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# Self-similar Solutions to a Parabolic System Modeling Chemotaxis

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**Abstract.** We study the forward self-similar solutions to a parabolic system modeling chemotaxis

$$u_t = \nabla \cdot (\nabla u - u \nabla v), \quad \tau v_t = \Delta v + u$$

in the whole space  $\mathbb{R}^2$ , where  $\tau$  is a positive constant. Using the Liouville type result and the method of moving planes, it is proved that self-similar solutions  $(u, v)$  must be radially symmetric about the origin. Then the structure of the set of self-similar solutions is investigated. As a consequence, it is shown that there exists a threshold in  $\int_{\mathbb{R}^2} u$  for the existence of self-similar solutions. In particular, for  $0 < \tau \leq 1/2$ , there exists a self-similar solution  $(u, v)$  if and only if  $\int_{\mathbb{R}^2} u < 8\pi$ .

**Key Words:** self-similar solution; parabolic system; chemotaxis; radial symmetry; blow-up analysis.

## 1. Introduction

We are concerned with the parabolic system of the form

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u \nabla v) \\ \tau \frac{\partial v}{\partial t} = \Delta v + u \end{cases} \quad (1.1)$$

for  $x \in \mathbb{R}^N$  and  $t > 0$ , where  $\tau > 0$  is a constant. This is a simplified system of the one given by Keller and Segel [16] describing chemotactic feature of cellular slime molds sensitive to the gradient of a chemical substance secreted by themselves. The functions  $u(x, t) \geq 0$  and  $v(x, t) \geq 0$  denote the cell density of cellular slime molds and the concentration of the chemical substance at the place  $x$  and the time  $t$ , respectively.

Backward self-similar solutions are studied in [12] for  $\tau = 0$ . The present paper is devoted to the forward self-similar solutions. Namely, this system is invariant under the similarity transformation

$$u_\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^2 t) \quad \text{and} \quad v_\lambda(x, t) = v(\lambda x, \lambda^2 t)$$

for  $\lambda > 0$ , that is, if  $(u, v)$  is a solution of (1.1) globally in time, then so is  $(u_\lambda, v_\lambda)$ . A solution  $(u, v)$  is said to be *self-similar*, when the solution is invariant under this transformation, that is,  $u(x, t) = u_\lambda(x, t)$  and  $v(x, t) = v_\lambda(x, t)$  for all  $\lambda > 0$ . Letting  $\lambda = 1/\sqrt{t}$ , we see that  $(u, v)$  has the form

$$u(x, t) = \frac{1}{t} \phi\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad v(x, t) = \psi\left(\frac{x}{\sqrt{t}}\right) \quad (1.2)$$

for  $x \in \mathbb{R}^N$  and  $t > 0$ . It follows that

$$\int_{\mathbb{R}^N} u(x, t) dx = t^{(N-2)/2} \int_{\mathbb{R}^N} \phi(y) dy \quad (1.3)$$

for  $\phi \in L^1(\mathbb{R}^N)$ . Therefore, self-similar solution  $(u, v)$  preserves the mass  $\|u(\cdot, t)\|_{L^1(\mathbb{R}^2)}$  if and only if  $N = 2$ . On the other hand, the mass conservation of  $u(\cdot, t)$  follows formally in the original system (1.1) in any space dimensions. Regarding this fact, we study the case  $N = 2$  in this paper.

By a direct computation it is shown that  $(u, v)$  in (1.2) satisfies (1.1) if and only if  $(\phi, \psi)$  satisfies

$$\begin{cases} \nabla \cdot (\nabla \phi - \phi \nabla \psi) + \frac{1}{2} x \cdot \nabla \phi + \phi = 0, & x \in \mathbb{R}^2, \\ \Delta \psi + \frac{\tau}{2} x \cdot \nabla \psi + \phi = 0, & x \in \mathbb{R}^2. \end{cases} \quad (1.4)$$

We are concerned with the classical solutions  $(\phi, \psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$  of (1.4) satisfying

$$\phi, \psi \geq 0 \quad \text{in } \mathbb{R}^2 \quad \text{and} \quad \phi(x), \psi(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (1.5)$$

Define the solution set  $\mathcal{S}$  of (1.4) as

$$\mathcal{S} = \{(\phi, \psi) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2) : (\phi, \psi) \text{ is a solution of (1.4) with (1.5)}\}. \quad (1.6)$$

The existence of radial solutions  $(\phi, \psi) \in \mathcal{S}$  has been known by [20, Theorem 1] and [22, Theorem 1.1]. We investigate the structure of the solution set  $\mathcal{S}$ .

