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Abe, Shigeo

Gee, Andrew H

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Global Convergence of the Hopfield Neural Network with Nonzero Diagonal Elements

Shigeo Abe, *Senior Member, IEEE* and Andrew H. Gee

Abstract—**In this paper we derive stability conditions of local minima and their convergence regions of the Hopfield neural network when the diagonal elements of the coefficient matrix are all nonzero. Then for the traveling salesman problem (TSP) we clarify the ranges of the weight values in the energy function and the range of values of the diagonal elements, so that the feasible solutions become stable and infeasible solutions become unstable. Simulations of the TSP show that the above criteria are valid and, by gradually decreasing diagonal elements, quality of solutions is drastically improved, compared with that of zero diagonal elements.**

I. Introduction

SINCE the introduction of the Hopfield neural network [1] (hereafter called the Hopfield model), there have been many negative and positive discussions [2],[3] concerning the quality of solutions, especially for large-size combinatorial optimization problems. In his doctoral dissertation [4],[5], Aiyer discussed an innovative method to drastically improve the quality of solutions. He proposed a projection method in which a trajectory of a solution is confined in the valid subspace where the hypercube vertices included are all feasible. He also proposed the MGNC (matrix graduated nonconvexity) method, in which the diagonal elements are increased during integration, so that the solution moves in the direction of the eigenvector whose absolute eigenvalue is maximum. For the 30-city TSP problem used in [1], he showed that his method outperformed the mean field annealing method, on average.

He suggested that if the constraints are properly defined in the TSP, the projection method and the Hopfield model, where the energy is expressed by a weighted sum of the objective function and the constraints, are equivalent. Thus the MGNC method also works for the Hopfield model. If this is true, the Hopfield model has a wider applicability than the projection method because the latter is restricted to the problem where the valid subspace can be defined.

We have already clarified the convergence region of a local minimum in the Hopfield model with zero diagonal elements and a method to determine weights in the energy function so that the feasible solutions become stable [6]. In this paper our aim is to establish a theoretical basis for the Hopfield model when the diagonal elements are decreased during integration. (The sign of the energy function in this paper is opposite to that in [4]. Thus the diagonal elements are *decreased* instead of *increased.*) In the following, first we derive stability conditions of vertices and nonvertex equilibrium points. Then our theory is applied to the TSP, and the condition of weights which stabilize feasible solutions and destabilize infeasible solutions is derived. By computer simulations, validity of the condition is confirmed, and the performance is shown to be drastically improved by gradually decreasing the diagonal elements.

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S. Abe is with Hitachi Research Lab., Hitachi Ltd., Hitachi, Japan.

A. H. Gee is with the Department of Engineering, Cambridge University, Cambridge, England.

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II. Convergence Region of Local Minima

A. Local Minima of the Hopfield Model

Let the energy function E that should be minimized be given by

$$E = \frac{1}{2}\mathbf{x}^t T \mathbf{x} + \mathbf{b}^t \mathbf{x} \tag{1}$$

where $\mathbf{x} = (x_1, \cdots, x_n)^t$: the variable vector, $x_i = 1$ or 0, $i = 1, \cdots, n$;
$T = \{T_{ij}\}$: the $n \times n$ symmetric matrix; and
$\mathbf{b}$: the n dimensional input vector.

Then the Hopfield model for the energy function E is given by using a piece-wise linear function and extending the range of x_i ($i = 1, \cdots, n$) to [0, 1] [7]:

$$\frac{d\mathbf{x}}{dt} = -\frac{\partial E}{\partial \mathbf{x}} = -T\mathbf{x} - \mathbf{b}, 0 \le x_i \le 1, \text{ for } i = 1, \cdots, n. \tag{2}$$

As discussed in [6], the results obtained for the piece-wise linear function are also valid for the hyperbolic tangent function used in [1].

Let $\mathbf{s} = (s_1, \cdots, s_n)^t$ be an equilibrium point of (2). Namely, for set S_k which includes k integers selected from $\{1, \cdots, n\}$, let the following equations hold:

$$\begin{aligned} T_i\mathbf{s} + b_i = 0, 0 < s_i < 1 &\quad \text{for } i \in S_k \\ s_i = 0 \text{ or } 1 &\quad \text{for } i \notin S_k. \end{aligned} \tag{3}$$

where T_i is the l—th row vector of T. If $k = 0$ (i.e., $S_k = \emptyset$), s is a vertex, and if $k = n$, s is a solution of $T\mathbf{x} + \mathbf{b} = 0$.

The stability of s can be analyzed by substituting $\mathbf{x} = \mathbf{s} + \mathbf{x}'$ into (2):

$$\frac{d\mathbf{x}'}{dt} = -T\mathbf{s} - \mathbf{b} - T\mathbf{x}' \tag{4}$$

Thus, from (3), (4) becomes

$$\frac{dx'_i}{dt} = -T_i\mathbf{x}' \qquad \text{for } i \in S_k \tag{5}$$

$$\frac{dx'_i}{dt} = -T_i\mathbf{s} - b_i - T_i\mathbf{x}' \qquad \text{for } i \notin S_k. \tag{6}$$

Then it is evident that the following theorem holds.

