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**(Citation)**

SIAM Journal on Mathematical Analysis, 37(3):841-866

**(Issue Date)**

2006

**(Resource Type)**

journal article

**(Version)**

Version of Record

**(URL)**

<https://hdl.handle.net/20.500.14094/90000302>



## OPTIMAL RATE OF CONVERGENCE OF THE BENCE–MERRIMAN–OSHER ALGORITHM FOR MOTION BY MEAN CURVATURE\*

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**Abstract.** Bence, Merriman, and Osher proposed an algorithm for computing the motion a hypersurface by mean curvature in terms of solutions of the usual heat equation, continually reinitialized after short time steps. In this paper, applying some techniques of asymptotic analysis for the Allen–Cahn equation, we give a rate of convergence of their algorithm for the motion of a smooth and compact hypersurface by mean curvature. We also consider the special case of a circle evolving by curvature and show that our rate is optimal.

**Key words.** motion by mean curvature, numerical algorithm, rate of convergence, optimality

**AMS subject classifications.** 35K05, 35K55, 65M15

**DOI.** 10.1137/04061862X

**1. Introduction.** In 1992, Bence, Merriman, and Osher proposed in [2] an algorithm for computing the motion of a hypersurface by its mean curvature. It is described as follows.

Given a closed set  $C_0 \subset \mathbb{R}^N$ , we solve the initial-value problem for the heat equation

$$(1.1) \quad \begin{cases} u_t - \Delta u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = \begin{cases} 1, & x \in C_0, \\ -1, & x \in \mathbb{R}^N \setminus C_0. \end{cases} \end{cases}$$

Fix a time step  $h > 0$  and set

$$C_1 = \{x \in \mathbb{R}^N \mid u(h, x) \geq 0\}.$$

Next we solve (1.1) with  $C_0$  replacing  $C_1$  and define a new set  $C_2$  with the solution  $u$  replaced by that of (1.1) with the new initial data. Repeating this procedure, we have a sequence  $\{C_k\}_{k=0,1,\dots}$  of closed sets in  $\mathbb{R}^N$ . Then we define

$$C_t^h = C_k \quad \text{if } kh \leq t < (k+1)h, \quad k = 0, 1, \dots,$$

for  $t \geq 0$ . Letting  $h \searrow 0$ , we obtain

$$\partial C_t^h \longrightarrow \Gamma_t, \quad \Gamma_0 = \partial C_0,$$

and  $\Gamma_t$  moves by its  $((N-1)$ -times) mean curvature.

The convergence of the Bence–Merriman–Osher (BMO) algorithm was proved by Mascarenhas [19], Evans [5], Barles and Georgelin [1], and Goto, Ishii, and Ogawa [10]. The generalizations of this algorithm were considered by Ishii [14], Ishii, Pires,

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\*Received by the editors November 10, 2004; accepted for publication (in revised form) March 14, 2005; published electronically December 2, 2005. This work was partially supported by JSPS grant-in-aid for scientific research grants 14540117 and 15340051.

<http://www.siam.org/journals/sima/37-3/61862.html>

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and Souganidis [16], Ishii and Ishii [15], Vivier [23], and Leoni [18]. However, to the author's knowledge, there are a few results on the rate of convergence of the BMO algorithm. In [22] Ruuth gave an error estimate for the case of the planar graph on  $[0, h]$ . Ishii and Nakamura [17] proved that the Hausdorff distance between the motion by mean curvature  $\Gamma_t$  and the approximate interface  $\Gamma_t^h := \partial C_t^h$  is an order of  $h^{1/2}$  as  $h \searrow 0$ . This estimate is valid before the onset of singularities, but not optimal.

The purpose of this paper is to show the optimal rate of convergence of the BMO algorithm, valid before the onset of singularities, for the Hausdorff distance between  $\Gamma_t$  and  $\Gamma_t^h$ . In fact, assuming  $\{\Gamma_t\}_{0 \leq t < T_0}$  is the motion of a smooth and compact hypersurface by mean curvature, we prove that, for any  $T < T_0$ ,

$$\sup_{t \in [0, T]} d_H(\Gamma_t, \Gamma_t^h) \leq Lh,$$

where  $L$  is a constant depending on  $T$ , but independent of small  $h > 0$ , and  $d_H$  denotes the Hausdorff distance. This estimate is optimal and improves that of [17].

Both of the order in  $h$  and the optimality are the consequence of the maximum principle and the explicit constructions of sub- and supersolutions of (1.1), which are inspired by the asymptotic analysis of solutions of the Allen–Cahn equation (see, e.g., Fife [7] and de Mottoni and Schatzman [4]). As for the relation between the BMO algorithm and the Allen–Cahn equation, from the viewpoint of the splitting methods in numerical analysis, Vivier [23] first pointed out that we may think the Allen–Cahn equation is an approximation of the BMO algorithm. Leoni [18] and Goto, Ishii, and Ogawa [10] gave the proofs of the convergence of the BMO algorithm and a generalized scheme by applying some techniques of the asymptotic analysis for the Allen–Cahn equation. The arguments in this paper also rely on them.

This paper is organized as follows. In section 2 we discuss the formal asymptotic expansion of the radially symmetric solution of (1.1). To derive formally the equation of the motion of the interface, we consider the asymptotic behavior as  $t \searrow 0$  of the zero of the solution of (1.1). In section 3 we construct sub- and supersolutions of (1.1) by using some functions provided by the formal asymptotic expansion. We treat the nonradial case in subsection 3.2 and the radial case in subsection 3.3. Section 4 is devoted to the rate of convergence of the BMO algorithm to the motion of a smooth and compact hypersurface by mean curvature. The arguments in sections 3–4 are very similar to those in [18], although Leoni considered in her paper a different situation from ours. In section 5, we return to the special case of a circle evolving by curvature. By the radial symmetry, we have only to consider the asymptotic behavior of the radius  $R_h$  of the approximate circle as  $h \searrow 0$ . In subsection 5.1 we obtain the short-time asymptotics of  $R_h$ . In subsection 5.2 we formally derive a corrector for  $R_h$ . Based on these results, in subsection 5.3 we obtain the behavior of  $R_h$  and show the optimality of our estimate obtained in section 4. The considerations of section 5 are motivated by Nochetto, Paolini, and Verdi [20]. In [20] they obtained the optimal error estimate of the approximate interface given by the solution of a variational inequality to the smooth motion by mean curvature. The appendix is devoted to the proof of a lemma given in section 2.

In the following of this paper, we denote by  $K$  various constants depending only on known ones, and the notation  $g = O(f)$  means that  $|g| \leq K'f$  for some constant  $K' > 0$  independent of small  $t > 0$ .

**2. Formal asymptotic expansion to the radial case.** In this section we briefly discuss the formal asymptotic expansion to the simplest situation, the radially

symmetric solution of (1.1) as  $t \searrow 0$ . Even though this presentation is only formal, it shows us several crucial aspects for the constructions of sub- and supersolutions of (1.1) in the next section. For each  $x_0 \in \mathbb{R}^N$ , put  $B(x_0, R) = \{x \in \mathbb{R}^N \mid |x - x_0| < R\}$ .

If  $u = u(t, r)$  ( $r = |x|$ ) and  $C_0 = \overline{B(0, R)}$ , then the problem (1.1) turns to

$$(2.1) \quad \begin{cases} \mathcal{L}u := u_t - u_{rr} - \frac{N-1}{r}u_r = 0 & \text{in } (0, +\infty) \times (0, +\infty), \\ u_r(t, 0) = 0, & t > 0, \\ u(0, r) = \begin{cases} 1, & r \in [0, R], \\ -1, & r \in (R, +\infty). \end{cases} \end{cases}$$

Set  $\tilde{\Gamma}_0 = \partial B(0, R)$  and  $\tilde{\Gamma}_t = \{x \in \mathbb{R}^N \mid u(t, |x|) = 0\}$  for  $t > 0$ . Then we can easily verify that  $\tilde{\Gamma}_t$  is a sphere in  $\mathbb{R}^N$ .

As to the behavior of the solution of (2.1) away from  $\tilde{\Gamma}_t$ , we have the following.

LEMMA 2.1. *For any  $\delta \in (0, R/2)$ , there exist  $M_0 > 0$  and  $t_0 \in (0, 1)$  such that, for all  $t \in (0, t_0)$ ,*

$$\begin{aligned} 1 - M_0 e^{-\delta^2/32t} &\leq u(t, r) \leq 1 && \text{for all } 0 \leq r \leq R - \delta, \\ -1 &\leq u(t, r) \leq -1 + M_0 e^{-\delta^2/32t} && \text{for all } r \geq R + \delta. \end{aligned}$$

See Goto, Ishii, and Ogawa [10, Proposition 6.1] for the proof. From this lemma, it is sufficient for us to consider the asymptotics of the solution of (2.1) near  $\tilde{\Gamma}_t$ . Fix  $\delta \in (0, R/2)$  and take  $t_0 > 0$  so small that Lemma 2.1 holds. Let  $\tilde{\phi}(t)$  be the radius of  $\tilde{\Gamma}_t$ . We assume that the solution  $u$  of (2.1) is approximated by the following formal series in  $(t, r) \in (0, t_0) \times (R - \delta, R + \delta)$ :

$$(2.2) \quad u(t, r) = \sum_{j=0}^{+\infty} t^{j/2} U_j \left( t, \frac{\tilde{d}(t, r)}{2\sqrt{t}} \right), \quad \tilde{\phi}(t) = \sum_{j=0}^{+\infty} t^j \phi_j(t), \quad \tilde{\phi}(0) = \phi_0(0) = R.$$

Here  $U_j$  and  $\phi_j$  are assumed to be bounded in  $(0, t_0) \times (R - \delta, R + \delta)$  for each  $j \in \mathbb{N} \cup \{0\}$  and  $\tilde{d}(t, r)$  is the signed distance function to  $\tilde{\Gamma}_t$  given by

$$\tilde{d}(t, r) = \tilde{\phi}(t) - r.$$

Before choosing  $U_j$  and  $\phi_j$  ( $j \in \mathbb{N} \cup \{0\}$ ), we give a lemma on the effect of the diffusion on  $\partial B(0, R)$ . Select  $\tilde{r} \geq 0$  so that  $u(t, R - \tilde{r}) = 0$ . Then  $\tilde{r} = \tilde{\phi}(0) - \tilde{\phi}(t)$  and it is the normal distance between  $\partial B(0, R)$  and  $\tilde{\Gamma}_t$ .

LEMMA 2.2. *We have*

$$\tilde{r} = \frac{(N-1)t}{R} + \frac{(N-1)(3N-1)t^2}{6R^3} + O(t^3) \quad \text{as } t \searrow 0.$$

See the appendix for the proof. By this lemma, we formally have

$$\tilde{\phi}' \Big|_{t=0} = -\tilde{r}' \Big|_{t=0} = -\frac{N-1}{R},$$

where  $' = d/dt$ . This suggests that we may set  $\phi_0(t) = \phi(t)$ , where  $\phi(t) = \sqrt{R^2 - 2(N-1)t}$  and it solves

$$(2.3) \quad \phi'(t) = -\frac{N-1}{\phi(t)} \quad \text{for } t > 0, \quad \phi(0) = R.$$

Using Lemma 2.2 and the definition of  $\phi$ , we can estimate the distance between  $\partial B(0, \phi(t))$  and  $\tilde{\Gamma}_t$  for small  $t > 0$ .

PROPOSITION 2.3. *Let  $\phi(t) = \sqrt{R^2 - 2(N - 1)t}$  and let  $\tilde{r}$  be the normal distance between  $\partial B(0, R)$  and  $\tilde{\Gamma}_t$ . Then*

$$\tilde{r} - (R - \phi(t)) = \frac{(N - 1)t^2}{3R^3} + O(t^3) \quad \text{as } t \searrow 0.$$

Remark 2.1. Ruuth [22, Chapter 4] obtained a similar result to this proposition in the case of the graph in  $\mathbb{R}^2$ .

Proof of Proposition 2.3. It is easily seen by Taylor expansion to  $\phi$  around  $t = 0$  that

$$\phi(t) = R - \frac{(N - 1)t}{R} - \frac{(N - 1)^2 t^2}{2R^2} + O(t^3) \quad \text{as } t \searrow 0.$$

Hence, we have the result by using Lemma 2.2 and this expansion.  $\square$

We choose  $U_j$  and  $\phi_j$  ( $j \in \mathbb{N} \cup \{0\}$ ) of (2.2). First, we do some  $\phi_j$ 's. It is seen that, as  $t \searrow 0$ ,

$$\begin{aligned} \tilde{r} - (R - \phi(t)) &= - \sum_{j=1}^{+\infty} t^j \phi_j(t), \\ \frac{1}{R} &\approx \frac{1}{\tilde{\phi}(t)} = \frac{1}{\phi(t)} \left\{ 1 - \sum_{j=1}^{+\infty} t^j \frac{\phi_j(t)}{\phi(t)} + \left( \sum_{j=1}^{+\infty} t^j \frac{\phi_j(t)}{\phi(t)} \right)^2 - \dots \right\}. \end{aligned}$$

It follows from these relations and Proposition 2.3 that

$$- \sum_{j=1}^{+\infty} t^j \phi_j(t) = \frac{t^2(N - 1)}{3(\phi(t))^3} \left\{ 1 - \sum_{j=1}^{+\infty} t^j \frac{\phi_j(t)}{\phi(t)} + \left( \sum_{j=1}^{+\infty} t^j \frac{\phi_j(t)}{\phi(t)} \right)^2 - \dots \right\}^3 + O(t^3)$$

for sufficiently small  $t > 0$ . Comparing the coefficients of  $t^j$  ( $j = 1, 2$ ) on both sides, we have

$$t\text{-term} : \phi_1(t) = 0, \quad t^2\text{-term} : \phi_2(t) = -\frac{N - 1}{3(\phi(t))^3}.$$

We omit the choices of  $\phi_j$ 's ( $j \geq 3$ ).