**Theorem 1.** *Any  $(\phi, \psi) \in \mathcal{S}$  is radially symmetric about the origin, and satisfies  $\phi, \psi \in L^1(\mathbb{R}^2)$ .*

**Theorem 2.** *The solution set  $\mathcal{S}$  is expressed as a one parameter family:*

$$\mathcal{S} = \{(\phi(s), \psi(s)) : s \in \mathbb{R}\}.$$

*If  $\lambda(s) = \|\phi(s)\|_{L^1(\mathbb{R}^2)}$ , then  $(\phi(s), \psi(s))$  and  $\lambda(s)$  satisfy the following properties:*

- (i)  $s \mapsto (\phi(s), \psi(s)) \in C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$  and  $s \mapsto \lambda(s) \in \mathbb{R}$  are continuous;
- (ii)  $(\phi(s), \psi(s)) \rightarrow (0, 0)$  in  $C^2(\mathbb{R}^2) \times C^2(\mathbb{R}^2)$  and  $\lambda(s) \rightarrow 0$  as  $s \rightarrow -\infty$ ;
- (iii)  $\|\psi(s)\|_{L^\infty(\mathbb{R}^2)} \rightarrow \infty$ ,

$$\lambda(s) \rightarrow 8\pi, \quad \text{and} \quad \phi(s)dx \rightarrow 8\pi\delta_0(dx) \quad \text{in the sense of measure as } s \rightarrow \infty,$$

*where  $\delta_0(dx)$  denotes Dirac's delta function with the support in origin;*

- (iv)  $0 < \lambda(s) < 8\pi$  for  $s \in \mathbb{R}$ , if  $0 < \tau \leq 1/2$ , and  $0 < \lambda(s) \leq \max\{4\pi^3/3, 4\pi^3\tau^2/3\}$  for  $s \in \mathbb{R}$ , if  $\tau > 1/2$ .

As a consequence of Theorem 2 we obtain the following:

**Corollary.** *There exists a constant  $\lambda^*$  satisfying  $\lambda^* = 8\pi$ , if  $0 < \tau \leq 1/2$ , and  $8\pi \leq \lambda^* \leq \max\{4\pi^3/3, 4\pi^3\tau^2/3\}$ , if  $\tau > 1/2$ , such that*

- (i) *for every  $\lambda \in (0, \lambda^*)$ , there exists a solution  $(\phi, \psi) \in \mathcal{S}$  satisfying  $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$ ;*
- (ii) *for  $\lambda > \lambda^*$ , there exists no solution  $(\phi, \psi) \in \mathcal{S}$  satisfying  $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$ .*

**Remark.** Biler [1] has shown that the system (1.4) with  $\tau = 1$  has a radial solution  $(\phi, \psi)$  satisfying  $\|\phi\|_{L^1(\mathbb{R}^2)} = \lambda$  for every  $\lambda \in (0, 8\pi)$ , and has no radial solutions  $(\phi, \psi)$  satisfying  $\|\phi\|_{L^1(\mathbb{R}^2)}/2\pi \geq 7.82 \dots$

Theorem 1 is a consequence of the following:

**Theorem 3.** *Assume that  $(\phi, \psi)$  is a nonnegative solution of (1.4) satisfying  $\phi, \psi \in L^\infty(\mathbb{R}^2)$ . Then  $\phi$  and  $\psi$  are positive, and there exists a constant  $\sigma > 0$  such that*

$$\phi(x) = \sigma e^{-|x|^2/4} e^{\psi(x)}. \tag{1.7}$$

*Assume furthermore that  $\psi(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then  $\phi$  and  $\psi$  are radially symmetric about the origin, and satisfy  $\partial\phi/\partial r < 0$  and  $\partial\psi/\partial r < 0$  for  $r = |x| > 0$ , and*

$$\phi(x) = O(e^{-|x|^2/4}) \quad \text{and} \quad \psi(x) = O(e^{-\min\{\tau, 1\}|x|^2/4}) \quad \text{as } |x| \rightarrow \infty.$$

The proof of Theorem 3 consists of two steps. First we show that (1.7) holds by employing the Liouville type result essentially due to Meyers and Serrin [19]. Then we show the radial symmetry of solutions by the method of moving planes. This device was first developed by Serrin [28] in PDE theory, and later extended and generalized by Gidas, Ni, and Nirenberg [7, 8]. We will obtain a symmetry result for the Eq. (1.8) below with a change of variables as in [23].