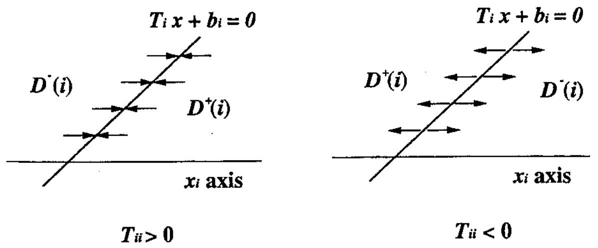

Fig. 1. Relationship between the hyperplane and the diagonal element.

Theorem 1: The equilibrium point s given by (3) is stable if for $i \notin S_k$

$$T_i\mathbf{s} + b_i > 0 \text{ for } s_i = 0 \text{ or } T_i\mathbf{s} + b_i < 0 \text{ for } s_i = 1 \quad (7)$$

and all the eigenvalues of the $k \times k$ submatrix $\{T_{ij}\}$, where i and j are included in S_k, are non-negative.

Since in [6] we have discussed the stability of equilibrium points when $T_{ii} = 0$, in our following study, we assume that T_{ii} are all nonzero.

B. Stability of Vertexes

Let the hypercube and the interior of the hypercube be defined by

$$H = \{\mathbf{x} \mid 0 \le x_i \le 1 \text{ for } i = 1, \cdots, n\} \quad (8)$$

and

$$H^- = \{\mathbf{x} \mid 0 < x_i < 1 \text{ for } i = 1, \cdots, n\}, \quad (9)$$

respectively. And for vertex $\mathbf{c} = (c_1, \cdots, c_n)^t$ in H we define n adjacent vertices $\mathbf{c}(i) = (c_1, \cdots, c_{i-1}, 1 - c_i, c_{i+1}, \cdots, c_n)^t$ where $c_i = 1$, or 0 for $i = 1, \cdots, n$.

Let H^- be divided into three parts by the hyperplane $T_i\mathbf{x} + b_i = 0$ as follows:

$$\begin{aligned} D^+(i) &= \{\mathbf{x} \mid 0 < x_j < 1, j = 1, \cdots, n, T_i\mathbf{x} + b_i > 0\} \\ D^-(i) &= \{\mathbf{x} \mid 0 < x_j < 1, j = 1, \cdots, n, T_i\mathbf{x} + b_i < 0\} \\ D^0(i) &= \{\mathbf{x} \mid 0 < x_j < 1, j = 1, \cdots, n, T_i\mathbf{x} + b_i = 0\}. \end{aligned} \quad (10)$$

For $\mathbf{x} \in D^+(i)$, as time elapses component x_i always decreases so long as $\mathbf{x}$ is in domain $D^+(i)$. If $\mathbf{x} \in D^0(i)$, for $\mathbf{x}' = (x_1, \cdots, x_i + \varepsilon, x_{i+1}, \cdots, x_n)^t$, $\varepsilon > 0$ and $T_{ii} > 0$, $T_i\mathbf{x}' + b_i > 0$ holds, namely $\mathbf{x}' \in D^+(i)$. Thus if T_{ii} is positive, $\mathbf{x}'$ moves to the hyperplane $T_i\mathbf{x} + b_i = 0$. Likewise, if T_{ii} is negative, $\mathbf{x}'$ moves away from the hyperplane $T_i\mathbf{x} + b_i = 0$ (see Fig. 1). Thus, there is a great disparity between T_{ii} being zero and nonzero since, if T_{ii} is zero, $T_i\mathbf{x} + b_i = 0$ is parallel to x_i axis. Also because of this, the two adjacent vertices and the edge connecting them cannot have the same energy (degeneracy of energy) for nonzero T_{ii} [6].

First we consider stability of vertex $\mathbf{c}$ in which $\mathbf{c} \notin D^0(i)$ for all i holds, or even if $\mathbf{c} \in D^0(i)$ for some i holds, $D^0(i)$ and H^- do not intersect. For vertex $\mathbf{c}$ in hypercube H, we define domain $D_\mathbf{c}$ as follows:

$$D_\mathbf{c} = \bigcap_{\substack{i = 1, \cdots, n \\ \mathbf{c} \notin D^0(i)}} Z(i) \quad (11)$$

where

$$Z(i) = \begin{cases} D^+(i) & \text{for } T_i\mathbf{c} + b_i > 0 \\ D^-(i) & \text{for } T_i\mathbf{c} + b_i < 0 \end{cases} \quad (12)$$