Second, we select some  $U_j$ 's. We set  $\tilde{\phi}(t) = \phi(t) + t^2\phi_2(t)$  for simplicity. Put  $\rho = \tilde{d}/2\sqrt{t}$ . Since  $r = \phi - (2\sqrt{t}\rho - t^2\phi_2)$ , we get

$$(2.4) \quad \frac{1}{r} = \frac{1}{\phi - (2\sqrt{t}\rho - t^2\phi_2)} = \frac{1}{\phi} \sum_{j=0}^{+\infty} \left\{ \frac{1}{\phi} (2\sqrt{t}\rho - t^2\phi_2) \right\}^j \quad \text{for small } t > 0.$$

Besides we easily see that

$$d_t = \phi' + 2t\phi_2 + t^2\phi_2', \quad d_r = -1, \quad d_{rr} = 0.$$

Thus we use (2.2), (2.4) and these identities to compute that

$$\begin{aligned} \mathcal{L}u &= -\frac{1}{4t} (U_{0,\rho\rho} + 2\rho U_{0,\rho}) + \frac{1}{\sqrt{t}} \left\{ \frac{U_{0,\rho}}{2} \left( \phi' + \frac{N-1}{\phi} \right) - \left( \frac{1}{4} U_{1,\rho\rho} + \frac{\rho}{2} U_{1,\rho} - \frac{1}{2} U_1 \right) \right\} \\ &\quad + \left\{ \frac{(N-1)\rho U_{0,\rho}}{\phi^2} + \frac{U_{1,\rho}}{2} \left( \phi' + \frac{N-1}{\phi} \right) - \left( \frac{1}{4} U_{2,\rho\rho} + \frac{\rho}{2} U_{2,\rho} - U_2 \right) \right\} \\ &\quad + \sqrt{t} \left\{ U_{0,\rho} \phi_2 + U_{1,t} + (N-1) \left( \frac{2\rho^2}{\phi^3} U_{0,\rho} + \frac{\rho U_{1,\rho}}{\phi^2} \right) + \frac{U_{2,\rho}}{2} \left( \phi' + \frac{N-1}{\phi} \right) \right. \\ &\quad \left. - \left( \frac{1}{4} U_{3,\rho\rho} + \frac{\rho}{2} U_{3,\rho} - \frac{3}{2} U_3 \right) \right\} + \dots \\ &= 0, \end{aligned}$$

where  $U_{i,\rho} = \partial U_i / \partial \rho$  and  $U_{i,\rho\rho} = \partial^2 U_i / \partial \rho^2$ .

We compare the coefficients of  $t^{j/2}$  ( $j = -2, -1, 0, 1, 2, \dots$ ). In the case of the  $t^{-1}$ -term, we can derive

$$(2.5) \quad U_{0,\rho\rho} + 2\rho U_{0,\rho} = 0 \quad \text{on } \mathbb{R}^1.$$

Taking Lemma 2.1 into account, we impose the following condition on  $U_0$ :

$$(2.6) \quad U_0(t, \rho) \longrightarrow \begin{cases} 1 & \text{as } \rho \rightarrow +\infty, \\ -1 & \text{as } \rho \rightarrow -\infty \end{cases} \quad \text{for any small } t > 0.$$

Then we have

$$(2.7) \quad U_0 = U_0(\rho) = \frac{2}{\sqrt{\pi}} \int_0^\rho e^{-s^2} ds.$$

As for the  $t^{-1/2}$ -term, by (2.3) we obtain

$$(2.8) \quad \frac{1}{4} U_{1,\rho\rho} + \frac{\rho}{2} U_{1,\rho} - \frac{1}{2} U_1 = 0 \quad \text{on } \mathbb{R}^1.$$

Since the rate of convergence (2.6) is faster than the exponential one, combining Lemma 2.1 with this fact, we have the following condition on  $U_1$ :

$$(2.9) \quad U_1(t, \rho) \longrightarrow 0 \quad \text{as } \rho \rightarrow \pm\infty \quad \text{for any small } t > 0.$$

Therefore we have  $U_1 \equiv 0$  because the uniqueness of solutions of (2.8) under (2.9) holds in the class of bounded functions.

In the case of the  $t^{j/2}$ -term ( $j = 0, 1$ ), from (2.3) and the fact that  $U_1 \equiv 0$  we get

$$(2.10) \quad \frac{1}{4} U_{2,\rho\rho} + \frac{\rho}{2} U_{2,\rho} - U_2 = \frac{(N-1)\rho U_{0,\rho}}{\phi^2} \quad \text{on } \mathbb{R}^1,$$

$$(2.11) \quad \frac{1}{4} U_{3,\rho\rho} + \frac{\rho}{2} U_{3,\rho} - \frac{3}{2} U_3 = (N-1) \left( 2\rho^2 - \frac{1}{3} \right) \frac{U_{0,\rho}}{\phi^3} \quad \text{on } \mathbb{R}^1.$$

By the same reason as above, the following condition is imposed on  $U_j$ 's ( $j = 2, 3$ ):

$$(2.12) \quad U_j(t, \rho) \longrightarrow 0 \quad \text{as } \rho \rightarrow \pm\infty \quad \text{for any small } t > 0.$$

Solving (2.10) and (2.11) under (2.12), we obtain

$$(2.13) \quad U_2(t, \rho) = -\frac{(N-1)\rho e^{-\rho^2}}{\sqrt{\pi}(\phi(t))^2}, \quad U_3(t, \rho) = -\frac{4(N-1)\rho^2 e^{-\rho^2}}{3\sqrt{\pi}(\phi(t))^3}.$$

We omit selecting  $U_j$  ( $j \geq 4$ ).

*Remark 2.2.* In the above discussion, we have not applied the Fredholm alternative to derive the equation of the motion of the interface, which is used to do so in the case of the Allen–Cahn equation (see, e.g., Fife [7], de Mottoni and Schatzman [4], and Nochetto, Paolini, and Verdi [20]), because such equations as (2.8) under (2.9) have only the trivial solution. For this reason, we have used other methods such as Lemma 2.2 and Proposition 2.3 to determine  $\phi_0$ ,  $\phi_1$ , and  $\phi_2$ .

**3. Subsolutions and supersolutions.** We construct sub- and supersolutions of (1.1) in  $(kh, (k + 1)h) \times \mathbb{R}^N$  for  $h > 0$  and  $k \in \mathbb{N} \cup \{0\}$ . These functions will be used in sections 4 and 5 to derive the optimal rate of convergence of the BMO algorithm.

In this and the next section we assume that  $\{\Gamma_t\}_{0 \leq t < T_0}$  is a motion of a smooth and compact hypersurface by mean curvature. The precise assumption on  $\{\Gamma_t\}_{0 \leq t < T_0}$  is given in subsection 3.1. In addition, the existence, uniqueness, and behavior of  $\{\Gamma_t\}_{0 \leq t < T_0}$  are mentioned in Remark 4.1 of section 4.

**3.1. Signed distance function.** For each  $t \in [0, T_0)$ , the signed distance function  $d = d(t, x)$  to  $\Gamma_t$  is defined by

$$(3.1) \quad d(t, x) = \begin{cases} \text{dist}(x, \Gamma_t) & \text{for } x \in D_t^+, \\ 0 & \text{for } x \in \Gamma_t, \\ -\text{dist}(x, \Gamma_t) & \text{for } x \in D_t^-, \end{cases}$$

where  $D_t^+$  denotes the bounded domain enclosed by  $\Gamma_t$  and  $D_t^- = \mathbb{R}^N \setminus (D_t^+ \cup \Gamma_t)$ . Then  $d$  satisfies

$$(3.2) \quad d_t = \Delta d \quad \text{on } \Gamma_t, \quad t > 0.$$

For any  $T \in (0, T_0)$  and  $\delta > 0$ , let  $\mathcal{N}_\delta$  be the tubular neighborhood of  $\{(t, x) \in [0, T] \times \mathbb{R}^N \mid x \in \Gamma_t\}$ :

$$\mathcal{N}_\delta := \{(t, x) \in [0, T] \times \mathbb{R}^N \mid |d(t, x)| \leq \delta\}.$$

We assume that  $\{\Gamma_t\}_{0 \leq t < T_0}$  is so smooth that, for any  $T < T_0$ , there exists a  $\delta > 0$  satisfying

$$(3.3) \quad d_t, d_{x_i}, d_{x_i x_j}, d_{x_i x_j t}, d_{x_i x_j x_k}, d_{x_i x_j x_k x_l} \in L^\infty(\mathcal{N}_{5\delta}) \quad \text{for } i, j, k, l = 1, \dots, N.$$

It follows from this condition that for any  $(t, x) \in \mathcal{N}_{5\delta}$ , there is a unique  $y(t, x) \in \Gamma_t$  satisfying

$$(3.4) \quad |d(t, x)| = |x - y(t, x)|.$$

Let  $\tilde{\kappa} = \tilde{\kappa}(t, x)$  be the sum of the square of all principal curvatures at  $x \in \Gamma_t$ . Define

$$\kappa^s = \kappa^s(t, x) := \tilde{\kappa}(t, y(t, x)) \quad \text{for } (t, x) \in \mathcal{N}_{5\delta}.$$

Then (3.3) yields that

$$(3.5) \quad \kappa^s, \kappa^s_t, \kappa^s_{x_i}, \kappa^s_{x_i x_j} \in L^\infty(\mathcal{N}_{5\delta}) \quad \text{for } i, j = 1, 2, \dots, N.$$

Moreover, we observe by this property that there exists a  $\kappa_1 > 0$  such that

$$(3.6) \quad |d_t - \Delta d - \kappa^s d| \leq \kappa_1 d^2 \quad \text{on } \mathcal{N}_{5\delta}.$$

For the details to (3.2)–(3.6), see, e.g., Chen [3], Gilbarg and Trudinger [9], and Paolini and Verdi [21].

**3.2. Nonradial case.** In this subsection we construct a sub- and a super-solution of (1.1) in  $(kh, (k + 1)h) \times \mathbb{R}^N$  for each  $h > 0$  and  $k \in \mathbb{N} \cup \{0\}$ .

We modify slightly the signed distance function  $d$  in (3.1). For any  $(t, x) \in \mathcal{N}_{5\delta}$ ,  $k \in \mathbb{N} \cup \{0\}$ , and  $\alpha_k \geq 0$ , set

$$\underline{d}_k(t, x) = d(t, x) - \alpha_k h^2, \quad \bar{d}_k(t, x) = d(t, x) + \alpha_k h^2.$$

We introduce smooth functions  $\eta$  and  $\zeta$  satisfying

$$(3.7) \quad \eta(r) = \begin{cases} r, & s \leq \delta, \\ 2\delta, & s \geq 3\delta, \\ -2\delta, & s \leq -3\delta, \end{cases} \quad 0 \leq \eta' \leq 1, \quad |\eta''| \leq \frac{M_1}{\delta},$$

$$(3.8) \quad \zeta(r) = \begin{cases} 1, & |s| \leq \delta, \\ 0, & |s| \geq 3\delta, \end{cases} \quad 0 \leq \zeta \leq 1, \quad |\zeta'| \leq \frac{M_1}{\delta}, \quad |\zeta''| \leq \frac{M_1}{\delta^2},$$

where  $M_1$  is a constant independent of  $\delta$ . Motivated by the formal discussion in section 2, we define  $\underline{u}$  and  $\bar{u}$  by

$$(3.9) \quad \underline{u}(t, x) = U_0 \left( \frac{\eta(\underline{d}_k(t, x))}{2\sqrt{t - kh}} \right) + (t - kh)\zeta(\underline{d}_k(t, x))U_2 \left( t, x, \frac{\underline{d}_k(t, x)}{2\sqrt{t - kh}} \right) - (t - kh)^{3/2}U_3 - U_4\alpha_k h^2 \sqrt{t - kh},$$

$$(3.10) \quad \bar{u}(t, x) = U_0 \left( \frac{\eta(\bar{d}_k(t, x))}{2\sqrt{t - kh}} \right) + (t - kh)\zeta(\bar{d}_k(t, x))U_2 \left( t, x, \frac{\bar{d}_k(t, x)}{2\sqrt{t - kh}} \right) + (t - kh)^{3/2}U_3 + U_4\alpha_k h^2 \sqrt{t - kh}$$

in  $(kh, (k + 1)h) \times \mathbb{R}^N$  and  $k \in \mathbb{N} \cup \{0\}$ . Here  $U_2 = U_2(t, x, \rho)$  is given by

$$(3.11) \quad U_2(t, x, \rho) = -\frac{1}{\sqrt{\pi}}\kappa^s(t, x)\rho e^{-\rho^2},$$

and  $U_3$  and  $U_4$  are positive constants selected later. At  $t = kh$ , we set

$$(3.12) \quad \underline{u}(kh, x) = \begin{cases} 1 & \text{if } \underline{d}_k(kh, x) \geq 0, \\ -1 & \text{if } \underline{d}_k(kh, x) < 0, \end{cases} \quad \bar{u}(kh, x) = \begin{cases} 1 & \text{if } \bar{d}_k(kh, x) \geq 0, \\ -1 & \text{if } \bar{d}_k(kh, x) < 0. \end{cases}$$

We note that  $U_2$  satisfies

$$(3.13) \quad \frac{1}{4}U_{2,\rho\rho} + \frac{\rho}{2}U_{2,\rho} - U_2 = \kappa^s \rho U_{0,\rho} \quad \text{for } (t, x) \in \mathcal{N}_{5\delta}, \quad \rho \in \mathbb{R}^1.$$

**PROPOSITION 3.1.** *Let  $d$  satisfy (3.3) for some  $\delta > 0$ . Then there exist  $h_1 > 0$ ,  $U_3 > 0$ , and  $U_4 > 0$  such that for each  $h \in (0, h_1)$ ,  $k \in \mathbb{N} \cup \{0\}$ , and  $\alpha_k \geq 0$ ,  $\underline{u}$  and  $\bar{u}$  are, respectively, a subsolution and a supersolution of (1.1) in  $(kh, (k + 1)h) \times \mathbb{R}^N$ .*

*Proof.* We set  $k = 0$  and prove the subsolution case.

