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Existence of type II blowup solutions for a semilinear heat equation with critical nonlinearity

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Abstract.
We consider the blowup rate of solutions for a semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u, \quad x \in \Omega \subset \mathbf{R}^N, \ t > 0,$$

with critical power nonlinearity $p = (N+2)/(N-2)$ and $N \geq 3$. First we investigate the profiles of backward self-similar solutions by making use of the variational method, and then, by employing the intersection comparison argument with a particular self-similar solution, we derive the criteria of the blowup rate of solutions, assuming the positivity of solutions in backward space-time parabola. In particular, we show the existence of the so called type II blowup solutions for the Cauchy-Dirichlet problems on suitable shrinking domains in the case $N = 3$.

MSC: 35K55; 35B33; 35J20

Keywords: Semilinear heat equation; Critical nonlinearity; Blow-up rate; Type II blow up; Self-similar solution

1. Introduction

We are concerned with the blowup rate of solutions for a semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u, \quad x \in \Omega \subset \mathbf{R}^N, \ t > 0, \tag{1.1}$$

with critical power nonlinearity

$$p = \frac{N+2}{N-2} \quad \text{and} \quad N \geq 3. \tag{1.2}$$

In order to recall some notations, let us consider the solution u of (1.1) satisfying the initial condition

$$u(x,0) = u_0 \in L^\infty(\Omega) \cap C(\Omega), \tag{1.3}$$

where either $\Omega = \mathbf{R}^N$ or Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. In the latter case, we impose the Dirichlet boundary condition $u = 0$ on $\partial\Omega$ and $t > 0$.

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It is well known that, for each initial data u_0, there exists $T = T(u_0) \in (0, \infty]$ such that the problem (1.1) and (1.3) has a unique solution $u \in C([0,T), L^\infty(\Omega))$ which is classical for $0 < t < T$, and that if $T < \infty$ then $\lim_{t\to T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. In the case where $T < \infty$, we say that the solution blows up in finite time, and T is called the blow up time. Let u be a solution which blows up at a finite time T. A simple comparison argument (cf. [9]) shows that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \geq ((p-1)(T-t))^{-1/(p-1)} \quad \text{for } 0 \leq t < T.$$

Here the function $((p-1)(T-t))^{-1/(p-1)}$ is a solution of the corresponding ordinary differential equation $w_t = |w|^{p-1}w$. We say that the blowup is of *type I* if u satisfies

$$\limsup_{t\to T} (T-t)^{1/(p-1)} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

The blowup is called *type II* if it is not of type I.

We briefly review some known results concerning blowup rate estimates. In the case $p > 1$, Friedman and McLeod [9] showed that the solution of (1.1) and (1.3) blows up in type I rate when Ω is a bounded convex domain and u_0 satisfies $u_0 \geq 0$ and $\Delta u_0 + u_0^p \geq 0$. In the case $1 < p < (N+2)/(N-2)$, Giga and Kohn [13, 14] proved that the solution of (1.1) and (1.3) blows up in type I rate when Ω is a bonded convex domain or $\Omega = \mathbf{R}^N$ under the assumption that either $u_0 \geq 0$ or $1 < p < (3N+8)/(3N-4)$. Later, Giga, Matsui, and Sasayama [16, 17] improved this result by removing such assumptions. In the case $(N+2)/(N-2) < p < p_c$, where

$$p_c = \begin{cases} \infty, & 3 \leq N \leq 10, \\ 1 + \dfrac{4}{N-4-2\sqrt{N-1}}, & N \geq 11, \end{cases}$$

Matano and Merle [25] showed that solutions of the problem (1.1) and (1.3) blow up in type I rate when Ω is a finite ball and u is radially symmetric. A similar result for $\Omega = \mathbf{R}^N$ can also be found in [25] under the additional condition. On the other hand, in the case $N \geq 11$ and $p > p_c$, Herrero and Velázquez [19, 20] obtained a radially symmetric positive solution in $\mathbf{R}^N$ which blows up in type II rate. A new simpler proof of this result was shown by Mizoguchi [26]. In the critical case $p = (N+2)/(N-2)$, Matano and Merle [25] showed that the solution of (1.1) and (1.3) blow up in type I rate when Ω is a finite ball or $\Omega = \mathbf{R}^N$ and u_0 is radially symmetric and positive. Filippas, Herrero, and Velázquez [8] showed by a formal analysis that there exists a sign-changing solution which blows up in type II rate when $3 \leq N \leq 6$. We refer to [16, Remark 2.3] and a survey [7] for details.

Our main concern in this paper is to characterize the blowup rate of solutions in the critical case, assuming the positivity of solutions in backward space-time parabola. In particular, we will show the existence of the type II blowup solutions for the Cauchy-Dirichlet problems on suitable shrinking domains. In the rest of the paper, we assume that (1.2) holds.

The basis of our method is the scaling property of (1.1), the fact that (1.1) is invariant under the scaling

$$u^\lambda(x,t) = \lambda^{2/(p-1)} u(x_0 + \lambda(x - x_0), t_0 + \lambda^2(t - t_0)) \quad \text{for } \lambda > 0$$

with each $(x_0, t_0) \in \mathbf{R}^N \times (0, \infty)$. A solution is called *self-similar* about (x_0, t_0) if $u^\lambda \equiv u$ for each $\lambda > 0$, and is called *backward* if the solution u is defined for all $t < t_0$. Choosing $(x_0, t_0) = (0, T)$, a backward self-similar solution of (1.1) has the form

$$u(x,t) = (T-t)^{-1/(p-1)} v\left(x/(T-t)^{1/2}\right),$$

where v satisfies the elliptic equation

$$\Delta v - \frac{1}{2} x \cdot \nabla v - \frac{1}{p-1} v + |v|^{p-1} v = 0 \quad \text{in } \mathbf{R}^n.$$

First we consider the existence of positive solutions of the Dirichlet problem

$$(1.4) \qquad \begin{cases} \Delta v - \dfrac{1}{2} x \cdot \nabla v - \dfrac{1}{p-1} v + v^p = 0 & \text{in } B_R, \\ v = 0 & \text{on } \partial B_R, \end{cases}$$

where $B_R = \{x \in \mathbf{R}^N : |x| < R\}$ with $R > 0$. We obtain the following result.

Theorem 1. (i) *Let $N \geq 4$. Then the problem* (1.4) *has a positive radially symmetric solution for any $R > 0$.*

(ii) *Let $N = 3$. Then there exists $R_0 > 0$ such that* (1.4) *has a positive radially symmetric solution for $R > R_0$, and that* (1.4) *has no positive radially symmetric solution for $R \in (0, R_0]$.*

Remark 1.1. (i) The constant R_0 is given as a first zero of the solution ϕ of

$$\phi'' + \left(\frac{1}{2} - \frac{1}{16} r^2\right) \phi = 0, \quad r > 0,$$

satisfying $\phi'(0) = 0$ and $\phi(0) > 0$. It is easy to see that the solution ϕ is given by $\phi(r) = F(-1/4, 1/2; r^2/4)$, where $F(\alpha, \gamma; s)$ is a Kummer's function. (See Remark 2.1 below.) We see that $R_0 = 2.2325....$

(ii) We denote by v_α a solution of the ordinary differential equation

$$v'' + \left(\frac{N-1}{r} - \frac{r}{2}\right) v' - \frac{1}{p-1} v + |v|^{p-1} v = 0 \quad \text{for } r > 0$$

satisfying $v'(0) = 0$ and $v(0) = \alpha$, where $\alpha > 0$ is a parameter. Denote by $r_0 = r_0(\alpha) > 0$ the first zero of $v_\alpha(r)$. We will find that there exists a sequence $\alpha_k \to \infty$ such that

$$\begin{cases} r_0(\alpha_k) \to 0 & \text{as } k \to \infty \quad \text{if } N \geq 4, \ \textit{and} \\ r_0(\alpha_k) \to R_0 & \text{as } k \to \infty \quad \text{if } N = 3. \end{cases}$$

Furthermore, for any constants A and B with $0 < A < B$, we have

$$\inf\{-v'_{\alpha_k}(r) : A \leq v_{\alpha_k}(r) \leq B\} \to \infty \quad \text{as } k \to \infty.$$

(See Proposition 3.1 below.) A closely related result was obtained in [8, Section 6] by a different method.

We will show Theorem 1 by applying the variational method due to Brezis-Nirenberg [5]. The idea is to compare the best Sobolev constant S of the embedding $H^1 \subset L^{2N/(N-2)}$ and $S_{\sigma,R}$ given by (2.2) below. One can prove that if $S_{\sigma,R} < S$ then (1.4) has a positive solution, and that $S_{\sigma,R} < S$ holds when either $N \geq 4$ or $N = 3$ and $R > R_0$. Non-existence part will be shown by the Pohozaev-type argument following the idea by Atkinson-Peletier [2]. By an ODE argument we will show the properties of solutions mentioned in (ii) of Remark 1.1.

Next, we consider the blowup rate of solutions, assuming the positivity of solutions in backward space-time parabola. We say that a solution u of (1.1) blows up at $(x,t) = (a,T)$ if there exist sequence $a_n \in \Omega$ and $t_n \in (0,T)$ such that $a_n \to a$, $t_n \to T$, and $u(a_n, t_n) \to \infty$ as $n \to \infty$.

Theorem 2. *Let u be a radially symmetric solution of* (1.1) *such that u blows up at $(x,t) = (0,T)$, and that u satisfies*

$$u > 0 \quad \textit{in} \bigcup_{0<t<T} B_{\rho(t)} \times \{t\}$$

with some $\rho \in C[0,T]$ satisfying $\rho > 0$ on $[0,T)$.

(i) *Assume that $u(\cdot,t)$ is nonincreasing in $r = |x| \in (0, \rho(t))$. If ρ satisfies*

$$\liminf_{t\to T} (T-t)^{-1/2} \rho(t) > \begin{cases} 0 & \textit{if } N \geq 4, \\ R_0 & \textit{if } N = 3, \end{cases} \tag{1.5}$$

where R_0 is the constant in Theorem 1, then

$$\limsup_{t\to T} (T-t)^{1/(p-1)} \|u(\cdot,t)\|_{L^\infty(B_{\rho(t)})} < \infty, \tag{1.6}$$

namely, u blows up in type I rate.

(ii) *Assume that u satisfies*

$$u = 0 \quad on \quad \bigcup_{0<t<T} \partial B_{\rho(t)} \times \{t\}.$$

If ρ satisfies $\lim_{t\to T}(T-t)^{-1/2}\rho(t) = 0$ *then*

$$\limsup_{t\to T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty(B_{\rho(t)})} = \infty,$$

namely, u blows up in type II rate.

Remark 1.2. (i) Let us consider the case where Ω is a finite ball in (1.1). If a solution u in Ω is positive, radially symmetric, and nonincreasing in $r = |x|$, then the blowup of the solution u is type I by (i). Here, we do not need to require the boundary condition on $\partial\Omega$.

(ii) The assumption on ρ in (ii) implies that $\rho(T) = 0$. Then we require in (ii) that u blows up and the domain $B_{\rho(t)}$ disappears at the simultaneous time. At this time we do not know whether there exists a solution u satisfying these assumptions.

Let us consider the case where $N = 3$ and $\liminf_{t\to T}(T-t)^{-1/2}\rho(t) \in (0, R_0]$ in Theorem 2. Define ρ by

$$\rho(t) = R(T-t)^{1/2} \tag{1.7}$$

with $R > 0$ and $T > 0$, and consider the problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in B_{\rho(t)},\ t \in (0,T), \\ u = 0, & x \in \partial B_{\rho(t)},\ t \in (0,T), \\ u(x,0) = \lambda u_0, & x \in B_{\rho(0)}, \end{cases} \tag{1.8}$$

where $\lambda > 0$ is a parameter and $u_0 \in C(\overline{B}_{\rho(0)})$. In Section 5 below, we will find that there exists a maximal existence time denoted by $T_{\max} = T_{\max}(\lambda u_0) \in (0, T]$ such that (1.8) has a unique solution $u \in C([0, T_{\max}), L^\infty(B_{\rho(t)}))$ which is classical for $0 < t < T_{\max}$, and that if $T_{\max} < T$ then $\lim_{t\to T_{\max}} \|u(\cdot,t)\|_{L^\infty(B_{\rho(t)})} = \infty$.

Theorem 3. *Let $N = 3$ and take $T > 0$ and $R \in (0, R_0]$ in* (1.7)*. Assume, in* (1.8)*, that $u_0 \not\equiv 0$ is nonnegative, radially symmetric, and nonincreasing in $r = |x|$. Then there exists $\lambda^* > 0$ such that $T_{\max}(\lambda^* u_0) = T$ and a solution u of* (1.8) *with $\lambda = \lambda^*$ satisfies*

$$\lim_{t\to T}(T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty(B_{\rho(t)})} = \infty, \tag{1.9}$$

namely, u blows up in type II rate.

Remark 1.3. Let us consider the cases (i) $N \geq 4$ and $R > 0$ in (1.7) and (ii) $N = 3$ and $R > R_0$ in (1.7). In either case, there also exists $\lambda^* > 0$ such that, if $\lambda = \lambda^*$ in (1.8), then $T_{\max}(\lambda^* u_0) = T$ and a solution blows up at $(x, t) = (0, T)$. (See Proposition 5.2 below.) However, this blowup must be of type I by (i) of Theorem 2.

The main idea of the proof of Theorem 2 is to employ the intersection comparison argument with a particular self-similar solution obtained in Theorem 1. Concerning the zero numbers properties of solutions for linear parabolic equations, we refer to [24, 11, 1, 6]. In the proof of Theorem 3, we will employ the method of backward self-similar variables, introduced by [13, 14, 10]. We rescale u by similarity variables around $(0, T)$ by setting

$$w(y, s) = (T - t)^{1/(p-1)} u(y(T - t)^{1/2}, t), \quad s = -\log(T - t) + \log T.$$

This function w solves the parabolic equation

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{1}{p-1} w + w^p, \quad y \in B_R, \ s > 0, \tag{1.10}$$

and the blowup time $t = T$ corresponds to $s = \infty$. Then, studying solutions of (1.8) near blowup time is equivalent to analysing the large-time asymptotics of solutions of (1.10). We will verify the existence of solution w of (1.10) such that w is time-global and unbounded as $s \to \infty$ by employing the energy method due to Giga and Kohn [13, 14].

This paper is organized as follows: In Section 2 we will prove Theorem 1. In Section 3 we will show asymptotic properties of solutions obtained in Theorem 1. Section 4 is devoted to the proof of Theorems 2. Finally, in Section 5, we will give the proof of Theorem 3.