By Theorem 3 it follows that under the condition  $\phi, \psi \in L^\infty(\mathbb{R}^2)$  the system (1.4) is reduced to the equation

$$\Delta\psi + \frac{\tau}{2}x \cdot \nabla\psi + \sigma e^{-|x|^2/4}e^\psi = 0 \quad \text{in } \mathbb{R}^2 \quad (1.8)$$

for some positive constant  $\sigma$ . Moreover,  $(\phi, \psi) \in \mathcal{S}$  if and only if  $\psi$  satisfies (1.8) with

$$\psi(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \quad (1.9)$$

and  $\phi$  is given by (1.7). Let  $\lambda = \|\phi\|_{L^1(\mathbb{R}^2)}$ . From (1.7) we see that

$$\lambda = \sigma \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(y)} dy.$$

Then (1.8) is rewritten as the elliptic equation with nonlocal term,

$$\Delta\psi + \frac{\tau}{2}x \cdot \nabla\psi + \lambda e^{-|x|^2/4}e^\psi \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4}e^{\psi(y)} dy = 0 \quad \text{in } \mathbb{R}^2. \quad (1.10)$$

The proof of Theorem 2 is based on the ODE arguments to Eqs. (1.8) and (1.10). Furthermore, we employ the results by Brezis and Merle [2] concerning the asymptotic behavior of sequences of solutions of

$$-\Delta u_k = V_k(x)e^{u_k} \quad \text{in } \Omega, \quad (1.11)$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain and  $V_k$  is a nonnegative continuous functions. We also need Theorem 4 below in order to prove Theorem 2. Here we recall Theorem 3 in [2].

**Theorem A [2].** *Suppose that*

$$0 \leq V_k(x) \leq C_0, \quad x \in \Omega, \quad (1.12)$$

*for some positive constant  $C_0$ . Let  $\{u_k\}$  be a sequence of solutions of (1.11) satisfying*

$$\limsup_{k \rightarrow \infty} \int_{\Omega} e^{u_k} dx < \infty. \quad (1.13)$$

Then there exists a subsequence (still denoted by  $\{u_k\}$ ) satisfying one of the following alternatives:

- (i)  $\{u_k\}$  is bounded in  $L^\infty_{\text{loc}}(\Omega)$ ;
- (ii)  $u_k \rightarrow -\infty$  uniformly on compact subset of  $\Omega$ ;
- (iii) there exists a finite blow-up set  $\mathcal{B} = \{a_1, \dots, a_\ell\} \subset \Omega$  such that, for any  $1 \leq i \leq \ell$ , there exists  $\{x_k\} \subset \Omega$ ,  $x_k \rightarrow a_i$ ,  $v_k(x_k) \rightarrow \infty$ , and  $v_k \rightarrow -\infty$  uniformly on compact subsets of  $\Omega \setminus \mathcal{B}$ . Moreover,  $V_k e^{u_k} dx \rightharpoonup \sum_{i=1}^\ell \alpha_i \delta_{a_i}(dx)$  in the sense of measure with  $\alpha_i \geq 4\pi$ , where  $\delta_{a_i}(dx)$  is Dirac's delta function with the support in  $x = a_i$ .

It was conjectured in [2] that each  $\alpha_i$  can be written as  $\alpha_i = 8\pi m_i$  for some positive integer  $m_i$ . This was established by Li and Shafrir in [18]. Chen has shown in [3] that any positive integer  $m_i$  can occur in the case  $V \equiv 1$  and  $\Omega$  is a unit disc. On the other hand, under more restrictive assumption that  $V_k \in C^1(\Omega)$  we obtain the following theorem. It is related to Theorem 0.3 of Li [17] and is proven in the appendix of the present paper.

**Theorem 4.** *Suppose that  $V_k \in C^1(\Omega)$  satisfies (1.12) and*

$$\|\nabla V_k\|_{L^\infty(\Omega)} \leq C_1 \quad (1.14)$$

for some positive constants  $C_0$  and  $C_1$ . Let  $\{u_k\}$  be a sequence of solutions of (1.11) satisfying (1.13) and

$$\max_{\partial\Omega} u_k - \min_{\partial\Omega} u_k \leq C_2 \quad (1.15)$$

for some positive constant  $C_2$ . Assume that the alternative (iii) in Theorem A holds. Then  $\alpha_i = 8\pi$  for each  $i \in \{1, 2, \dots, \ell\}$ .