For the notational simplicity, put  $\underline{\rho} = \underline{d}_0/2\sqrt{t}$ ,  $\underline{z}(t, x) = \eta(\underline{d}_0(t, x))$ , and  $\underline{\rho}_z = \underline{z}/2\sqrt{t}$ . We denote by  $\mathbb{L}$

$$\mathbb{L}u = u_t - \Delta u \quad \text{for } u = u(t, x).$$

It is seen by calculations that

$$\begin{aligned} \mathbb{L}\underline{u} = & -\frac{1}{4t}(U_{0,\rho\rho}|D\underline{z}|^2 + 2\rho_z U_{0,\rho}) + \frac{U_{0,\rho}}{2\sqrt{t}}(\underline{z}_t - \Delta \underline{z}) - \zeta \left\{ \frac{1}{4}U_{2,\rho\rho}|D\underline{d}_0|^2 + \frac{\rho}{2}U_{2,\rho} - U_2 \right\} \\ & + \sqrt{t} \{ \zeta U_{2,\rho}(\underline{d}_{0,t} - \Delta \underline{d}_0) + \zeta' U_{2,\rho}|D\underline{d}_0|^2 + \zeta \langle DU_{2,\rho}, D\underline{d}_0 \rangle \} \\ & + t \{ \zeta(U_{2,t} - \Delta U_2) + \zeta' U_2(\underline{d}_{0,t} - \Delta \underline{d}_0) \} - \frac{3}{2}\sqrt{t}U_3 - \frac{U_4\alpha_0 h^2}{2\sqrt{t}}, \end{aligned}$$

where  $Df = (f_{x_1}, \dots, f_{x_N})$ . We divide our consideration into two cases.



Case 1.  $|\underline{d}_0(t, x)| \leq \delta$ .

In this case,  $\eta = \zeta = 1$ . Thus  $\underline{z} = \underline{d}_0$ ,  $\rho_{\underline{z}} = \underline{\rho}$  and

$$(3.14) \quad |D\underline{z}| = |D\underline{d}_0| = |Dd| = 1, \quad \underline{z}_t - \Delta \underline{z} = \underline{d}_{0,t} - \Delta \underline{d}_0 = d_t - \Delta d \quad \text{on } \mathcal{N}_{5\delta}.$$

Moreover, we observe that

$$(3.15) \quad \sup_{\substack{\rho \in \mathbb{R}^1 \\ t=0,2}} |\rho^l U_{0,\rho}| + \sup_{(t,x) \in \mathcal{N}_{5\delta}, \rho \in \mathbb{R}^1} (|U_2| + |U_{2,\rho}| + |U_{2,t}| + |\Delta U_2| + |DU_{2,\rho}|) \leq K.$$

Here we have used (3.3) and (3.5) to obtain the boundedness for the second term on the left-hand side of this inequality.

It follows from (2.5), (3.3), (3.14), and (3.15) that

$$\mathbb{L}\underline{u} \leq \frac{U_{0,\rho}}{2\sqrt{t}}(d_t - \Delta d) - \left\{ \frac{1}{4}U_{2,\rho\rho} + \frac{\rho}{2}U_{2,\rho} - U_2 \right\} + \sqrt{t} \left\{ K(1 + \sqrt{t}) - \frac{3}{2}U_3 \right\} - \frac{U_4\alpha_0 h^2}{2\sqrt{t}}.$$

We see by the positivity of  $U_{0,\rho}$ , (3.6) and (3.13) that

$$\frac{U_{0,\rho}}{2\sqrt{t}}(d_t - \Delta d) - \left\{ \frac{1}{4}U_{2,\rho\rho} + \frac{\rho}{2}U_{2,\rho} - U_2 \right\} \leq 2\sqrt{t}\kappa_1 \rho^2 U_{0,\rho} + \frac{U_{0,\rho}}{2\sqrt{t}}(\kappa^s + 2\kappa_1 d)\alpha_0 h^2.$$

Using (3.5), (3.15) and this inequality, we get

$$\mathbb{L}\underline{u} \leq \sqrt{t} \left( K(1 + \sqrt{t}) - \frac{3}{2}U_3 \right) + \frac{h^2}{2\sqrt{t}}(U_{4,1}(U_{4,2} + 2\kappa_1\delta) - U_4)\alpha_0,$$

where  $U_{4,1} = \|U_{0,\rho}\|_{L^\infty(\mathbb{R})}$  and  $U_{4,2} = \|\kappa^s\|_{L^\infty(\mathcal{N}_{5\delta})}$ . Therefore we can take

$$U_3 \geq \frac{4}{3}K, \quad U_4 \geq U_{4,1}(U_{4,2} + 2\kappa_1\delta)$$

to obtain

$$\mathbb{L}\underline{u} \leq 0 \quad \text{in } \{(t, x) \in (0, h) \times \mathbb{R}^N \mid |\underline{d}_0(t, x)| \leq \delta\}.$$

Case 2.  $|\underline{d}_0(t, x)| \geq \delta$ .

In this case,  $|\underline{z}| \geq \delta$ . Then we see by (3.3), (3.5), and (3.7) that

$$(3.16) \quad \begin{aligned} & |\rho_{\underline{z}}^l U_{0,\rho}(\rho_{\underline{z}})| + |U_2(t, x, \underline{\rho})| + |U_{2,\rho}(t, x, \underline{\rho})| + |U_{2,t}(t, x, \underline{\rho})| \\ & + |\Delta U_2(t, x, \underline{\rho})| + |DU_{2,\rho}(t, x, \underline{\rho})| \leq Ke^{-\delta^2/8t} \end{aligned}$$

for small  $t > 0$  and  $l = 0, 1, 2, 3$ . By (3.3) and (3.7) we get

$$|w_t - \Delta w| \leq K \quad (w = z, \underline{d}_0).$$

Using (2.5), (3.8), (3.13), (3.16) and this estimate, we obtain

$$\mathbb{L}\underline{u} \leq Ke^{-\delta^2/8t} - \frac{3\sqrt{t}}{2}U_3.$$

We choose  $h_1 > 0$  so small that  $e^{-\delta^2/8t} \leq \sqrt{t}$  for all  $t \in (0, h_1)$ . Hence, letting  $U_3 \geq 2K/3$  and  $U_4 \geq 0$ , we have

$$\mathbb{L}\underline{u} \leq 0 \quad \text{in } \{(t, x) \in (0, h) \times \mathbb{R}^N \mid |\underline{d}_0(t, x)| \geq \delta\}$$

for all  $h \in (0, h_1)$ .

Consequently, taking a large  $U_3 > 0$  and setting  $U_4 = U_{4,1}(U_{4,2} + 2\kappa_1\delta)$ , we obtain

$$\mathbb{L}\underline{u} \leq 0 \quad \text{in } (0, h) \times \mathbb{R}^N$$

for all  $h \in (0, h_1)$ . The supersolution case can be shown by a method similar to the above.

For the case  $k \geq 1$ , we can show the assertion of this proposition by using the same  $h_1$ ,  $U_3$ , and  $U_4$  as in the case  $k = 0$ .  $\square$

**3.3. Radial case.** This subsection is devoted to the construction of a sub- and a supersolution of (1.1) which are radially symmetric. We recall that if  $u = u(t, r)$  ( $r = |x|$ ) and  $C_0 = B(0, R)$ , then the problem (1.1) turns to (2.1). We assume  $N = 2$  to simplify our arguments.

For any  $R > 0$ , put

$$(3.17) \quad \phi(t) = \sqrt{R^2 - 2t}, \quad \phi_1(t) = \frac{1}{3(\phi(t))^3} \quad \text{on } [0, h],$$

$$\tilde{d}(t, r) = \phi(t) - t^2\phi_1(t) - r \quad \text{on } [0, h] \times \mathbb{R}^1.$$

Let  $\eta$  and  $\zeta$  be the same functions as (3.7) and (3.8), respectively. Set  $z(t, r) = \eta(\tilde{d}(t, r))$  and define  $\underline{u}$  and  $\bar{u}$  by

$$(3.18) \quad \underline{u}(t, r) = U_0\left(\frac{z(t, r)}{2\sqrt{t}}\right) + \zeta(\tilde{d}(t, r)) \left\{ tU_2\left(t, \frac{\tilde{d}(t, r)}{2\sqrt{t}}\right) + t^{3/2}U_3\left(t, \frac{\tilde{d}(t, r)}{2\sqrt{t}}\right) \right\} - t^2U_4,$$

$$(3.19) \quad \bar{u}(t, r) = U_0\left(\frac{z(t, r)}{2\sqrt{t}}\right) + \zeta(\tilde{d}(t, r)) \left\{ tU_2\left(t, \frac{\tilde{d}(t, r)}{2\sqrt{t}}\right) + t^{3/2}U_3\left(t, \frac{\tilde{d}(t, r)}{2\sqrt{t}}\right) \right\} + t^2U_4$$

for  $t > 0$ ,  $r \in \mathbb{R}^1$ . Here  $U_2, U_3$  are given by (2.13) and  $U_4$  is a constant selected later. At  $t = 0$ , we put

$$(3.20) \quad \underline{u}(0, r) = \bar{u}(0, r) = \begin{cases} 1, & \text{if } \tilde{d}(0, r) \geq 0, \\ -1, & \text{if } \tilde{d}(0, r) < 0. \end{cases}$$

**PROPOSITION 3.2.** Fix  $\delta \in (0, R/5 \wedge 1)$ . Then there exist  $h_2 > 0$  and  $U_4 > 0$  such that for each  $h \in (0, h_2)$ ,  $\underline{u}$  and  $\bar{u}$  are, respectively, a subsolution and a supersolution of (2.1) in  $(0, h) \times \mathbb{R}^1$ . In addition, they satisfy the boundary condition of (2.1).

*Proof.* We assume  $k = 0$  and prove the subsolution case. Set  $\rho_z = z/2\sqrt{t}$  and  $\tilde{\rho} = \tilde{d}/2\sqrt{t}$  for the notational simplicity.

It is observed by calculations that

$$\begin{aligned} \mathcal{L}\underline{u} = & -\frac{1}{4t} (U_{0,\rho\rho}z_r^2 + 2\rho_z U_{0,\rho}) + \frac{U_{0,\rho}}{2\sqrt{t}} \left( z_t - z_{rr} - \frac{1}{r}z_r \right) - \zeta \left( \frac{1}{4}U_{2,\rho\rho}\tilde{d}_r^2 + \frac{\tilde{\rho}}{2}U_{2,\rho} - U_2 \right) \\ & + \sqrt{t}\zeta \left\{ U_{2,\rho} \left( \tilde{d}_t - \tilde{d}_{rr} - \frac{1}{r}\tilde{d}_r \right) - \left( \frac{1}{4}U_{3,\rho\rho}\tilde{d}_r^2 + \frac{\tilde{\rho}}{2}U_{3,\rho} - \frac{3}{2}U_3 \right) \right\} \\ & + t \left[ \zeta \left\{ U_{2,t} + \sqrt{t}U_{3,t} + \frac{U_{3,\rho}}{2} \left( \tilde{d}_t - \tilde{d}_{rr} - \frac{1}{r}\tilde{d}_r \right) \right\} - 2U_4 \right] \\ & - 2\zeta' \tilde{d}_r \left\{ tU_{2,\rho} \frac{\tilde{d}_r}{2\sqrt{t}} + t^{3/2}U_{3,\rho} \frac{\tilde{d}_r}{2\sqrt{t}} \right\} \\ & + t \left\{ \zeta' \left( \tilde{d}_t - \tilde{d}_{rr} - \frac{1}{r}\tilde{d}_r \right) - \zeta'' \tilde{d}_r^2 \right\} (U_2 + \sqrt{t}U_3). \end{aligned}$$

We divide our consideration into two cases.

Case 1.  $|\tilde{d}(t, r)| \leq \delta$ .

