{13}^2 & \Delta_{12}s_{12}c_{12} & 0 \\ \Delta_{12}s_{12}c_{12} & \Delta_{12}c_{12}^2 & 0 \\ 0 & 0 & \Delta_{13} + As_{13}^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & As_{13}c_{13} \\ 0 & 0 & 0 \\ As_{13}c_{13} & 0 & 0 \end{pmatrix} \right\} \xi. \quad (7)$$

Under the hierarchy $\Delta_{12}, A(x) \ll \Delta_{13}$, the second matrix in the right-hand side (RHS) may be treated as a perturbation, while the 2×2 submatrix in the first matrix should not be dealt with as a small perturbation, because neglect of the submatrix causes a degeneracy of eigenvalues. Thus, neglecting the second matrix, we obtain a block diagonalized equation, and the leading order reduction formula Eq. (1) is easily obtained. In this approximation, the discarded terms have the magnitude of order $O(As_{13}c_{13})$, which handles the error of this formula. We then derive a next-to-leading order reduction formula to reduce the magnitude of the error.

For this purpose, let us rewrite Eq. (7) as

$$i\frac{d\xi}{dt} = \left\{ \begin{pmatrix} 0 & \Delta_{12}s_{12}c_{12} & 0 \\ \Delta_{12}s_{12}c_{12} & \Delta_{12}c_{12}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \Delta_{12}s_{12}^2 + Ac_{13}^2 & 0 & As_{13}c_{13} \\ 0 & 0 & 0 \\ As_{13}c_{13} & 0 & \Delta_{13} + As_{13}^2 \end{pmatrix} \right\} \xi. \quad (8)$$

In order to incorporate the effect of the perturbation $As_{13}c_{13}$, we then diagonalize the second matrix by moving to a new base $\eta = V_\epsilon^\dagger \xi$, with V_ϵ being a time dependent unitary matrix,

$$V_\epsilon = \begin{pmatrix} \cos \epsilon(t) & 0 & \sin \epsilon(t) \\ 0 & 1 & 0 \\ -\sin \epsilon(t) & 0 & \cos \epsilon(t) \end{pmatrix}. \quad (9)$$

The diagonalization goes as

$$V_\epsilon^\dagger \begin{pmatrix} \Delta_{12}s_{12}^2 + Ac_{13}^2 & 0 & As_{13}c_{13} \\ 0 & 0 & 0 \\ As_{13}c_{13} & 0 & \Delta_{13} + As_{13}^2 \end{pmatrix} V_\epsilon = \begin{pmatrix} k_- & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_+ \end{pmatrix}, \quad (10)$$

where the two eigenvalues are given as

$$k_\pm = \frac{1}{2} [A + \Delta_{13} + s_{12}^2 \Delta_{12} \pm \sqrt{(A + \Delta_{13} + s_{12}^2 \Delta_{12})^2 - 4\{\Delta_{13} \Delta_{12} s_{12}^2 + A(c_{13}^2 \Delta_{13} + s_{13}^2 s_{12}^2 \Delta_{12})\}}], \quad (11)$$

and the angle ϵ satisfies the relation

$$\tan 2\epsilon(t) = \frac{A(t) \sin 2\theta_{13}}{\Delta_{13} - s_{12}^2 \Delta_{12} - A(t) \cos 2\theta_{13}}. \quad (12)$$

Although the hierarchy $\Delta_{12}, A \ll \Delta_{13}$ implies that the time dependent angle is small,

$$\epsilon \simeq \frac{A \sin 2\theta_{13}}{2\Delta_{13}}, \quad (13)$$

the angle plays an important role in the improvement of the reduction formula. While the second matrix on the RHS of Eq. (8) is diagonalized, we have additional terms in the time evolution equation in the base of η . First, since the base η depends on time, the left-hand side of Eq. (8) yields the following extra contribution:

$$-iV_\epsilon^\dagger \frac{dV_\epsilon}{dt} = i\epsilon'(t) \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (14)$$

This term has a magnitude of the following order:

$$|\epsilon'(t)| \simeq \left| \frac{A' \sin 2\theta_{13}}{2\Delta_{13}} \right| \sim \frac{A \sin 2\theta_{13}}{2\Delta_{13}L}, \quad (15)$$

where L denotes the typical length in which the matter effect A changes. In the case of the sun, the typical length corresponds to the scale height, $L = R_\odot \sim R_\odot/10.54$ [24].

Second, the first matrix in Eq. (8) is also slightly altered due to the change of base to

$$\begin{aligned}
& V_\epsilon^\dagger \begin{pmatrix} 0 & \Delta_{12}s_{12}c_{12} & 0 \\ \Delta_{12}s_{12}c_{12} & \Delta_{12}c_{12}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V_\epsilon \\
&= \Delta_{12} \begin{pmatrix} 0 & s_{12}c_{12}c_\epsilon & 0 \\ s_{12}c_{12}c_\epsilon & c_{12}^2 & s_{12}c_{12}s_\epsilon \\ 0 & s_{12}c_{12}s_\epsilon & 0 \end{pmatrix} \quad (16)
\end{aligned}$$

$$= \Delta_{12} \begin{pmatrix} 0 & s_{12}c_{12}(1+O(\epsilon^2)) & 0 \\ s_{12}c_{12}(1+O(\epsilon^2)) & c_{12}^2 & s_{12}c_{12}O(\epsilon) \\ 0 & s_{12}c_{12}O(\epsilon) & 0 \end{pmatrix}. \quad (17)$$

Ignoring the small extra off-diagonal terms, we approximately obtain the propagation equation in the time dependent base η

$$\begin{aligned}
i \frac{d\eta}{dt} &= \begin{pmatrix} k_- & \Delta_{12}s_{12}c_{12} + O(\Delta_{12}\sin 2\theta_{12}\epsilon^2) & O\left(\frac{A \sin 2\theta_{13}}{L\Delta_{13}}\right) \\ \Delta_{12}s_{12}c_{12} + O(\Delta_{12}\sin 2\theta_{12}\epsilon^2) & \Delta_{12}c_{12}^2 & O(\Delta_{12}\sin 2\theta_{12}\epsilon) \\ O\left(\frac{A \sin 2\theta_{13}}{L\Delta_{13}}\right) & O(\Delta_{12}\sin 2\theta_{12}\epsilon) & k_+ \end{pmatrix} \eta \\
&\simeq \begin{pmatrix} \tilde{A}_{eff} + \Delta_{12}s_{12}^2 & \Delta_{12}s_{12}c_{12} & 0 \\ \Delta_{12}s_{12}c_{12} & \Delta_{12}c_{12}^2 & 0 \\ 0 & 0 & k_+ \end{pmatrix} \eta, \quad (18)
\end{aligned}$$

where

$$\tilde{A}_{eff} \equiv k_- - \Delta_{12}s_{12}^2. \quad (19)$$

This effective matter effect is well approximated under the hierarchy $\Delta_{12}, A(x) \ll \Delta_{13}$ as

$$\tilde{A}_{eff} \rightarrow c_{13}^2 A - \frac{\sin^2 2\theta_{13} A^2}{4\Delta_{13}}, \quad (20)$$

and reduces to A_{eff} used in the leading formula in the limit $A/\Delta_{13} \rightarrow 0$. The off-diagonal terms discarded in Eq. (18) are smaller than the second matrix of Eq. (7) by the order $\Delta_{12}(\sin 2\theta_{12})/\Delta_{13}$ in the case

$$\frac{A \sin 2\theta_{13}}{L\Delta_{13}} \lesssim \Delta_{12} \sin 2\theta_{12}\epsilon, \quad (21)$$

which can be rewritten as

$$E \lesssim L \delta m_{12}^2 \sin 2\theta_{12}. \quad (22)$$

This condition is well satisfied for the solar neutrinos, since the parameters are $E \lesssim 10$ MeV, $L = R_s \approx 3 \times 10^{14}$ eV⁻¹, and $\delta m_{12}^2 \sim 10^{-4}$ eV². We also discard the element of $O(\Delta_{12} \sin 2\theta_{12}\epsilon^2)$ in the 2×2 submatrix because of its smallness. Thus, the order of the neglected matrix elements in Eq. (18) is much reduced, compared with that in the leading order formula, and a considerable improvement is expected in the next-to-leading order formula obtained below.