Recently, attentions have been paid to blowup problems for the system

$$\begin{cases} \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u \nabla v), & x \in \Omega, \ t > 0, \\ \tau \frac{\partial v}{\partial t} = \Delta v - \gamma v + u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0, \quad v(x, 0) = v_0, & x \in \Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^2$  is a bounded domain with smooth boundary  $\partial\Omega$ ,  $\tau$  and  $\gamma$  are positive constants, and  $\nu$  is the outer normal unit vector. Childress and Percus [5] and Childress [4] have studied the stationary problem and have conjectured that there exists a threshold

in  $\|u_0\|_{L^1(\Omega)}$  for the blowup of the solution  $(u, v)$ . Their arguments were heuristic, while recent studies are supporting their validity rigorously, see, [11], [13], [24], [26], and [27].

On the other hand, it is asserted that self-similar solutions take an important role in the asymptotic behavior of the solution to the Cauchy problem for the semilinear parabolic equation, see, e.g., [6], [14], and [15]. From Corollary, we are led to the following conjectures for the problem (1.1) subject to the initial condition  $u(x, 0) = u_0$  and  $v(x, 0) = v_0$  in  $\mathbb{R}^2$ .

*For  $0 < \tau \leq 1/2$ , if  $\|u_0\|_{L^1(\mathbb{R}^2)} < 8\pi$  then the solution of the Cauchy problem to (1.1) exists globally in time, and if  $\|u_0\|_{L^1(\Omega)} > 8\pi$  then the solution can blowup in a finite time.*

We organize this paper as follows. In Section 2 we show that (1.7) holds by employing the Liouville type result. In Section 3 we show the radial symmetry of solutions by the method of moving planes, and then give the proof of Theorem 3. In Section 4 we give the ODE arguments to investigate the properties of radial solutions of (1.8). We study the behavior of sequences  $\{(\phi_k, \psi_k)\} \subset \mathcal{S}$  satisfying  $\|\psi_k\|_{L^\infty(\mathbb{R}^2)} \rightarrow \infty$  in Section 5. In Section 6 we investigate the upper bounds of  $\|\phi\|_{L^1(\mathbb{R}^2)}$ . Finally, in Section 7 we prove Theorems 2 by using of the results in Sections 4-6. In the appendixes, we are concerned with the existence of solutions to the problem (1.8) and (1.9), and give the proof of Theorem 4.

## 2. Reduction to the single equation

In this section we show that the system (1.4) is reduced to Eq. (1.8) if  $\phi, \psi \in L^\infty(\mathbb{R}^2)$ . More precisely, we have the following:

**Proposition 2.1.** *Let  $(\phi, \psi)$  be a nonnegative solution of (1.4) with  $\phi, \psi \in L^\infty(\mathbb{R}^2)$ . Then the relation (1.7) holds with some constant  $\sigma > 0$ .*

To prove this proposition we use the Liouville type result for second order elliptic inequalities essentially due to Meyers and Serrin [19].

**Lemma 2.1.** *Let  $u$  satisfy*

$$\Delta u + \nabla b \cdot \nabla u \geq 0 \quad \text{in } \mathbb{R}^2. \quad (2.1)$$

*Assume that  $x \cdot \nabla b(x) \leq 0$  for large  $|x|$ . If  $\sup_{x \in \mathbb{R}^2} u(x) < \infty$  then  $u$  must be a constant function.*

*Proof.* Take a function  $\mu$  as  $\mu(r) = 1/\log(1+r)$ . Then  $\mu$  satisfies the Meyers-Serrin condition

$$\int_1^\infty \frac{k(t)}{t} dt = \infty, \quad \text{where } k(t) = \exp\left(-\int_1^t \frac{\mu(s)}{s} ds\right).$$

Define  $v$  as

$$v(r) = \int_1^r \frac{k(t)}{t} dt, \quad r \geq 1.$$

Then  $v(r)$  is positive and increasing for  $r \in (1, \infty)$ , and satisfies  $v(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . Furthermore  $v = v(|x|)$  solves

$$\Delta v + \nabla b \cdot \nabla v = \frac{k(|x|)}{|x|^2} (-\mu(|x|) + x \cdot \nabla b(x)).$$