In this case,  $\eta = \zeta = 1$ . Thus  $z = \tilde{d}$ ,  $\rho_z = \tilde{\rho}$  and

$$(3.21) \quad w_r = -1, \quad w_t - w_{rr} - \frac{1}{r}w_r = \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \quad \text{for } w = z, \tilde{d}.$$

Moreover, we see from (2.13) and (3.17) that

$$(3.22) \quad \sup_{\substack{\rho \in \mathbb{R}^1 \\ t=0,1,2,3}} |\rho^l U_{0,\rho}| + \sup_{\substack{t \in [0, R^2/4], \rho \in \mathbb{R}^1 \\ j=2,3, l=0,1,2,3}} (|\rho^l U_j| + |\rho^l U_{j,\rho}| + |U_{j,t}|) \leq K.$$

By (2.5), (3.21), and this boundedness, we have

$$\begin{aligned} \mathcal{L}u \leq & \frac{U_{0,\rho}}{2\sqrt{t}} \left( \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \right) - \left( \frac{1}{4}U_{2,\rho\rho} + \frac{\tilde{\rho}}{2}U_{2,\rho} - U_2 \right) \\ & + t^{1/2} \left\{ U_{2,\rho} \left( \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \right) - \left( \frac{1}{4}U_{3,\rho\rho} + \frac{\tilde{\rho}}{2}U_{3,\rho} - \frac{3}{2}U_3 \right) \right\} \\ & + \frac{tU_{3,\rho}}{2} \left( \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \right) + t(K - 2U_4). \end{aligned}$$

We estimate  $\phi' - 2t\phi_1 - t^2\phi'_1 + 1/r$ . We remark that  $\tilde{d} = 2\sqrt{t}\tilde{\rho}$  and  $r = \phi - (2\sqrt{t}\tilde{\rho} + t^2\phi_1)$ . It follows from (3.17) that there exists an  $h_2 \in (0, \delta)$  such that  $\phi(t) \geq 4\delta$  and  $t^2\phi_1(t) \leq \delta$  for  $t \in (0, h_2)$ . Thus we get  $2\sqrt{t}|\tilde{\rho}| + t^2\phi_1 \leq 2\delta$  and

$$\frac{1}{r} \leq \frac{1}{\phi} \left\{ 1 + \frac{1}{\phi}(2\sqrt{t}\tilde{\rho} + t^2\phi_1) + \frac{1}{\phi^2}(2\sqrt{t}\tilde{\rho} + t^2\phi_1)^2 \right\} + \frac{1}{3\delta\phi^3}(2\sqrt{t}|\tilde{\rho}| + t^2\phi_1)^3$$

for all  $t \in (0, h_2)$ . Since  $\phi$  satisfies  $\phi' + 1/\phi = 0$  and  $\phi'_1$  is bounded, we observe that

$$(3.23) \quad \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \leq \frac{2\sqrt{t}\tilde{\rho}}{\phi^2} + \frac{4t\tilde{\rho}^2}{\phi^3} - 2t\phi_1 + \frac{8t^{3/2}|\tilde{\rho}|^3}{3\delta\phi^3} + Kt^2.$$

We observe from the positivity of  $U_{0,\rho}$  on  $\mathbb{R}^1$ , (2.10), (2.11), (3.22), and this estimate that

$$\begin{aligned} & \frac{U_{0,\rho}}{2\sqrt{t}} \left( \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \right) - \left( \frac{1}{4}U_{2,\rho\rho} + \frac{\tilde{\rho}}{2}U_{2,\rho} - U_2 \right) \\ & - \sqrt{t} \left( \frac{1}{4}U_{3,\rho\rho} + \frac{\tilde{\rho}}{2}U_{3,\rho} - \frac{3}{2}U_3 \right) \leq U_{0,\rho}(Kt\tilde{\rho}^3 + Kt^{3/2}) \leq tK(1 + \sqrt{t}). \end{aligned}$$

In addition, (3.22) and (3.23) yield that

$$t^{1/2}U_{2,\rho} \left( \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \right) + \frac{tU_{3,\rho}}{2} \left( \phi' - 2t\phi_1 - t^2\phi'_1 + \frac{1}{r} \right) \leq Kt.$$

Therefore we obtain

$$\mathcal{L}u \leq t(K(1 + \sqrt{t}) - 2U_4).$$

Consequently, taking  $U_4$  sufficiently large, we obtain

$$\mathcal{L}u \leq 0 \quad \text{in } \{(t, r) \in (0, h) \times \mathbb{R}^1 \mid |\tilde{d}(t, r)| \leq \delta\}$$

for all  $h \in (0, h_2)$ .

Case 2.  $|\tilde{d}(t, r)| \geq \delta$ .

In this case,  $|z| \geq \delta$ . Then we see by (2.13) that

$$(3.24) \quad |\rho_z^l U_{0,\rho}(\rho_z)| + |\tilde{\rho}^l U_j(t, \tilde{\rho})| + |\tilde{\rho}^l U_{j,\rho}(t, \tilde{\rho})| + |U_{j,t}(t, \tilde{\rho})| \leq Ke^{-\delta^2/8t}$$

for small  $t > 0$ ,  $j = 2, 3$ , and  $l = 0, 1, 2, 3$ . Then it is easily observed by the fact that  $\delta \leq r \leq R + 3\delta$  and by (3.7) and (3.17) that

$$\left| w_t - w_{rr} - \frac{1}{r}w_r \right| \leq K \quad \text{for } w = z, \tilde{d}.$$

Thus we apply (3.7), (3.8), (3.24), and this inequality to obtain

$$\mathcal{L}\underline{u} \leq Ke^{-\delta^2/8t} - 2tU_4.$$

Taking  $h_2 > 0$  smaller if necessary, we have  $e^{-\delta^2/8t} \leq t$  for all  $t \in (0, h_2)$ . Thus we choose  $U_4 \geq K/2$  to have

$$\mathcal{L}\underline{u} \leq 0 \quad \text{in } \{(t, r) \in (0, h) \times \mathbb{R}^1 \mid |\tilde{d}(t, r)| \geq \delta\}$$

for all  $h \in (0, h_2)$ .

Consequently, taking  $U_4 > 0$  large and  $h_2 > 0$  small, we obtain

$$\mathcal{L}\underline{u} \leq 0 \quad \text{in } (0, h) \times \mathbb{R}^1$$

for all  $h \in (0, h_2)$ . Since  $\underline{u}(t, r) = U_0(\delta/\sqrt{t}) - t^2U_4$  for  $0 \leq r \ll 1$ , it is easily verified that  $\underline{u}_r(t, 0) = 0$ . The supersolution case can be shown in a similar way.  $\square$

**4. Rate of convergence.** In this section we consider the rate of convergence of the BMO algorithm to the motion of a smooth and compact hypersurface by mean curvature.

To state our theorem, we rewrite the BMO algorithm as follows. Let  $\Gamma_0 \subset \mathbb{R}^N$  be a smooth and compact hypersurface and  $C_0 \subset \mathbb{R}^N$  the compact set such that  $\partial C_0 = \Gamma_0$ . Fix a time step  $h > 0$ . Let  $u^h = u^h(t, x)$  be the solution of

$$(4.1) \quad \begin{cases} u_t^h = \Delta u^h & \text{in } (kh, (k+1)h) \times \mathbb{R}^N, \\ u^h(kh, x) = \begin{cases} 1, & x \in C_k, \\ -1, & x \in \mathbb{R}^N \setminus C_k, \end{cases} \\ C_k = \begin{cases} \text{the above set } C_0 & \text{for } k = 0, \\ \left\{ x \in \mathbb{R}^N \mid \lim_{t \rightarrow kh-} u^h(t, x) \geq 0 \right\} & \text{for } k = 1, 2, \dots \end{cases} \end{cases}$$

Set

$$(4.2) \quad \begin{aligned} C_t^h &= \begin{cases} \{x \in \mathbb{R}^N \mid u^h(t, x) \geq 0\} & \text{for } t \neq kh, \\ C_k & \text{for } t = kh, \end{cases} \\ \Gamma_t^h &= \partial C_t^h (= \{x \in \mathbb{R}^N \mid u^h(t, x) = 0\}). \end{aligned}$$

We note that  $C_{kh}^h$  coincides with  $C_k$  defined in the introduction and that  $\Gamma_t^h$  is a smooth and compact hypersurface for each  $t \geq 0$ ,  $h > 0$ . Furthermore, there exists an  $R_0 > 0$  such that  $\Gamma_t^h \subset \overline{B(0, R_0)}$  for all  $t \geq 0$  and  $h > 0$  (see Barles and Georgelin [1, Lemma 5.1]). Using this formulation, we have the following theorem.

THEOREM 4.1. Let  $\{\Gamma_t\}_{0 \leq t < T_0}$  be a smooth and compact motion by mean curvature satisfying (3.3). Let  $\Gamma_t^h$  be defined by (4.2). Then, for any  $T \in (0, T_0)$ , there exist  $h_0 > 0$  and  $L > 0$  such that

$$(4.3) \quad \sup_{t \in [0, T]} d_H(\Gamma_t^h, \Gamma_t) \leq Lh$$

for all  $h \in (0, h_0)$ . Here  $d_H(A, B)$  denotes the Hausdorff distance between  $A, B \subset \mathbb{R}^N$ .

Remark 4.1. On the existence, uniqueness, and behavior of a motion by mean curvature  $\{\Gamma_t\}_{0 \leq t < T_0}$ , the following results are known. Assume that  $\Gamma_0$  is the boundary of class  $C^{k, \alpha}$  of a bounded domain ( $k \geq 2, 0 < \alpha < 1$ ).

- (i) For some  $T_0 = T_0(\Gamma_0) > 0$ , there uniquely exists a smooth and compact motion by mean curvature  $\{\Gamma_t\}_{0 \leq t < T_0}$  starting from  $\Gamma_0$ . Moreover, the signed distance function  $d$  defined by (3.1) is of class  $C^{(k+\alpha)/2, k+\alpha}(\mathcal{N}_{\delta_0})$  for some small  $\delta_0 > 0$  (see Evans and Spruck [6]).
- (ii) If  $N = 2$  or  $\Gamma_0$  is convex, then the motion  $\{\Gamma_t\}_{0 \leq t < T_0}$  can be extended up to  $T_0 = T_{max}$ , where  $T_{max}$  is the extinction time for  $\Gamma_t$  (see Gage and Hamilton [8], Grayson [11], and Huisken [13]). In other cases the singularities may appear before  $\Gamma_t$  shrinks to a point (see, e.g., Grayson [12]).

Therefore (4.3) is valid before  $\Gamma_t$  shrinks to a point or develops the singularities.

Proof of Theorem 4.1. Set  $k = 0, u = u^h$  and let  $\underline{u}$  and  $\bar{u}$  be defined by (3.9) and (3.10), respectively. Define

$$\begin{aligned} \underline{\Sigma}_t^h &:= \{x \in \mathbb{R}^N \mid \underline{u}(t, x) = 0\}, & \underline{\Theta}_t^h &:= \{x \in \mathbb{R}^N \mid \underline{d}_0(t, x) = 0\}, \\ \bar{\Sigma}_t^h &:= \{x \in \mathbb{R}^N \mid \bar{u}(t, x) = 0\}, & \bar{\Theta}_t^h &:= \{x \in \mathbb{R}^N \mid \bar{d}_0(t, x) = 0\}. \end{aligned}$$

Note that these sets are smooth and compact hypersurfaces.

Step 1. We prove that there exist  $h_{0,1} > 0, L_1$  and  $L_2 > 0$  such that

$$(4.4) \quad d_H(\underline{\Theta}_t^h, \underline{\Sigma}_t^h), d_H(\bar{\Theta}_t^h, \bar{\Sigma}_t^h) \leq (L_1 h \alpha_0 + L_2) h^2$$

for all  $t \in (0, h)$  and  $h \in (0, h_{0,1})$ .

We easily see from (3.15) that there exists an  $h_{0,1} = h_{0,1} > 0$  such that

$$(4.5) \quad |\underline{d}_0(t, x)| < 2\sqrt{t} \quad \text{on } \underline{\Sigma}_t^h \cup \underline{\Theta}_t^h$$

for all  $t \in [0, h)$  and  $h \in (0, h_{0,1})$ . Moreover, taking  $h_{0,1}$  smaller if necessary, we observe that for any  $t \in (0, h_{0,1})$  and  $x \in \mathbb{R}^N$  satisfying  $|\underline{d}_0(t, x)| \leq 2\sqrt{t}$ ,

$$(4.6) \quad \langle D\underline{u}(t, x), D\underline{d}_0(t, x) \rangle \geq \frac{U_{0,\rho}(1)}{2\sqrt{t}} - Kt \geq \frac{1}{10\sqrt{\pi t}}.$$

Let  $\underline{x} \in \underline{\Theta}_t^h$  and take  $\underline{y} \in \underline{\Sigma}_t^h$  so that  $\underline{y} = \underline{x} + |\underline{x} - \underline{y}| D\underline{d}_0(t, \underline{x})$ . Applying the mean value theorem, we obtain

$$\begin{aligned} 0 &= \underline{u}(t, \underline{y}) = \underline{u}(t, \underline{x}) + \langle D\underline{u}(t, \theta \underline{x} + (1 - \theta) \underline{y}), \underline{y} - \underline{x} \rangle \quad (0 < \theta < 1) \\ &= -t^{3/2} U_3 - U_4 \alpha_0 h^2 \sqrt{t} + |\underline{x} - \underline{y}| \langle D\underline{u}(t, \theta \underline{y} + (1 - \theta) \underline{x}), D\underline{d}_0(t, \underline{x}) \rangle. \end{aligned}$$

It is easily seen that  $D\underline{d}_0(t, \underline{x}) = Dd(t, \theta \underline{y} + (1 - \theta) \underline{x})$ . Hence we can use (4.6) to have  $|\underline{x} - \underline{y}| \leq 10\sqrt{\pi t}(U_4 \alpha_0 h^2 + tU_3)$  and thus

$$\sup_{x \in \underline{\Theta}_t^h} \text{dist}(x, \underline{\Sigma}_t^h) \leq (L_1 h \alpha_0 + L_2) h^2,$$

where  $L_1 = 10\sqrt{\pi}U_4$ ,  $L_2 = 10\sqrt{\pi}U_3$  and  $U_3, U_4$  are the same constants as in Proposition 3.1. Similarly, we can show that

$$\sup_{x \in \underline{\Sigma}_t^h} \text{dist}(x, \underline{\Theta}_t^h), \sup_{x \in \bar{\Theta}_t^h} \text{dist}(x, \bar{\Sigma}_t^h), \sup_{x \in \bar{\Sigma}_t^h} \text{dist}(x, \bar{\Theta}_t^h) \leq (L_1 h \alpha_0 + L_2) h^2.$$

Hence we obtain (4.4).