2. Proof of Theorem 1

We will show the existence of solutions of (1.4) by making use of the variational argument due to Brezis-Nirenberg [5]. We rewrite the equation in (1.4) in the form

$$\frac{1}{\sigma} \nabla \cdot (\sigma \nabla v) - \frac{1}{p-1} v + v^p = 0$$

with $\sigma(x) = e^{-|x|^2/4}$. Then a positive solution of (1.4) can be found as a minimizer of the problem

$$\min \left\{ \int_{B_R} \left(|\nabla u|^2 + \frac{1}{p-1} u^2 \right) \sigma dx : u \in H_0^1(B_R), \ \int_{B_R} |u|^{p+1} \sigma dx = 1 \right\}. \tag{2.1}$$

A minimizer u of (2.1) can be taken to be nonnegative, and u satisfies

$$\int_{B_R} \left(\nabla u \cdot \nabla \psi + \frac{1}{p-1} v\psi \right) \sigma dx = \mu \int_{B_R} u^p \psi \sigma dx$$

for any $\psi \in H_0^1(B_R)$, where μ is a Lagrange multiplier. Then $v = \mu^{1/(p-1)} u$ is a weak solution of (1.4) with $v \geq 0$ and $v \not\equiv 0$. By Lemma A.1 in Appendix below, we see that $v \in C^2(B_R)$ and $v > 0$ in B_R.

For $R > 0$ we define $S_{\sigma,R}$ by

$$S_{\sigma,R} = \inf_{u \in H^1(B_R) \setminus \{0\}} \frac{\displaystyle\int_{B_R} \left(|\nabla u|^2 + \frac{1}{p-1} u^2 \right) \sigma dx}{\left(\displaystyle\int_{B_R} |u|^{2N/(N-2)} \sigma dx \right)^{(N-2)/N}}. \tag{2.2}$$

Let us denote by S the best Sobolev constant of the embedding $H_0^1 \subset L^{2N/(N-2)}$, which is given by

$$S = \inf_{u \in H_0^1(B_R) \setminus \{0\}} \frac{\displaystyle\int_{B_R} |\nabla u|^2 dx}{\left(\displaystyle\int_{B_R} |u|^{2N/(N-2)} dx \right)^{(N-2)/N}}. \tag{2.3}$$

Recall that S is independent of the domain B_R and depend only on N.

In this section, we will show the following propositions.

Proposition 2.1. *If $S_{\sigma,R} < S$ then the problem* (1.4) *has a positive radially symmetric solution.*

Proposition 2.2. (i) *Let $N \geq 4$. Then $S_{\sigma,R} < S$ for any $R > 0$.*
(ii) *Let $N = 3$. Then there exists $R_0 > 0$ such that $S_{\sigma,R} < S$ for $R > R_0$.*

Proposition 2.3. *Let $N = 3$. If $R \in (0, R_0]$ then the problem* (1.4) *has no positive radial solution.*

As a consequence of Propositions 2.1 and 2.2 we obtain the existence part of Theorem 1. It is clear that Proposition 2.3 implies the non-existence part of (ii) of Theorem 1.

First we will prove Proposition 2.1. To this end we need some lemmas as follows.

Lemma 2.1. *Put $v(x) = e^{-|x|^2/8} u(x)$ for $u \in H_0^1(B_R)$. Then $v \in H_0^1(B_R)$ and*

$$\int_{B_R} |\nabla u|^2 \sigma dx = \int_{B_R} |\nabla v|^2 dx - \int_{B_R} \left(\frac{N}{4} - \frac{|x|^2}{16} \right) v^2 dx. \tag{2.4}$$

Proof. A direct calculation yields

$$\int_{B_R} |\nabla u|^2 \sigma dx = \int_{B_R} |\nabla v|^2 dx + \frac{1}{16}\int_{B_R} |x|^2 v^2 dx + \frac{1}{2}\int_{B_R} (x\cdot\nabla v)v dx.$$

By using Green's formula, we have

$$\frac{1}{2}\int_{B_R} (x\cdot\nabla v)v dx = \frac{1}{4}\int_{B_R} x\cdot\nabla(v^2)dx = -\frac{N}{4}\int_{B_R} v^2 dx.$$

Thus (2.4) holds. □

Let $v(x) = e^{-|x|^2/8}u(x)$ for $u \in H_0^1(B_R)$. By Lemma 2.1 we have

$$\frac{\int_{B_R}\left(|\nabla u|^2 + \frac{1}{p-1}u^2\right)\sigma dx}{\left(\int_{B_R}|u|^{p+1}\sigma dx\right)^{2/(p+1)}} = \frac{\int_{B_R}\left(|\nabla v|^2 - av^2\right)dx}{\left(\int_{B_R} b|v|^{p+1}dx\right)^{2/(p+1)}}, \tag{2.5}$$

where

$$a(x) = \frac{1}{2} - \frac{|x|^2}{16} \quad \text{and} \quad b(x) = e^{|x|^2/(2N-4)}. \tag{2.6}$$

It is clear that u achieves the infimum $S_{\sigma,R}$ of (2.2), if and only if, v achieves $S_{\sigma,R}$ in (2.5).

Lemma 2.2. *Let $\{w_k\}$ be a sequence such that w_k is radially symmetric about the origin for each $k = 1, 2, \ldots$. Assume that $\{w_k\}$ is bounded in $H_0^1(B_R)$ and $w_k \to 0$ strongly in $L^2(B_R)$ as $k \to \infty$. Then*

$$\int_{B_R} (b-1)|w_k|^{p+1}dx \to 0 \quad \text{as } k\to\infty,$$

where b is the function given in (2.6).

Proof. Note that $b(x) - 1 \le C|x|^2$ for $x \in B_R$ with some constant $C > 0$. Then it suffices to show

$$\int_{B_R} |x|^2|w_k|^{p+1}dx \to 0 \quad \text{as } k\to\infty. \tag{2.7}$$

Since we have

$$N - 1 + \frac{2}{1-\theta} - \frac{(N-2)(p+1-2\theta)}{2(1-\theta)} \to 1 \quad \text{as } \theta \to 0,$$

we can take $\theta \in (0,1)$ so small that

$$\int_0^R r^{N-1+\frac{2}{1-\theta}-\frac{(N-2)(p+1-2\theta)}{2(1-\theta)}} dr < \infty.$$

By the Hölder inequality, we have

$$(2.8)\qquad \int_{B_R} |x|^2|w_k|^{p+1}dx \le \left(\int_{B_R} w_k^2 dx\right)^{\theta}\left(\int_{B_R} |x|^{2/(1-\theta)}|w_k|^{(p+1-2\theta)/(1-\theta)}dx\right)^{1-\theta}.$$

Since $w_k \in H_0^1(B_R)$ is radially symmetric, there exists a constant $C_1 > 0$ which is independent of k such that $w_k(r)$ with $r = |x|$ satisfies

$$|w_k(r)| \le C_1\|\nabla w_k\|_{L^2(B_R)} r^{-(N-2)/2} \quad \text{for } 0 < r \le R.$$

(See, e.g., [29, Radial Lemma].) Then it follows that

$$\begin{aligned}&\int_{B_R} |x|^{2/(1-\theta)}|w_k|^{(p+1-2\theta)/(1-\theta)}dx \\ &\le C\int_0^R r^{N-1+\frac{2}{1-\theta}-\frac{(N-2)(p+1-2\theta)}{2(1-\theta)}}dr\|\nabla w_k\|_{L^2(B_R)}^{(p+1-2\theta)/(1-\theta)}\end{aligned}$$

with some constant $C > 0$. From (2.8) we obtain

$$\int_{B_R} |x|^2 w_k^{p+1}dx \le C\|w_k\|_{L^2(B_R)}^{2\theta}\|\nabla w_k\|_{L^2(B_R)}^{p+1-2\theta}.$$

By the assumptions that $\{w_k\}$ is bounded in $H_0^1(B_R)$ and $w_k \to 0$ in $L^2(B_R)$ as $k \to \infty$, we obtain (2.7). This completes the proof. □

Proof of Proposition 2.1. We easily see that $u \in H_0^1(B_R)$ is a minimizer of the problem (2.1) if and only if u achieves the infimum in (2.2). Then it suffices to show that there exists a radially symmetric function in $H_0^1(B_R)$ which achieves the infimum $S_{\sigma,R}$ of (2.2). Let $\{u_k\} \subset H_0^1(B_R)$ be a minimizing sequence of $S_{\sigma,R}$ which is radially symmetric about the origin, and put $v_k(x) = e^{-|x|^2/8}u_k(x)$ for each $k = 1, 2, \ldots$. Then, from (2.5) we may assume that

$$(2.9)\qquad \int_{B_R}\left(|\nabla v_k|^2 - av_k^2\right)dx \to S_{\sigma,R} \quad \text{as } k \to \infty$$

and

$$\int_{B_R} b|v_k|^{p+1}dx = 1 \quad \text{for } k = 1, 2, \ldots.$$

Since $\{u_k\}$ is bounded in $H_0^1(B_R)$, $\{v_k\}$ is also bounded in $H_0^1(B_R)$. Then there exist a subsequence, still denoted by $\{v_k\}$, and $v \in H_0^1(B_R)$ such that

$$v_k \rightharpoonup v \quad \text{weakly in } H_0^1(B_R),$$

$$v_k \to v \quad \text{strongly in } L^2(B_R).$$

Put $w_k = v_k - v$. Then

$$w_k \rightharpoonup 0 \quad \text{weakly in } H_0^1(B_R),$$

$$w_k \to 0 \quad \text{strongly in } L^2(B_R).$$

By the Brezis-Lieb lemma [4] we have

$$1 = \int_{B_R} b|v_k|^{p+1} dx = \int_{B_R} b|v|^{p+1} dx + \int_{B_R} b|w_k|^{p+1} dx + o(1).$$

Lemma 2.2 implies that

$$\begin{aligned} \int_{B_R} b|w_k|^{p+1} dx &= \int_{B_R} |w_k|^{p+1} dx + \int_{B_R} (b-1)|w_k|^{p+1} dx \\ &= \int_{B_R} |w_k|^{p+1} dx + o(1). \end{aligned}$$

Thus we obtain

$$1 = \int_{B_R} b|v|^{p+1} dx + \int_{B_R} |w_k|^{p+1} dx + o(1). \tag{2.10}$$

From $2/(p+1) < 1$ and the Sobolev embeddings, we have

$$\begin{aligned} 1 &\le \left(\int_{B_R} b|v|^{p+1} dx \right)^{2/(p+1)} + \left(\int_{B_R} |w_k|^{p+1} dx \right)^{2/(p+1)} + o(1) \\ &\le \left(\int_{B_R} b|v|^{p+1} dx \right)^{2/(p+1)} + \frac{1}{S} \int_{B_R} |\nabla w_k|^2 dx + o(1), \end{aligned} \tag{2.11}$$

where $S > 0$ is the constant in (2.3). From (2.9) it follows that

$$\begin{aligned} & \int_{B_R} \left(|\nabla v_k|^2 - a v_k^2 \right) dx \\ = & \int_{B_R} \left(|\nabla v|^2 - a v^2 \right) dx + \int_{B_R} |\nabla w_k|^2 dx + o(1) = S_{\sigma,R} + o(1). \end{aligned}$$

Then, from (2.11) we obtain

$$\begin{aligned} & \int_{B_R} \left(|\nabla v|^2 - a v^2 \right) dx + \int_{B_R} |\nabla w_k|^2 dx \\ \le & \ S_{\sigma,R} \left(\int_{B_R} b|v|^{p+1} dx \right)^{2/(p+1)} + \frac{S_{\sigma,R}}{S} \int_{B_R} |\nabla w_k|^2 dx + o(1). \end{aligned}$$

This implies that

$$\begin{aligned} & \left(1 - \frac{S_{\sigma,R}}{S} \right) \int_{B_R} |\nabla w_k|^2 dx \\ \le & \ S_{\sigma,R} \left(\int_{B_R} b|v|^{p+1} dx \right)^{2/(p+1)} - \int_{B_R} \left(|\nabla v|^2 - a v^2 \right) dx + o(1). \end{aligned}$$

From $S_{\sigma,R} < S$ and the definition of $S_{\sigma,R}$, we have $\int_{B_R} |\nabla w_k|^2 dx \to 0$ as $k \to \infty$. This implies that $v_k \to v$ strongly in $H^1(B_R)$. Thus, from (2.10), we obtain $\int_{B_R} b v^{p+1} dx = 1$.

Then, from (2.5), $u(x) = e^{|x|^2/8}v(x)$ achieves the infimum $S_{\sigma,R}$ of (2.2), and hence u is a minimizer of the problem (2.1). As has been mentioned above, we obtain a positive solution of (1.4). This completes the proof of Proposition 2.1. □

Next, we will prove Proposition 2.2. For $v \in H_0^1(B_R)$ we define $Q(v)$ by

$$Q(v) = \frac{\displaystyle\int_{B_R} |\nabla v|^2 dx - \int_{B_R} av^2 dx}{\left(\displaystyle\int_{B_R} bv^{p+1} dx\right)^{2/(p+1)}}.$$

First we consider the case $N \geq 4$. Put $m_R = \min\{2, R/2\}$. For $\varepsilon > 0$ we set

$$v_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}},$$

where $\phi \in C_0^\infty(B_R)$, $0 \leq \phi \leq 1$, is a cut off function such that $\phi(x) \equiv 1$ if $|x| \leq m_R/2$ and $\phi(x) \equiv 0$ if $|x| \geq m_R$.