By the assumption, there exists a large  $R > 0$  such that

$$\Delta v + \nabla b \cdot \nabla v < 0 \quad \text{for } |x| \geq R. \quad (2.2)$$

Now assume to the contrary that  $u$  is not a constant function. Without loss of generality we may assume that  $u$  is not a constant function in  $|x| \leq R$ . Define

$$U(r) = \sup\{u(x) : |x| = r\}.$$

Then  $U(r)$  is strictly increasing for  $r \geq R$ . To see why, suppose  $R \leq r_1 < r_2$  and  $U(r_1) \geq U(r_2)$ . Then  $u$  attains its maximum for  $|x| \leq r_2$  at an interior point and by the strong maximum principle  $u$  is constant, which contradicts the assumption. Therefore  $U(r)$  is strictly increasing, and we have  $U(R+1) > U(R)$ . Choose  $\delta > 0$  so small that

$$0 < \delta < \frac{U(R+1) - U(R)}{v(R+1) - v(R)}. \quad (2.3)$$

Put  $w(x) = u(x) - \delta v(|x|)$ . Then it follows from (2.1) and (2.2) that

$$\Delta w + \nabla b \cdot \nabla w > 0 \quad \text{for } |x| \geq R. \quad (2.4)$$

From (2.3) we obtain  $U(R+1) - \delta v(R+1) > U(R) - \delta v(R)$ . This implies

$$\sup_{|x|=R+1} w(x) > \sup_{|x|=R} w(x).$$

Since  $w(x) \rightarrow -\infty$  as  $|x| \rightarrow \infty$ ,  $w$  has the maximum at a point  $x_0 \in \mathbb{R}^2$ ,  $|x_0| > R$ . Then we have  $\Delta w + \nabla b \cdot \nabla w \leq 0$  at  $x = x_0$ . This contradicts (2.4). Hence,  $u$  must be a constant function.  $\square$

**Lemma 2.2.** *Let  $(\phi, \psi)$  be a nonnegative solution of (1.4) with  $\phi, \psi \in L^\infty(\mathbb{R}^2)$ . Then  $\nabla\psi \in L^\infty(\mathbb{R}^2)$ .*

*Proof.* Define  $u$  and  $v$  by (1.2), respectively. Then  $(u, v)$  solves (1.1), and it holds that

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} = \frac{1}{t}\|\phi\|_{L^\infty(\mathbb{R}^2)} \quad \text{and} \quad \|v(t)\|_{L^\infty(\mathbb{R}^2)} = \|\psi\|_{L^\infty(\mathbb{R}^2)}.$$

Take  $t_0 > 0$ . From the second equation of (1.1) we have

$$v(t) = e^{((t-t_0)/\tau)\Delta}v(t_0) + \frac{1}{\tau} \int_{t_0}^t e^{((t-s)/\tau)\Delta}u(s)ds \equiv v_1(t) + v_2(t), \quad t > t_0,$$

where  $\{e^{t\Delta}\}$  is the heat semigroup. We recall the  $L^p$ - $L^q$  estimates for the linear heat equation,

$$\|\nabla e^{(t/\tau)\Delta}w\|_{L^q(\mathbb{R}^2)} \leq Ct^{1/q-1/p-1/2}\|w\|_{L^p(\mathbb{R}^2)} \quad (2.5)$$

for  $t > 0$  with  $1 \leq p \leq q \leq \infty$ , where  $C = C(\tau)$  is a positive constant. See, e.g., [10]. In particular we have

$$\|\nabla e^{(t/\tau)\Delta}w\|_{L^\infty(\mathbb{R}^2)} \leq Ct^{-1/2}\|w\|_{L^\infty(\mathbb{R}^2)} \quad \text{for } t > 0.$$

Then it follows that

$$\|\nabla v_1(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(t-t_0)^{-1/2}\|v(t_0)\|_{L^\infty(\mathbb{R}^2)} \leq C(t-t_0)^{-1/2}\|\psi\|_{L^\infty(\mathbb{R}^2)}$$

and

$$\|\nabla v_2(t)\|_{L^\infty(\mathbb{R}^2)} \leq C \int_{t_0}^t (t-s)^{-1/2}\|u(s)\|_{L^\infty}ds \leq C\|\phi\|_{L^\infty} \int_{t_0}^t (t-s)^{-1/2}s^{-1}ds$$

for  $t > t_0$ . Consequently, we obtain  $\|\nabla v(t)\|_{L^\infty(\mathbb{R}^2)} < \infty$  for each  $t > t_0$ . By the definition of  $v$  it follows that  $\|\nabla v(t)\|_{L^\infty(\mathbb{R}^2)} = t^{-1/2}\|\nabla\psi\|_{L^\infty(\mathbb{R}^2)}$ . Thus we have  $\nabla\psi \in L^\infty(\mathbb{R}^2)$ .  $\square$