*Step 2.* We show that there exists an  $h_0 > 0$  such that

$$(4.7) \quad d_H(\Gamma_t, \Gamma_t^h) \leq \{(1 + L_1 h)\alpha_0 + L_2\} h^2 \quad \text{for all } t \in (0, h) \text{ and } h \in (0, h_0).$$

Let  $h_1 > 0$  be given in Proposition 3.1. Set  $h_0 = \min\{h_{0,1}, h_1\}$  and fix  $h \in (0, h_0)$ . Since it is easily verified by (3.12) that  $\underline{u}(0, x) \leq u(0, x) \leq \bar{u}(0, x)$  on  $\mathbb{R}^N$ , we have  $\underline{u}(t, x) \leq u(t, x) \leq \bar{u}(t, x)$  on  $[0, h] \times \mathbb{R}^N$  by Proposition 3.1 and the comparison principle for the heat equation. This implies that

$$(4.8) \quad \Gamma_t^h \subset \{x \in \mathbb{R}^N \mid \underline{u}(t, x) \leq 0 \leq \bar{u}(t, x)\} \quad \text{for all } t \in [0, h),$$

that is,  $\Gamma_t^h$  lies between  $\underline{\Sigma}_t^h$  and  $\bar{\Sigma}_t^h$ .

For any  $x \in \Gamma_t$ , we can find an  $\underline{x} \in \underline{\Theta}_t^h$  such that  $\text{dist}(x, \underline{\Theta}_t^h) = \alpha_0 h^2 = |x - \underline{x}|$ . From Step 1, we have

$$\text{dist}(x, \underline{\Sigma}_t^h) \leq |x - \underline{x}| + \text{dist}(\underline{x}, \underline{\Sigma}_t^h) \leq \{(1 + L_1 h)\alpha_0 + L_2\} h^2$$

for all  $t \in [0, h)$ . Since  $x \in \Gamma_t$  is arbitrary, we get

$$(4.9) \quad \sup_{x \in \Gamma_t} \text{dist}(x, \underline{\Sigma}_t^h) \leq \{(1 + L_1 h)\alpha_0 + L_2\} h^2 \quad \text{for all } t \in [0, h).$$

Similarly, we can show that

$$(4.10) \quad \sup_{x \in \Gamma_t} \text{dist}(x, \bar{\Sigma}_t^h) \leq \{(1 + L_1 h)\alpha_0 + L_2\} h^2 \quad \text{for all } t \in [0, h),$$

with the same  $L_1, L_2$  as above.

Hence, using (4.8)–(4.10), we obtain

$$\sup_{x \in \Gamma_t} \text{dist}(x, \Gamma_t^h) \leq \max \left\{ \sup_{x \in \Gamma_t} \text{dist}(x, \bar{\Sigma}_t^h), \sup_{x \in \Gamma_t} \text{dist}(x, \underline{\Sigma}_t^h) \right\} \leq \{(1 + L_1 h)\alpha_0 + L_2\} h^2$$

for all  $t \in (0, h)$ .

Since  $\Gamma_t$  also lies between  $\underline{\Sigma}_t^h$  and  $\bar{\Sigma}_t^h$ , by the same argument as above, we get

$$\sup_{x \in \Gamma_t^h} \text{dist}(x, \Gamma_t) \leq \max \left\{ \sup_{x \in \bar{\Sigma}_t^h} \text{dist}(x, \Gamma_t), \sup_{x \in \underline{\Sigma}_t^h} \text{dist}(x, \Gamma_t) \right\} \leq \{(1 + L_1 h)\alpha_0 + L_2\} h^2$$

for all  $t \in (0, h)$ . Therefore we obtain (4.7).

*Step 3.* We consider the case  $k = 1$ . Put  $\alpha_1 = (1 + L_1 h)\alpha_0 + L_2$  and fix  $h \in (0, h_0)$ . Then we can see by Proposition 3.1 that  $\underline{u}$  and  $\bar{u}$  are, respectively, a subsolution and a supersolution of (1.1) in  $(h, 2h) \times \mathbb{R}^N$ . Since  $\Gamma_t^h$  moves continuously in  $t$  in the sense of the Hausdorff distance (cf. Goto, Ishii, and Ogawa [10, Corollary 3.1]), we observe by (4.7) that

$$\Gamma_h^h \subset \{x \in \mathbb{R}^N \mid d_1(h, x) \leq 0 \leq \bar{d}_1(h, x)\}.$$

It is easily seen by (3.12) and this inclusion that  $\underline{u}(h, x) \leq u(h, x) \leq \bar{u}(h, x)$  on  $\mathbb{R}^N$ , and hence we obtain  $\underline{u}(t, x) \leq u(t, x) \leq \bar{u}(t, x)$  on  $[h, 2h) \times \mathbb{R}^N$  by the comparison principle for the heat equation. Therefore applying the argument in Step 2, we have

$$d_H(\Gamma_t, \Gamma_t^h) \leq \{(1 + L_1 h)\alpha_1 + L_2\}h^2 \quad \text{for all } t \in [h, 2h).$$

*Step 4.* We select  $m \in \mathbb{N}$  satisfying  $mh \leq T < (m + 1)h$  for each  $h \in (0, h_0)$  and repeat the arguments in Steps 2–3 inductively. Set

$$\alpha_k = (1 + L_1 h)\alpha_{k-1} + L_2 \quad \text{for } k = 1, 2, \dots, m.$$

Then it follows from Proposition 3.1 that  $\underline{u}$  and  $\bar{u}$  are, respectively, a subsolution and a supersolution of (1.1) in  $(kh, (k + 1)h) \times \mathbb{R}^N$ . Since we can verify from (3.12) that  $\underline{u}(kh, x) \leq u(kh, x) \leq \bar{u}(kh, x)$  on  $\mathbb{R}^N$ , we have  $\underline{u}(t, x) \leq u(t, x) \leq \bar{u}(t, x)$  on  $[kh, (k + 1)h) \times \mathbb{R}^N$  by the comparison principle for the heat equation. Thus we obtain

$$d_H(\Gamma_t, \Gamma_t^h) \leq \{(1 + L_1 h)\alpha_k + L_2\}h^2 \quad \text{for all } t \in [kh, (k + 1)h) \text{ and } k = 0, 1, 2, \dots, m$$

by an argument similar to Step 2.

*Step 5.* We estimate the sequence  $\{\alpha_k\}_{1 \leq k \leq m}$ . Since  $\alpha_0 \geq 0$  is arbitrary, we can take  $\alpha_0 = 0$ . Then we observe from the definition of  $\alpha_k$  that

$$\begin{aligned} \alpha_k &= (1 + L_1 h)\alpha_{k-1} + L_2 = (1 + L_1 h)^2\alpha_{k-2} + L_2 \{1 + (1 + L_1 h)\} \\ &= \dots \\ &= L_2 \sum_{l=1}^k (1 + L_1 h)^{l-1} \leq L_2 \frac{(1 + L_1 h)^m - 1}{L_1 h}. \end{aligned}$$

By the choice of  $m$ , we get

$$\alpha_k \leq \frac{L_2(e^{L_1 T_0} - 1)}{L_1 h} \leq \frac{L_2 T_0 e^{L_1 T_0}}{h} \quad \text{for } k = 0, 1, \dots, m.$$

Thus we obtain

$$\sup_{t \in [0, T]} d_H(\Gamma_t, \Gamma_t^h) \leq \{(1 + L_1)L_2 T_0 e^{L_1 T_0} + L_2\}h$$

for all  $h \in (0, h_0)$ . Therefore the proof is completed. □

**5. Optimality.** This section is devoted to the optimality for the estimate in Theorem 4.1. For this purpose, we consider a circle evolving by curvature.

Let  $C_0 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$  and fix a time step  $h > 0$ . Let  $u^h$  be the radially symmetric solution of (4.1). Then we can easily verify that for any  $t > 0$ ,  $\Gamma_t^h$  defined in (4.2) is a circle centered at the origin, and we denote by  $R_h(t)$  the radius of  $\Gamma_t^h$ . Put  $\phi(t) = \sqrt{1 - 2t}$  and let  $\Gamma_t = \partial B(0, \phi(t))$ . Take  $\delta \in (0, 1/5)$  and set

$$(5.1) \quad T_{max} = \frac{1}{2}, \quad T_\delta = \frac{1}{2} - \frac{25\delta^2}{2}, \quad m = \left\lceil \frac{T_\delta}{h} \right\rceil,$$

where  $T_{max}$  is the extinction time for  $\Gamma_t$  and  $[s]$  denotes the Gauss symbol for  $s \in \mathbb{R}$ . Applying Theorem 4.1, we see that for each  $\delta \in (0, 1/5)$ , there exist  $h_2 > 0$  and  $M_1 > 0$  such that

$$(5.2) \quad \sup_{t \in [0, T_\delta]} |R_h(t) - \phi(t)| \leq M_1 h \quad \text{for all } h \in (0, h_2).$$

In the remainder of this section we consider a more precise behavior of  $R_h$  as  $h \searrow 0$  than (5.2) and show that the estimate of Theorem 4.1 is optimal. Our main result of this section is stated as follows.

**THEOREM 5.1.** *For each  $\delta \in (0, 1/5)$ , there exist  $h_0 > 0$  and  $L > 0$  such that*

$$(5.3) \quad |R_h(t) - (\phi(t) - t^2\phi_1^0(t))| \leq Lt^{5/2} \quad \text{for } t \in [t, h],$$

$$(5.4) \quad |R_h(t) - (\phi(t) - h\varphi(t))| \leq Lh^{3/2} \quad \text{for } t \in [h, T_\delta]$$

for all  $h \in (0, h_0)$ . Here  $\phi_1^0(t), \varphi(t)$  are given by

$$\phi_1^0(t) = \frac{1}{3(\phi(t))^3}, \quad \varphi(t) = -\frac{\log \phi(t)}{3\phi(t)}.$$

This theorem shows that  $\Gamma_t^h$  moves faster than  $\Gamma_t$ .

As a corollary of Theorem 5.1, we obtain an estimate of the distance between  $\Gamma_t$  and  $\Gamma_t^h$ .

**COROLLARY 5.2.** *For each  $\delta \in (0, 1/5)$ , there exist  $h_1 \in (0, h_0)$ ,  $\bar{L} > 0$ , and  $\underline{L} > 0$  such that*

$$(5.5) \quad \underline{L}t^2 \leq d_H(\Gamma_t, \Gamma_t^h) \leq \bar{L}t^2 \quad \text{for } t \in [0, h],$$

$$(5.6) \quad \underline{L}th \leq d_H(\Gamma_t, \Gamma_t^h) \leq \bar{L}th \quad \text{for } t \in [h, T_\delta]$$

for all  $h \in (0, h_1)$ .

This corollary shows that in the case where  $\{\Gamma_t\}_{t \geq 0}$  is a motion of a smooth and compact hypersurface by mean curvature, the linear rate in  $h$  is optimal to the convergence of the BMO algorithm.

We prepare some functions which will be used in the following subsections. For  $k = 0, 1, 2, \dots, m$ , we define

$$(5.7) \quad \phi^k(t) = \sqrt{(R_h(kh))^2 - 2t}, \quad \phi_1^k(t) = \frac{1}{3(\phi^k(t))^3} \quad \text{for } t \in [0, h].$$

Note that  $\phi^0 = \phi$  and  $(\phi^k)' = -1/\phi^k$  in  $(0, h)$ . It is easily seen by (5.2) and these facts that, for any  $\delta > 0$ , there exists an  $h_0 > 0$  such that

$$(5.8) \quad \frac{1}{3(1+\delta)^3} \leq \phi_1^k(t) \leq \frac{1}{3 \cdot (3\delta)^3}, \quad \frac{1}{(1+\delta)^5} \leq (\phi_1^k)'(t) \leq \frac{1}{(3\delta)^5}$$

for all  $t \in [0, h]$ ,  $k = 0, 1, \dots, m$ , and  $h \in (0, h_0)$ .

**5.1. Short-time asymptotics of  $R_h$ .** In this subsection, we prove the following theorem suggested by Proposition 2.3.

**THEOREM 5.3.** *There exist  $h_3 > 0$  and  $L_1 > 0$  such that*

$$|R_h(t+kh) - (\phi^k(t) - t^2\phi_1^k(t))| \leq L_1t^{5/2}$$

for all  $t \in [0, h]$ ,  $k = 0, 1, \dots, m$ , and  $h \in (0, h_3)$ .