The survival rate for the electron neutrino is calculated, by solving Eq. (18), to be

$$\begin{aligned}
S(\nu_e \rightarrow \nu_e) &= \left| (1 \quad 0 \quad 0) V_{23} V_{13} V_\epsilon(t) \begin{pmatrix} \mathbf{R}_{2 \times 2}(t, 0) & \mathbf{0} \\ \mathbf{0}^T & \exp\left(-i \int_0^t k_+ dt\right) \end{pmatrix} V_\epsilon(0)^\dagger V_{13}^\dagger V_{23}^\dagger \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right|^2 \\
&= \cos^2 \theta_{13} \cos^2[\theta_{13} + \epsilon(0)] \left| (1 \quad 0) \mathbf{R}_{2 \times 2}(t, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 + \sin^2 \theta_{13} \sin^2[\theta_{13} + \epsilon(0)].
\end{aligned} \tag{23}$$

Here, $\mathbf{R}_{2 \times 2}(t, 0)$ is the resolvent matrix for the two-flavor neutrino oscillation with the matter effect \tilde{A}_{eff} and mass squared difference Δ_{12} . Thus we get the next-to-leading order reduction formula

$$\begin{aligned}
S_{3\nu}(\nu_e \rightarrow \nu_e) &= \cos^2 \theta_{13} \cos^2[\theta_{13} + \epsilon(0)] \tilde{S}_{2\nu,eff}(\tilde{A}_{eff}, \Delta_{12}) \\
&\quad + \sin^2 \theta_{13} \sin^2[\theta_{13} + \epsilon(0)].
\end{aligned} \tag{24}$$

Here, $\tilde{S}_{2\nu,eff}(\tilde{A}_{eff}, \Delta_{12})$ is the survival rate for the electron neutrino in the effective two-flavor system with the matter effect \tilde{A}_{eff} and mass squared difference Δ_{12} . This reduction formula has a twofold improvement compared with the leading order formula Eq. (1); the matter effect A_{eff} is replaced by \tilde{A}_{eff} and the angle is corrected from θ_{13} to $\theta_{13} + \epsilon$ at the production point of the neutrino, $t=0$.³

This next-to-leading formula is valid no matter whether or not the resonance due to the matter effect occurs. We can, however, further simplify this relation if the effective survival rate $\tilde{S}_{2\nu,eff}(\tilde{A}_{eff}, \Delta_{12})$ is represented using the jump probability between the two mass eigenstates at the resonance point $P_c(\tilde{A}_{eff}, \theta, \Delta)$, which is almost the same as that for A_{eff} . The two-flavor survival rate is represented using the jump probability as [2]⁴

$$\tilde{S}_{2\nu,eff} = \frac{1}{2} + \left\{ \frac{1}{2} - P_c(\tilde{A}_{eff}, \theta_{12}, \Delta_{12}) \right\} \cos 2\tilde{\theta}_{m0} \cos 2\theta_{12} \tag{25}$$

where $\tilde{\theta}_{m0}$ is the mixing angle in the matter at the neutrino production point,

³Although this *angle* correction is pointed out in Ref. [13], it is neglected because of its smallness.

⁴This relation is valid only if the survival rate is averaged over with respect to the initial time. Strictly speaking, we therefore cannot use the obtained simplified reduction formula without averaging over the survival rate with respect to the initial time. This is not the case for the reduction formula obtained in the present paper, Eq. (24).

$$\cos 2\tilde{\theta}_{m0} = \frac{\Delta_{12} \cos 2\theta_{12} - \tilde{A}_{eff}(0)}{\sqrt{[\Delta_{12} \cos 2\theta_{12} - \tilde{A}_{eff}(0)]^2 + (\Delta_{12} \sin 2\theta_{12})^2}}. \tag{26}$$

We say that the resonance is complete in the case that the electron neutrino is produced far above the resonance point. In this case, the survival rate is represented by a jump probability as in Eq. (25). On the other hand, we say that the resonance is incomplete in the case that the neutrino is produced too near the resonance point. In this case, we cannot represent the survival rate by the jump probability. The survival rate for an electron neutrino produced at a point with infinite matter effect, $A_{eff} = c_{13}^2 A \rightarrow \infty$, is represented by

$$S_{2\nu,eff} = \frac{1}{2} - \left\{ \frac{1}{2} - P_c(A_{eff}, \theta_{12}, \Delta_{12}) \right\} \cos 2\theta_{12}. \tag{27}$$

For the case $P_c(\tilde{A}_{eff}, \theta_{12}, \Delta_{12}) \simeq P_c(A_{eff}, \theta_{12}, \Delta_{12})$, the survival rate $\tilde{S}_{2\nu,eff}$ is rewritten as

$$\tilde{S}_{2\nu,eff} = \frac{1}{2} - \left(S_{2\nu,eff} - \frac{1}{2} \right) \cos 2\tilde{\theta}_{m0}. \tag{28}$$

Substituting this relation into Eq. (24), we obtain the simplified formula

$$\begin{aligned}
S_{3\nu}(\nu_e \rightarrow \nu_e) &= \cos^2 \theta_{13} \cos^2(\theta_{13} + \epsilon) \\
&\quad \times \left\{ \frac{1}{2} - \left(S_{2\nu,eff} - \frac{1}{2} \right) \cos 2\tilde{\theta}_{m0} \right\} \\
&\quad + \sin^2 \theta_{13} \sin^2(\theta_{13} + \epsilon).
\end{aligned} \tag{29}$$

In some cases, the survival rate $S_{2\nu,eff}$ is known exactly, and the simplified formula can be represented in completely