*Proof of Proposition 2.1.* Put  $w(x) = -\phi(x)e^{|x|^2/4}e^{-\psi(x)} \leq 0$ . Then  $e^{-|x|^2/4}e^\psi \nabla w = -\nabla\phi - x\phi/2 + \phi\nabla\psi$ . From the first equation of (1.4) we have

$$\nabla \cdot (e^{-|x|^2/4}e^\psi \nabla w) = 0, \quad \text{or} \quad \Delta w + \nabla b \cdot \nabla w = 0 \quad \text{in } \mathbb{R}^2,$$

where  $\nabla b(x) = -x/2 + \nabla\psi(x)$ . From Lemma 2.2 we have

$$x \cdot \nabla b(x) = \left( -\frac{|x|^2}{2} + x \cdot \nabla\psi(x) \right) \leq 0$$

for large  $|x|$ . As a consequence of Lemma 2.1,  $w$  must be a constant function. This completes the proof of Proposition 2.1.  $\square$

### 3. Radial symmetry: Proof of Theorem 3

In this section we investigate the radial symmetry of solutions to (1.8) and prove Theorem 3. Namely, we show the following:

**Proposition 3.1.** *Let  $\psi \in C^2(\mathbb{R}^2)$  be a positive solution of (1.8) with (1.9). Then  $\psi$  must be radially symmetric about the origin.*

We prepare several lemmas.

**Lemma 3.1.** *We have*

$$\psi(x) \leq Ce^{-\min\{\tau, 1\}|x|^2/4} \quad \text{for } x \in \mathbb{R}^2 \quad (3.1)$$

with some constant  $C > 0$ .

*Proof.* Define

$$Lu = -\Delta u - \frac{\tau}{2}x \cdot \nabla u$$

and put  $\kappa_\tau = \min\{1, \tau\}$ . Let  $C$  be a positive constant and let  $v(x) = Ce^{-\kappa_\tau|x|^2/4}$ . Then

$$Lv = C\kappa_\tau \left(1 + \frac{(\tau - \kappa_\tau)}{4}|x|^2\right) e^{-\kappa_\tau|x|^2/4} \geq C\kappa_\tau e^{-\kappa_\tau|x|^2/4}.$$

Since  $L\psi = \sigma e^{-|x|^2/4}e^\psi$ , if we choose  $C$  so large that  $C\kappa_\tau > \sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}}$ , then  $Lv > L\psi$  in  $\mathbb{R}^2$ . Since  $v, \psi \rightarrow 0$  as  $|x| \rightarrow \infty$ , by the maximum principle we have  $v \geq \psi$  in  $\mathbb{R}^2$ . This implies (3.1).  $\square$

We define  $w(x, t)$  by

$$w(x, t) = t^{-\alpha} \psi\left(\frac{x}{\sqrt{t}}\right), \quad \text{where } \alpha = \frac{\sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}}}{\tau}. \quad (3.2)$$

**Lemma 3.2** (i) *For every  $T > 0$  we have  $\sup_{0 < t < T} w(x, t) \rightarrow 0$  as  $|x| \rightarrow \infty$ .*

(ii) *For every  $\mu > 0$  we have  $\sup_{|x| > \mu} w(x, t) \rightarrow 0$  as  $t \rightarrow 0$ .*

*Proof.* From Lemma 3.1 we have  $|y|^{2\alpha}\psi(y) \rightarrow 0$  as  $|y| \rightarrow \infty$ , that is, for all  $\varepsilon > 0$  there exists  $R > 0$  such that

$$|y|^{2\alpha}\psi(y) < \varepsilon \quad \text{for } |y| \geq R. \quad (3.3)$$

From (3.2) we have

$$|x|^{2\alpha}w(x, t) = \left(\frac{|x|}{\sqrt{t}}\right)^{2\alpha} \psi\left(\frac{x}{\sqrt{t}}\right). \quad (3.4)$$