*Proof.* Set  $k = 0$  and  $u = u^h$  for simplicity. Define  $\tilde{\phi}(t) = \phi(t) - t^2\phi_1^0(t)$  and  $\tilde{d}(t, r) = \tilde{\phi}(t) - r$ . Let  $\underline{u}$  and  $\bar{u}$  be defined by (3.18) and (3.19), respectively. Let  $h_2$  be given in Proposition 3.2. Since  $\underline{u}(0, r) = \bar{u}(0, r) = u(0, r)$  on  $\mathbb{R}^1$  by (3.20), applying Proposition 3.2 and the comparison principle for the heat equation, we get

$$(5.9) \quad \underline{u}(t, r) \leq u(t, r) \leq \bar{u}(t, r) \quad \text{in } [0, h] \times \mathbb{R}^1$$



for all  $h \in (0, h_2)$ . Let  $\underline{\phi} = \underline{\phi}(t)$  and  $\bar{\phi} = \bar{\phi}(t)$  be the zero of  $\underline{u}(t, \cdot)$  and  $\bar{u}(t, \cdot)$ , respectively. Then it follows from (5.9) and (5.12) that

$$(5.10) \quad \underline{\phi}(t) \leq R_h(t) \leq \bar{\phi}(t) \quad \text{for all } t \in [0, h) \text{ and } h \in (0, h_2)$$

for all  $h \in (0, h_2)$ .

We estimate  $\underline{\phi}$  and  $\bar{\phi}$ . At first, it is easily seen by (3.22) that there exists an  $h_3 \in (0, \delta^2/4 \wedge h_2)$  such that

$$(5.11) \quad -t^2 U_4 = \underline{u}(t, \tilde{\phi}(t)) < 0 = \underline{u}(t, \underline{\phi}(t)) < \underline{u}(t, \tilde{\phi}(t) - 2t^2),$$

$$(5.12) \quad \underline{u}_r(t, r) \leq -\frac{U_{0,\rho}(1)}{2\sqrt{t}} + Kt \leq -\frac{U_{0,\rho}(1)}{4\sqrt{t}} < 0$$

for all  $t \in (0, h_3)$  and  $r \in \mathbb{R}$  satisfying  $|\tilde{d}(t, r)| \leq 2\sqrt{t} (\leq \delta)$ . Here we have used (3.22) to derive these estimates. Thus we can observe from (5.11) and (5.12) that

$$(5.13) \quad \tilde{\phi}(t) - Kt^{5/2} \leq \underline{\phi}(t) \leq \tilde{\phi}(t) \quad \text{for all } t \in (0, h) \text{ and } h \in (0, h_3).$$

We can also show similarly that

$$(5.14) \quad \tilde{\phi}(t) \leq \bar{\phi}(t) \leq \tilde{\phi}(t) + Kt^{5/2} \quad \text{for all } t \in (0, h) \text{ and } h \in (0, h_3).$$

Combining (5.13), (5.14) with (5.10) and setting  $L_1 = K$ , we obtain the result for  $k = 0$ .

In the case  $k \geq 1$ , let  $\eta$  be defined by (3.7) and set

$$\tilde{d}^k(t, r) = \phi^k(t) - t^2 \phi_1^k(t) - r, \quad z^k(t, r) = \eta(\tilde{d}^k(t, r)).$$

We define  $\underline{u}^k$  and  $\bar{u}^k$  by (3.18)–(3.20) with replacing  $\tilde{d}, z$  with  $\tilde{d}^k, z^k$ , respectively. Then we can check that Proposition 3.2 holds for these  $\underline{u}^k$  and  $\bar{u}^k$  for any  $h \in (0, h_2)$  and small  $h_2 > 0$ . Since we can easily verify that the choices of  $h_2$  and  $h_3$  depend only on  $\delta \in (0, 1/5)$ , we can apply the above argument to obtain the result.  $\square$

**5.2. Derivation of a corrector for  $R_h$ .** In this subsection we formally calculate  $R_h(t) - \phi(t)$  and find a corrector for  $R_h(t)$  on each time interval  $[kh, (k + 1)h)$  ( $k \in \mathbb{N} \cup \{0\}$ ). By Theorem 5.3, we see that

$$(5.15) \quad |R_h(\bar{t}) - (\phi(\bar{t}) - \bar{t}^2 \phi_1^0(\bar{t}))| \leq L_1 \bar{t}^{5/2} \quad \text{for all } \bar{t} \in [0, h] \text{ and } h \in (0, h_3).$$

Next we compute  $R_h(\bar{t} + h) - \phi(\bar{t} + h)$  for  $\bar{t} \in [0, h]$ . Theorem 5.3 yields that  $|R_h(\bar{t} + h) - (\phi^1(\bar{t}) - \bar{t}^2 \phi_1^1(\bar{t}))| \leq L_1 \bar{t}^{5/2}$  for all  $\bar{t} \in [0, h_3]$ . From (5.15) and this estimate, we have

$$(5.16) \quad \begin{aligned} R_h(\bar{t} + h) - \phi(\bar{t} + h) &\geq \phi^1(\bar{t}) - \bar{t}^2 \phi_1^1(\bar{t}) - \phi(\bar{t} + h) - L_1 \bar{t}^{5/2} \\ &= \sqrt{(R_h(h))^2 - 2\bar{t}} - \sqrt{1 - 2(\bar{t} + h)} - \bar{t}^2 \phi_1^1(\bar{t}) + L_1 \bar{t}^{5/2} \\ &\geq \sqrt{(\phi(h) - h^2 \phi_1^0(h) - L_1 h^{5/2})^2 - 2\bar{t}} - \sqrt{(\phi(h))^2 - 2\bar{t}} - \bar{t}^2 \phi_1^1(\bar{t}) - L_1 \bar{t}^{5/2} \\ &=: I_1 - I_2 - \bar{t}^2 \phi_1^1(\bar{t}) - L_1 \bar{t}^{5/2}. \end{aligned}$$

We observe by Taylor expansion to  $I_1$  and  $I_2$  around  $\bar{t} = 0$  that

$$(5.17) \quad I_1 - I_2 = -h^2\phi_1^0(h) - L_1h^{5/2} - \frac{(h^2\phi_1^0(h) + L_1h^{5/2})\bar{t}}{\phi(h)(\phi(h) - h^2\phi_1^0(h) - L_1h^{5/2})} \\ - \int_0^{\bar{t}} \left( \frac{(\bar{t} - s)}{\{(\phi(h) - h^2\phi_1^0(h) - L_1h^{5/2})^2 - 2s\}^{3/2}} - \frac{(\bar{t} - s)}{\{(\phi(h))^2 - 2s\}^{3/2}} \right) ds.$$

It follows from

$$(5.18) \quad \left( \frac{1}{1-r} \right)^3 \leq 1 + 8r \quad \text{for all } |r| \ll 1$$

that

$$\frac{1}{\{(\phi(h) - h^2\phi_1^0(h) - L_1h^{5/2})^2 - 2s\}^{3/2}} \\ \leq \frac{1}{\{(\phi(h))^2 - 2s\}^{3/2}} \left( 1 + \frac{8(h^2\phi_1^0(h) + L_1h^{5/2})(2\phi(h) - h^2\phi_1^0(h) - L_1h^{5/2})}{(\phi(h))^2 - 2s} \right)$$

for any  $s \in [0, h]$  and small  $h > 0$ . By using this inequality, we have

$$I_1 - I_2 \geq -h^2\phi_1^0(h) - L_1h^{5/2} - \frac{\bar{t}(h^2\phi_1^0(h) + L_1h^{5/2})}{\phi(h)(\phi(h) - h^2\phi_1^0(h) - L_1h^{5/2})} \\ - \frac{4\bar{t}^2(h^2\phi_1^0(h) + L_1h^{5/2})(2\phi(h) - h^2\phi_1^0(h) - L_1h^{5/2})}{\{(\phi(h))^2 - 2\bar{t}\}^{5/2}}.$$

In addition, since we also see by (5.18) that

$$\frac{1}{\phi(h) - h^2\phi_1^0(h) - L_1h^{5/2}} \leq \frac{1}{\phi(h)} \left( 1 + \frac{8(h^2\phi_1^0(h) + L_1h^{5/2})}{\phi(h)} \right)$$

for any small  $h > 0$ , noting that  $h^2\phi_1^0(h) + L_1h^{5/2} > 0$  and  $(\phi(h))^2 - 2\bar{t} \geq (\phi(2h))^2$  on  $[0, h]$ , we get

$$(5.19) \quad I_1 - I_2 \geq -h^2\phi_1^0(h) - L_1h^{5/2} \\ - \bar{t}h^2 \left\{ \left( \frac{1}{(\phi(h))^2} + \frac{8h\phi(h)}{(\phi(2h))^5} \right) \phi_1^0(h) + \frac{8h^2(\phi_1^0(h))^2}{(\phi(h))^3} \right\} \\ - \bar{t} \left\{ \left( \frac{1}{(\phi(h))^2} + \frac{8h\phi(h)}{(\phi(2h))^5} + \frac{16h^2\phi_1^0(h)}{(\phi(h))^3} \right) L_1h^{5/2} + \frac{8(L_1h^{5/2})^2}{(\phi(h))^3} \right\}.$$

Setting

$$\phi_1^1 = \left( \frac{1}{(\phi(h))^2} + \frac{8h\phi(h)}{(\phi(2h))^5} \right) \phi_1^0(h) + \frac{8h^2(\phi_1^0(h))^2}{(\phi(h))^3}, \\ \mathcal{L}^1 = \left( \frac{1}{(\phi(h))^2} + \frac{8h\phi(h)}{(\phi(2h))^5} + \frac{16h^2\phi_1^0(h)}{(\phi(h))^3} \right) L_1h^{5/2} + \frac{8(L_1h^{5/2})^2}{(\phi(h))^3},$$

we obtain

$$R_h(\bar{t} + h) - \phi(\bar{t} + h) \geq -h^2\phi_1^0(h) - \bar{t}^2\phi_1^1(\bar{t}) - \bar{t}h^2\phi_2^1 - L_1h^{5/2} - L_1\bar{t}^{5/2} - \mathcal{L}^1\bar{t}$$

for all  $\bar{t} \in [0, h]$  and small  $h > 0$ .

To consider the case  $k = 2, 3, \dots, m$ , we define

$$(5.20) \quad \psi^k = \psi_1^k + h\psi_2^k, \quad \psi_1^k = \sum_{l=0}^k \phi_1^l(h), \quad \psi_2^k = \sum_{l=0}^k \phi_2^l, \quad L_2^k = h \sum_{l=0}^k \mathcal{L}^l$$

$$(\phi_2^0 = 0, \mathcal{L}^0 = 0),$$

$$(5.21) \quad \phi_2^l = \left( \frac{1}{(\phi(lh))^2} + \frac{8h\phi(lh)}{(\phi((l+1)h))^5} \right) \psi^{l-1} + \frac{8h^2(\psi^{l-1})^2}{(\phi(lh))^3},$$

$$(5.22) \quad \mathcal{L}^l = \left( \frac{1}{(\phi(lh))^2} + \frac{8h\phi(lh)}{(\phi((l+1)h))^5} + \frac{16h^2\psi^{l-1}}{(\phi(lh))^3} \right) (L_1lh^{5/2} + L_2^{l-1}) + \frac{8(L_1lh^{5/2} + L_2^{l-1})^2}{(\phi(lh))^3}.$$

Assume that for  $k \geq 2$ ,

$$\begin{aligned} &R_h(\bar{t} + (k-1)h) - (\phi(\bar{t} + (k-1)h)) \\ &\geq -h^2\psi^{k-2} - \bar{t}^2\phi_1^{k-1}(\bar{t}) - \bar{t}h^2\phi_2^{k-1} - L_1(k-1)h^{5/2} - L_1\bar{t}^{5/2} - L_2^{k-2} - \mathcal{L}^{k-1}\bar{t} \end{aligned}$$

for all  $\bar{t} \in [0, h]$ . Since by Theorem 5.3 we have  $|R_h(\bar{t}+kh) - (\phi^k(\bar{t}) - \bar{t}^2\phi_1^k(\bar{t}))| \leq L_1\bar{t}^{5/2}$  for all  $\bar{t} \in [0, h_2]$ , similar calculations to (5.16) yield

$$\begin{aligned} &R_h(\bar{t} + kh) - \phi(\bar{t} + kh) \\ &\geq \sqrt{(\phi(kh) - h^2\psi^{k-1} - L_1kh^{5/2} - L_2^{k-1})^2 - 2\bar{t}} - \sqrt{(\phi(kh))^2 - 2\bar{t} - \bar{t}^2\phi_1^k(\bar{t}) - L_1\bar{t}^{5/2}} \\ &=: I_4 - \bar{t}^2\phi_1^k(\bar{t}) - L_1\bar{t}^{5/2}. \end{aligned}$$

Replacing  $h^2\psi^{k-1}$  and  $L_1h^{5/2}$  with, respectively,  $h^2\psi^{k-1}$  and  $L_1kh^{5/2} + L_2^{k-1}$  in the case  $k = 1$ , we get

$$I_4 \geq -h^2\psi^{k-1} - \bar{t}^2\phi_1^k(\bar{t}) - \bar{t}h^2\phi_2^k - L_1kh^{5/2} - L_1\bar{t}^{5/2} - L_2^{k-1} - \mathcal{L}^k\bar{t}.$$

Thus we have

$$\begin{aligned} R_h(\bar{t} + kh) - \phi(\bar{t} + kh) &\geq -h^2\psi^{k-1} - \bar{t}^2\phi_1^k(\bar{t}) - \bar{t}h^2\phi_2^k \\ &\quad - kL_1h^{5/2} - L_1\bar{t}^{5/2} - L_2^{k-1} - \mathcal{L}^k\bar{t} \end{aligned}$$

for all  $\bar{t} \in [0, h]$  and small  $h > 0$ .