It is easy to see that for any $\mathbf{x} \in D_\mathbf{c}$, $\mathbf{x}$ moves monotonically so long as $\mathbf{x}$ remains in $D_\mathbf{c}$. If for $i = 1, \cdots, n$ vertex $\mathbf{c}$ satisfies

$$T_i\mathbf{c} + b_i > 0 \text{ for } c_i = 0 \text{ or } T_i\mathbf{c} + b_i < 0 \text{ for } c_i = 1 \quad (13)$$

or for $T_i\mathbf{c} + b_i = 0$ and $D^0(i) \cap H^- = \emptyset$,

$$T_i\mathbf{c}(i) + b_i > 0 \text{ for } c_i = 0 \text{ or } T_i\mathbf{c}(i) + b_i < 0 \text{ for } c_i = 1, \quad (14)$$

vertex $\mathbf{c}$ is stable, since $\mathbf{x}$ ($\in D_\mathbf{c}$), which is in the neighborhood of $\mathbf{c}$, moves to $\mathbf{c}$. Here notice that if $T_i\mathbf{c} + b_i = 0$ and $D^0(i) \cap \mathbf{H}^- = \emptyset$, the sign of $T_i\mathbf{x} + b_i$ is the same on the edge $\mathbf{c}$ and $\mathbf{c}(i)$, except for $\mathbf{c}$.

Now for vertex $\mathbf{c}$ we define the convergent domain $D'_\mathbf{c}$ as follows:

$$\begin{aligned} D'_\mathbf{c} = \{\mathbf{x} \mid \mathbf{x} \in D_\mathbf{c} \text{ and } \\ &(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_n)^t \in D_\mathbf{c}, \\ &x'_i = \gamma c_i + (1-\gamma)x_i, 0 \le \gamma < 1 \text{ for } i = 1, \cdots, n\} \end{aligned} \quad (15)$$

The condition that $(x_1, \cdots, x_{i-1}, x'_i, x_{i+1}, \cdots, x_n)^t \in D_\mathbf{c}$ means that a line segment connecting $\mathbf{x}$ and $(x_1, \cdots, x_{i-1}, c_i, x_{i+1}, \cdots, x_n)^t$ excluding the latter point, which is parallel to the x_i axis, does not intersect with hyperplanes $T_i\mathbf{x} + b_i = 0$ ($i = 1, \cdots, n$). Also it is easy to see that $D'_\mathbf{c}$ is a nonempty, connected domain. Therefore, so long as $\mathbf{x}$ is in $D'_\mathbf{c}$, a corrected $\mathbf{x}$ does not go back to $H^- - D'_\mathbf{c}$.

From the above definitions it is easy to see that the following theorem holds which is similar to Theorem 1 in [6]:

Theorem 2: If vertex $\mathbf{c}$ satisfies (13) or (14), for any $\mathbf{x} \in D'_\mathbf{c}$, $\mathbf{x}$ converges monotonically to $\mathbf{c}$.

Now consider the situation where $\mathbf{c} \in D^0(i)$ for some i and $D^0(i)$ and H^- intersect. Without loss of generality, we can assume

$$\begin{aligned} &\mathbf{c} \in D^0(i), D^0(i) \cap H^- \ne \emptyset \text{ for } i = 1, \cdots, k, \text{ and} \\ &\mathbf{c} \in D^0(i), \text{ or } \mathbf{c} \in D^0(i) \text{ and } D^0(i) \cap H^- = \emptyset \\ &\quad \text{for } i = k+1, \cdots, n. \end{aligned} \quad (16)$$

Then the following theorem holds.

Theorem 3: For vertex $\mathbf{c}$ which satisfies (16), let

$$V = \bigcap_{i=1,\cdots,k} D^{f(c_i)}(i) \ne \emptyset$$

where $f(0) = +$ and $f(1) = -$. Then vertex $\mathbf{c}$ is stable and for $\mathbf{x} \in D'_\mathbf{c} \cap V$, $\mathbf{x}$ converges to $\mathbf{c}$.

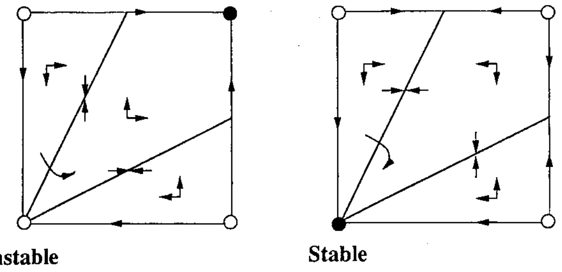

Fig. 2. Stability of a vertex when hyperplanes intersect it.

Proof: Since $\mathbf{x} \in D'_{\mathrm{c}}$, x_i $(i = k+1, \cdots, n)$ moves to c_i. Also since $\mathbf{x} \in V$, $x_i (i = 1, \cdots, k)$ moves to c_i. Thus the corrected $\mathbf{x}$ remains in D'_{c}. Let $T_i\mathbf{x}' + b_i = 0$ for some i $(i = 1, \cdots, k)$ and $\mathbf{x}' \in \overline{D'_{\mathrm{c}} \cap V} - (D'_{\mathrm{c}} \cap V)$ where $\overline{S}$ is a closure of set S. Since V is nonempty, we can take $T_j\mathbf{x}' + b_j \neq 0$ where $j \in \{1, \cdots, k\}$ and $j \neq i$. Thus $\mathbf{x}' \in D^{f(c_j)}(j)$. Namely, x'_j moves to c_j. Therefore $\mathbf{x}'$ moves into $D^{f(c_i)}(i)$ through $T_i\mathbf{x}' + b_i = 0$ (see Fig. 2). Thus if $\mathbf{x} \in D'_{\mathrm{c}} \cap Z$, the corrected $\mathbf{x}$ is confined in $D'_{\mathrm{c}} \cap Z$. Thus the theorem holds. (End of proof.)