(i) Fix  $T > 0$ . From (3.3) and (3.4) it follows that

$$\sup_{0 < t < T} |x|^{2\alpha}w(x, t) < \varepsilon \quad \text{for } |x| \geq R\sqrt{T}.$$

Since  $\varepsilon > 0$  is arbitrary, we obtain  $\sup_{0 < t < T} w(x, t) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

(ii) From (3.3) and (3.4) it follows that

$$\mu^{2\alpha} \sup_{|x| > \mu} w(x, t) \leq \sup_{|x| > \mu} |x|^{2\alpha}w(x, t) < \varepsilon \quad \text{for } 0 < t < (\mu/R)^2.$$

Then we have  $\sup_{|x| > \mu} w(x, t) \rightarrow 0$  as  $t \rightarrow 0$ . □

For  $\mu \in \mathbb{R}$  we define  $T_\mu$  and  $\Sigma_\mu$  by

$$T_\mu = \{x = (x_1, x_2) \in \mathbb{R}^2 | x_1 = \mu\} \quad \text{and} \quad \Sigma_\mu = \{x \in \mathbb{R}^2 | x_1 < \mu\},$$

respectively. For  $x \in \mathbb{R}^2$  and  $\mu \in \mathbb{R}$  let  $x^\mu$  be the reflection of  $x$  with respect to  $T_\mu$ , that is,  $x^\mu = (2\mu - x_1, x_2)$ . It is easy to see that if  $\mu > 0$ ,

$$|x^\mu| > |x| \quad \text{for } x \in \Sigma_\mu \quad \text{and} \quad \{x^\mu : x \in \Sigma_\mu\} = \{x : x_1 > \mu\} \subset \{x : |x| \geq \mu\}.$$

By Lemma 3.2 we have the following:

**Lemma 3.3.** (i) For every  $T > 0$  we have  $\sup_{0 < t < T} w(x^\mu, t) \rightarrow 0$  as  $|x| \rightarrow \infty$ ,  $x \in \Sigma_\mu$ .

(ii) For every  $\mu > 0$  we have  $\sup_{x \in \Sigma_\mu} w(x^\mu, t) = 0$  as  $t \rightarrow 0$ .

**Lemma 3.4.** Let  $\mu > 0$ . Define  $z(x, t) = w(x, t) - w(x^\mu, t)$ . Then

$$\tau z_t \geq \Delta z + c_\mu(x, t)z \quad \text{in } \Sigma_\mu \times (0, \infty) \quad \text{and} \quad z = 0 \quad \text{on } T_\mu \times (0, \infty), \quad (3.5)$$

where

$$c_\mu(x, t) = \frac{1}{t} \left( -\alpha\tau + \sigma e^{-|x|^2/(4t)} \int_0^1 e^{s\psi(x/\sqrt{t}) + (1-s)\psi(x^\mu/\sqrt{t})} ds \right). \quad (3.6)$$

We have  $c_\mu(x, t) \leq 0$  in  $\mathbb{R}^2 \times (0, \infty)$ .

*Proof.* By virtue of (3.2) we have

$$\tau w_t = \Delta w - \frac{\alpha\tau}{t}w + \sigma t^{-\alpha-1}e^{-|x|^2/4t}e^{t^\alpha w}.$$

Let  $w^\mu(x, t) = w(x^\mu, t)$ . Then  $w^\mu$  satisfies

$$\tau w_t^\mu = \Delta w^\mu - \frac{\alpha\tau}{t}w^\mu + \sigma t^{-\alpha-1}e^{-|x^\mu|^2/4t}e^{t^\alpha w^\mu}.$$

Since  $|x^\mu| \geq |x|$ , we obtain

$$\tau w_t^\mu \leq \Delta w^\mu - \frac{\alpha\tau}{t}w^\mu + \sigma t^{-\alpha-1}e^{-|x|^2/4t}e^{t^\alpha w^\mu}.$$

Then we obtain  $\tau z_t \geq \Delta z + c_\mu z$ , where  $c_\mu$  is the function in (3.6). Since  $\alpha$  satisfies  $\alpha\tau = \sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}}$ , we have  $tc_\mu(x, t) \leq -\alpha\tau + \sigma e^{\|\psi\|_{L^\infty(\mathbb{R}^2)}} = 0$  for  $(x, t) \in \mathbb{R}^2 \times (0, \infty)$ .  $\square$