Proof of (i) *of Proposition* 2.2. We note that $a(x) \geq 1/4$ for $0 \leq |x| \leq m_R$ and $b(x) \geq 1$ for $|x| \geq 0$. From the proof of Lemma 1.1 in [5] we have, as $\varepsilon \to 0$,

$$\int_{B_R} |\nabla v_\varepsilon|^2 dx = \varepsilon^{-(N-2)/2} K_1 + O(1), \tag{2.12}$$

$$\left(\int_{B_R} bv_\varepsilon^{p+1} dx\right)^{2/(p+1)} \geq \left(\int_{B_R} v_\varepsilon^{p+1} dx\right)^{2/(p+1)} = \varepsilon^{-(N-2)/2} K_2 + O(\varepsilon), \tag{2.13}$$

$$\int_{B_R} av_\varepsilon^2 dx \geq \frac{1}{4}\int_{B_R} v_\varepsilon^2 dx = \begin{cases} K_3 \varepsilon^{-(N-4)/2} + O(1) & \text{if } N \geq 5, \\ K_3 |\log \varepsilon| + O(1) & \text{if } N = 4, \end{cases} \tag{2.14}$$

where K_1, K_2, and K_3 denote positive constants which depend only on N and such that $K_1/K_2 = S$. From (2.12)-(2.14) it follows that

$$\int_{B_R} |\nabla v_\varepsilon|^2 dx - \int_{B_R} av_\varepsilon^2 dx \leq \begin{cases} \varepsilon^{-(N-2)/2}\left(K_1 - K_3\varepsilon + O(\varepsilon^{(N-2)/2})\right) & \text{if } N \geq 5, \\ \varepsilon^{-1}\left(K_1 - K_3 \varepsilon |\log \varepsilon| + O(\varepsilon)\right) & \text{if } N = 4, \end{cases}$$

and that

$$Q(v_\varepsilon) \leq \begin{cases} S - (K_3/K_2)\varepsilon + O(\varepsilon^{(N-2)/2}), & \text{if } N \geq 5, \\ S - (K_3/K_2)\varepsilon|\log\varepsilon| + O(\varepsilon), & \text{if } N = 4. \end{cases}$$

Then we have $Q(v_\varepsilon) < S$ for sufficient small $\varepsilon > 0$. This implies that $S_{\sigma,R} < S$ for any $R > 0$. □

Let us consider the case $N = 3$. In this case we put

$$v_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{1/2}},$$

where $\phi \in C^1[0, R]$ satisfies $\phi(r) > 0$ for $0 \le r < R$ and $\phi'(0) = \phi(R) = 0$. By the similar argument as in the proof of Lemma 1.3 in [5], as $\varepsilon \to 0$, we have

$$\int_{B_R} |\nabla v_\varepsilon|^2 dx = \tilde{K}_1 \varepsilon^{-1/2} + \omega \int_0^R |\phi'(r)|^2 dr + O(\varepsilon^{1/2}), \tag{2.15}$$

$$\int_{B_R} v_\varepsilon^{p+1} dx = \tilde{K}_2 \varepsilon^{-3/2} + O(\varepsilon^{-1/2}), \tag{2.16}$$

$$\int_{B_R} v_\varepsilon^2 dx = \omega \int_0^R \phi^2(r) dr + O(\varepsilon^{1/2}), \tag{2.17}$$

where $\tilde{K}_1$ and $\tilde{K}_2$ are positive constants such that $\tilde{K}_1/\tilde{K}_2^{1/3} = S$ and ω is the area of S^2. Furthermore, we obtain the following:

Lemma 2.3. *As $\varepsilon \to 0$ we have*

$$\int_{B_R} |x|^2 v_\varepsilon^2 dx = \omega \int_0^R r^2 \phi^2(r) dr + O(\varepsilon), \tag{2.18}$$

$$\int_{B_R} b v_\varepsilon^{p+1} dx = \tilde{K}_2 \varepsilon^{-3/2} + O(\varepsilon^{-1/2}), \tag{2.19}$$

where $\tilde{K}_2$ and ω are constants given above.

Proof. We see that

$$\int_{B_R} |x|^2 v_\varepsilon^2 dx = \omega \int_0^R \frac{r^4 \phi^2(r)}{\varepsilon + r^2} dr = \omega \int_0^R \left(r^2 \phi^2(r) - \frac{\varepsilon r^2 \phi^2(r)}{\varepsilon + r^2} \right) dr.$$

Since

$$\int_0^R \frac{r^2 \phi^2(r)}{\varepsilon + r^2} dr \le \int_0^R \frac{r^2}{\varepsilon + r^2} dr = O(1),$$

we obtain (2.18).

Note that $|b(x) - 1| \le C|x|^2$ for $0 \le |x| \le R$ with some constant $C > 0$. Then

$$\left| \int_{B_R} b v_\varepsilon^{p+1} dx - \int_{B_R} v_\varepsilon^{p+1} dx \right| \le C \int_{B_R} |x|^2 v_\varepsilon^{p+1} dx. \tag{2.20}$$

We have

$$\begin{aligned} \int_{B_R} |x|^2 v_\varepsilon^{p+1} dx &= \omega \int_0^R \frac{r^4 \phi^6(r)}{(\varepsilon + r^2)^3} dr \le \omega \int_0^R \frac{r^4}{(\varepsilon + r^2)^3} dr \\ &\le \omega \varepsilon^{-1/2} \int_0^\infty \frac{t^4}{(1 + t^2)^3} dt = O(\varepsilon^{-1/2}). \end{aligned}$$

From (2.16) and (2.20) we obtain (2.19). □

Proof of (ii) *of Proposition* 2.2. From (2.15) and (2.17)-(2.19) we obtain

$$\int_{B_R} |\nabla v_k|^2 - a v_k^2 dx = \tilde{K}_1 \varepsilon^{-1/2} + \omega \int_0^R \left(|\phi'|^2 - \frac{1}{2}\phi^2 + \frac{r^2}{16}\phi^2 \right) dr + O(\varepsilon^{1/2})$$

and

$$\left(\int_{B_R} b v_\varepsilon^{p+1} dx \right)^{2/(p+1)} = \varepsilon^{-1/2} \left(\tilde{K}_2^{1/3} + O(\varepsilon) \right).$$

Thus it follows that

$$Q(v_\varepsilon) = S + \omega \tilde{K}_2^{-1/3} \varepsilon^{1/2} \int_0^R \left(|\phi'|^2 - \frac{1}{2}\phi^2 + \frac{r^2}{16}\phi^2 \right) dr + O(\varepsilon).$$

Let us consider the minimization problem

$$\lambda_R = \inf_{\phi \in \Phi} \frac{\displaystyle\int_0^R \left(|\phi'|^2 - \frac{1}{2}\phi^2 + \frac{r^2}{16}\phi^2 \right) dr}{\displaystyle\int_0^R \phi^2 dr}, \tag{2.21}$$

where $\Phi = \{\phi \in H^1([0, R]) : \phi'(0) = \phi(R) = 0\}$. Then there exists $R_0 > 0$ such that $\lambda_R = 0$ if $R = R_0$ and $\lambda_R < 0$ if $R > R_0$. Let $R > R_0$, and let ϕ be a minimizer of the problem (2.21). Then we obtain $Q(v_\varepsilon) < S$ for sufficient small $\varepsilon > 0$. This implies that $S_{\sigma,R} < S$ if $R > R_0$. □

Remark 2.1. Since $\lambda_R = 0$ in (2.21) if $R = R_0$, we see that R_0 is given by the first zero of the solution ϕ of

$$\phi'' + \left(\frac{1}{2} - \frac{1}{16} r^2 \right) \phi = 0, \quad r > 0, \tag{2.22}$$

satisfying $\phi'(0) = 0$ and $\phi(0) > 0$. It is easy to see that the solution ϕ is given by ϕ_1 in (2.24) below.

We will prove Proposition 2.3 following the idea by Atokinson and Peletier [2]. Let us denote by $F(\alpha, \gamma; s)$ a Kummer's function, which are given by

$$F(\alpha, \gamma; s) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{s^n}{n!},$$

where $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$. Define F_1 and F_2, respectively, by

$$F_1(s) = F\left(-\frac{1}{4}, \frac{1}{2}; s \right) = 1 - \frac{1}{2}s - \sum_{n=2}^{\infty} \prod_{k=2}^{n} \left(\frac{4k-5}{2k-1} \right) \frac{s^n}{2^n n!}$$

and

$$F_2(s) = F\left(\frac{1}{4}, \frac{3}{2}; s\right) = 1 + \frac{1}{6}s + \frac{1}{3}\sum_{n=2}^{\infty}\prod_{k=2}^{n}\left(\frac{4k-3}{2k+1}\right)\frac{s^n}{2^n n!}.$$

Lemma 2.4. *We have $(F_1(s)F_2(s))' \le 0$ for $s \ge 0$, where $' = d/ds$.*

Proof. By the definitions of F_1 and F_2, we have $F_1(s) \le 1$, $F_1'(s) \le 0$, $F_2(s) \ge 1$, and $F_2'(s) \ge 0$. Then it follows that

$$(F_1(s)F_2(s))' = F_1'(s)F_2(s) + F_1(s)F_2'(s) \le F_1'(s) + F_2'(s). \tag{2.23}$$

We see that

$$F_1'(s) + F_2'(s) = -\frac{1}{2} + \frac{1}{6} + \sum_{n=2}^{\infty}\left(-\prod_{k=2}^{n}\left(\frac{4k-5}{2k-1}\right) + \frac{1}{3}\prod_{k=2}^{n}\left(\frac{4k-3}{2k+1}\right)\right)\frac{s^{n-1}}{2^n(n-1)!}.$$

From the fact that

$$\frac{4k-5}{2k-1} \ge \frac{4k-3}{2k+1} \quad \text{for } k = 2, 3, \dots,$$

we have $F_1' + F_2' \le 0$. From (2.23) we obtain $(F_1(s)F_2(s))' \le 0$ for $s \ge 0$. □

Recall that $F = F(\alpha, \gamma; s)$ is a solution of the equation $sF'' + (\gamma - s)F' - \alpha F = 0$, and that $s^{1-\gamma}F(\alpha - \gamma + 1, 2 - \gamma, s)$ is an independent solution when γ is not an integer. Put

$$\phi_1(r) = e^{-r^2/8}F_1(r^2/4) \quad \text{and} \quad \phi_2(r) = e^{-r^2/8}rF_2(r^2/4). \tag{2.24}$$

Then we easily see that ϕ_1 and ϕ_2 are solutions of (2.22) and satisfy $\phi_1'(0) = 0$ and $\phi_2(0) = 0$, respectively. In particular, the constant R_0 is given by the first zero of ϕ_1. By Strum's comparison theorem, we find that $\phi_2(r) > 0$ for $0 < r \le R_0$. Put $\psi(r) = \phi_1(r)\phi_2(r)$. Then $\psi(r) > 0$ for $0 < r < R_0$.

Proof of Proposition 2.3. Assume to the contrary that the problem (1.4) has a positive radial solution v with some $R \in (0, R_0]$. Put $w(r) = e^{-r^2/8}v(r)$. Then w satisfies

$$w'' + \frac{2}{r}w' + \left(\frac{1}{2} - \frac{1}{16}r^2\right)w + e^{r^2/2}w^5 = 0 \quad \text{for } 0 < r < R.$$

Put $\psi(r) = \phi_1(r)\phi_2(r)$, where ϕ_1 and ϕ_2 are functions given by (2.24). Then ψ satisfies $\psi(r) > 0$ for $0 < r < R_0$ and $\psi(0) = 0$. By Lemma A.3 in Appendix below, we obtain the following identity:

$$\begin{aligned} \frac{1}{2}R^2\psi(R)w'(R)^2 \;&=\; \frac{1}{4}\int_0^R r^2w^2\left\{\psi''' + 2\left(1 - \frac{1}{8}r^2\right)\psi' - \frac{1}{4}r\psi\right\}dr \\ &\quad + \frac{2}{3}\int_0^R re^{r^2/2}w^6\left\{r\psi' - \left(1 - \frac{1}{4}r^2\right)\psi\right\}dr. \end{aligned} \tag{2.25}$$

By a direct calculation we have

$$\psi''' + 2\left(1 - \frac{1}{8}r^2\right)\psi' - \frac{1}{4}r\psi = 0 \quad \text{for } 0 < r < R_0 \tag{2.26}$$

and

$$r\psi' - \left(1 - \frac{1}{4}r^2\right)\psi = -\frac{1}{4}r^2\psi + \frac{1}{2}r^3 e^{-r^2/4}\frac{d}{ds}(F_1(s)F_2(s)), \tag{2.27}$$

where $s = r^2/4$. Here we have used relations

$$\psi' = \phi_1'\phi_2 + \phi_1\phi_2', \quad \psi'' = -\left(1 - \frac{1}{8}r^2\right)\psi + 2\phi_1'\phi_2',$$
$$r\psi' = -\frac{r^2}{2}\psi + \psi + \frac{1}{2}r^3 e^{-r^2/4}\frac{d}{ds}(F_1(s)F_2(s)).$$

From (2.27) and Lemma 2.4 we have

$$r\psi' - \left(1 - \frac{1}{4}r^2\right)\psi < 0 \quad \text{for } 0 < r < R_0. \tag{2.28}$$

Substituting (2.26) and (2.28) into (2.25), we have a contradiction. Therefore, the problem (1.4) has no positive radial solution. □

3. Asymptotic properties of solutions for (1.4)

In this section we discuss the asymptotic properties of solutions for (1.4) by employing the ODE argument. Let us consider the initial value problem for the ordinary differential equation

$$\begin{cases} v'' + \left(\dfrac{N-1}{r} - \dfrac{r}{2}\right)v' - \dfrac{1}{p-1}v + |v|^{p-1}v = 0, \quad r > 0, \\ v'(0) = 0 \quad \text{and} \quad v(0) = \alpha, \end{cases} \tag{3.1}$$

where $\alpha > 0$ is a parameter. We denote by v_α a solution of (3.1). It is clear that v_α satisfies

$$v_\alpha'(r) = r^{1-N}e^{r^2/4}\int_0^r s^{N-1}e^{-s^2/4}\left(\frac{1}{p-1}v_\alpha(s) - v_\alpha(s)^p\right)ds. \tag{3.2}$$

Here and henceforth, we put $\kappa = (p-1)^{-1/(p-1)}$. First we give the following lemma.

Lemma 3.1. *Let $\alpha > \kappa$. Then there exists $r_0 = r_0(\alpha) > 0$ such that $v_\alpha(r) > 0$ for $0 \le r < r_0$, $v_\alpha(r_0) = 0$, and $v_\alpha'(r) < 0$ for $0 < r \le r_0$.*

Remark 3.1. In the case $\alpha \in (0, \kappa)$, the solution $v_\alpha(r)$ is increasing for a while, and may have a zero at some $r > 0$. But we do not need to consider the case $\alpha \in (0, \kappa)$ in our argument.

Proof. From (3.2) we see that $v'_\alpha(r) < 0$ for sufficient small $r > 0$. First, we will show that $v'_\alpha(r) < 0$ for $r > 0$ as far as $v_\alpha(r) > 0$. Assume to the contrary that there exists some $r_1 > 0$ such that $v'_\alpha(r) < 0$ for $0 < r < r_1$, $v'_\alpha(r_1) = 0$, and $v_\alpha(r_1) > 0$. By the Pohozaev-type identity due to Mizoguchi [27, Lemma 2.1], we have

$$\frac{p-1}{4(p+1)}\int_0^r s^{N+1}(v'_\alpha)^2\sigma ds = \frac{N}{p+1}r^{N-1}v_\alpha v'_\alpha\sigma + \frac{1}{2}r^N(v'_\alpha)^2\sigma$$
$$-\frac{1}{2(p+1)}r^{N+1}v_\alpha v'_\alpha\sigma - \frac{1}{(p+1)(p-1)}r^N v_\alpha^2\sigma + \frac{1}{p+1}r^N v_\alpha^{p+1}\sigma,$$

where $\sigma(r) = e^{-r^2/4}$. Putting $r = r_1$, we have

$$0 < \frac{p-1}{4(p+1)}\int_0^{r_1} s^{N+1}(v'_\alpha)^2\sigma ds = \frac{1}{p+1}r_1^N v_\alpha(r_1)^2\left(v_\alpha(r_1)^{p-1} - \frac{1}{p-1}\right)\sigma.$$

This implies that $v_\alpha(r_1) > \kappa$. Then $v_\alpha(r) > \kappa$ for $0 \le r \le r_1$. From (3.2) we have $v'_\alpha(r_1) > 0$. This is a contradiction. Therefore, $v'_\alpha(r) < 0$ for $r > 0$ as far as $v_\alpha(r) > 0$.