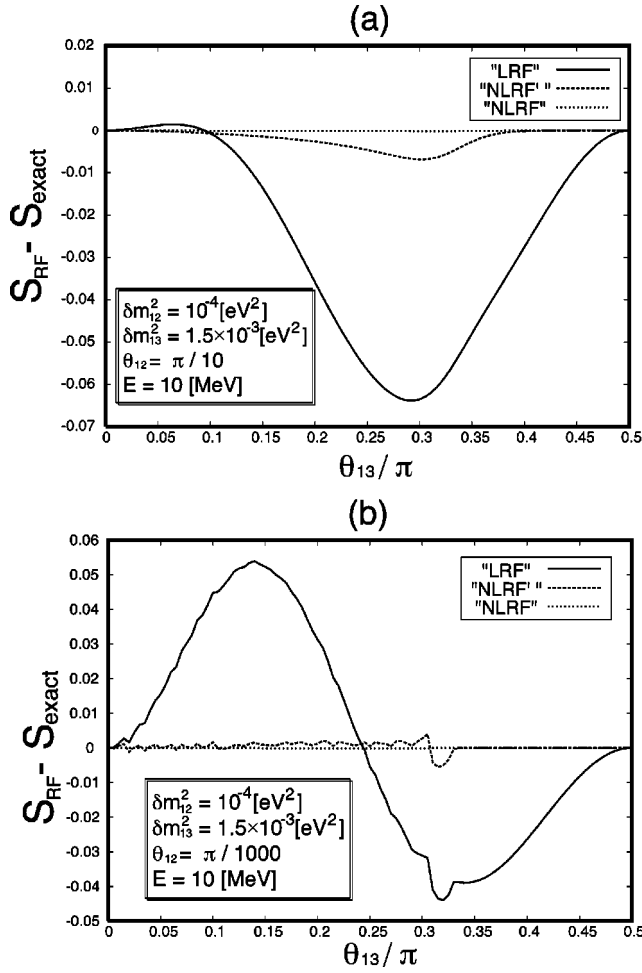


FIG. 1. The error of the reduction formula is shown for the parameters written in the figure as a function of θ_{13} in (a). The result using the leading formula Eq. (1), denoted by “LRF,” has the largest error of about 0.07. The next-to-leading order formula Eq. (24) without the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$, denoted by “NLRF’,” has an error of about 0.01. The next-to-leading order formula Eq. (24) with \tilde{A}_{eff} , denoted by “NLRF,” has the smallest error of order 10^{-4} . Larger errors arise for larger $\sin 2\theta_{13}$. (b) The same (a) for smaller value of θ_{12} . The magnitudes of each error are almost the same as those of (a). They arise, however, at relatively small values of θ_{13} .

analytic form. For example, when the matter density distribution is of the exponential type, $A = A_0 \exp(-r/r_0)$, which we adopt in the numerical analysis in the next section, the survival rate is represented analytically using the jump probability,

$$P_c(A, \theta, \Delta) = \frac{\exp(2\pi\Delta r_0 \cos \theta^2) - 1}{\exp(2\pi\Delta r_0) - 1}. \quad (30)$$

Using Eqs. (19),(24),(26),(27),(28),(30), the survival rate for the three-flavor neutrino oscillation $S_{3\nu}$ is represented analytically under the condition $P_c(\tilde{A}_{eff}, \theta_{12}, \Delta_{12}) \simeq P_c(A_{eff}, \theta_{12}, \Delta_{12})$, in this case. We notice that this simpli-

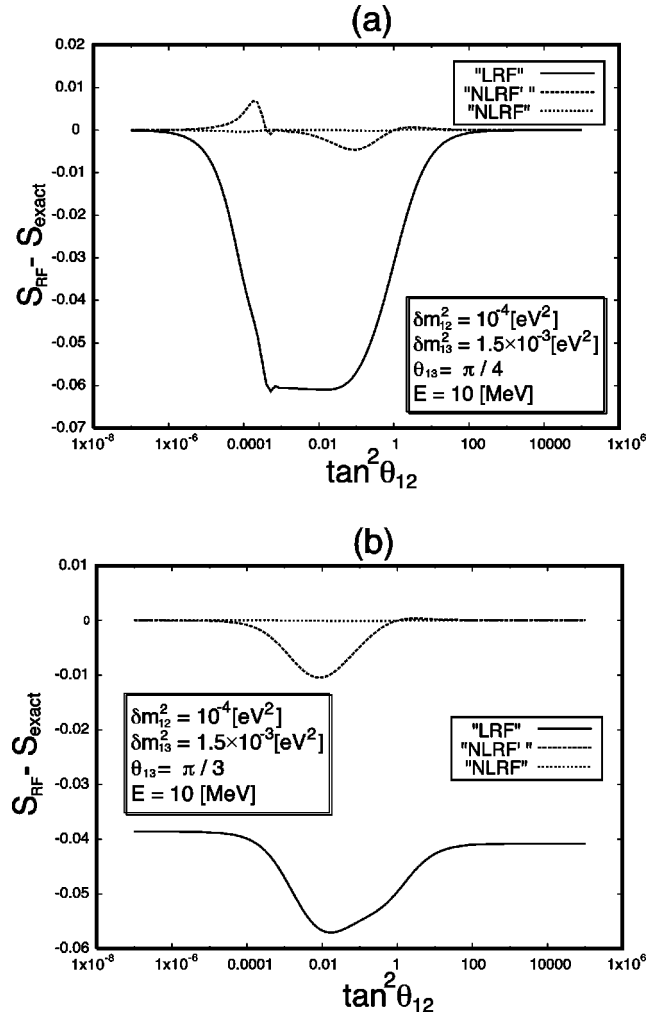


FIG. 2. (a) The error of the reduction formula for the parameters written in the figure as a function of θ_{12} . The leading formula Eq. (1) has the largest error of about 0.06. The next-to-leading order reduction formula Eq. (24) without the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$ has error around 0.01. The next-to-leading order reduction formula Eq. (24) with the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$ has an error of only order 10^{-4} . Although the error of the leading order approximation vanishes for extremely large or small values of θ_{12} , this is accidental one. (b) Same as (a) for $\theta_{13} = \pi/3$. Although the magnitudes of the errors are almost the same as in (a) there remains a finite error for extremely large or small values of θ_{12} . (b) shows the result for the same parameter set except for $\theta_{13} = \pi/3$.

fication is valid only in the case that the survival rate can be represented by the jump probability. It cannot be used in the case of incomplete resonance as is seen from the numerical analysis in the next subsection.

B. Numerical confirmation of the validity of the next-to-leading order reduction formula

In the present subsection, we numerically examine the precision of the next-to-leading order reduction formula. As an example, we use an exponential type electron density dis-

tribution $N(r) = 245N_A \exp(-10.54r/R_\odot)$, which is a good approximation for the solar neutrino propagation in most regions [24].⁵ Here, N_A is Avogadro's number. We consider the mass squared differences $10^{-10} < \delta m_{12}^2 < 10^{-4} \text{ eV}^2$, $1.5 \times 10^{-3} < \delta m_{13}^2 < 4 \times 10^{-3} \text{ eV}^2$ and the whole range of mixing angles $0 < \theta_{12} < \pi/2, 0 < \theta_{13} < \pi/2$. We average over the survival rate with respect to the final time, and do not do it with respect to the initial time, since we do not use averaging in deriving Eq. (24).

Since we have a vast parameter region, we do not investigate the whole parameter region. Instead, we investigate the parameter region where the error of the reduction formula is expected to be large, since we would like to be conservative concerning the precision of the reduction formula. We first fix the values of the mass squared differences as $\delta m_{12}^2 = 10^{-4} \text{ eV}^2$ and $\delta m_{13}^2 = 1.5 \times 10^{-3} \text{ eV}^2$, where the hierarchy between the mass differences is minimum in the region we consider and the error is expected to be maximally enhanced.

We show the error of the reduction formula, the deviation from the exact result $S_{RF} - S_{exact}$, as a function of θ_{13} in Fig. 1(a). The result derived from the leading formula Eq. (1) is known to have the largest error of about 0.07. The next-to-leading order formula Eq. (24) without the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$ has an error of about 0.01. The next-to-leading order formula Eq. (24) with the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$ has the smallest error, of order 10^{-4} . We show the same figure for smaller value of θ_{12} in Fig. 1(b). The magnitudes of each error are almost the same as those of Fig. 1(a). They arise, however, at relatively small value of θ_{13} in this case. Generally, the errors are small for small $\sin 2\theta_{13}$ as is expected from the fact that the angle correction ϵ is proportional to $\sin 2\theta_{13}$ [Eq. (13)] and the correction to the matter effect is proportional to $\sin^2 2\theta_{13}$ [Eq. (20)].