In order that V be nonempty, all T_{ii} need to be positive. At present whether Theorem 3 is equivalent to Theorem 1 under (16) is an open question.

C. Stability of Nonvertex Equilibrium Points

As for stability of the equilibrium point $\mathbf{s}$ which satisfies $T\mathbf{s} + \mathbf{b} = 0$, the following theorems hold.

Theorem 4: Assume that T_{ii} is negative for some i. Then $\mathbf{s}$ is an unstable equilibrium point.

Proof: Since x_i goes away from $T_i\mathbf{s} + b_i = 0$, $\mathbf{s}$ cannot be stable. (End of proof.)

Theorem 5: Assume that T_{ii} are all positive, that $\mathbf{s}$ satisfies $0 \leq s_i \leq 1$ for $i = 1, \cdots, n$, where a strict inequality holds for some i, and that there is a stable vertex $\mathbf{c}$. Then $\mathbf{s}$ is an unstable equilibrium point.

Proof: Since $0 < s_i < 1$ holds for some i, $\mathbf{s}$ is not a vertex. Take $\mathbf{x} \in D_{\mathrm{c}}$ in the neighborhood of $\mathbf{s}$. According to the definition of D_{c}, $\mathbf{x}$ moves towards $\mathbf{c}$, namely it goes away from $\mathbf{s}$. Thus $\mathbf{s}$ is unstable. (End of proof.)

The remaining equilibrium points are on the surface of H. The following corollary is evident.

Corollary 1: If for the equilibrium point $\mathbf{s}$, given by (3), (7) does not hold for some $i \notin S_k$, an unstable vertex exists in the reduced dimension spanned by i axes $(i \in S_k)$, or T_{ii} is negative for some $i \in S_k$, $\mathbf{s}$ is unstable.

According to the above discussions and [6], differences of T_{ii} being zero and nonzero are as follows:

Diagonal Elements are Zero: Adjacent vertices may have the same energy and they and the edge connecting them may become stable. Only under this condition do nonvertex equilibrium points become stable. If the energies of the adjacent vertices are different, those vertices cannot be stable at the same time.

Diagonal Elements are Nonzero: An edge connecting two adjacent vertexes cannot be stable. But adjacent vertices may become stable at the same time even though the corresponding energies are different. Nonvertex equilibrium points may be stable when some or all of the diagonal elements are positive.

D. Suppression of Spurious States

The weights in the energy function need to be determined so that feasible solutions become stable and infeasible solutions become unstable. We use only (13) to stabilize feasible solutions or destabilize infeasible solutions, since we can avoid the situation where the hyperplanes include vertices by proper selection of the weights.

If the diagonal elements of T are decreased from positive to negative during integration of (2) until all the diagonal elements become negative, spurious states at nonvertex equilibrium points are automatically suppressed. But this may not always be possible, as will be shown for the TSP.

III. Traveling Salesman Problem

A. Problem Formulation

The traveling salesman problem is to travel to n cities within the minimum distance with the constraint that all the cities must be visited only once. For each city n neurons are assigned, and the visit of a city at the i-th order is defined as the i-th neuron output to be one, and the remaining $n-1$ neuron outputs of the city to be zero. We use the following energy function which is shown to be the best formulation in [4]:

$$E = \frac{A}{2}\sum_x\Big(\sum_i V_{xi} - 1\Big)^2 + \frac{B}{2}\sum_i\Big(\sum_y V_{yi} - 1\Big)^2 + \frac{D}{2}\sum_x\sum_{y\neq x}\sum_i d_{xy}V_{xi}(V_{y,i+1} + V_{y,i-1}) + \frac{F}{2}\sum_x\sum_i V_{xi}^2 \tag{17}$$

where V_{xi}: i-th neuron for city x, and $0 \leq V_{xi} \leq 1$;
d_{xy}: distance between city x and city y;
A, B, and D (> 0): weights;
subscripts $i-1$ and $i+1$ are calculated by modulo n; and
F: a weight which is decreased during integration.

The first and the second terms impose the constraint that $V_{xi} = 1$ holds for one x or i while varying them from 1 to n. The fourth term is to implement the MGNC method.

Taking a partial derivative of (17) with respect to V_{xi} gives

$$\frac{\partial E}{\partial V_{xi}} = A\Big(\sum_j V_{xj} - 1\Big) + B\Big(\sum_y V_{yi} - 1\Big) + D\sum_{y\neq x} d_{xy}(V_{y,i+1} + V_{y,i-1}) + FV_{xi}. \tag{18}$$

Now for a feasible solution, the first and second terms of (18) are zero. Thus from (13), the feasible solution is stable if

$$D(d_{xy} + d_{xz}) + F < 0 \quad \text{for } V_{xi} = 1 \tag{19}$$

$$Dd_{xy} > 0 \text{ or } D(d_{xy} + d_{xz}) > 0 \quad \text{for } V_{xi} = 0 \tag{20}$$

where x, y and z are all different. Since (20) always holds, if F is 0, feasible solutions are all unstable. Also if

$$F < -D \max_{x,y,z}(d_{xy} + d_{xz}), \tag{21}$$

all the feasible solutions are stable.