By [13] the problem (3.1) has no bounded positive solution $u \not\equiv \kappa$ on $[0,\infty)$. Therefore, there exists $r_0 = r_0(\alpha) > 0$ such that $v_\alpha(r_0) = 0$. □

We denote by $r_0 = r_0(\alpha) > 0$ the first zero of $v_\alpha(r)$ for $\alpha > \kappa$. We remark that $r_0(\alpha) > R_0$ for any $\alpha > \kappa$ if $N = 3$, where R_0 is the constant in Theorem 1.

In this section, we will show the following proposition.

Proposition 3.1. *There exists a sequence $\alpha_k \to \infty$ such that*

$$\begin{cases} r_0(\alpha_k) \to 0 & \text{as } k \to \infty \quad \text{if } N \ge 4, \text{ and} \\ r_0(\alpha_k) \to R_0 & \text{as } k \to \infty \quad \text{if } N = 3. \end{cases} \tag{3.3}$$

Furthermore, for any constants A and B with $0 < A < B$, we have

$$\inf\{-v'_{\alpha_k}(r) : A \le v_{\alpha_k}(r) \le B\} \to \infty \quad \text{as } k \to \infty. \tag{3.4}$$

First we show the following lemma.

Lemma 3.2. *For any $\widetilde{R} > 0$ there exists $\delta = \delta(\widetilde{R}) > 0$ such that, if $\alpha \in (0,\delta]$, then $v_\alpha(r) > 0$ for $0 \le r \le \widetilde{R}$.*

Proof. We note that the problem (3.1) has a trivial solution $v \equiv 0$. Put $\varepsilon \in (0,\kappa)$. By the continuous dependence of solutions with respect to the initial values, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\alpha \in (0,\delta]$ then $|v_\alpha(r)| \le \varepsilon$ for $0 \le r \le \widetilde{R}$. We will show that $v_\alpha(r) > 0$ for $0 \le r \le \widetilde{R}$. Assume to the contrary that $v_\alpha(r) > 0$ for $0 \le r < r_1$ and $v_\alpha(r_1) = 0$ for some $r_1 \in (0, \widetilde{R}]$. Then

$$\frac{1}{p-1} v_\alpha(r) - v_\alpha(r)^p \ge 0 \quad \text{for } 0 \le r \le r_1.$$

It follows from (3.2) that $v'_\alpha(r) \ge 0$ for $0 \le r \le r_1$, and hence $v_\alpha(r_1) > 0$. This is a contradiction. Thus we obtain $v_\alpha(r) > 0$ for $0 \le r \le \widetilde{R}$. □

Let $\{R_k\}$ be a sequence such that

$$\text{(3.5)} \qquad \begin{cases} R_k > 0 \quad \text{and} \quad R_k \to 0 & \text{as } k \to \infty \quad \text{if } N \ge 4, \ \textit{and} \\ R_k > R_0 \quad \text{and} \quad R_k \to R_0 & \text{as } k \to \infty \quad \text{if } N = 3, \end{cases}$$

where R_0 is the constant in Theorem 1. We denote by $v(r; R_k)$ the positive radially symmetric solution of (1.4) with $R = R_k$ obtained by Theorem 1.

Lemma 3.3. *We have* $\lim_{k\to\infty} v(0; R_k) = \infty$.

Proof. Put $\alpha_k = v(0; R_k)$ for each $k = 1, 2, \ldots$. Assume to the contrary that $\liminf_{k\to\infty} \alpha_k < \infty$. Then there exists a subsequence, still denoted by $\{\alpha_k\}$, such that $\lim_{k\to\infty} \alpha_k = \alpha^* \in [0, \infty)$. By the uniqueness of the initial value problem, we have $v_{\alpha_k}(r) \equiv v(r; R_k)$ for $0 \le r \le R_k$ and $v_{\alpha_k}(R_k) = 0$ for $k = 1, 2, \ldots$. Lemma 3.2 implies that $\alpha_k > \delta$ with some $\delta > 0$ for $k = 1, 2, \ldots$. Then we have $\alpha^* \ge \delta > 0$. By the continuous dependence of solutions on the initial values, for any $r > 0$, we obtain

$$\text{(3.6)} \qquad \|v_{\alpha_k}(\cdot) - v_{\alpha^*}(\cdot)\|_{L^\infty([0,r])} \to 0 \quad \text{as } k \to \infty.$$

In the case $N \ge 4$, take $r_1 > 0$ so small that $v_{\alpha^*}(r) > \alpha^*/2$ for $0 \le r \le r_1$. Put $r = r_1$ in (3.6). Then, for sufficient large k, we have $v_{\alpha_k}(r) > 0$ on $r \in [0, r_1]$. This contradicts the facts that $v_{\alpha_k}(R_k) = 0$ and $R_k \to 0$ as $k \to \infty$. In the case $N = 3$, take $r > R_0$ in (3.6). Since $v_k(R_k) = 0$ and $R_k \to R_0$ as $k \to \infty$, we have $v_{\alpha^*}(R_0) = 0$. From $\alpha^* > 0$, v_{α^*} is a positive radial solution of (1.4) with $R = R_0$. This contradicts (ii) of Theorem 1. Therefore, we obtain $\alpha_k \to \infty$ as $k \to \infty$ in either cases $N \ge 4$ and $N = 3$. □

For $\alpha > \kappa$, let $w_\alpha(s) = r^{2/(p-1)} v_\alpha(r)$ and $s = \log r$. Then w_α satisfies

$$\text{(3.7)} \qquad w''_\alpha - \frac{e^{2s}}{2} w'_\alpha - L^{p-1} w_\alpha + w_\alpha^p = 0 \qquad \text{for } s \in \mathbf{R},$$

where

$$\text{(3.8)} \qquad L = \left[\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right]^{1/(p-1)}.$$

Furthermore, w_α satisfies $w_\alpha(s) \to 0$ and $w'_\alpha(s) \to 0$ as $s \to -\infty$. Define $E[w_\alpha]$ by

$$E[w_\alpha](s) = \frac{w'_\alpha(s)^2}{2} + F(w_\alpha(s)),$$

where

$$F(w) = -\frac{L^{p-1}}{2}w^2 + \frac{1}{p+1}w^{p+1}. \tag{3.9}$$

Then, from (3.7) we observe that

$$\frac{d}{ds}E[w_\alpha](s) = \frac{1}{2}e^{2s}w'_\alpha(s)^2 \geq 0 \quad \text{for } s \in \mathbf{R}.$$

Since $w_\alpha(s) \to 0$ and $w'_\alpha(s) \to 0$ as $s \to -\infty$, we have $E[w_\alpha](s) \geq 0$ for $s \in \mathbf{R}$.

For $\alpha > \kappa$, put $s_0 = s_0(\alpha) = \log r_0(\alpha)$. Then $w_\alpha(s) > 0$ for $(-\infty, s_0)$ and $w_\alpha(s_0) = 0$. We see that F defined by (3.9) satisfies

$$F(0) = F(w^*) = 0, \quad \text{where } w^* = \left(\frac{p+1}{2}\right)^{1/(p-1)} L.$$

We obtain the following results.

Lemma 3.4. (i) *If $w_\alpha(s) < w^*$ then $w'_\alpha(s) \neq 0$.*
(ii) *We have* $\sup_{s\in(-\infty,s_0]} w_\alpha(s) \geq w^*$.
(iii) *There exists a unique $s^* = s^*(\alpha) \in (-\infty, s_0)$ such that*

$$w_\alpha(s^*) = \sup_{s\in(-\infty,s_0]} w_\alpha(s).$$

Furthermore, $w'_\alpha(s) > 0$ for $-\infty < s < s^*$ and $w'_\alpha(s) < 0$ for $s^* < s < s_0$.

Proof. (i) We see that $E[w_\alpha](s) \geq 0$ for $s > -\infty$, and that $F(w_\alpha(s)) < 0$ if $w_\alpha(s) < w^*$. Then we obtain $w'_\alpha(s) \neq 0$.

(ii) Since $\lim_{s\to-\infty} w_\alpha(s) = 0$ and $w_\alpha(s_0) = 0$, there exists at least one $s_1 \in (-\infty, s_0)$ such that $w'_\alpha(s_1) = 0$. From (i) we have $w_\alpha(s_1) \geq w^*$. This implies that (ii) holds.

(iii) It suffices to show that there exists a unique $s_1 \in (-\infty, s_0)$ such that $w'_\alpha(s_1) = 0$. Assume to the contrary that $w'_\alpha(s_1) = w'_\alpha(s_2) = 0$ with some $-\infty < s_1 < s_2 < s_0$. From (i) we have $w_\alpha(s_1) \geq w^*$ and $w_\alpha(s_2) \geq w^*$. We note that (3.7) can be written as

$$(p_0(s)w'_\alpha)' + p_0(s)\left(-L^{p-1}w_\alpha + w_\alpha^p\right) = 0 \quad \text{for } s \in \mathbf{R}, \tag{3.10}$$

where $p_0(s) = \exp(-e^{2s}/4)$. Integrating (3.10) on $[s_1, s_2]$, we have

$$\int_{s_1}^{s_2} p_0(s)\left(-L^{p-1}w_\alpha(s) + w_\alpha(s)^p\right) ds = 0.$$

From $w_\alpha(s_i) \geq w^* > L$ for $i = 1, 2$, there exists $s_3 \in (s_1, s_2)$ such that

$$\min_{s\in[s_1,s_2]} w_\alpha(s) = w_\alpha(s_3) < L \quad \text{and} \quad w'_\alpha(s_3) = 0.$$

This contradicts (i). Then there exists a unique $s_1 \in (-\infty, s_0)$ such that $w'_\alpha(s_1) = 0$. Thus (iii) holds with $s^* = s_1$. □

Let

$$z_\alpha(t) = \frac{v_\alpha(r)}{\alpha} \quad \text{and} \quad t = \alpha^{(p-1)/2} r. \tag{3.11}$$

Then z_α satisfies

$$\begin{cases} z''_\alpha + \left(\dfrac{N-1}{t} - \alpha^{1-p}\dfrac{t}{2}\right) z'_\alpha - \dfrac{\alpha^{1-p}}{p-1} z_\alpha + |z_\alpha|^{p-1} z_\alpha = 0 \quad \text{for } t > 0, \\ z_\alpha(0) = 1 \quad \text{and} \quad z'_\alpha(0) = 0. \end{cases} \tag{3.12}$$

We denote by Z the solution of the problem

$$\begin{cases} Z'' + \dfrac{N-1}{t} Z' + Z^p = 0 \quad \text{for } t > 0, \\ Z(0) = 1 \quad \text{and} \quad Z'(0) = 0. \end{cases} \tag{3.13}$$

Recall that L is the constant defined by (3.8). We obtain the following:

Lemma 3.5. (i) *For any fixed $\tilde{t} > 0$, we have*

$$\|z_\alpha(\cdot) - Z(\cdot)\|_{L^\infty([0,\tilde{t}])} \to 0 \quad \text{as } \alpha \to \infty. \tag{3.14}$$

(ii) *Let $\varepsilon \in (0, L)$. There exist $0 < t_1 < t_2$ and $\alpha_1 > 0$ such that, if $\alpha \geq \alpha_1$, then*

$$t_1^{2/(p-1)} z_\alpha(t_1) > L \quad \text{and} \quad t_2^{2/(p-1)} z_\alpha(t_2) < \varepsilon. \tag{3.15}$$

Proof. (i) Let $\{\alpha_k\}$ be a sequence satisfying $\alpha_k \to \infty$ as $k \to \infty$. For simplicity, one sets $z_k = z_{\alpha_k}$. Without loss of generality, we may assume that $\alpha_k \geq 1$ and satisfies

$$\frac{N-1}{t} - \frac{\alpha_k^{1-p} t}{2} \geq 0 \quad \text{for } 0 < t \leq \tilde{t}.$$

Define $\widetilde{E}_\alpha[z]$ by

$$\widetilde{E}_\alpha[z](t) = \frac{z'(t)^2}{2} - \frac{\alpha^{1-p}}{2(p-1)} z^2 + \frac{1}{p+1}|z|^{p+1}.$$

Then, from the equation in (3.12) we have

$$\frac{d}{dt}\widetilde{E}_{\alpha_k}[z_k](t) = -\left(\frac{N-1}{t} - \frac{\alpha_k^{1-p} t}{2}\right) z_k'^2 \leq 0 \quad \text{for } 0 < t \leq \tilde{t}.$$

Then we obtain $\widetilde{E}_{\alpha_k}[z_k](t) \le \widetilde{E}_{\alpha_k}[z_k](0) < 1/(p+1)$ for $0 \le t \le \tilde{t}$. We note here that, for $\alpha \ge 1$, we have

$$\begin{aligned} \inf_{z\in\mathbf{R}} \left\{ -\frac{\alpha^{1-p}}{2(p-1)} z^2 + \frac{1}{p+1}|z|^{p+1} \right\} &\ge \inf_{z\in\mathbf{R}} \left\{ -\frac{1}{2(p-1)} z^2 + \frac{1}{p+1}|z|^{p+1} \right\} \\ &= -\frac{1}{2(p+1)} \left(\frac{1}{p-1} \right)^{2/(p-1)}. \end{aligned}$$

Then $\widetilde{E}_{\alpha_k}[z_k](t)$ is uniformly bounded on $[0, \tilde{t}\,]$, and hence $\{z_k\}$ and $\{z_k'\}$ are also uniformly bounded on $[0, \tilde{t}\,]$. By the Ascolli-Arzela, a subsequence in $\{z_k\}$ converges to some function z uniformly on $[0, \tilde{t}\,]$. We easily see that z_k satisfies

$$z_k(t) = 1 - \int_0^t \tau^{1-N} e^{-\alpha_k^{1-p}\tau^2/4} \int_0^\tau \xi^{N-1} e^{\alpha_k^{1-p}\xi^2/4} \left(\frac{\alpha_k^{1-p}}{p-1} z_k(\xi) + z_k^p(\xi) \right) d\xi d\tau$$

for each $k = 1, 2, \dots$. Letting $k \to \infty$, we obtain

$$z(t) = 1 - \int_0^t \tau^{1-N} \int_0^\tau \xi^{N-1} z^p(\xi) d\xi d\tau.$$

Then z is a solution of the problem (3.13), that is, $z \equiv Z$. This implies that (3.14) holds.