We show the results of the error of the reduction formula as a function of θ_{12} for $\sin^2 2\theta_{13} = 1$ in Fig. 2(a), where the error is expected to be large. The leading order formula Eq. (1) has the largest error of about 0.06. The next-to-leading order reduction formula Eq. (24) without the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$ has an error around 0.01. The next-to-leading order reduction formula Eq. (24) with the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$ has an error of order 10^{-4} . Although the error of the leading order approximation vanishes for extremely large or small values of θ_{12} , this is accidental. Since the survival rate $\tilde{S}_{2\nu,eff}$ is apparently unity for extremely small or large θ_{12} , the error comes from the angle correction ϵ in this region. The correction of $S_{3\nu}$ due to the mixing angle correction is estimated to be

⁵This approximation is not so good near the core and the surface of the sun. The density around the core causes the error of the reduction formula through the angle, ϵ . Since the real density is smaller than the value obtained by the exponential approximation, the real error of the reduction formula for the sun is expected to be smaller than the value obtained in the present numerical calculation.

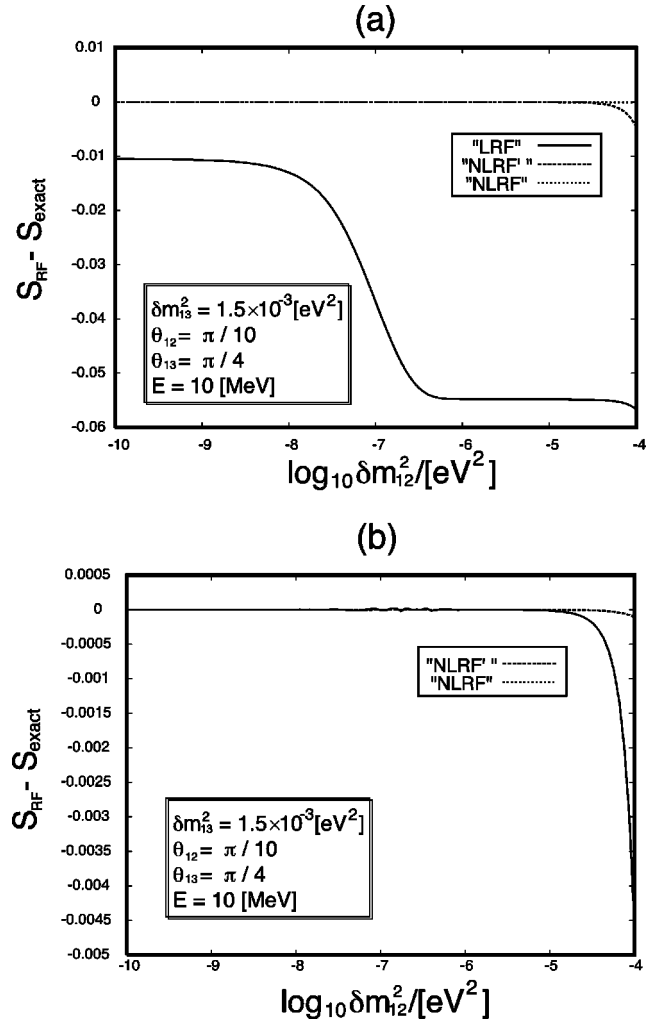


FIG. 3. The error of the reduction formula is shown for the parameters written in the figure as a function of δm_{12}^2 . (a) and (b) are the same except for the scale of the vertical axis. The leading formula Eq. (1) has the largest error of about 0.05. The other lines, which have errors less than 0.01, are the next-to-leading order formula Eq. (24) with and without the replacement, $A_{eff} \rightarrow \tilde{A}_{eff}$. For the leading formula, the error remains even for small δm_{12}^2 , although it tends to vanish for the next-to-leading order formula.

$$\frac{\sin^2 2\theta_{13} A(0)}{2\Delta_{13}} (s_{13}^2 - c_{13}^2 \tilde{S}_{2\nu,eff}) \quad (31)$$

from Eqs. (13),(24). For $\theta_{13} = \pi/4$ and $\tilde{S}_{2\nu,eff} = 1$, this quantity accidentally vanishes. This error therefore remains for other values of θ_{13} as shown in Fig. 2(b), which is the same figure as Fig. 2(a) except for $\theta_{13} = \pi/3$.

We next confirm that the error tends to be smaller for a larger hierarchy, i.e., smaller δm_{12}^2 and larger δm_{13}^2 , as is suggested by the fact that the errors are handled by the relative importance of the neglected off-diagonal elements compared with the dominant matrix element Δ_{13} , i.e., $A(0) \times (\sin 2\theta_{13})/2\Delta_{13}$ for the leading order formula and $\Delta_{12}(\sin 2\theta_{12})\epsilon/2\Delta_{13} \sim \Delta_{12}(\sin 2\theta_{12})A(0)(\sin 2\theta_{13})/4\Delta_{13}^2$ for

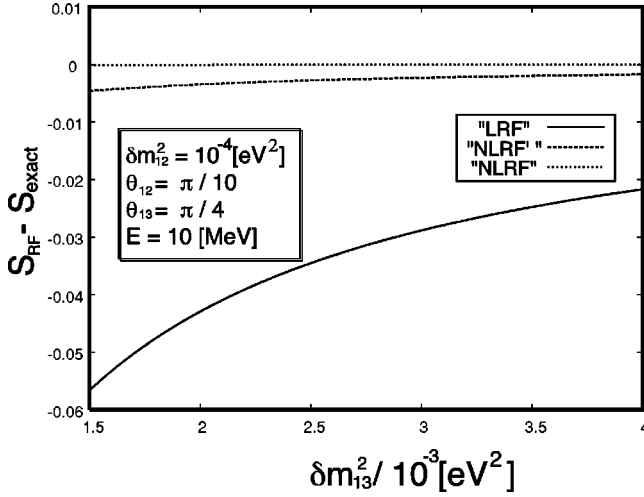


FIG. 4. The error of the reduction formula is shown for the parameters written in the figure as a function of δm^2_{13} . The leading formula Eq. (1) has the largest error of about 0.06. The other lines, which have errors less than 0.01, are the next-to-leading order formula Eq. (24), with and without the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$. All errors tend to be reduced for larger δm^2_{13} .

the next-to-leading order formula. We first show the δm^2_{12} dependence of the error in Fig. 3. The leading formula Eq. (1) has the largest error as in the above cases. There remains a finite error, even for extremely small values of δm^2_{12} . The errors for the next-to-leading order approximations tend to vanish for smaller values of δm^2_{12} . This is because the error is handled by $\Delta_{12}(\sin 2\theta_{12})A(0)(\sin 2\theta_{13})/4\Delta_{13}^2$, and sug-

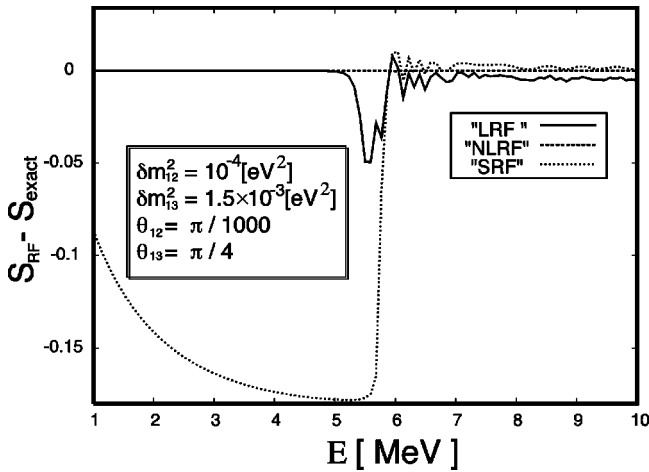


FIG. 5. The error of the reduction formula is shown for the parameters written in the figure as a function of the neutrino energy E . The error of the simplified formula is very large for smaller values of the neutrino energies. On the other hand, the leading order reduction formula has an error around 0.06. The next-to-leading order reduction formula has an error of order 10^{-4} .