Now to suppress spurious states, we assume that

$$A = B. \tag{22}$$

a) Change states of one or more neurons in a feasible solution from one to zero.
   Let column and row sums corresponding to V_{xi} be zero. Then from (13) and (18) all the infeasible solutions become unstable if

$$A > \frac{D}{2} \max_{x,y,z}(d_{xy} + d_{xz}) \tag{23}$$

   where x, y and z are different.
b) Change states of one or more neurons in a feasible solution from zero to one.
   Let column and row sums corresponding to V_{xi} be two. Then from (13) and (18) all the infeasible solutions become unstable if

$$F > -2A - D \max_{x,y,z}(d_{xy} + d_{xz}) \tag{24}$$

   where x, y and z are different.
c) Change states of two or more neurons in a feasible solution from zero to one and one to zero at the same time.
   In this case more than one neuron fires at some column or row. Assume $V_{xi} = V_{xj} = 1$ $(i \neq j)$. If

$$A + F > 0, \tag{25}$$

   (18) for V_{xi} is positive. Thus from (13), the infeasible solutions are unstable.

If (25) holds, (24) also holds. Thus if (21), (23), and (25) hold, stable vertices are all feasible. But since the diagonal elements of T are all $2A + F$ and from (23) and (25), $2A + F$ needs to be positive. Thus the nonvertex equilibrium points may be stable.

B. Computer Simulations

Using the 10- and 30-city TSP in [1], we demonstrate the validity of the criteria of the weights (21), (23), and (25), and investigate the quality of the solution of the Hopfield model when diagonal elements are gradually decreased.

We applied the Euler method to (2) to obtain combinatorial solutions [6]. The initial values were set as

$$V_{xi} = 1/n + \alpha \mathit{Rand} \tag{26}$$

where α is a small positive value and *Rand* is a random value in $[-0.5, 0.5]$, so that the initial values approximately satisfy the equality constraints [4].

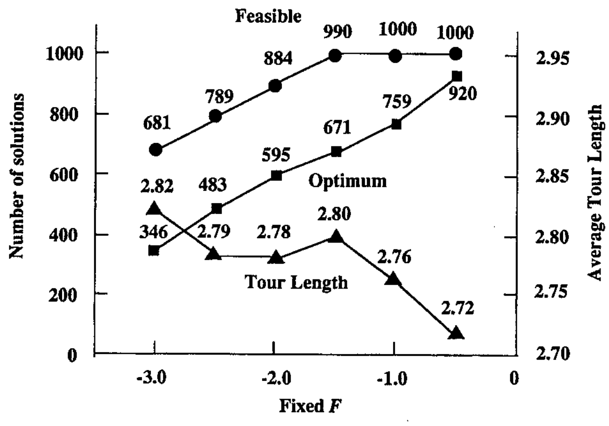

Fig. 3. Number of solutions and average tour length of 10-city TSP with fixed F during integration (1000 trials, $A = 2$, $\alpha = 0.0001$, minimum tour length $= 2.709$).

The diagonal elements are decreased by a method similar to that discussed in [4]. Namely, when

$$\sum_x \sum_i |V_{xi}| < 0.0001 \tag{27}$$

is satisfied, we decrease F as follows:

$$F \leftarrow F - 0.1. \tag{28}$$

If F does not reach a predetermined value until after a predetermined time of integration, we decrease F according to (28) in the time periods that follow, until the predetermined value is reached. This is to ensure that a feasible solution is obtained.

The integration is terminated when all V_{xi} reach 1 or 0, or all the differences of current and previous corrections of V_{xi} are within a specified value. If some V_{xi} are between 1 and 0, we digitize them to 1 or 0, according to whether they are greater than, or equal to 0.5, or not.

Ten-city TSP Setting $D = 1$, (21), (23), and (25) become

$$-1.667 > F > -A \tag{29}$$
$$A > 0.834. \tag{30}$$

Fig. 3 shows the number of solutions and average tour length for 1000 trials against F when $A = 2$, which satisfies (23). During integration we did not change F to see whether (29) is a valid criterion. According to the theory, if F is equal to, or less than -2, infeasible solutions, in which two neurons in a column or row fire, may become stable. For $F = -2$, -2.5, and -3, all the infeasible solutions were of this type of solution. Infeasible solutions for $F = -1.5$ were those in which one or no neuron fires in a row or column. Thus, the second inequality in (29) was a valid criterion. But the first inequality in (29) was too strict. When $F = -0.5$, the quality of the solution was best. This is because we need not stabilize the feasible solutions which are of bad solution quality. Also, if feasible solutions are destabilized, nonvertex solutions near