(ii) Recall that the graph of $t^{2/(p-1)}Z(t)$ intersect (transversely) the value L exactly twice, and that $t^{2/(p-1)}Z(t) \to 0$ as $t \to \infty$. (See, e.g., [30, Propositions 3.4 and 3.7].) Then there exists $0 < t_1 < t_2$ such that

$$t_1^{2/(p-1)} Z(t_1) > L \quad \text{and} \quad t_2^{2/(p-1)} Z(t_2) < \varepsilon.$$

From (3.14) with $\tilde{t} > t_2$, there exists $\alpha_1 > 0$ such that if $\alpha \ge \alpha_1$ then (3.15) holds. □

From (ii) and (iii) of Lemma 3.4, for any $\varepsilon \in (0, w^*)$, there exists a unique $s_\varepsilon = s_\varepsilon(\alpha) \in (s^*, s_0)$ such that $w_\alpha(s_\varepsilon) = \varepsilon$. Define r^* and r_ε by

$$r^* = r^*(\alpha) = e^{s^*(\alpha)} \quad \text{and} \quad r_\varepsilon = r_\varepsilon(\alpha) = e^{s_\varepsilon(\alpha)},$$

respectively. Then $r_\varepsilon(\alpha) > r^*(\alpha)$ and $r_\varepsilon^{2/(p-1)} v_\alpha(r_\varepsilon) = \varepsilon$. From (iii) of Lemma 3.4, we see that $r^{2/(p-1)} v_\alpha(r)$ is increasing for $0 < r < r^*$ and decreasing for $r^* < r < r_0$. In particular,

$$(r^{2/(p-1)} v_\alpha(r))' < 0 \quad \text{for } r^* < r < r_0. \tag{3.16}$$

We have the following results.

Lemma 3.6. (i) *We have $r_\varepsilon(\alpha) \to 0$ as $\alpha \to \infty$. In particular, $r^*(\alpha) \to 0$ as $\alpha \to \infty$.*

(ii) *We have* $v_\alpha(r^*(\alpha)) \to \infty$ *as* $\alpha \to \infty$.

(iii) *Let* $N = 3$. *Then, for any* $\tilde{r} \in (0, R_0)$, $v_\alpha(\tilde{r}) \to 0$ *as* $\alpha \to \infty$.

Proof. (i) Let $\varepsilon \in (0, L)$. From (ii) of Lemma 3.5, there exist $0 < t_1 < t_2$ and $\alpha_1 > 0$ such that if $\alpha \geq \alpha_1$ then (3.15) holds. Put $r_i(\alpha) = \alpha^{-2/(p-1)} t_i$ for $i = 1, 2$. Then, for $\alpha \geq \alpha_1$, it follows from (3.11) that

$$r_1^{2/(p-1)} v_\alpha(r_1) = t_1^{2/(p-1)} z_\alpha(t_1) > L \quad \text{and} \quad r_2^{2/(p-1)} v_\alpha(r_2) = t_2^{2/(p-1)} z_\alpha(t_2) < \varepsilon.$$

This implies that $r_2(\alpha) \in (r^*(\alpha), r_0(\alpha))$ and $r_2(\alpha) > r_\varepsilon(\alpha)$ for $\alpha \geq \alpha_1$. Since $r_2(\alpha) = \alpha^{-2/(p-1)} t_2 \to 0$ as $\alpha \to \infty$, we have $r_\varepsilon(\alpha) \to 0$ as $\alpha \to \infty$.

(ii) From (ii) of Lemma 3.4, we have

$$r^{*2/(p-1)} v_\alpha(r^*) = w_\alpha(s^*) \geq w^*.$$

This implies that $v_\alpha(r^*) \geq w^* r^{*-2/(p-1)}$. Since $r^*(\alpha) \to 0$ as $\alpha \to \infty$ from (i), we have $v_\alpha(r^*) \to \infty$ as $\alpha \to \infty$.

(iii) Take $\varepsilon \in (0, w^*)$ arbitrarily. From (i) we can take $\tilde{\alpha} > 0$ so large that $r^*(\alpha) < r_\varepsilon(\alpha) < \tilde{r}$ for $\alpha \geq \tilde{\alpha}$. From (3.16) we have

$$\varepsilon = r_\varepsilon^{2/(p-1)} v_\alpha(r_\varepsilon) > \tilde{r}^{2/(p-1)} v_\alpha(\tilde{r}).$$

This implies that $v_\alpha(\tilde{r}) < \tilde{r}^{-2/(p-1)} \varepsilon$. Since $\varepsilon > 0$ is arbitrariry, we obtain $v_\alpha(\tilde{r}) \to 0$ as $\alpha \to \infty$. □

Let A and B be constants with $0 < A < B$. Define $r_A = r_A(\alpha)$ and $r_B = r_B(\alpha)$ by

$$v_\alpha(r_A) = A \quad \text{and} \quad v_\alpha(r_B) = B,$$

respectively. From $v_\alpha'(r) < 0$ for $0 < r \leq r_0$, it follows that $0 < r_B < r_A$.

Lemma 3.7. (i) *Let* $N = 3$. *Then* $r_A(\alpha) \to 0$ *as* $\alpha \to \infty$.

(ii) *Let* $\alpha_k \to \infty$ *be a sequence such that*

$$r_A(\alpha_k) \to 0 \quad \text{as } k \to \infty. \tag{3.17}$$

Then (3.4) *holds.*

Proof. (i) Take $\tilde{r} \in (0, R_0)$ arbitrariry. From (iii) of Lemma 3.6, there exists $\tilde{\alpha} > 0$ such that $v_\alpha(\tilde{r}) < A$ for $\alpha \geq \tilde{\alpha}$. Since $v_\alpha(r)$ is strictly decreasing in $r > 0$, we have $r_A(\alpha) < \tilde{r}$ for $\alpha \geq \tilde{\alpha}$. Since $\tilde{r} \in (0, R_0)$ is arbitrariry, we obtain $\lim_{\alpha \to \infty} r_A(\alpha) = 0$.

(ii) Since $v_{\alpha_k}(r)$ is strictly decreasing in $r \in (0, r_0)$ and $v_{\alpha_k}(r^*) \to \infty$ as $k \to \infty$ by (ii) of Lemma 3.6, we have $r^*(\alpha_k) < r_B(\alpha_k) < r_A(\alpha_k)$ for sufficient large k. It follows from (3.16) that

$$\frac{2}{p-1} v_{\alpha_k}(r) + r v'_{\alpha_k}(r) < 0 \quad \text{for } r^* < r \leq r_0.$$

In particular, we have

$$-v'_{\alpha_k}(r) > \frac{2}{(p-1)r} v_{\alpha_k}(r) \geq \frac{2}{(p-1)r_A(\alpha_k)} A \quad \text{for } r_B(\alpha_k) \leq r \leq r_A(\alpha_k).$$

From (3.17) we obtain (3.4). □

Proof of Proposition 3.1. Recall that $\{R_k\}$ be the sequence satisfying (3.5), and that $v(r; R_k)$ be the positive radial solution of (1.4) with $R = R_k$. Put $\alpha_k = v(0; R_k)$ for $k = 1, 2, \ldots$. Lemma 3.3 implies that $\alpha_k \to \infty$ as $k \to \infty$. It is clear that $v_{\alpha_k}(r) \equiv v(r; R_k)$ and $r_0(\alpha_k) = R_k$ for $k = 1, 2, \ldots$. Thus (3.3) holds.

From (i) and (ii) of Lemma 3.7, we obtain (3.4) in the case $N = 3$. In the case $N \geq 4$, from $r_A(\alpha_k) \in (0, r_0(\alpha_k))$ and (3.3), we have (3.17). Then, form (ii) of Lemma 3.7, we obtain (3.4). This completes the proof of Proposition 3.1. □

4. Proof of Theorem 2

For a continuous function ψ defined on an interval J, we define the zero number of the function ψ on J by

$$\mathcal{Z}_J[\psi] = \#\{r \in J : \psi(r) = 0\}.$$

Let us recall the zero number properties of solutions for linear parabolic equations. The following result is proved by Chen-Polačik [6].

Lemma 4.1. Let $u = u(|x|, t)$ be a radially symmetric solution of the linear parabolic equation

$$u_t = \Delta u + a(|x|, t)u, \quad |x| < R, \ t \in (t_1, t_2),$$

where $0 < R < \infty$, $0 \leq t_1 < t_2 \leq \infty$, and $a(r, t)$ is continuous on $[0, R] \times (t_1, t_2)$. Assume that $u(r, t)$ is not identically equal to 0 and satisfies $u(R, t) \neq 0$ for $t \in (t_1, t_2)$. The the following properties hold:

(i) $t \mapsto \mathcal{Z}_{[0,R]}[u(\cdot, t)]$ takes finite values and it is nonincreasing;

(ii) if $u_r(\tilde{r}, \tilde{t}) = u(\tilde{r}, \tilde{t}) = 0$ for some $\tilde{r} \in [0, R)$, $\tilde{t} \in (t_1, t_2)$, then

$$\mathcal{Z}_{[0,R]}[u(\cdot, t)] > \mathcal{Z}_{[0,R]}[u(\cdot, s)] \quad \text{for } t_1 < t < \tilde{t} < s < t_2.$$

Remark 4.1. The conclusion of the above lemma remains true if the boundary point R depends on t smoothly. See Remark 2.8 in [25].

For $\alpha > \kappa$, let v_α be a solution of the problem (3.1), and let $r_0 = r_0(\alpha) > 0$ be the first zero of $v_\alpha(r)$. Take $T > 0$, and put

$$U_\alpha(r,t) = (T-t)^{-1/(p-1)} v_\alpha(r/(T-t)^{1/2}) \tag{4.1}$$

for $0 \le r \le r_0(T-t)^{1/2}$ and $0 < t < T$. Then $U_\alpha = U_\alpha(|x|,t)$ is a self-similar solution of (1.1) and blows up at $(r,t) = (0,T)$ with type I rate.

Lemma 4.2. *Let u be a radially symmetric solution of* (1.1) *such that u blows up at $(r,t) = (0,T)$ and satisfies*

$$u > 0 \quad \text{in} \bigcup_{0<t<T} B_{\rho(t)} \times \{t\},$$

where $\rho \in C[0,T]$ and $\rho(t) > 0$ on $[0,T)$. Put $U_\alpha(r,t)$ by (4.1) *for $0 \le r \le r_0(T-t)^{1/2}$ and $0 < t < T$.*

(i) *Assume that $\rho(t) > r_0(T-t)^{1/2}$ for $t \in [\tau, T)$ with some $\tau \ge 0$. Then, for each fixed $t \in [\tau,T)$, we have $\mathcal{Z}_{J_1(t)}[u(\cdot,t) - U_\alpha(\cdot,t)] \ge 1$, where $J_1(t) = [0, r_0(T-t)^{1/2}]$.*

(ii) *Assume that $\rho(t) < r_0(T-t)^{1/2}$ for $t \in [\tau,T)$ with some $\tau \ge 0$, and that*

$$u = 0 \quad \text{on} \bigcup_{0<t<T} \partial B_{\rho(t)} \times \{t\}.$$

Then, for each fixed $t \in [\tau,T)$, we have $\mathcal{Z}_{[0,\rho(t)]}[U_\alpha(\cdot,t) - u(\cdot,t)] \ge 1$.

Proof. (i) Assume to the contrary that $\mathcal{Z}_{J_1(t_0)}[u(\cdot,t_0) - U_\alpha(\cdot,t_0)] = 0$ with some $t_0 \in [\tau,T)$, that is,

$$u(r,t_0) > U_\alpha(r,t_0) \quad \text{for } 0 \le r \le r_0(T-t_0)^{1/2}.$$

Then, from (4.1), there exists some $\widetilde{T} \in (t_0,T)$ such that

$$u(r,t_0) > (\widetilde{T}-t_0)^{-1/(p-1)} v_\alpha(r/(\widetilde{T}-t_0)^{1/2}) \quad \text{for } 0 \le r \le r_0(\widetilde{T}-t_0)^{1/2}.$$

Put $\widetilde{U}_\alpha(r,t) = (\widetilde{T}-t)^{-1/(p-1)} v_\alpha(r/(\widetilde{T}-t)^{1/2})$. Then $\widetilde{U}_\alpha = \widetilde{U}_\alpha(|x|,t)$ satisfies

$$\begin{cases} (\widetilde{U}_\alpha)_t = \Delta \widetilde{U}_\alpha + \widetilde{U}_\alpha^p, & |x| < r_0(\widetilde{T}-t)^{1/2},\ t_0 < t < T, \\ \widetilde{U}_\alpha = 0, & |x| = r_0(\widetilde{T}-t)^{1/2},\ t_0 < t < T, \\ \widetilde{U}_\alpha(r,t_0) < u(r,t_0), & |x| \le r_0(\widetilde{T}-t_0)^{1/2}. \end{cases}$$

By the comparison principle, we obtain

$$\widetilde{U}_\alpha(r,t) < u(r,t) \quad \text{for } 0 \le r \le r_0(\widetilde{T}-t)^{1/2},\ t_0 \le t < \widetilde{T}.$$

On the other hand, we have $\infty > u(0,\widetilde{T}) > \lim_{t\to\widetilde{T}} \widetilde{U}_\alpha(0,t) = \infty$. This is a contradiction. Thus (i) holds.