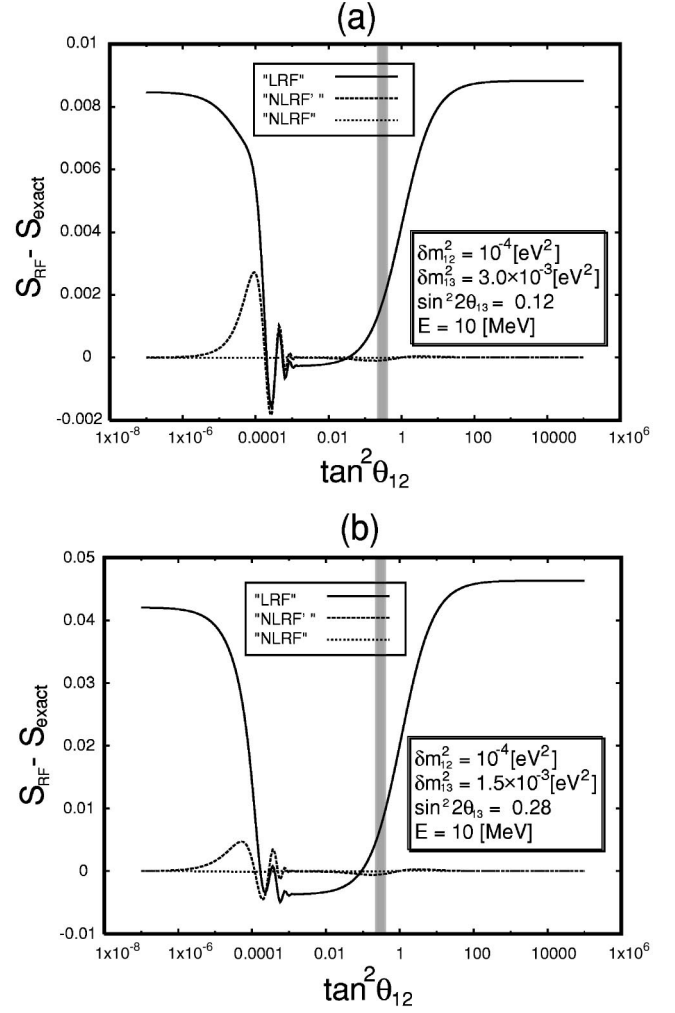


FIG. 6. The error of the reduction formula for the realistic parameters (a) $\delta m^2_{13} = 3.0 \times 10^{-3} \text{ eV}^2$, $\sin^2 2\theta_{13} = 0.12$ and (b) $\delta m^2_{13} = 1.5 \times 10^{-3} \text{ eV}^2$, $\sin^2 2\theta_{13} = 0.28$. The error of the leading formula Eq. (1) is shown as “LRF.” The errors of the next-to-leading order formula Eq. (24) without and with the replacement $A_{eff} \rightarrow \tilde{A}_{eff}$ are shown as “NLRF” and “NRF,” respectively. The allowed region from the solar neutrino data is shaded. The error for the case (b) is found to be larger than that of (a), and the errors are rather small for the shaded region in both cases.

gests that the correction to the matter effect Eq. (19) is negligible in this case. Therefore, the correction to the matter effect will be negligible for the LOW and vacuum oscillation (VO) solutions of the solar neutrino oscillation.

We next show the dependence of the error on the mass difference δm^2_{13} in Fig. 4. The leading formula Eq. (1) again has the largest error. The error tends to be reduced for larger values of δm^2_{13} , as is expected.

All of the above numerical results strongly suggest that the reduction formula at the leading order potentially has an error around 0.1. This corresponds to the fact that the error due to the neglect of the mixing angle correction $\epsilon(0)$ is estimated as

$$\frac{\sin^2 2\theta_{13}A(0)}{2\Delta_{13}}(s_{13}^2 - c_{13}^2 \tilde{s}_{2\nu,eff}) \leq 0.1, \quad (32)$$

from Eqs. (13),(24). The numerical calculations also suggest that the next-to-leading order correction drastically improves the reduction formula and the error is reduced to be of order 10^{-4} . These results indicate that the leading order reduction formula is pretty good for a rough estimation of the allowed parameter region and the next-to-leading order formula is necessary for its precise determination. A detailed analysis of the allowed parameter region from the solar neutrino data is not within the scope of the present paper.

Finally, we examine how precise the simplified reduction formula is.⁶ Since this formula is derived under the condition that the survival rate can be expressed by the jump probability, it will not be precise when the resonance is incomplete. For the case of the above example, $N(r) = 245N_A \exp(-10.54r/R_\odot)$, the resonance is incomplete for neutrinos which have smaller energies ~ 1 MeV. The simplified reduction formula therefore is not expected to be valid in this case. On the other hand, the simplified formula is expected to be precise for neutrinos that have larger energies ~ 10 MeV, because in this case the resonance is expected to occur almost completely. We show in Fig. 5 the error of the reduction formula as a function of the neutrino energy E . One can observe that the simplified formula is not valid for smaller energies of the neutrino from this figure. The errors of both the leading and the next-to-leading order reduction formulas are also shown in the figure. According to the above result, we learn that we should not use the simplified formula without ensuring that the resonance is complete, although the formula is attractive because of its simplicity.

So far we have not restricted the values of the mixing angles, since our aim was to conservatively clarify the precision of the next-to-leading order reduction formula. However, there exist meaningful upper bounds on the mixing angle θ_{13} from the reactor experiments CHOOZ [31] and Palo Verde [32]. The smaller the mixing angle θ_{13} becomes, the better the precision of reduction formulas is expected to be. Thus we have calculated the errors of the reduction formulas for $\sin^2 2\theta_{13} = 0.12$, the upper bound corresponding to the best fit value $\delta m_{13}^2 = 3 \times 10^{-3} \text{ eV}^2$ to account for the atmospheric neutrino data [33]. The results are shown in Fig. 6(a), as functions of the remaining mixing angle $\tan^2 \theta_{12}$. Also shown by a shaded area is the region allowed at 90% C.L. by the LMA MSW solution [34]. We learn from this figure that the leading order formula has an error up to 1% or so for some values of $\tan^2 \theta_{12}$, while the error of the next-to-leading order formula Eq. (24) is essentially negligible. The error of the leading order formula is, however, less than 0.2% if we remain in the shaded region. In Fig. 6(b), we have also shown the errors of the reduction formulas for $\sin^2 2\theta_{13} = 0.28$, the upper bound corresponding to the value

$\delta m_{13}^2 = 1.5 \times 10^{-3} \text{ eV}^2$, the lowest mass squared difference to account for the atmospheric neutrino data [33] at 90% C.L. We now learn that the error of the leading order formula is enhanced by both smaller δm_{13}^2 and larger θ_{13} ; the error reaches 5% or so. Even for the shaded region, the error can be up to 1%.