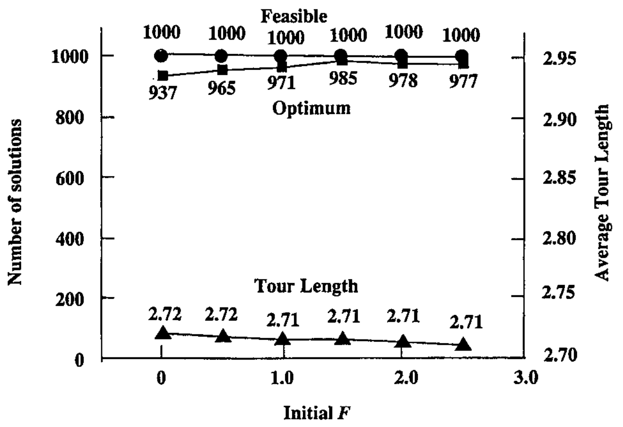

Fig. 4. Number of solutions and average tour length of 10-city TSP with decreased F during integration (1000 trials, $A = 2$, final $F = -0.5$, $\alpha = 0.0001$, minimum tour length = 2.709).

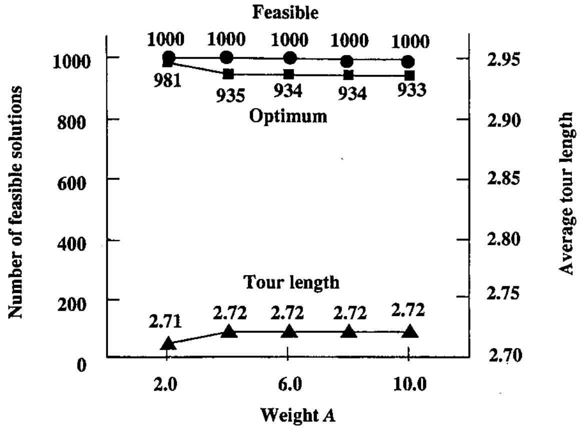

Fig. 6. Number of feasible solutions and average tour length of 10-city TSP with decreased F during integration (1000 trials, initial $F = 1.5$, final $F = -0.5$, $\alpha = 0.0001$, minimum tour length = 2.709).

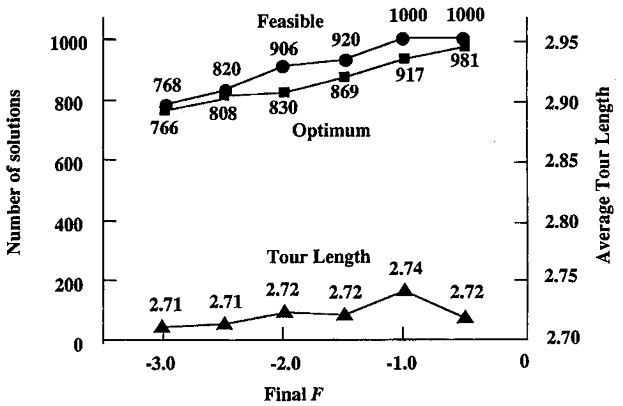

Fig. 5. Number of solutions and average tour length of 10-city TSP with decreased F during integration (1000 trials, initial $F = 1.5$, increment of $F = 0.1$, $A = 2$, $\alpha = 0.0001$, minimum tour length = 2.709).

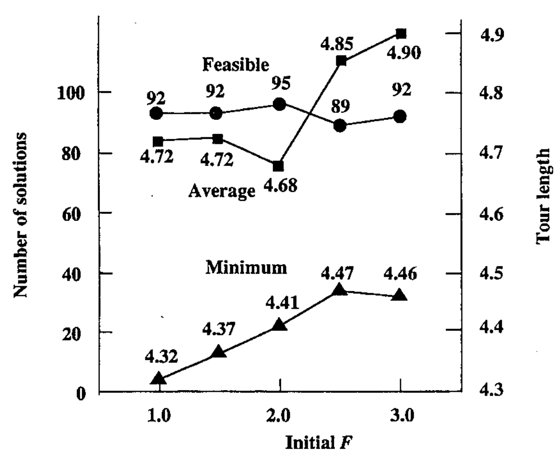

Fig. 7. Number of solutions and average tour length of 30-city TSP with decreased F during integration (100 trials, final $F = -0.5$, increment of $F = 0.01$, $A = 2$, $\alpha = 0.0001$, minimum tour length = 4.312).

the feasible solutions may become stable. If these solutions are obtained, they become feasible solutions by the digitization process after integration is terminated. If $F = 0$ and (30) holds, the solution of $T\mathbf{s} + \mathbf{b} = 0$ becomes the only stable solution [6]. Thus it seems that the absolute value of F needs to be as small as possible under the condition that at least stable solutions exist on the surface of the hypercube.

Comparing the results with the zero-diagonal elements [6], the improvement of the solution quality was drastic. With the zero-diagonal elements, the optimal solutions were obtained 432 times out of 1000 trials, at best.

Fig. 4 shows the number of feasible solutions and average tour length against initial F when $A = 2$. Weight F was lowered to -0.5. The MGNC method worked well to improve the quality of solutions. The best performance was obtained when initial $F = 1.5$.