(ii) Assume to the contrary that $\mathcal{Z}_{[0,\rho(t_0)]}[U_\alpha(\cdot,t_0)-u(\cdot,t_0)] = 0$ with some $t_0 \in [\tau, T)$, that is,

$$u(r,t_0) < U_\alpha(r,t_0) \quad \text{for } 0 \le r \le \rho(t_0).$$

Then there exists some $\widehat{T} > T$ such that

$$u(r,t_0) < (\widehat{T}-t_0)^{-1/(p-1)} v_\alpha(r/(\widehat{T}-t_0)^{1/2}) \quad \text{for } 0 \le r \le \rho(t_0).$$

Put $\widehat{U}_\alpha(r,t) = (\widehat{T}-t)^{-1/(p-1)} v_\alpha(r/(\widehat{T}-t)^{1/2})$. Then $\widehat{U}_\alpha = \widehat{U}_\alpha(|x|,t)$ satisfies

$$\begin{cases} (\widehat{U}_\alpha)_t = \Delta \widehat{U}_\alpha + \widehat{U}_\alpha^p, & |x| < \rho(t),\ t_0 < t < T, \\ \widehat{U}_\alpha > 0, & |x| = \rho(t),\ t_0 < t < T, \\ \widehat{U}_\alpha(r,t_0) > u(r,t_0), & |x| \le \rho(t_0). \end{cases}$$

By the comparison principle, we obtain

$$\widehat{U}_\alpha(r,t) > u(r,t) \quad \text{for } 0 \le r \le \rho(t),\ t_0 \le t < T.$$

On the other hand, we have $\infty > \widehat{U}_\alpha(0,T) > \lim_{t\to T} u(0,t) = \infty$. This is a contradiction. Thus (ii) holds. □

Proof of Theorem 2. (i) By the assumption (1.5), there exist $t_0 \ge 0$ and $\rho_0 > 0$, with $\rho_0 > R_0$ if $N = 3$, such that

$$\rho(t) > \rho_0(T-t)^{1/2} \quad \text{for } t_0 \le t < T.$$

Let v_α be a solution of the problem (3.1) with $\alpha > \kappa$, and let $r_0(\alpha)$ be the first zero of $v_\alpha(r)$. Put $U_\alpha(r,t)$ by (4.1). Since there exists a sequence $\{\alpha_k\}$ satisfying (3.3) and (3.4) by Proposition 3.1, we can take $\alpha > 0$ such that $r_0 = r_0(\alpha) < \rho_0$ and $\mathcal{Z}_{J_1(t)}[u(\cdot,t_0) - U_\alpha(\cdot,t_0)] = 1$, where $J_1(t) = [0, r_0(T-t_0)^{1/2}]$. We will verify that

$$u(0,t) < U_\alpha(0,t) \quad \text{for } t_0 \le t < T. \tag{4.2}$$

It is clear that (4.2) holds at $t = t_0$. Assume to the contrary that $u(0,t_1) = U_\alpha(0,t_1)$ with some $t_1 \in (t_0, T)$. Since $u_r(0,t) = (U_\alpha)_r(0,t) = 0$ by the symmetry, $u(r,t_1) - U_\alpha(r,t_1)$ has a multiple zero at $r = 0$. Then, by (ii) of Lemma 4.1, $\mathcal{Z}_{J_1(t_1)}[u(\cdot,t_1) -$

$U_\alpha(\cdot,t_1)]$ must drop by at least 1, and hence $\mathcal{Z}_{J_1(t)}[u(\cdot,t)-U_\alpha(\cdot,t)]=0$ for $t\in(t_1,T)$. This contradicts (i) of Lemma 4.2. Thus (4.2) holds. Then

$$\limsup_{t\to T}(T-t)^{1/(p-1)}u(0,t)\le\lim_{t\to T}(T-t)^{1/(p-1)}U_\alpha(0,t)=v_\alpha(0)<\infty.$$

Since u is nonincreasing in $r\in[0,\rho(t)]$, we obtain (1.6).

(ii) Assume to the contrary that u blows up in type I rate, that is, there exists some constant $M>0$ such that

$$(T-t)^{1/(p-1)}\|u(\cdot,t)\|_{L^\infty(B_{\rho(t)})}<M\quad\text{for } t\in(0,T). \tag{4.3}$$

Let v_α be a solution of the problem (3.1) with $\alpha>M$, and let $r_0=r_0(\alpha)$ be the first zero of $v_\alpha(r)$. Since v_α is decreasing, there exists $r_M\in(0,r_0)$ such that

$$v_\alpha(r)>M\quad\text{for } 0\le r<r_M\quad\text{and}\quad v_\alpha(r_M)=M. \tag{4.4}$$

By the assumption on $\rho(t)$, there exists $t_1\in(0,T)$ such that

$$\rho(t)(T-t)^{-1/2}<r_M\quad\text{for } t\in[t_1,T). \tag{4.5}$$

Then it is clear that $\rho(t)<r_0(T-t)^{1/2}$ for $t\in[t_1,T)$. From (4.3) and (4.5) we have

$$(T-t_1)^{1/(p-1)}u(r,t_1)<M\quad\text{for } 0\le r\le\rho(t_1)<r_M(T-t_1)^{1/2}. \tag{4.6}$$

From (4.4), the function U_α defined by (4.1) satisfies

$$(T-t_1)^{1/(p-1)}U_\alpha(r,t_1)=v_\alpha(r/(T-t_1)^{1/2})\ge M\quad\text{for } 0\le r\le r_M(T-t_1)^{1/2}. \tag{4.7}$$

From (4.6) and (4.7) we obtain $\mathcal{Z}_{[0,\rho(t_1)]}[u(\cdot,t_1)-U_\alpha(\cdot,t_1)]=0$. This contradicts (ii) of Lemma 4.2. This implies that u blows up with type II rate. □

5. Proof of Theorem 3

Define ρ by (1.7) with $R>0$ and $T>0$. For a solution u of (1.8), define w by

$$w(y,s)=(T-t)^{1/(p-1)}u(x,t) \tag{5.1}$$

with

$$y=\frac{x}{(T-t)^{1/2}}\quad\text{and}\quad s=-\log(T-t)+\log T. \tag{5.2}$$

Then w satisfies the problem

$$\begin{cases} w_s=\Delta w-\frac{1}{2}y\cdot\nabla w-\frac{1}{p-1}w+w^p, & y\in B_R,\ s\in(0,\infty),\\ w=0, & y\in\partial B_R,\ s\in(0,\infty),\\ w(y,0)=\lambda w_0, & y\in B_R,\end{cases} \tag{5.3}$$

where $w_0 = T^{1/(p-1)}u_0(T^{1/2}y)$. By the standard argument, there exists a maximal existence time $S_{\max} = S_{\max}(\lambda w_0) \in (0,\infty]$ such that the problem (5.3) has a unique solution $w \in C([0,S_{\max}), L^\infty(B_R))$ which is classical for $0 < s < S_{\max}$, and that if $S_{\max} < \infty$ then $\lim_{s\to S_{\max}} \|w(\cdot,s)\|_{L^\infty(B_R)} = \infty$. This implies that, for (1.8), there exists a maximal existence time denoted by $T_{\max} = T_{\max}(\lambda u_0) \in (0,T]$ such that the problem (1.8) has a unique solution $u \in C([0,T_{\max}); L^\infty(B_{\rho(t)}))$ which is classical for $0 < t < T_{\max}$, and that if $T_{\max} < T$ then $\lim_{t\to T_{\max}} \|u(\cdot,t)\|_{L^\infty(B_{\rho(t)})} = \infty$.

Theorem 3 is a consequence of the following two propositions.

Proposition 5.1. *Let $N = 3$, and take $T > 0$ and $R \in (0,R_0]$ in* (1.7). *Let u be a radially symmetric solution of* (1.8) *such that $T_{\max} = T$ and u blows up at $(x,t) = (0,T)$. Then u satisfies* (1.9), *namely, u blows up in type II rate.*

Proposition 5.2. *Assume, in* (1.8), *that $u_0 \not\equiv 0$ is nonnegative, radially symmetric, and nonincreasing in $r = |x|$. Then there exists $\lambda^* = \lambda^*(u_0) > 0$ such that $T_{\max}(\lambda^* u_0) = T$ and a solution of* (1.8) *with $\lambda = \lambda^*$ blows up at $(x,t) = (0,T)$.*

Remark 5.1. In Proposition 5.2 we do not require neither $N = 3$ or $R \in (0,R_0]$ in (1.7).

For a solution w of (5.3), we define $\mathcal{E}[w]$ by

$$\mathcal{E}[w](s) = \frac{1}{2}\int_{B_R} |\nabla w|^2 \sigma dy + \frac{1}{2(p-1)}\int_{B_R} w^2\sigma dy - \frac{1}{p+1}\int_{B_R} w^{p+1}\sigma dy,$$

where $\sigma(y) = e^{-|y|^2/4}$. Multiplying the equation in (5.3) by w_s and w, and integrating on B_R, we obtain

$$\frac{d}{ds}\mathcal{E}[w](s) = -\int_{B_R} w_s^2 \sigma dy \tag{5.4}$$

and

$$\frac{1}{2}\frac{d}{ds}\int_{B_R} w^2\sigma dy = -\int_{B_R}|\nabla w|^2\sigma dy - \frac{1}{p-1}\int_{B_R} w^2\sigma dy + \int_{B_R} w^{p+1}\sigma dy \tag{5.5}$$

respectively. It follows from (5.4) that $\mathcal{E}[w](s)$ is monotonically decreasing in $s > 0$.

The following lemma play a key role in the proofs of Propositions 5.1 and 5.2. We call a solution w of (5.3) is *global* if $S_{\max} = \infty$, that is, $w \in C([0,\infty), L^\infty(B_R))$.

Lemma 5.1. (i) *Let w be a global solution of* (5.3). *Then $\mathcal{E}[w](s) > 0$ for any $s > 0$.*
(ii) *Let w be a global solution of* (5.3) *satisfying*

$$\liminf_{s\to\infty} \|w(\cdot,s)\|_{L^\infty(B_R)} < \infty. \tag{5.6}$$

Then there exist a sequence $s_k \to \infty$ *and* $w^* \in C^2(\overline{B}_R)$ *such that*

$$w(\cdot, s_k) \to w^*(\cdot) \quad \text{in } C^2(\overline{B}_R) \quad \text{as } k \to \infty. \tag{5.7}$$

Furthermore, w^* *is a stationary solution of* (5.3).

Proof. (i) We will show (i) following the idea by [13, 14]. Set

$$f(s) = \int_{B_R} w^2(y,s)\sigma dy.$$

From (5.5) it follows that

$$\frac{1}{2}f'(s) = -2\mathcal{E}[w](s) + \frac{p-1}{p+1}\int_{B_R} w^{p+1}\sigma dy \quad \text{for } s \geq 0.$$

Assume to the contrary that $\mathcal{E}[w](s_0) \leq 0$ for some $s_0 \geq 0$. Since $\mathcal{E}[w](s)$ is monotonically decreasing, $\mathcal{E}[w](s) \leq 0$ for $s \geq s_0$. Then

$$\frac{1}{2}f'(s) \geq \frac{p-1}{p+1}\int_{B_R} w^{p+1}\sigma dy \quad \text{for } s \geq s_0. \tag{5.8}$$

By the Hölder inequality, we obtain

$$f = \int_{B_R} w^2 \sigma dy \leq \left(\int_{B_R} \sigma dy\right)^{(p-1)/(p+1)} \left(\int_{B_R} w^{p+1}\sigma dy\right)^{2/(p+1)}.$$

Hence,

$$\int_{B_R} w^{p+1}\sigma dy \geq C_\sigma f^{(p+1)/2} \quad \text{with} \quad C_\sigma = \left(\int_{B_R} \sigma dy\right)^{-(p-1)/2}.$$

From (5.8) it follows that

$$\frac{1}{2}f'(s) \geq \frac{p-1}{p+1}C_\sigma f(s)^{(p+1)/2} \quad \text{for } s \geq s_0.$$

From $(p+1)/2 > 1$ and $f(s_0) > 0$, the function $f(s)$ must blow up in finite time. This contradicts $w \in C([0,\infty), L^\infty(B_R))$. Thus $\mathcal{E}[w](s) > 0$ for any $s \geq 0$.

(ii) From (5.6), there exist a sequence $s_k \to \infty$ and a constant $C_1 > 0$ such that $\|w(\cdot, s_k)\|_{L^\infty(B_R)} \leq C_1$ for $k = 1, 2, \ldots$. Let us consider the problem (5.3) with an initial value $w(\cdot, 0) = w(\cdot, s_k)$ for each $k = 1, 2, \ldots$. Then, by Ladyženskaja et. al. [23], there exists $\tau > 0$ such that, for any $\delta \in (0, \tau)$, we have $C_2 = C_2(\tau, \delta) > 0$ satisfying

$$\|w\|_{C^{2+\theta, 1+\theta/2}(B_R \times (s_k+\delta, s_k+\tau))} \leq C_2 \quad \text{for } k = 1, 2, \ldots,$$

where $\theta \in (0,1)$. Take $\tilde{s_k} \in (s_k + \delta, s_k + \tau)$. Then $\|w(\cdot, \tilde{s_k})\|_{C^{2+\theta}(B_R)} \leq C_2$ for $k = 1, 2, \ldots$. By Ascoli-Arzela, there exists a subsequence, denoted by $\{s_k\}$, and $w^* \in C^2(\overline{B}_R)$ such that (5.7) holds.

Integrate (5.4) on $[s_k, s]$, and let $s \to \infty$. Since $\mathcal{E}[w](s) > 0$ for $s \geq 0$ from (i), we have

$$\int_{s_k}^{\infty} \int_{B_R} w_s^2 \sigma dy ds \leq \mathcal{E}[w](s_k) < \infty. \tag{5.9}$$

Take $\tilde{s} > 0$ arbitrarily. By the Hölder inequality, we have

$$|w(y, s_k + \tilde{s}) - w(y, s_k)|^2 = \left| \int_{s_k}^{s_k+\tilde{s}} w_s(y, s) ds \right|^2 \leq \left(\int_{s_k}^{s_k+\tilde{s}} w_s(y, s)^2 ds \right) \tilde{s}$$

for $y \in B_R$. Then it follows that

$$\int_{B_R} |w(y, s_k + \tilde{s}) - w(y, s_k)|^2 \sigma dy \leq \tilde{s} \int_{s_k}^{s_k+\tilde{s}} \int_{B_R} w_s^2 \sigma dy ds.$$

From (5.9) we obtain

$$\int_{s_k}^{s_k+\tilde{s}} \int_{B_R} w_s^2 \sigma dy ds \to 0 \quad \text{as } k \to \infty,$$

and hence,

$$\int_{B_R} |w(y, s_k + \tilde{s}) - w(y, s_k)|^2 \sigma dy \to 0 \quad \text{as } k \to \infty. \tag{5.10}$$

Denote by $S(s)w(\cdot, 0)$ a solution of (5.3). By continuous dependence of solutions of (5.3) over initial data in $L^\infty(B_R)$, it follows from (5.7) that

$$w(\cdot, s_k + \tilde{s}) = S(\tilde{s})w(\cdot, s_k) \to S(\tilde{s})w^*(\cdot) \quad \text{in } L^\infty(B_R) \quad \text{as } k \to \infty. \tag{5.11}$$

Letting $k \to \infty$ in (5.10), we deduce from (5.7) and (5.11) that

$$\int_{B_R} |S(\tilde{s})w^* - w^*|^2 \sigma dy = 0.$$

Then $S(\tilde{s})w^* \equiv w^*$. Since $\tilde{s} > 0$ is arbitrarily, w^* is a stationary solution of (5.3). □

Proof of Proposition 5.1. Assume to the contrary that u satisfies

$$\liminf_{t \to T} (T - t)^{1/(p-1)} \|u(\cdot, t)\|_{L^\infty(B_{\rho(t)})} < \infty.$$

Define w by (5.1) with (5.2). Then w is a global solution of (5.3) satisfying (5.6). By (ii) of Lemma 5.1, there exist a sequence $s_k \to \infty$ and $w^* \in C^2(\overline{B}_R)$ such that (5.7) holds. It is clear that w^* is radially symmetric and $w^* \geq 0$ in B_R.