From these analyses, we can say that for the values of θ_{13} implied by the reactor experiments [31] the error of the leading order formula can be rather small, while that of the next-to-leading order formula is completely negligible. If we further impose the condition suggested by the LMA MSW solution, the error is even smaller, i.e., at most 1%. Such an error, however, will be problematic for precision tests of neutrino experiments, such as the (indirect) search for CP violation [20–23], which require the precise determination of the size of the unitarity triangle with an accuracy of a few percent [23], and therefore a better precision of the reduction formula itself. We hope that the next-to-leading order reduction formula proposed here, being a simple formula to use, will be useful for study of the precision tests of neutrino experiments.

III. REDUCTION FORMULA IN THE CASE OF TWO RESONANCES

We derive a reduction formula when ν_e experiences two successive resonances in Sec. III A. The expression we derive is slightly different from previously proposed ones and is applicable for wider situations. This formula is relevant for investigation into the supernova neutrino data or the hypothetical very high energy solar neutrino data due to the annihilation of WIMPs [25–30]. We then verify the validity of the reduction formula using the numerical calculation in Sec. III B.

A. Derivation of the reduction formula in the case of two resonances

Suppose ν_e is produced at time 0 and detected at time t , going through “higher” and “lower” resonances, caused by matching the matter effect with Δ_{13} and Δ_{12} , respectively. We divide the time interval into $[0, t_M]$ and $[t_M, t]$ ($0 < t_M < t$), where the conditions $\Delta_{12} \ll \Delta_{13}, A$ and $\Delta_{12}, A \ll \Delta_{13}$ are met, i.e., the higher and lower resonances are operative. The intermediate time t_M is chosen so that $\Delta_{12} \ll A(t_M) \ll \Delta_{13}$.

We first consider the time range where the higher resonance occurs, i.e., $\Delta_{12} \ll \Delta_{13}, A(t)$. A convenient base to describe this region is

$$\omega \equiv U\phi, \quad U = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_{23}^\dagger, \quad (33)$$

where the time evolution equation Eq. (3) can be cast into

⁶Strictly speaking, we cannot use the simplified reduction formula without averaging over the survival rate with respect to the initial time. The error due to this, however, seems to be very small numerically.

$$i\frac{d\omega}{dt} = \left\{ UV \begin{pmatrix} 0 & & \\ & \Delta_{12} & \\ & & \Delta_{13} \end{pmatrix} V^\dagger U^\dagger + U \begin{pmatrix} A(t) & & \\ & 0 & \\ & & 0 \end{pmatrix} U^\dagger \right\} \omega \quad (34)$$

$$= \begin{pmatrix} c_{12}^2 \Delta_{12} & -s_{12} c_{12} c_{13} \Delta_{12} & s_{12} c_{12} s_{13} \Delta_{12} \\ -s_{12} c_{12} c_{13} \Delta_{12} & s_{12}^2 c_{13}^2 \Delta_{12} + s_{13}^2 \Delta_{13} + A(t) & s_{13} c_{13} \Delta_{13} - s_{12}^2 s_{13} c_{13} \Delta_{12} \\ s_{12} c_{12} s_{13} \Delta_{12} & s_{13} c_{13} \Delta_{13} - s_{12}^2 s_{13} c_{13} \Delta_{12} & s_{12}^2 s_{13}^2 \Delta_{12} + c_{13}^2 \Delta_{13} \end{pmatrix} \omega \quad (35)$$

$$\cong \begin{pmatrix} \cos 2\theta_{12} \Delta_{12} & 0 & 0 \\ 0 & s_{13}^2 (\Delta_{13} - s_{12}^2 \Delta_{12}) + A(t) & s_{13} c_{13} (\Delta_{13} - s_{12}^2 \Delta_{12}) \\ 0 & s_{13} c_{13} (\Delta_{13} - s_{12}^2 \Delta_{12}) & c_{13}^2 (\Delta_{13} - s_{12}^2 \Delta_{12}) \end{pmatrix} \omega + s_{12}^2 \Delta_{12} I \omega \quad (36)$$

$$\equiv \begin{pmatrix} \cos 2\theta_{12} \Delta_{12} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{M}_H(t) \end{pmatrix} \omega. \quad (37)$$

Here, we neglect the term proportional to the unit matrix I .
The resolvent matrix in this base R^H is

$$R_H(t, 0) \cong \begin{pmatrix} e^{-i \cos 2\theta_{12} \Delta_{12} t} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}_{2 \times 2}^H(t, 0) \end{pmatrix}, \quad (38)$$

where $\mathbf{R}_{2 \times 2}^H$ is the resolvent matrix for the effective two-flavor neutrino system with matter effect $A(t)$, mass squared difference $\Delta_{13} - s_{12}^2 \Delta_{12}$, and mixing angle θ_{13} .

We next consider the time range, $[t_M, t]$, where the lower resonance occurs, i.e., $\Delta_{12}, A(t) \ll \Delta_{13}$. A convenient base in this region is the time dependent base $\kappa(t) \equiv Z(t)\phi$, by which the Hamiltonian for fixed t is diagonalized,

$$Z(t) \left\{ V \begin{pmatrix} 0 & & \\ & \Delta_{12} & \\ & & \Delta_{13} \end{pmatrix} V^\dagger + \begin{pmatrix} A(t) & & \\ & 0 & \\ & & 0 \end{pmatrix} \right\} Z^\dagger(t) \\ = \begin{pmatrix} k_1(t) & 0 & 0 \\ 0 & k_2(t) & 0 \\ 0 & 0 & k_3(t) \end{pmatrix} \quad (k_1 < k_2 < k_3). \quad (39)$$

Because of the hierarchy $\Delta_{12}, A \ll \Delta_{13}$, the Hamiltonian and

therefore the resolvent are approximately block diagonalized:

$$R_L(t, t_M) \cong \begin{pmatrix} \mathbf{R}_{2 \times 2}^L(t, t_M) & \mathbf{0} \\ \mathbf{0}^T & e^{-i \int_{t_M}^t k_3(t') dt'} \end{pmatrix}. \quad (40)$$

Since the lower resonance is nearly complete, i.e., $\Delta_{12} \ll A(t_M) \ll \Delta_{13}$, the 2×2 resolvent matrix can be generally written as

$$\mathbf{R}_{2 \times 2}^L(t, t_M) \cong \begin{pmatrix} 1 & 0 \\ 0 & e^{-i \Delta_{12}(t - t_L)} \end{pmatrix} \\ \times \begin{pmatrix} \sqrt{1 - P_L} e^{i\alpha} & \sqrt{P_L} e^{i\beta} \\ \sqrt{P_L} e^{i\gamma} & -\sqrt{1 - P_L} e^{-i(\alpha - \beta - \gamma)} \end{pmatrix}, \quad (41)$$

where t_L is some time far after the lower resonance, i.e., $A(t_L) \ll \Delta_{12}$, and P_L is the jump probability between adiabatic states with respect to the lower resonance.