In contrast to Fig. 3, Fig. 5 shows the performance when F was decreased from 1.5 to the specified values. The performance improvement was drastic especially when the absolute value of $F$is small. Fig. 6 shows the performance against weight A when F was changed from 1.5 to -0.5. The performance was almost the same, except for $A = 2.0$. It seemed that when A was smaller, the performance was better.

Thirty-city TSP Setting $D = 1$, (21), (23), and (25) become

$$-2.31 > F > -A \quad (31)$$

$$A > 1.16 \quad (32)$$

According to the simulation results of the 10-city TSP, to obtain good solution quality, A needs to be as small as possible under (32), andthe absolute value of final F needs to be as small as possible, under the condition that there are stable solutions on the surface of H. So we set $A = 2$ and final $F = -0.5$. Fig. 7 shows the results for 100 trials when initial F was changed down to -0.5. Comparing the results with zero-diagonal elements in [6], the improvement was remarkable. But the rates of feasible solution obtained were not 100%. The infeasible solutions were all those which were adjacent to feasible solutions, namely, those with one fired neuron in the feasible solutions being unfired. This is because we set $F = -0.5$, which makes some or all of the feasible solutions unstable. The performance changed according to the initial F. The shortest average tour length was obtained when the initial

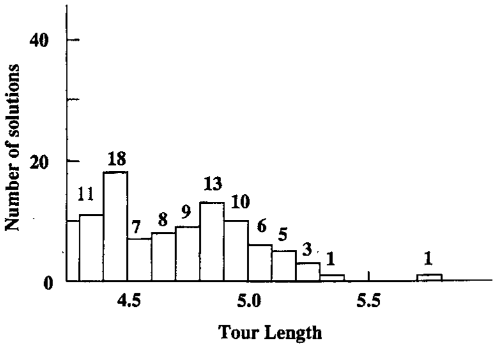

Fig. 8. Number of solutions and tour length of 30-city TSP with decreased F during integration (100 trials, initial $F = 1$, increment of $F = 0.01$, final $F = -0.5$, $A = 2$, $\alpha = 0.0001$, minimum tour length = 4.312).

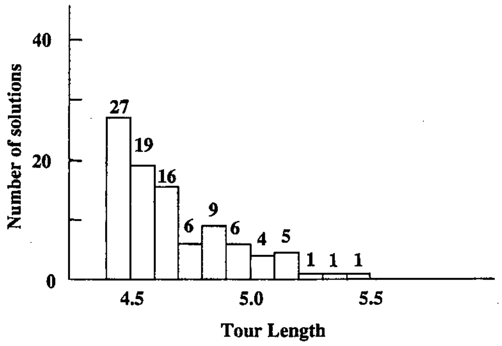

Fig. 9. Number of solutions and tour length of 30-city TSP with decreased F during integration (100 trials, initial $F = 2$, increment of $F = 0.01$, final $F = -0.5$, $A = 2$, $\alpha = 0.0001$, minimum tour length = 4.312).

F was 2.0, while the minimum tour length was obtained when the initial F was 1.0

Figs. 8, 9, and 10 show the distributions of solutions for the initial $F = 1$, 2, and 3. It is seen from these figures, when $F = 2$, the MGNC method worked best in that near optimal solutions were frequently obtained, although the tour length less than 4.4 was not obtained.

IV. Discussion of Results

Many researchers had been obsessed with the idea that zero-diagonal elements are necessary to put the Hopfield model to work. This works, but very poorly for a large-size optimization problem. The obsession was dispelled by the Hopfield model with nonzero diagonal elements and with the MGNC method. Although the convergence rate to feasible solutions was a little bit worse compared to the zero diagonal elements, the performance improvement was drastic.

Essentially the MGNC method is analogous (in fact almost identical) to the temperature annealing process used in the mean field annealing algorithm [8], or the hysteretic annealing process used in [9]. This process makes the network's behavior much less sensitive to the random starting position and, by moving $\mathbf{x}$ initially along particular eigenvectors of T, can help find better solutions [10].

The performance of the Hopfield model with MGNC for the 30-city TSP was between that of the projection method with

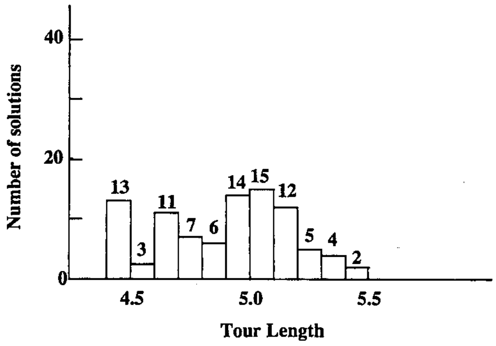

Fig. 10. Number of solutions and tour length of 30-city TSP with decreased F during integration (100 trials, initial $F = 3$, increment of $F = 0.01$, final $F = -0.5$ $A = 2$, $\alpha = 0.0001$, minimum tour length = 4.312).