We will show that $w^* \not\equiv 0$. Assume to the contrary that $w^* \equiv 0$. Let v_α be a solution of (3.1) with $\alpha > \kappa$, and let $r_0(\alpha)$ be the first zero of v_α. From $r_0(\alpha) > R_0 \geq R$, we have $\min\{v_\alpha(r) : 0 \le r \le R\} > 0$. Since (5.7) holds with $w^* \equiv 0$, there exists some $s_k > 0$ such that

$$\|w(\cdot, s_k)\|_{L^\infty(B_R)} < \min\{v_\alpha(r) : 0 \le r \le R\}. \tag{5.12}$$

Put U_α by (4.1). Note that $\rho(t) < r_0(\alpha)(T-t)^{1/2}$ for $0 \le t < T$. Then, by (ii) of Lemma 4.2, we have

$$\mathcal{Z}_{[0,\rho(t)]}[U_\alpha(\cdot,t) - u(\cdot,t)] \ge 1 \quad \text{for any } t \in [0,T). \tag{5.13}$$

Define

$$W_\alpha(y,s) = (T-t)^{1/(p-1)} U_\alpha(|x|,t)$$

with (5.2). Then, from (4.1) we have $W_\alpha(y,s) \equiv v_\alpha(|y|)$ for $s \ge 0$. Then (5.13) implies that

$$\mathcal{Z}_{[0,R]}[v_\alpha(\cdot) - w(\cdot,s)] \ge 1 \quad \text{for any } s \in [0,\infty).$$

This contradicts (5.12). Therefore, $w^* \not\equiv 0$.

By the maximum principle, we have $w^* > 0$ in B_R. Thus, the problem (1.4) with $R \le R_0$ has a radial positive solution. This contradicts (ii) of Theorem 1. Therefore, u satisfies (1.9). □

Next, we will show Proposition 5.2. Let $u_0 \in C(\overline{B}_{\rho(0)})$ satisfy the assumptions in Proposition 5.2, and fix u_0 in the problem (1.8). Now denote by u_λ a solution of (1.8) and by $T_m(\lambda)$ a maximal existence time of u_λ. Define Λ_0 and Λ_∞, respectively, by

$$\Lambda_0 = \left\{ \lambda > 0 : T_m(\lambda) = T \text{ and } \sup_{t\in[0,T)} \|u_\lambda(\cdot,t)\|_{L^\infty(B_{\rho(t)})} < \infty \right\}$$

and $\Lambda_\infty = \{\lambda > 0 : T_m(\lambda) < T\}$.

Lemma 5.2. *Λ_0 is nonempty and open.*

Proof. First we will show that Λ_0 is nonempty. Put $z_1(t) = \kappa(T+1-t)^{-1/(p-1)}$, where $\kappa = (p-1)^{-1/(p-1)}$. Then z_1 satisfies $dz_1/dt = z_1^p$ for $0 \le t < T+1$. Take $\lambda > 0$ so small that

$$\lambda \|u_0\|_{L^\infty(B_{\rho(0)})} \le z_1(0) = \kappa(T+1)^{-1/(p-1)}.$$

Then, by the comparison theorem, we have

$$\|u_\lambda(\cdot,t)\|_{L^\infty(B_{\rho(t)})} \le z_1(t) \quad \text{for } 0 \le t < T.$$

This implies that $\lambda \in \Lambda_0$.

Next we will show that Λ_0 is open. Let $\lambda_0 \in \Lambda_0$, and put

$$M = \sup_{t\in[0,T)} \|u_{\lambda_0}(\cdot,t)\|_{L^\infty(B_{\rho(t)})}.$$

Define $z(t) = \kappa(T-t)^{-1/(p-1)}$. Then there exists $t_1 \in (0,T)$ such that $z(t_1) > M$. We can take $\hat{T} > T$ so that $\kappa(\hat{T}-t_1)^{-1/(p-1)} > M$. Define $\hat{z}(t) = \kappa(\hat{T}-t)^{-1/(p-1)}$. Then

$$\|u_{\lambda_0}(\cdot,t_1)\|_{L^\infty(B_{\rho(t_1)})} < \hat{z}(t_1).$$

By the continuous dependence with respect to $\lambda > 0$, there exists $\varepsilon > 0$ such that if $|\lambda - \lambda_0| < \varepsilon$ then

$$\|u_\lambda(\cdot,t_1)\|_{L^\infty(B_{\rho(t_1)})} < \hat{z}(t_1).$$

Note that $\hat{z}$ satisfies $d\hat{z}/dt = \hat{z}^p$ for $t_1 < t < \hat{T}$. By the comparison theorem, if $|\lambda - \lambda_0| < \varepsilon$ we obtain

$$\|u_\lambda(\cdot,t)\|_{L^\infty(B_{\rho(t)})} < \hat{z}(t) \quad \text{for } t_1 \le t < T.$$

This implies that $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subset \Lambda_0$. Thus Λ_0 is open. □

Lemma 5.3. Λ_∞ *is nonempty. Moreover, if* $\lambda \in \Lambda_\infty$ *then* u_λ *blows up at* $x = 0$ *and this blowup is of type I.*

Proof. First we show that Λ_∞ is nonempty. By the assumption on u_0, there exists $\tilde{R} \in (0, \rho(0)]$ such that $u_0(x) > 0$ for $|x| < \tilde{R}$. Take $T_1 \in (0,T)$ so that $\rho(T_1) < \tilde{R}$, and consider the problem

$$(5.14) \qquad \begin{cases} z_t = \Delta z + z^p, & x \in B_{\rho(T_1)},\ t \in (0,T_1), \\ z = 0, & x \in \partial B_{\rho(T_1)},\ t \in (0,T_1), \\ z(x,0) = \mu z_0(|x|), & x \in B_{\rho(T_1)}, \end{cases}$$

where $\mu > 0$ is a parameter and $z_0 \not\equiv 0$ is a nonnegative continuous function on $[0, \rho(T_1)]$ with $z_0(\rho(T_1)) = 0$. We denote by z_μ a unique solution of (5.14). By [28, Theorem 2], there exists $\mu > 0$ such that z_μ blows up in finite time $t = T_2$ with $T_2 \le T_1$. Take $\lambda > 0$ so large that $\lambda u_0(r) \ge \mu z_0(r)$ for $0 \le r \le \rho(T_1)$. Then, by the comparison theorem, u_λ blows up at $t = T_m(\lambda) \le T_2$. This implies that $\lambda \in \Lambda_\infty$, and hence Λ_∞ is nonempty.

Let $\lambda \in \Lambda_\infty$. By the maximum principle, $u(r,t)$ is nonincreasing in $r = |x|$ for each fixed $0 \le t < T_m(\lambda)$. (See, e.g. [9].) Then u_λ blows up at $x = 0$. Furthermore, $u_\lambda > 0$ in $B_{\rho(T_m(\lambda))} \times (0, T_m(\lambda))$. Then, by applying (i) of Theorem 2 with $\rho(t) \equiv \rho(T_m(\lambda))$, this blowup is of type I. □

Lemma 5.4. Λ_∞ *is open.*

We will prove Lemma 5.4 following the proof of Proposition 5.2.

Proof of Proposition 5.2. Put $\lambda_0 = \sup \Lambda_0$ and $\lambda_\infty = \inf \Lambda_\infty$. By the comparison argument, we have $0 < \lambda_0 \le \lambda_\infty$. Take $\lambda^* \in [\lambda_0, \lambda_\infty]$. Since Λ_0 and Λ_∞ are open by Lemma 5.2 and Lemma 5.4, we have $\lambda^* \notin \Lambda_0$ and $\lambda^* \notin \Lambda_\infty$. This implies that $T_m(\lambda^*) = T$ and $\sup_{t\in[0,T)} \|u_{\lambda^*}(\cdot,t)\|_{L^\infty(B_{\rho(t)})} = \infty$, that is, u_{λ^*} blows up at $t = T$. □

For the solution u_λ of (1.8), we define w_λ by

$$w_\lambda(y,s) = (T-t)^{1/(p-1)} u_\lambda(x,t) \tag{5.15}$$

with (5.2). Then $w = w_\lambda$ satisfies (5.3). It is clear that u_λ blows up at $t = T_1 < T$, if and only if, w_λ blows up at $s = S_1$ with $S_1 = -\log(T - T_1) + \log T$. To prove Lemma 5.4, we need the following lemma.

Lemma 5.5. *Assume that* $\lambda \in \Lambda_\infty$ *and* u_λ *blows up at* $t = T_1 < T$. *Put* w_λ *by* (5.15) *with* (5.2). *Then*

$$\int_0^s \int_{B_R} \left(\frac{\partial w_\lambda}{\partial s}\right)^2 \sigma dy ds \to \infty \quad \text{as } s \to S_1 \tag{5.16}$$

with $S_1 = -\log(T - T_1) + \log T$.

Remark 5.2. In the subcritical case $1 < p < (N+2)/(N-2)$, a closely related result was shown by Giga [12].

Proof. We will show (5.16) by making use of the idea in [21]. From (5.2) and (5.15) we see that

$$\begin{aligned} &\int_0^s \int_{B_R} \left(\frac{\partial w_\lambda}{\partial s}\right)^2 \sigma dy ds \ge C_0 \int_0^s \int_{B_R} \left(\frac{\partial w_\lambda}{\partial s}\right)^2 dy ds \\ &= C_0 \int_0^t \int_{B_{\rho(\tau)}} \left(-\frac{u_\lambda}{(p-1)(T-\tau)} - \frac{x\cdot\nabla u_\lambda}{2(T-\tau)} + \frac{\partial u_\lambda}{\partial t}\right)^2 dx d\tau, \end{aligned} \tag{5.17}$$

where $C_0 = e^{-R^2/4} > 0$. From Lemma 5.3, u_λ blows up at $(x,t) = (0, T_1)$ with type I rate, and hence,

$$\sup_{t\in[0,T_1)} (T_1 - t)^{1/(p-1)} \|u_\lambda(\cdot,t)\|_{L^\infty(B_{\rho(t)})} < \infty.$$

Furthermore, by [15, Theorem 2.1] we obtain

$$\liminf_{t\to T_1} (T_1 - t)^{1/(p-1)} \|u_\lambda(\cdot,t)\|_{L^\infty(B_{\rho(t)})} > 0.$$

Define $\tilde{w}$ by

$$\tilde{w}(y,s) = (T_1 - t)^{1/(p-1)} u_\lambda(x,t) \tag{5.18}$$

with $y = x/(T_1 - t)^{1/2}$ and $s = -\log(T_1 - t)$. By [13, Corollary], we have

$$\tilde{w}(y,s) \to \kappa = (p-1)^{-1/(p-1)} \quad \text{as } s \to \infty \tag{5.19}$$

uniformly in $|y| \le C$ for each $C > 0$. We may assume here that $C = 1$. By the parabolic regularity, we obtain

$$\nabla \tilde{w}(y,s) \to 0 \quad \text{and} \quad \tilde{w}_s(y,s) \to 0 \quad \text{as } s \to \infty \tag{5.20}$$

uniformly in $|y| \le 1$. From (5.18) and (5.19) we obtain

$$(T_1 - t)^{1/(p-1)} u_\lambda(x,t) \to \kappa \quad \text{as } t \to T_1 \tag{5.21}$$

uniformly in $|x| \le (T_1 - t)^{1/2}$. We note here that

$$(T_1 - t)^{1/(p-1)} x \cdot \nabla_x u_\lambda(x,t) = y \cdot \nabla_y \tilde{w}$$

and

$$(T_1 - t)^{p/(p-1)} \frac{\partial u_\lambda}{\partial t} = \frac{1}{p-1}\tilde{w} + \frac{1}{2} y \cdot \nabla_y \tilde{w} + \frac{\partial \tilde{w}}{\partial s}.$$

Then it follows from (5.20) that, as $t \to T_1$,

$$(T_1 - t)^{1/(p-1)} x \cdot \nabla u_\lambda(x,t) \to 0 \quad \text{and} \quad (T_1 - t)^{p/(p-1)} \frac{\partial u_\lambda}{\partial t} \to \frac{1}{p-1}\kappa \tag{5.22}$$

uniformly in $|x| \le (T_1 - t)^{1/2}$. Then, from (5.21) and (5.22), there exist $t_0 < T_1$ and $C_1 > 0$ such that

$$\left(-\frac{u_\lambda}{(p-1)(T-t)} - \frac{x \cdot \nabla u_\lambda}{2(T-t)} + \frac{\partial u_\lambda}{\partial t} \right)^2 \ge C_1 (T_1 - t)^{-2p/(p-1)}$$

for $t_0 \le t < T_1$ and $|x| \le (T_1 - t)^{1/2}$. Take $\tilde{t}_0 \in [t_0, T_1)$ so that $\rho(t) > (T_1 - t)^{1/2}$ for $\tilde{t}_0 \le t \le T_1$. Then

$$\begin{aligned}
&\int_0^t \int_{B_{\rho(\tau)}} \left(-\frac{u_\lambda}{(p-1)(T-\tau)} - \frac{x \cdot \nabla u_\lambda}{2(T-\tau)} + \frac{\partial u_\lambda}{\partial t} \right)^2 dx d\tau \\
\ge\ & \int_{\tilde{t}_0}^t \int_{|x| \le (T_1 - \tau)^{1/2}} \left(-\frac{u_\lambda}{(p-1)(T-\tau)} - \frac{x \cdot \nabla u_\lambda}{2(T-\tau)} + \frac{\partial u_\lambda}{\partial t} \right)^2 dx d\tau \\
\ge\ & C \int_{\tilde{t}_0}^t (T_1 - \tau)^{N/2 - 2p/(p-1)} d\tau = C(-\log(T_1 - t) - \log(T_1 - \tilde{t}_0)) \to \infty
\end{aligned}$$

as $t \to T_1$. From (5.17) we obtain (5.16). □

Proof of Lemma 5.4. Assume that $\lambda_1 \in \Lambda_\infty$ and u_{λ_1} blows up at $t = T_1 < T$. Define w_{λ_1} by (5.15) with $\lambda = \lambda_1$, and put $S_1 = -\log(T - T_1) + \log T$. Integrating (5.4) on $[0, s]$ for $s < S_1$, we have

$$\mathcal{E}[w_{\lambda_1}](s) - \mathcal{E}[w_{\lambda_1}](0) = -\int_0^s \int_{B_R} \left(\frac{\partial w_{\lambda_1}}{\partial s}\right)^2 \sigma dyds.$$

From Lemma 5.5, we obtain $\mathcal{E}[w_{\lambda_1}](s) \to -\infty$ as $s \to S_1$. Then, there exists $s_2 < S_1$ such that $\mathcal{E}[w_{\lambda_1}](s_2) < 0$. Since $\lambda \mapsto w_\lambda(\cdot, s) \in C^2(B_R)$ is continuous for each $s > 0$, there exists $\varepsilon > 0$ such that if $|\lambda - \lambda_1| < \varepsilon$ then $\mathcal{E}[w_\lambda](s_2) < 0$. By (i) of Lemma 5.1, w_λ must blowup in finite time. This implies that $\lambda \in \Lambda_\infty$ for $\lambda \in (\lambda_1 - \varepsilon, \lambda_1 + \varepsilon)$, and hence Λ_∞ is open. □

Appendix.