Using these resolvent matrices, the survival rate of ν_e is given as

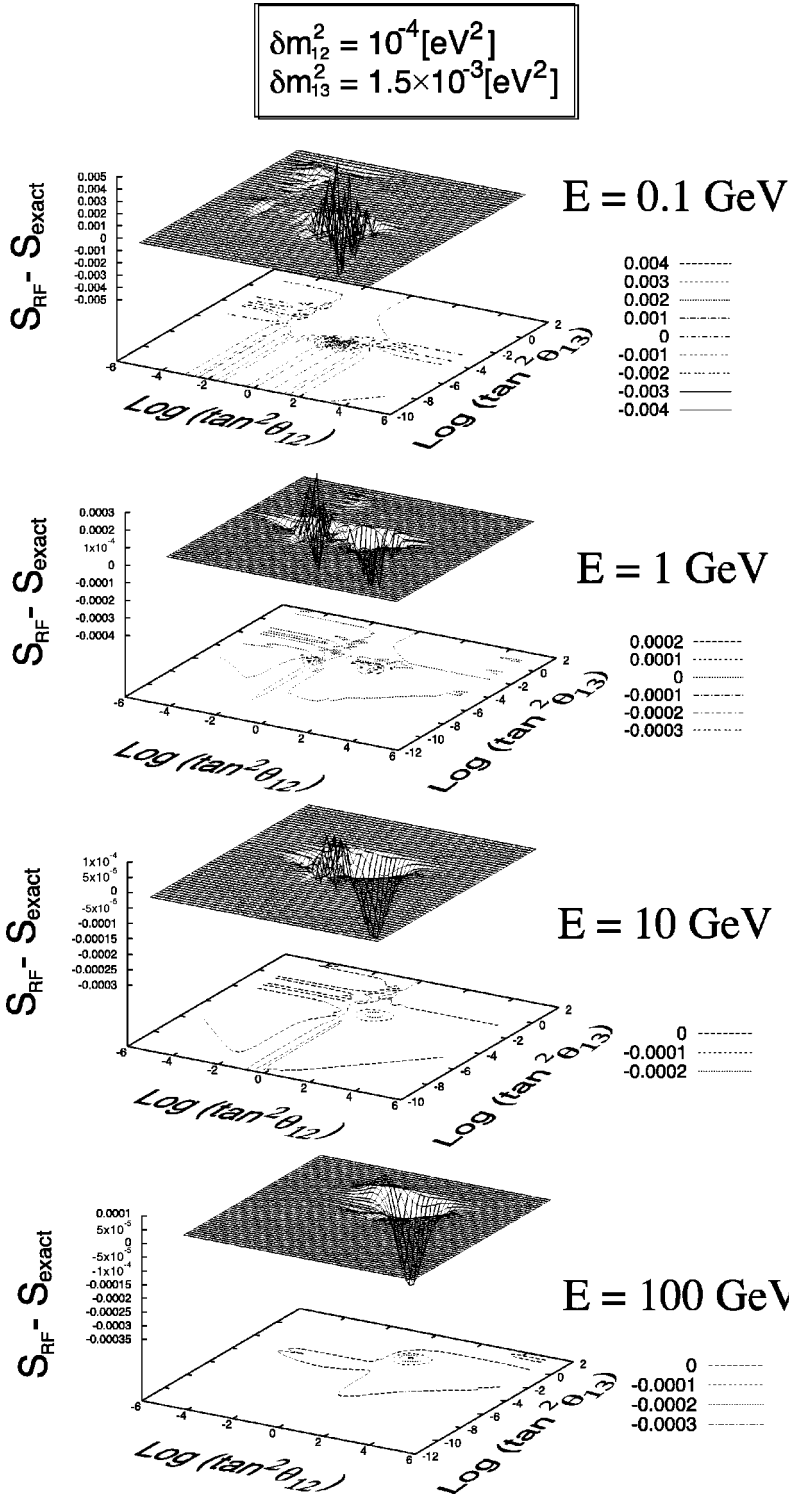


FIG. 7. The error of the reduction formula Eq. (54) is shown for the parameters written in the figure as a function of θ_{12} and θ_{13} for various values of the neutrino energy $E = 0.1, 1, 10, 100$ GeV. The largest error occurs in the case of the lowest energy $E = 0.1$ GeV. Even in this case, the error has magnitude less than 0.005.

$$S = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} Z^\dagger(t) \begin{pmatrix} \mathbf{R}_{2 \times 2}^L(t, t_M) & \mathbf{0} \\ \mathbf{0}^T & \exp\left(-i \int_{t_M}^t k_3(t') dt'\right) \end{pmatrix} Z(t_M) U^\dagger \begin{pmatrix} e^{-i \cos 2\theta_{12} \Delta_{12} t_M} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{R}_{2 \times 2}^H(t_M, 0) \end{pmatrix} U \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right|^2. \quad (42)$$

Here, the matrix $Z(t_M)U^\dagger$ is approximately written as

$$Z(t_M)U^\dagger \simeq \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{V}_{3m}^\dagger(t_M) \end{pmatrix} \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{13} & -s_{13} \\ 0 & s_{13} & c_{13} \end{pmatrix}, \quad (43)$$

where $\mathbf{V}_{3m}^\dagger(t)$ is the matrix that diagonalizes the 2×2 mass matrix \mathbf{M}_H as

$$\mathbf{V}_{3m}^\dagger(t)\mathbf{M}_H(t)\mathbf{V}_{3m}(t) = \begin{pmatrix} k_2(t) & 0 \\ 0 & k_3(t) \end{pmatrix}. \quad (44)$$

From Eq. (42) and Eq. (43) and $U(1,0,0)^T = (0,1,0)^T$,

$$S \simeq \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} Z^\dagger(t) \begin{pmatrix} \mathbf{R}_{2 \times 2}^L(t, t_M) & \mathbf{0} \\ \mathbf{0}^T & \exp\left(-i \int_{t_M}^t k_3(t') dt'\right) \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{V}_{3m}^\dagger(t_M) \mathbf{R}_{2 \times 2}^H(t_M, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \right|^2. \quad (45)$$

In order to express the survival rate in terms of the two-flavor survival rate, we set

$$|a|^2 = \frac{S_{2 \times 2}^H - \sin^2 \theta_{13}}{\cos 2 \theta_{13}}, \quad |b|^2 = \frac{\cos^2 \theta_{13} - S_{2 \times 2}^H}{\cos 2 \theta_{13}}. \quad (51)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \mathbf{V}_{3m}^\dagger(t_M) \mathbf{R}_{2 \times 2}^H(t_M, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (46)$$

The survival rate is thus expressed as

$$S = \left| \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} Z^\dagger(t) \begin{pmatrix} \mathbf{R}_{2 \times 2}^L(t, t_M) \begin{pmatrix} 0 \\ a \end{pmatrix} \\ b \exp\left(-i \int_{t_M}^t k_3(t') dt'\right) \end{pmatrix} \right|^2 \quad (52)$$

The survival rate for the two-flavor system with respect to the higher resonance is expressed by these quantities as

$$S_{2 \times 2}^H = \left| (10) \mathbf{R}_{2 \times 2}^H(t, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 \quad (47)$$

$$= \left| (10) \mathbf{V}_{3m}(t) \mathbf{V}_{3m}^\dagger(t) \mathbf{R}_{2 \times 2}^H(t, 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right|^2 \quad (48)$$

$$\simeq \left| (10) \mathbf{V}_{3m}(t) \begin{pmatrix} e^{-i \int_{t_M}^t k_2(t') dt'} & 0 \\ 0 & e^{-i \int_{t_M}^t k_3(t') dt'} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 \quad (49)$$

$$\rightarrow |a|^2 \cos^2 \theta_{13} + |b|^2 \sin^2 \theta_{13}. \quad (50)$$

In Eq. (50), we averaged with respect to the final time t and used

$$\mathbf{V}_{3m}(t) \simeq \mathbf{V}_{3m}(t_M) \simeq \begin{pmatrix} c_{13} & s_{13} \\ -s_{13} & c_{13} \end{pmatrix}.$$

Using the unitarity condition $|a|^2 + |b|^2 = 1$, we get

$$\begin{aligned} & \rightarrow |a|^2 \cos^2 \theta_{13} \{P_L \cos^2 \theta_{12} + (1 - P_L) \sin^2 \theta_{12}\} \\ & + |b|^2 \sin^2 \theta_{13}. \end{aligned} \quad (53)$$

In Eq. (53), we averaged the survival rate with respect to the final time and used $Z^\dagger(t) = V$. Using Eq. (51), we get the reduction formula, which is our final result,

$$S = S_{2 \times 2, \text{inf}}^L \frac{S_{2 \times 2}^H - \sin^2 \theta_{13}}{\cos 2 \theta_{13}} \cos^2 \theta_{13} + \frac{\cos^2 \theta_{13} - S_{2 \times 2}^H}{\cos 2 \theta_{13}} \sin^2 \theta_{13}. \quad (54)$$

Here, $S_{2 \times 2, \text{inf}}^L = [P_L \cos^2 \theta_{12} + (1 - P_L) \sin^2 \theta_{12}]$ is the two-flavor survival rate for the lower resonance in the case that the electron neutrino is produced at a point with $A \rightarrow \infty$. We verify the validity of this formula by a numerical method in the next subsection.