Similarly, we can observe that

$$\begin{aligned} R_h(\bar{t} + kh) - \phi(\bar{t} + kh) &\leq -h^2\psi^{k-1} - \bar{t}^2\phi_1^k(\bar{t}) - \bar{t}h^2\phi_2^k \\ &\quad + L_1kh^{5/2} + L_1\bar{t}^{5/2} + L_2^k + \mathcal{L}^k\bar{t} \end{aligned}$$

for all  $\bar{t} \in [0, h]$ ,  $k \in \mathbb{N} \cup \{0\}$  and small  $h > 0$ . Therefore we obtain

$$(5.23) \quad |R_h(\bar{t} + kh) - \{\phi(\bar{t} + kh) - (h^2\psi^{k-1} + \bar{t}^2\phi_1^k(\bar{t}) + \bar{t}h^2\phi_2^k)\}| \leq L_1kh^{5/2} + L_1\bar{t}^{5/2} + L_2^k + \mathcal{L}^k\bar{t}$$

for all  $t \in [0, h]$ ,  $k \in \mathbb{N} \cup \{0\}$  and small  $h > 0$ . This inequality shows that the term  $h^2\psi^{k-1} + \bar{t}^2\phi_1^k(\bar{t}) + \bar{t}h^2\phi_2^k$  is a (formal) corrector for  $R_h(t)$ .

**5.3. Proofs of Theorem 5.1 and Corollary 5.2.** This subsection is devoted to the estimates and the limits of  $\psi_1^k, \psi_2^k, L_2^k$ , and  $\mathcal{L}^k$  and the proofs of Theorem 5.1 and Corollary 5.2. Remember that we have taken  $\delta \in (0, 1/5)$  and set  $T_{max}, T_\delta$ , and  $m$  as in (5.1).

PROPOSITION 5.4. *Let  $\varphi_1(t) = (1/\phi(t) - 1)/3$ . Then there exist  $h_3 > 0$  and  $M_3 > 0$  such that*

$$\sup_{t \in [0, T_\delta]} |\varphi_1(t) - h\psi_1^{\lfloor t/h \rfloor}| \leq M_3 h \quad \text{for all } h \in (0, h_3).$$

*Proof.* We remark that  $\varphi_1(t) = \frac{1}{3} \int_0^t \frac{1}{(\phi(s))^3} ds$ . Set  $k = \lfloor t/h \rfloor$ . It is easily seen by the definition of  $\phi_1^k$  in (5.20) and  $kh \leq T_{max}$  that

$$\begin{aligned} |\varphi_1(t) - h\psi_1^{\lfloor t/h \rfloor}| &\leq \frac{1}{3} \left| \int_0^{kh} \frac{1}{(\phi(s))^3} ds - h \sum_{l=0}^k \frac{1}{(\phi(lh))^3} \right| + \frac{h}{3(\phi(T_\delta))^3} \\ &\quad + \frac{T_{max}}{3} \max_{0 \leq l \leq k} \left| \frac{1}{(\phi(lh))^3} - \frac{1}{\{(R_h(lh))^2 - 2h\}^{3/2}} \right|. \end{aligned}$$

Since  $1/(\phi(t))^3$  is increasing in  $t$ , we easily observe that

$$\left| \int_0^{kh} \frac{1}{(\phi(s))^3} ds - h \sum_{l=0}^k \frac{1}{(\phi(lh))^3} \right| \leq \left( \frac{1}{(\phi(T_\delta))^3} - \frac{1}{(\phi(0))^3} \right) h.$$

Combining (5.2) with this inequality, we have the result.  $\square$

We obtain the estimates for  $\psi_2^k$  by the following lemma.

LEMMA 5.5. *There exist  $h_4 > 0, M_4 > 0$ , and  $M_5 > 0$  such that*

$$M_4(kh)^2 \leq h^2\psi_2^k \leq M_5$$

for  $k = 0, 1, \dots, m$  and  $h \in (0, h_4)$ .

*Proof.* The definition of  $\psi_1^k$  in (5.20) yields that

$$(5.24) \quad 0 < \psi_1^1 \leq \dots \leq \psi_1^m.$$

Besides we easily see that

$$\alpha := \sup_{0 \leq l \leq m, h > 0} \max \left\{ \frac{1}{(\phi(lh))^2}, \frac{8\phi(lh)}{(\phi((l+1)h))^5}, \frac{8}{(\phi(lh))^3} \right\} < +\infty.$$

Thus, for  $l = 1, 2, \dots, m$ ,  $\phi_2^l$  satisfies

$$(5.25) \quad \phi_2^l \leq \alpha \{ (1+h)\psi^{l-1} + h^2(\psi^{l-1})^2 \}.$$

We estimate  $\phi_2^l$  by using (5.21) and this inequality.

First, for sufficiently small  $h > 0$  we get  $\psi_2^0(h) = \phi_2^0(h) \leq 1$  and

$$\phi_2^1 \leq \alpha(2+h).$$

Fix  $k = 2, 3, \dots, m$  and let  $l = 2, 3, \dots, k$ . From the fact  $m = \lfloor T_\delta/h \rfloor$  we remark that

$$(5.26) \quad (1 + \alpha h(2+h))^l \leq (1 + \alpha h(2+h))^m \leq e^{3\alpha T_{max}}$$

for  $l = 2, 3, \dots, m$  and  $h \in (0, 1)$ . Taking (5.24) and this estimate into account, we choose  $h_4 \in (0, 1)$  such that for any  $h \in (0, h_4)$ ,

$$(5.27) \quad h(\psi_1^m + h\psi_1^1) \leq M_{5,1}, \quad M_{5,1}h \leq M_{5,1}e^{3\alpha T_{max}}h \leq 1,$$

where  $M_{5,1} = \varphi_1(T_\delta) + 1$ .

It follows from (5.21) with  $l = 2$  and (5.25) that

$$\phi_2^2 \leq \alpha\{(1+h)(\psi_1^1 + h\psi_2^1) + h^2(\psi_1^1 + h\psi_2^1)^2\}.$$

We easily see by (5.27) that  $h^2(\psi_1^1 + h\psi_2^1) \leq M_{5,1}h \leq 1$ . Thus we get

$$(5.28) \quad \phi_2^2 \leq \alpha(2+h)(\psi_1^1 + h\psi_2^1).$$

In the case of  $l = 3$ , since  $\psi^2 = \psi_1^2 + h\psi_2^1 + h\phi_2^2$ , we see by (5.24), (5.25), and (5.28) that

$$\phi_2^3 \leq \alpha\{(1+h)(1+\alpha h(2+h))(\psi_1^2 + h\psi_2^1) + h^2\{(1+\alpha h(2+h))(\psi_1^2 + h\psi_2^1)\}^2\}.$$

Using (5.26) and (5.27), we get

$$h^2(1+\alpha h(2+h))(\psi_1^2 + h\psi_2^1) \leq M_{5,1}e^{3\alpha T_{max}}h \leq 1.$$

Therefore we have

$$(5.29) \quad \phi_2^3 \leq \alpha(2+h)(1+\alpha h(2+h))(\psi_1^2 + h\psi_2^1).$$

As to the case of  $l = 4$ , note that  $\psi^3 = \psi_1^3 + h\psi_2^1 + h(\phi_2^2 + \phi_2^3)$ . Hence it is observed by (5.25)–(5.29) and a similar argument that

$$\phi_2^4 \leq \alpha(2+h)(1+\alpha h(2+h))^2(\psi_1^3 + h\psi_2^1).$$

By repeating this procedure, we can show that

$$\phi_2^l \leq \alpha(2+h)(1+\alpha h(2+h))^{l-2}(\psi_1^{l-1} + h\psi_2^1) \quad \text{for } l = 4, \dots, k.$$

Summing up  $l = 0$  to  $l = k$  and using (5.27), we get

$$h^2\psi_2^k = h^2 \sum_{l=0}^k \phi_2^l \leq 3\alpha(1 + M_{5,1}e^{3\alpha T_{max}}) \quad \text{for } h \in (0, h_4).$$

Setting  $M_5 = 3\alpha(1 + M_{5,1}e^{3\alpha T_{max}})$ , we have an upper bound for  $h^2\psi_2^k$ .

As for a lower bound for  $h^2\psi_2^k$ , we observe from the definition of  $\phi_2^l$  in (5.21) and (5.8) that

$$\phi_2^l \geq \frac{\psi^l}{(\phi(lh))^2} \geq \frac{l\phi_1^0(0)}{(\phi(0))^2} \geq \frac{l}{3(1+\delta)^3}.$$

Thus, putting  $M_4 = 1/6(1+\delta)^3$ , we obtain  $h^2\psi_2^k \geq M_4(kh)^2$ .  $\square$

We use this lemma to prove the following.

**PROPOSITION 5.6.** *Let  $\varphi_2(t) = -\log \phi(t)/3\phi(t) - \varphi_1(t)$ . Then there exist  $h_5 > 0$  and  $M_6 > 0$  such that*

$$\sup_{t \in [0, T_\delta]} |\varphi_2(t) - h^2\psi_2^{\lfloor t/h \rfloor}| \leq M_6h \quad \text{for all } h \in (0, h_5).$$

*Proof.* We easily see that  $\varphi_2$  is a unique solution of

$$\varphi_2(t) = \int_0^t \frac{\varphi_1(s) + \varphi_2(s)}{(\phi(s))^2} ds.$$

For each  $t \in [0, T_\delta]$ , set  $k = [t/h]$ . It is easily observed from the definition of  $\psi_2^k$  in (5.20) and Lemma 5.5 that

$$\begin{aligned} |\varphi_2(t) - h^2 \psi_2^{[t/h]}| &\leq (1 + Kh) \left\{ \left| \int_0^t \frac{\varphi_1(s)}{(\phi(s))^2} ds - h \sum_{l=0}^k \frac{h\psi_1^l}{(\phi(lh))^2} \right| \right. \\ &\quad \left. + \left| \int_0^t \frac{\varphi_2(s)}{(\phi(s))^2} ds - h \sum_{l=0}^k \frac{h^2\psi_2^l}{(\phi(lh))^2} \right| \right\} + Kh \\ &=: (1 + Kh)(I_1 + I_2) + Kh. \end{aligned}$$

Since  $1/(\phi(t))^2$  (resp.,  $\psi^k$ ) is increasing with respect to  $t$  (resp.,  $k$ ), we have

$$\int_{lh}^{(l+1)h} \frac{h\psi_1^{[s/h]}}{(\phi(s))^2} ds \leq \frac{h^2\psi_1^{l+1}}{(\phi((l+1)h))^2}.$$

Using Proposition 5.4 and this inequality, we calculate

$$\begin{aligned} I_1 &= \left| \int_0^{kh} \frac{\varphi_1(s) - h\psi_1^{[s/h]} + h\psi_1^{[s/h]}}{(\phi(s))^2} ds - h \sum_{l=0}^k \frac{h\psi_1^l}{(\phi(lh))^2} + \int_{kh}^t \frac{\varphi_1(s)}{(\phi(s))^2} ds \right| \\ &\leq Kh \int_0^{kh} \frac{ds}{(\phi(s))^2} + h \left| \sum_{l=0}^k \frac{h\psi_1^{l+1}}{(\phi((l+1)h))^2} - \sum_{l=0}^k \frac{h\psi_1^l}{(\phi(lh))^2} \right| + Kh \\ &\leq Kh. \end{aligned}$$

Similarly we can show that

$$I_2 \leq \int_0^t \frac{|\varphi_2(s) - h^2\psi_2^{[s/h]}|}{(\phi(s))^2} ds + Kh.$$

Therefore we obtain

$$|\varphi_2(t) - h^2\psi_2^{[t/h]}| \leq Kh + (1 + Kh) \int_0^t \frac{|\varphi_2(s) - h^2\psi_2^{[s/h]}|}{(\phi(s))^2} ds.$$

We apply the Gronwall inequality to get

$$|\varphi_2(t) - h^2\psi_2^{[t/h]}| \leq Kh \exp \left( (1 + Kh) \int_0^{T_\delta} \frac{ds}{(\phi(s))^2} \right)$$

for all  $t \in [0, T_\delta]$  and small  $h > 0$ . Thus we have the result.  $\square$

Finally we obtain the bounds for  $\mathcal{L}^k$  and  $L_2^k$ .

**PROPOSITION 5.7.** *There exist  $h_6 > 0$  and  $M_7 > 0$  such that*

$$\mathcal{L}^k \leq M_7(kh)h^{3/2}, \quad L_2^k \leq M_7(kh)^2h^{3/2}$$

for all  $k = 1, 2, \dots, m$  and  $h \in (0, h_6)$ .

*Proof.* The proof is similar to that of Lemma 5.5. It follows from (5.2) and Proposition 5.4 that

$$\alpha := \sup_{0 \leq l \leq m, h > 0} \max \left\{ \frac{1}{(\phi(lh))^2}, \frac{8\phi(lh)}{(\phi((l+1)h))^5}, \frac{16h\psi^l}{(\phi(lh))^3}, \frac{8}{(\phi(lh))^3} \right\} < +\infty.$$

Thus, for  $l = 1, 2, \dots, m$ ,  $\mathcal{L}^l$  satisfies

$$(5.30) \quad \mathcal{L}^l \leq \alpha \{ (1+h)(L_1lh^{5/2} + L_2^{l-1}) + (L_1lh^{5/2} + L_2^{l-1})^2 \}.$$

We estimate  $\mathcal{L}^l$  by using (5.22) and this inequality.