Lemma A.1. *Let $v \in H_0^1(B_R)$ be a weak solution of* (1.4) *with $v \geq 0$, $v \not\equiv 0$ in B_R. Then $v \in C^2(B_R)$ and $v > 0$ in B_R.*

To prove Lemma A.1 we need the following estimate, which is essentially due to Brezis-Kato [3].

Lemma A.2. Let $u \in H_0^1(B_R)$ satisfy

$$\Delta u - \frac{1}{2} x \cdot \nabla u - \frac{1}{p-1} u + c(x) u = 0 \qquad \text{in } B_R \tag{A.1}$$

with $c \in L^{N/2}(B_R)$. Then $u \in L^q(B_R)$ for any $q < \infty$.

Proof. Define $u_L = \min\{|u|, L\}$, where L is a positive constant. For $s \geq 2$, put $\phi_L = u_L^{s-2} u \in H_0^1(B_R)$. Multiplying (A.1) by ϕ_L and integrating on B_R, we obtain

$$\int_{B_R} \left(\nabla u \cdot \nabla (u_L^{s-2} u) + \frac{1}{p-1} u^2 u_L^{s-2}\right) \sigma dx = \int_{B_R} c u^2 u_L^{s-2} \sigma dx, \tag{A.2}$$

where $\sigma(x) = e^{-|x|^2/4}$. We see that

$$\int_{B_R} \nabla u \cdot \nabla (u_L^{s-2} u) \sigma dx = \int_{B_R} |\nabla u|^2 u_L^{s-2} \sigma dx + (s-2) \int_{u<L} |\nabla u|^2 u^{s-2} \sigma dx. \tag{A.3}$$

Observe that

$$\int_{B_R} \left|\nabla (u u_L^{(s-2)/2})\right|^2 \sigma dx = \int_{B_R} \left| u_L^{(s-2)/2} \nabla u + \frac{s-2}{2} u u_L^{(s-4)/2} \nabla u_L \right|^2 \sigma dx$$
$$= \int_{B_R} |\nabla u|^2 u_L^{s-2} \sigma dx + \frac{(s-2)(s+2)}{4} \int_{u<L} |\nabla u|^2 u^{s-2} \sigma dx.$$

Then it follows from $s \geq 2$ that

$$\int_{B_R} \left|\nabla(u u_L^{(s-2)/2})\right|^2 \sigma dx \leq \frac{s+2}{4} \int_{B_R} \nabla u \cdot \nabla(u_L^{s-2} u)\sigma dx.$$

From (A.2) and the fact that $e^{-R^2/4} \leq \sigma \leq 1$ in B_R, we have

$$e^{-R^2/4} \int_{B_R} \left|\nabla(u u_L^{(s-2)/2})\right|^2 dx \leq \frac{s+2}{4} \int_{B_R} |c| u^2 u_L^{s-2} dx.$$

By the Sobolev inequality, it follows that

$$S\|u u_L^{(s-2)/2}\|^2_{L^{2N/(N-2)}(B_R)} \leq \|\nabla(u u_L^{(s-2)/2})\|^2_{L^2(B_R)},$$

where $S > 0$ is the constant in (2.3). We therefore obtain

$$\|u u_L^{(s-2)/2}\|^2_{L^{2N/(N-2)}(B_R)} \leq C_1 \int_{B_R} |c(x)| u^2 u_L^{s-2} dx \tag{A.4}$$

with $C_1 = (s+2)e^{R^2/4}/(4S) > 0$. From $c \in L^{N/2}(B_R)$ we can choose $K > 0$ such that

$$\left(\int_{|c(x)|>K} |c|^{N/2} dx\right)^{2/N} < \frac{1}{2C_1}. \tag{A.5}$$

Observe that

$$\int_{B_R} |c| u^2 u_L^{s-2} dx \leq \int_{|c(x)|\leq K} |c| u^2 u_L^{s-2} dx + \int_{|c(x)|>K} |c| u^2 u_L^{s-2} dx. \tag{A.6}$$

We now claim that if $u \in L^s(B_R)$ with $s \geq 2$ then $u \in L^{Ns/(N-2)}(B_R)$. In fact, if $u \in L^s(B_R)$ with $s \geq 2$, then we have

$$\int_{|c(x)|\leq K} |c| u^2 u_L^{s-2} dx \leq K \int_{B_R} u^s dx. \tag{A.7}$$

From the Hölder inequality and (A.5) we obtain

$$\begin{aligned} \int_{|c(x)|>K} |c| u^2 u_L^{s-2} dx &\leq \left(\int_{|c(x)|>K} c^{N/2} dx\right)^{2/N} \|u u_L^{(s-2)/2}\|^2_{L^{2N/(N-2)}(B_R)} \\ &\leq \frac{1}{2C_1} \|u u_L^{(s-2)/2}\|^2_{L^{2N/(N-2)}(B_R)}. \end{aligned} \tag{A.8}$$

From (A.4) and (A.6)-(A.8) we obtain

$$\frac{1}{2}\|u u_L^{(s-2)/2}\|^2_{L^{2N/(N-2)}(B_R)} \leq C_1 K \int_{B_R} u^s dx.$$

Letting $L \to \infty$, we obtain $u \in L^{Ns/(N-2)}(B_R)$.

Now, from $u \in H^1(B_R)$ we have $u \in L^{s_0}(B_R)$ with $s_0 = 2N/(N-2)$. By the claim, $u \in L^{s_1}(B_R)$ with $s_1 = 2N^2/(N-2)^2$. Repeating this argument, we obtain

$u \in L^{s_k}(B_R)$ with $s_k = 2N^k/(N-2)^k$ for any k. This implies that $u \in L^q(B_R)$ for any $q < \infty$. □

Proof of Lemma A.1. We see that a solution $v \in H_0^1(B_R)$ of (1.4) satisfies (A.1) with $c = v^{4/(N-2)} \in L^{N/2}(B_R)$. By Lemma A.1 we have $v \in L^q(B_R)$ for any $q < \infty$. By applying the L^q estimate and the Sobolev embedding theorem, we obtain $v \in C^{1+\theta}(\overline{B_R})$ with $0 < \theta < 1$. Hence, by the Schauder estimate, it follows that $v \in C^{2+\theta}(\overline{B_R})$. By the strong maximum principle [18, Theorem 3.5], we obtain $v > 0$ in B_R. □

Lemma A.3. *Let* w *be a solution of*

$$w'' + \frac{2}{r}w' + \tilde{a}(r)w + \tilde{b}(r)w^5 = 0 \quad \text{for } 0 < r < R \tag{A.9}$$

satisfying $w'(0) = w(R) = 0$, *where* $\tilde{a}, \tilde{b} \in C^1[0, R]$. *Assume that* $\psi \in C^3[0, R]$ *satisfies* $\psi(0) = 0$. *Then we have*

$$\begin{aligned} \frac{1}{2}R^2\psi(R)w'(R)^2 &= \frac{1}{4}\int_0^R r^2 w^2 \left(\psi''' + 2\tilde{a}'\psi + 4\tilde{a}\psi'\right) dr \\ &\quad + \frac{2}{3}\int_0^R rw^6 \left(\tilde{b}(r\psi' - \psi) + \frac{1}{4}r\tilde{b}'\psi\right) dr. \end{aligned} \tag{A.10}$$

Proof. Multiplying (A.9) by $r^2\psi w'$, and integrating by parts on $[0, R]$, we get the identity

$$\begin{aligned} &\frac{1}{2}R^2\psi(R)w'(R)^2 + \int_0^R r\psi(w')^2 dr - \frac{1}{2}\int_0^R r^2\psi'(w')^2 dr \\ &\quad - \frac{1}{2}\int_0^R (r^2\tilde{a}\psi)'w^2 dr - \frac{1}{6}\int_0^R (r^2\tilde{b}\psi)'w^6 dr = 0. \end{aligned} \tag{A.11}$$

Note that (A.9) can be written as

$$r^{-2}(r^2 w')' + \tilde{a}(r)w + \tilde{b}(r)w^5 = 0 \quad \text{for } 0 < r < R. \tag{A.12}$$

Multiplying (A.12) by $r^2\psi' w$ and $r\psi w$, and integrating by parts on $[0, R]$, respectively, we obtain

$$-\int_0^R r^2\psi'(w')^2 dr + \frac{1}{2}\int_0^R (r^2\psi'')'w^2 dr + \int_0^R r^2\tilde{a}\psi' w^2 dr + \int_0^R r^2\tilde{b}\psi' w^6 dr = 0 \tag{A.13}$$

and

$$-\int_0^R r\psi(w')^2 dr + \frac{1}{2}\int_0^R r\psi'' w^2 dr + \int_0^R r\tilde{a}\psi w^2 dr + \int_0^R r\tilde{b}\psi w^6 dr = 0. \tag{A.14}$$

To obtain the Pohozaev type identity, we form the combination

$$(\text{A.11}) - \frac{1}{2}(\text{A.13}) + (\text{A.14}).$$

A simple calculation yields (A.10). □

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REFERENCES

[1] S. Angenent, The zero set of a solution of a parabolic equation, J. Reine Angew. Math. **390** (1988) 79–96.

[2] F. V. Atkinson and L. A. Peletier, Sur les solutions radiales de l'equation $\Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{2}\lambda u + |u|^{p-1}u = 0$, C. R. Acad. Sci. Paris Ser. I Math. **302** (1986) 99–101.

[3] H. Brezis and T. Kato, Remarks on the Schrodinger operator with singular complex potentials, J. Math. Pures Appl. (9) 58 (1979), 137–151.

[4] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. **88** (1983), 486-490.

[5] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. **36** (1983), 437-477.

[6] X.-Y. Chen and P. Polačik, Asymptotic periodicity of positive solutions of reaction diffusion equations on a ball, J. Reine Angew. Math. 472 (1996) 17–51.

[7] M. Fila, Blow-up of solutions of supercritical parabolic equations, Evolutionary Equations 2, 105–158, Handb. Differ. Eq. North-Holland, Amstrerdam 2005.

[8] S. Filippas, M. A. Herrero, J. J. L. Velazquez, Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456 (2000), no. 2004, 2957–2982.

[9] A. Friedman and B. McLeod, Blow-up of positive solutions of semilinear heat equations, Indiana Univ. Math. J. 34 (1985) 425–447.

[10] V. A. Galaktionov and S. A. Posashkov, The equation $u_t = u_{xx} + u^\beta$. Localization, asymptotic behavior of unbounded solutions, Akad. Nauk SSSR Inst. Prikl. Mat. Preprint 1985, no. 97, 30 pp (in Russian).

[11] V. A. Galaktionov and S. A. Posashkov, Application of new comparison theorems to the investigation of unbounded solutions of nonlinear parabolic equations, Differentsial'nye Uravneniya 22 (1986), 1165–1173.

[12] Y. Giga, A bound for global solutions of semilinear heat equations, Comm. Math. Phys. 103 (1986), 415–421.

[13] Y. Giga and R. V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations, Comm. Pure Appl. Math. 38 (1985), 297–319.

[14] Y. Giga and R. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J. 36 (1987), 1–40.

[15] Y. Giga and R. Kohn, Nondegeneracy of blowup for semilinear heat equations, Comm. Pure Appl. Math. 42 (1989), 845–884.

[16] Y. Giga, S. Matsui, and S. Sasayama, Blow up rate for semilinear heat equations with subcritical nonlinearity, Indiana Univ. Math. J. 53 (2004) 483–514.

[17] Y. Giga, S. Matsui, and S. Sasayama, On blow-up rate for sign-changing solutions in a convex domain, Math. Methods Appl. Sci. 27 (2004), 1771–1782.

[18] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 2001.

[19] M. A. Herrero and J. J. L. Velázquez, Explosion de solutions d'equations paraboliques semilineaires supercritiques, C. R. Acad. Sci. Paris Sér. I Math. 319 (1994) 141–145.

[20] M. A. Herrero and J. J. L. Velázquez, A blow up result for semilinear heat equations in the supercritical case, preprint.

[21] R. Ikehata and T. Suzuki, Semilinear parabolic equations involving critical Sobolev exponent: local and asymptotic behavior of solutions, Differential Integral Equations **13** (2000) 869–901.

[22] D. D. Joseph and T. S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Rational Mech. Anal. 49 (1972/73) 241–269.

[23] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, Linear and quasi-linear equations of parabolic type, Translations of Mathematical Monographs, 23. American Mathematical Society, Providence, R.I., 1996.

[24] H. Matano, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29** (1982), 401–441.

[25] H. Matano and F. Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation, Comm. Pure Appl. Math. 57 (2004) 1494–1541.

[26] N. Mizoguchi, Type-II blowup for a semilinear heat equation, Adv. Differential Equations **9** (2004) 1279–1316.

[27] N. Mizoguchi, Blowup behavior of solutions for a semilinear heat equation with supercritical nonlinearity, J. Differential Equations **205** (2004) 298–328.

[28] C. E. Mueller and F. B. Weissler, Single point blow-up for a general semilinear heat equation, Indiana Univ. Math. J. **34** (1985) 881–913.

[29] W-M. Ni, A nonlinear Dirichlet problem on the unit ball and its applications, Indiana Univ. Math. J. **31** (1982) 801–807.

[30] X. Wang, On the Cauchy problem for reaction-diffusion equations, Trans. Amer. Math. Soc. **337** (1993) 549–590.