This reduction formula coincides with the reduction formula obtained in Ref. [30]:

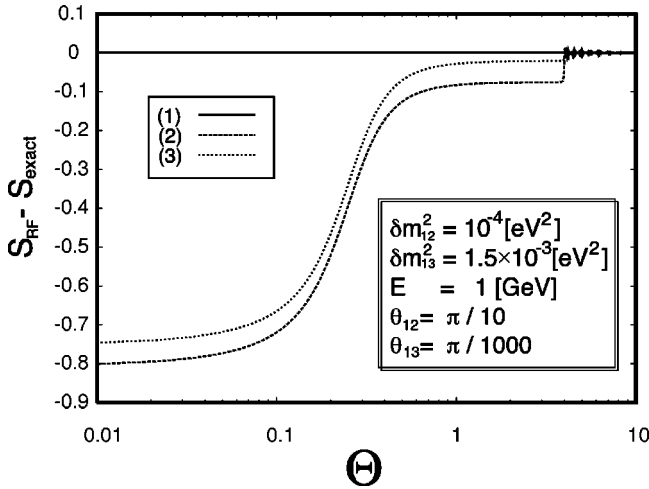


FIG. 8. The errors of the reduction formulas Eq. (54), Eq. (55), and Eq. (2) as a function of the initial electron density. The horizontal axis is Θ which is defined as $N(r) = 245\Theta N_A \exp(-10.54r/R_\odot)$. The lines (1), (2), and (3) correspond to Eqs. (54), (2), and (55), respectively. The reduction formula Eq. (54) is extremely accurate for all values of Θ . While the reduction formula Eq. (55) is better than Eq. (2), neither is accurate for smaller values of Θ .

$$S = P_2^H \cos^2 \theta_{13} S_{2 \times 2, \text{inf}}^L + (1 - P_2^H) \sin^2 \theta_{13},$$

$$\left(P_2^H = \frac{1}{2} + \left(\frac{1}{2} - P_H \right) \cos 2\theta_{13m} \right), \quad (55)$$

provided the higher resonance is complete and the two-flavor survival rate $S_{2 \times 2}^H$ can be written in terms of the jump probability P_H as

$$S_{2 \times 2}^H = \frac{1}{2} + \left(\frac{1}{2} - P_H \right) \cos 2\theta_{13m} \cos 2\theta_{13}. \quad (56)$$

This reduction formula Eq. (55) is the same as the reduction formula Eq. (2) obtained in Ref. [6], if θ_{13} is set to be $\pi/2$ and $S_{2 \times 2, \text{inf}}^L$ is further expressed by the jump probability P_L as $S_{2 \times 2, \text{inf}}^L = \frac{1}{2} - (\frac{1}{2} - P_L) \cos 2\theta_{12}$. Although the reduction formulas Eq. (54) and Eq. (55) are the same when the higher resonance is complete, they are different in case the higher resonance is incomplete.⁷ We also confirm the difference numerically in the next subsection.

B. Numerical confirmation of the validity of the reduction formula for the case of two resonances

In the present subsection, we examine the precision of the reduction formula for the case of two resonances Eq. (54) using a numerical calculation. As a typical example, we use the same electron density distribution $N(r) = 245N_A \exp(-10.54r/R_\odot)$ as in Sec. II B. Since we consider the case

⁷Since the higher resonance is complete at the parameters in Ref. [30], the use of the reduction formula Eq. (55) is valid there as shown in Sec. III B

where two resonances occur, we assume that the energy of the produced electron neutrino is very high compared to that of the solar neutrino, $E > 100$ MeV. This situation is quite similar to that considered in Ref. [30]. To get conservative results, we take the mass squared differences as $\delta m_{12}^2 = 10^{-4} \text{ eV}^2$ and $\delta m_{13}^2 = 1.5 \times 10^{-3} \text{ eV}^2$, where the hierarchy between the mass-squared differences is mildest in the experimentally allowed region and the error is expected to be enhanced maximally.

We show the error of the reduction formula Eq. (54) in Fig. 7 for various values of the electron energy $E = 0.1, 1, 10, 100$ GeV. We observe the largest error for the lowest energy $E = 0.1$ GeV. Even in this case, however, the error is less than 0.005. For higher energy neutrinos, the reduction formula is more accurate. This reduction formula is apparently accurate enough for the investigation into supernova neutrinos and the hypothetical $O(\text{GeV})$ solar neutrinos [25–30].

We finally compare the reduction formula obtained here [Eq. (54)] with that obtained in Ref. [30]. We compare the errors as a function of the initial electron density $N(0) = 245\Theta N_A$ in Fig. 8. The reduction formula Eq. (54) is extremely accurate for all values of Θ . While the reduction formula Eq. (55) is better than Eq. (2), it is not accurate for smaller values of Θ where the higher resonance becomes incomplete. The error, however, is small enough in the parameter range considered in Ref. [30] for their purpose. This result shows that the reduction formula Eq. (54) can be safely used even in the region where the higher resonance is incomplete.

IV. SUMMARY

In the present paper, we derive the next-to-leading order reduction formula for the ν_e survival rate Eq. (24) from the three-flavor neutrino oscillation to the two-flavor one in the case when there is only one resonance, as in the ordinary solar neutrino oscillation. Together with an analytic argument, we numerically verify the accuracy of the reduction formula, leaving the mixing angles free for generality. While we find that the leading order reduction formula Eq. (1) is accurate enough for a rough estimation, the next-to-leading order reduction formula is extremely accurate and adequate for precision tests of neutrino oscillations. Next, we study the accuracy of the reduction formulas in a realistic case, i.e., taking into account the current upper bound on θ_{13} . We find that the largest error of the leading order reduction formula is about 1% or so while the error of the next-to-leading order reduction formula is negligible. We thus point out that this precise next-to-leading order formula will be useful for precision tests of neutrino oscillations, for example the (indirect) study of CP violation [20–23]. We also verify the accuracy of the reduction formula written using the jump probability. This formula is accurate when the resonance is complete, i.e., for high energy neutrinos, although it is not valid when the resonance is incomplete, i.e., for low energy neutrinos.

We also derive the reduction formula Eq. (54) in the case of two resonances as in the oscillations of the supernova neutrinos and the hypothetical high energy $O(\text{GeV})$ solar

neutrinos due to the annihilation of WIMPs. We numerically verify that it is quite accurate and applicable for any parameter region. We then compare it to the reduction formulas obtained in Refs. [6,30]. Although the previously obtained formulas are valid only in case the higher resonance is complete, the formula obtained here is valid not only for the complete case but also for the incomplete case.

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