First, for sufficiently small  $h \in (0, 1)$ , we have

$$\mathcal{L}^1 \leq \alpha(2+h)L_1h^{5/2} \leq M_{7,1}h^{5/2}, \quad M_{7,1} = 3\alpha L_1.$$

Fix  $k = 2, 3, \dots, m$  and let  $l = 2, 3, \dots, k$ . In view of

$$(5.31) \quad (1 + \alpha h(2+h))^l \leq (1 + \alpha h(2+h))^m \leq e^{3\alpha T_{max}}$$

for  $l = 2, 3, \dots, m$  and  $h \in (0, 1)$ , we can choose  $h_6 \in (0, 1)$  such that for any  $h \in (0, h_6)$ ,

$$(5.32) \quad e^{3\alpha T_{max}}(T_{max}L_1 + M_{7,1}h)h^{3/2} \leq 1.$$

It is easily seen from (5.20) that  $L_2^1 = h\mathcal{L}^1 \leq M_{7,1}h^{7/2}$ . Thus we get, by (5.30),

$$(5.33) \quad \mathcal{L}^2 \leq \alpha \{ (1+h)(2L_1h^{5/2} + M_{7,1}h^{7/2}) + (2L_1h^{5/2} + M_{7,1}h^{7/2})^2 \}.$$

We easily observe by (5.32) that  $2L_1h^{5/2} + M_{7,1}h^{7/2} \leq 1$ . Hence we have

$$(5.34) \quad \mathcal{L}^2 \leq \alpha(2+h)(2L_1h^{5/2} + M_{7,1}h^{7/2}) \leq 3\alpha(L_1 + M_{7,1}h)(2h)h^{3/2}.$$

In the case of  $l = 3$ , since  $L_2^2 = h(\mathcal{L}^1 + \mathcal{L}^2)$ , we see by (5.30) that

$$\begin{aligned} \mathcal{L}^3 \leq & \alpha \{ (1+h)(1 + \alpha h(2+h))(3L_1h^{5/2} + M_{7,1}h^{7/2}) \\ & + \{ (1 + \alpha h(2+h))(3L_1h^{5/2} + M_{7,1}h^{7/2}) \}^2 \}. \end{aligned}$$

Using (5.31) and (5.32), we obtain

$$(1 + \alpha h(2+h))(3L_1h^{5/2} + M_{7,1}h^{7/2}) \leq e^{3\alpha T_{max}}(T_{max} + 1)L_1h^{3/2} \leq 1.$$

Thus we get

$$\mathcal{L}^3 \leq \alpha(2+h)(1 + \alpha h(2+h))(3L_1h^{5/2} + M_{7,1}h^{7/2}) \leq 3\alpha e^{3\alpha T_{max}}(L_1 + M_{7,1}h)(3h)h^{3/2}.$$

We repeat the above arguments to obtain

$$(5.35) \quad \mathcal{L}^l \leq \alpha(2+h)(1 + \alpha h(2+h))^{l-2}(L_1lh^{5/2} + M_{7,1}h^{7/2}) \leq M_7(lh)h^{3/2}$$

for  $M_7 = 3\alpha e^{3\alpha T_{max}}(L_1 + M_{7,1})$ . From this estimate, we get

$$L_2^k = h \sum_{l=1}^k \mathcal{L}^l \leq M_7(kh)^2 h^{3/2}$$

for all  $k = 0, 1, \dots, m$  and  $h \in (0, h_5)$ . □

We observe from Propositions 5.4–5.7 that

$$0 \leq h^2\psi^{k-1} + \bar{t}^2\phi_1^k(\bar{t}) + \bar{t}h^2\phi_2^k \leq Kh, \quad 0 \leq L_1kh^{5/2} + L_1\bar{t}^{5/2} + L_2^k + \mathcal{L}^k\bar{t} \leq Kh^{3/2}$$

for all  $\bar{t} \in [0, h)$ ,  $k = 0, 1, \dots, m$ . Thus (5.23) rigorously holds for sufficiently small  $h > 0$ .

*Proof of Theorem 5.1.* In the case  $k = 0$ , (5.3) is obtained by Theorem 5.3. Thus we assume  $k \geq 1$  and prove (5.4).

Noting that  $\varphi = \varphi_1 + \varphi_2$ , in view of Propositions 5.4–5.7, we can find an  $h_0 > 0$  so small that

$$(5.36) \quad |R_h(t) - (\phi(t) - h\varphi(t))| \leq Kh^2 + Kh^{3/2} \leq Lh^{3/2}$$

for some large  $L > 0$  and all  $t \in [0, T_\delta]$  and  $h \in (0, h_0)$ .  $\square$

*Proof of Corollary 5.2.* In the case  $k = 0$ , we have (5.5) by (5.3). Thus we may assume  $k \geq 1$  and  $kh \leq t < (k + 1)h$ . Let  $h_0 > 0$  be given in Theorem 5.1.

Using (5.8), (5.23), Lemma 5.5, and Proposition 5.7, we have

$$\begin{aligned} R_h(t) - \phi(t) &\leq -kKh^2 - M_4(kh)^2h - K(t - kh)^2 \\ &\quad + kL_1h^{5/2} + L_1(t - kh)^{5/2} + M_7(kh)^2h^{3/2} + M_7(t - kh)(kh)h^{3/2} \\ &\leq -(K - L_1h^{1/2})(kh^2 + (t - kh)^2) - (M_4 - 2M_7h^{1/2})(kh)^2h \\ &\leq -(K - L_1h^{1/2})(kh^2 + (t - kh)^2) \end{aligned}$$

for any  $h \in (0, h_0)$  satisfying  $M_4 \geq 2M_7h^{1/2}$ . Take  $h_1 \in (0, h_0)$  such that  $2M_7h^{1/2} \leq M_4$  and  $Lh^{1/2} \leq K/2$ . Since we get, from  $kh \geq t - h$ ,

$$kh^2 + (t - kh)^2 \geq \frac{1}{2}kh^2 + \frac{1}{2}kh^2 \geq \frac{1}{2}(t - h)h + \frac{1}{2}kh^2 \geq \frac{1}{2}th,$$

setting  $\underline{L} = K/4$ , we obtain

$$R_h(t) - \phi(t) \leq -\underline{L}th.$$

Similarly we can show that

$$R_h(t) - \phi(t) \geq -\bar{L}th \quad \text{for all } t \in [h, T_\delta] \text{ and } h \in (0, h_1)$$

for some  $\bar{L} > 0$ . From these two estimates, we have (5.6).  $\square$

**6. Appendix.** We give the proof of Lemma 2.2.

*Proof of Lemma 2.2.* Put  $\tilde{r} = vt$ . Then it follows from Evans [5, Theorem 4.1] that  $|v - (N - 1)/R| \leq Kt^{1/2}$  for any small  $t > 0$ . Hence we estimate  $v$  as  $t \searrow 0$  more precisely. In the following we always assume that  $t > 0$  is sufficiently small.

To simplify our consideration, we treat the following problem instead of (2.1):

$$(6.1) \quad \begin{cases} u_t - \Delta u = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, x) = \begin{cases} 1, & x \in \overline{B(z_0, R)}, \\ 0, & x \in \mathbb{R}^N \setminus \overline{B(z_0, R)}, \end{cases} \end{cases}$$

where  $z_0 = (0, \dots, 0, R) \in \mathbb{R}^N$ . We note that, in this setting,  $C_t^h = \{x \in \mathbb{R}^N \mid u(t, x) \geq 1/2\}$ . Then the solution  $u$  of (6.1) can be represented as

$$u(t, x) = \frac{1}{(4\pi t)^{N/2}} \int_{B(z_0, R)} e^{-|y-x|^2/4t} dy.$$



Set  $x_0 = (0, \dots, 0, vt)$  with  $v \geq 0$  and assume that  $u(t, x_0) = 1/2$ . Then

$$\frac{1}{2} = \frac{1}{(4\pi t)^{N/2}} \int_{\overline{B(z_0, R)}} e^{-(|y'|^2 + |y_N - vt|^2)/4t} dy.$$

Since the lower hemisphere of  $\partial B(z_0, R)$  can be written as

$$y_N = R - \sqrt{R^2 - |y'|^2} \quad (y = (y', y_N) \in \partial B(z_0, R), y' \in \overline{B'(0, R)}),$$

where  $B'(0, R) = \{x' \in \mathbb{R}^{N-1} \mid |x'| < R\}$ , we observe that

$$\frac{1}{2} = \frac{1}{(4\pi t)^{N/2}} \int_{\overline{B'(0, R)}} e^{-|y'|^2/4t} \int_{R - \sqrt{R^2 - |y'|^2}}^{+\infty} e^{-|y_N - vt|^2/4t} dy_N dy' + O(e^{-K/t}).$$

Changing the variable by setting  $z' = y'/2\sqrt{t}$ ,  $z_N = (y_N - vt)/2\sqrt{t}$ , we compute that

$$\frac{1}{2} = \frac{1}{\pi^{N/2}} \int_{\overline{B'(0, R/2\sqrt{t})}} e^{-|z'|^2} \left\{ \int_0^{+\infty} - \int_0^{\sqrt{t}(g(t, z') - v/2)} \right\} e^{-|z_N|^2} dz_N dz' + O(e^{-K/t}).$$

Here  $g(t, z')$  is defined by

$$(6.2) \quad g(t, z') = \frac{1}{2t} (R - \sqrt{R^2 - 4t|z'|^2}) = \frac{|z'|^2}{R} + \frac{t|z'|^4}{R^3} + O\left(\frac{t^2|z'|^6}{R^5}\right).$$

Since

$$\frac{1}{\pi^{N/2}} \int_{\overline{B'(0, R/2\sqrt{t})}} e^{-|z'|^2} \int_0^{+\infty} e^{-|z_N|^2} dz_N dz' = \frac{1}{2} - O(e^{-K/t}),$$

we deduce that

$$\frac{1}{\pi^{N/2}} \int_{\overline{B'(0, R/2\sqrt{t})}} e^{-|z'|^2} \int_0^{\sqrt{t}(g(t, z') - v/2)} e^{-|z_N|^2} dz_N dz' = O(e^{-K/t}).$$

Using Taylor expansion to the function  $\int_0^s e^{-|z_N|^2} dz_N$  around  $s = 0$ , we can observe that

$$(6.3) \quad \frac{1}{\pi^{N/2}} \int_{\overline{B'(0, R/2\sqrt{t})}} e^{-|z'|^2} \left\{ (2g(t, z') - v) - \frac{t}{12} (2g(t, z') - v)^3 \right\} = O(t^2).$$

By the way, lengthy calculations yield that

$$(6.4) \quad \int_{\overline{B'(0, R/2\sqrt{t})}} e^{-|z'|^2} |z'|^{2k} dz' = \pi^{(N-1)/2} \prod_{l=1}^k \frac{N + 2l - 3}{2} - O(e^{-K/t}).$$

We use this estimate to compute the left-hand side of (6.3). Combining (6.2) with (6.4) with  $k = 1, 2$ , we get

$$\frac{1}{\pi^{(N-1)/2}} \int_{\overline{B'(0, R/2\sqrt{t})}} e^{-|z'|^2} (2g(t, z') - v) dz' = \frac{N-1}{R} + \frac{(N^2-1)t}{2R^3} - v + O(e^{-K/t}).$$

We note that

$$(2g(t, z') - v)^3 = \frac{8|z'|^6}{R^3} - \frac{12v|z'|^4}{R^2} + \frac{6v^2|z'|^2}{R} - v^3 + tP(t, |z'|),$$

where  $tP(t, |z'|)$  is the remainder term satisfying

$$\int_{\mathbb{R}^{N-1}} e^{-|z'|^2} P(t, |z'|) dz' = O(1).$$

We use (6.4) with  $k = 0, 1, 2, 3$  and this estimate to obtain

$$\begin{aligned} & \frac{t}{12\pi^{(N-1)/2}} \int_{B'(0, R/2\sqrt{t})} e^{-|z'|^2} (2g(t, z') - v)^3 dz' \\ &= \frac{(N+3)(N^2-1)t}{12R^3} - \frac{(N^2-1)t}{4R^2}v + \frac{(N-1)t}{4R}v^2 - \frac{t}{12}v^3 + O(t^2). \end{aligned}$$

Therefore we obtain the following:

$$\frac{N-1}{R} - \frac{(N^2-1)(N-3)t}{12R^3} - v + \frac{(N^2-1)t}{4R^2}v - \frac{(N-1)t}{4R}v^2 + \frac{t}{12}v^3 = O(t^2).$$

Let  $G(v)$  be the left-hand side of this estimate. We find a root  $v^*$  of  $G(v) = 0$  near  $v_0 = (N-1)/R$  and consider  $v^* - v_0$ . We easily see that

$$(6.5) \quad G(v_0 + s) = \frac{(N-1)(3N-1)t}{6R^3} - \left(1 - \frac{(N-1)t}{2R^2}\right)s + \frac{t}{12}s^3.$$

Set

$$v_1 = v_1(t) := \frac{(N-1)(3N-1)t}{6R^3 - 3R(N-1)t}.$$

It is observed by (6.5) that

$$G(v_0 + 2v_1) < 0 < G(v_0 + v_1) \leq Kt^4, \quad -1 \leq \frac{dG}{dv}(v_0 + s) \leq -\frac{1}{2} \text{ for } s \in [0, 2v_1].$$

Hence there exists a root  $v^*$  of  $G(v) = 0$  satisfying  $0 < v^* - (v_0 + v_1) \leq Kt^4$ . Thus we conclude that

$$\left| v^* - \left( v_0 + \frac{(N-1)(3N-1)t}{6R^3} \right) \right| \leq Kt^2.$$

In the case of  $G(v) = O(t^2)$ , we can obtain the result of Lemma 2.2 by slightly modifying the above argument.  $\square$

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