MGNC (minimum tour length = 4.298, average tour length = 4.397) and that of the projection method without MGNC (minimum tour length = 4.350, average tour length = 4.763) [4]. (Because of reading errors of city coordinates from [1], the optimal tour length in our simulations was 4.312 while that in [4] was 4.268.) The reason that the convergence characteristics differ needs to be clarified in the future.

Since a simulation of the Hopfield model involves mostly large matrix multiplications and additions, it will run very rapidly on vector-parallel digital hardware, or a DSP chip. In the TSP simulations, we used Hitachi's super computer S-820. Without reprogramming the fortran program for the super computer, we could obtain the speedup gain of five to ten. For the case of analog implementations, while the circuits are in theory straightforward, in practice it is very difficult to make them work, as discussed in [10]. (We can replace the piece-wise linear function with the hyperbolic tangent function. Implementations of gradual decrease of the diagonal elements are discussed in [9].)

V. Conclusion

We derived stability conditions of local minima and their convergence regions in the Hopfield neural network when the diagonal elements are all nonzero. Then for the traveling salesman problem (TSP), we clarified the ranges of the weights in the energy function and the range of the diagonal elements to be changed, so that the feasible solutions became stable and spurious states were unstable. Simulations of the TSP by gradually increasing diagonal elements showed that the quality of solutions was improved drastically over that of zero diagonal elements.

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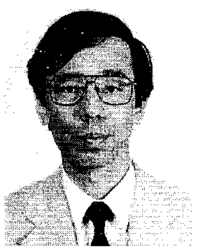
References

[1] J. J. Hopfield and D. W. Tank, "Neural computation of decisions in optimization problems," *Biolog. Cybern.*, Springer-Verlag, vol. 52, pp. 141–152, 1985.

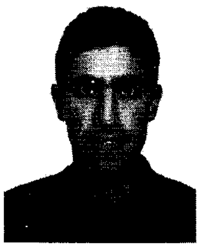
[2] V. Wilson and G. S. Pawley, "On the stability of the TSP problem algorithm of Hopfield and Tank," *Biolog. Cybern.*, vol. 58, pp. 63–70, 1988.
[3] C. Peterson and B. Söderberg, "A new method for mapping optimization problems onto neural networks," *Int. J. Neural Syst.*, vol. 1, no. 1, Jan. 1990.
[4] S. V. B. Aiyer, "Solving combinatorial optimization problems using neural networks with applications in speech recognition," in Tech. Rep. CUED/F-INFENG/TR 89, Dept. of Eng., Univ. of Cambridge, Oct. 1991.
[5] A. H. Gee, S. V. B. Aiyer, and R. W. Prager, "An analytical framework for optimizing neural networks," *Neural Networks*, vol. 6, no. 1, pp. 79–97, 1993.
[6] S. Abe, "Global convergence and suppression of spurious states of the Hopfield neural networks," *IEEE Trans. Circuits Syst. I*, vol. 40, no. 4, pp. 246–257, Apr. 1993.
[7] J.-H. Li, A. N. Michel, and W. Porod, "Analysis and synthesis of a class of neural networks: Linear systems operating on a closed hypercube," *IEEE Trans. Circuits Syst.*, vol. 36., pp. 1405–1422, Nov. 1989.
[8] D. E. Van den Bout and T. K. Miller III, "Graph partitioning using annealed neural networks," *IEEE Trans. Neural Networks*, vol. 1, no. 2, June 1990.
[9] S. P. Eberhardt, T. Daud, D. A. Kerns, T. X. Brown, and A. P. Thakoor, "Competitive neural architecture for hardware solution to the assignment problem," *Neural Networks*, vol. 4, pp. 431–442, 1991.
[10] A. H. Gee, "Problem solving with optimization networks," Ph.D. dissertation, also published as Tech. Rep. CUED/F-INFENG/TR 150, Dept. of Eng., Univ. of Cambridge, July 1993.



Shigeo Abe (M'79–SM'83) received the B.S. degree in electronics engineering, the M.S. degree in electrical engineering, and the Doctor of Engineering degree, all from Kyoto University, Kyoto, Japan, in 1970, 1972, and 1984, respectively.

Since 1972 he has been with Hitachi Research Laboratory, Hitachi, Ltd., engaged in power system analysis, and the development of a vector processor, a Prolog processor, neural network theories, and fuzzy system models. From 1978 to 1979, he was a visiting research associate with the University of Texas at Arlington. He is now a chief researcher at Hitachi Research Laboratory.

Dr. Abe was awarded an outstanding paper prize from the Institute of Electrical Engineers of Japan in 1984. He is a member of the International Neural Network Society, the Institute of Electrical Engineers of Japan, the Information Processing Society of Japan, the Institute of Electronics, Information, and Communication Engineers of Japan, and the Society of Instrument and Control Engineers of Japan.



Andrew Gee was born in London, England, in 1969. He received the B.A. degree in electrical and information science from the University of Cambridge, England, in 1990. He was recently awarded the Ph.D degree by the same university, for his thesis on the use of feedback neural networks for combinatorial optimization.

He is currently a Research Fellow of Queen's College, Cambridge. His research interests include neural networks and computer vision.