**Lemma 3.5.** *Let  $\mu > 0$ . We have  $w(x, t) \geq w(x^\mu, t)$  for  $(x, t) \in \Sigma_\mu \times (0, \infty)$ .*

*Proof.* Let  $z(x, t) = w(x, t) - w(x^\mu, t)$ . We show that  $z(x, t) \geq 0$  for  $(x, t) \in \Sigma_\mu \times (0, \infty)$ . Assume to the contrary that there exists a  $(x_0, t_0) \in \Sigma_\mu \times (0, \infty)$  such that  $z(x_0, t_0) < 0$ . Take  $\delta > 0$  so small that  $z(x_0, t_0) < -\delta$ . By (ii) of Lemma 3.3 we can take  $T_0 \in (0, t_0)$  so that  $w(x^\mu, T_0) < \delta$  for  $x \in \Sigma_\mu$ . Then it follows from  $w(x, t) > 0$  that

$$z(x, T_0) \geq -\delta \quad \text{for } x \in \Sigma_\mu. \quad (3.7)$$

Fix  $T > t_0$ . By (i) of Lemma 3.3 we can take  $R > |x_0|$  so large that  $w(x^\mu, t) < \delta$  for  $|x| \geq R$ ,  $x \in \Sigma_\mu$ ,  $t \in [T_0, T]$ . Then we obtain

$$z(x, t) \geq -\delta \quad \text{for } x \in \Sigma_\mu, |x| \geq R, t \in [T_0, T]. \quad (3.8)$$

Define  $Q = \{x \in \Sigma_\mu : |x| < R\}$ . Let  $\Gamma$  be a parabolic boundary of  $Q \times (T_0, T)$ , that is,

$$\Gamma = (Q \times \{T_0\}) \cup (\partial Q \times (T_0, T)).$$

From (3.5), (3.7), and (3.8) we have

$$\tau z_t \geq \Delta z + c(x, t)z \quad \text{in } Q \times (T_0, T) \quad \text{and} \quad z \geq -\delta \quad \text{on } \Gamma.$$

Put  $Z = z + \delta$ . Because  $c_\mu(x, t) \leq 0$ , it follows from the above inequality that

$$\tau Z_t \geq \Delta Z + c_\mu(x, t)Z \quad \text{in } Q \times (T_0, T) \quad \text{and} \quad Z \geq 0 \quad \text{on } \Gamma.$$

By the maximum principle [25] we have  $Z \geq 0$  on  $\overline{Q} \times [T_0, T]$ , which implies that

$$z(x, t) \geq -\delta \quad \text{on } \overline{Q} \times [T_0, T]. \quad (3.9)$$

On the other hand  $(x_0, t_0) \in Q \times (T_0, T)$  and  $z(x_0, t_0) < -\delta$ . This contradicts to (3.9). Hence  $z(x, t) \geq 0$  for  $(x, t) \in \Sigma_\mu \times (0, \infty)$ .  $\square$

*Proof of Proposition 3.1.* From Lemma 3.5 we have  $w(x, t) \geq w(x^\mu, t)$  for  $\mu > 0$  and  $(x, t) \in \Sigma_\mu \times (0, \infty)$ . From the continuity of  $w$  we have  $w(x, t) \geq w(x^0, t)$  for  $(x, t) \in \Sigma_0 \times (0, \infty)$ . We can repeat the previous arguments for the negative  $x_1$ -direction to conclude that  $w(x, t) \leq w(x^0, t)$  for  $(x, t) \in \Sigma_0 \times (0, \infty)$ . Hence  $w(x, t)$  is symmetric with respect to the plane  $x_1 = 0$ , which implies that  $\psi$  is symmetric with respect to the plane  $x_1 = 0$ . Since the equation (1.8) is invariant under the rotation, it follows that  $\psi$  is symmetric in every direction. Therefore  $\psi$  is radially symmetric with respect to the origin.  $\square$

*Proof of Theorem 3.* Let  $(\phi, \psi)$  be a nonnegative solution of (1.4) with  $\phi, \psi \in L^\infty(\mathbb{R}^2)$ . Then  $\phi$  is given by (1.7) for some constant  $\sigma > 0$  from Proposition 2.1. It follows that  $\phi > 0$  in  $\mathbb{R}^2$ , and  $\phi(x) = O(e^{-|x|^2/4})$  as  $|x| \rightarrow \infty$ . From the second equation of (1.4),  $\psi$  satisfies the equation (1.8). By the strong maximum principle,  $\psi > 0$  in  $\mathbb{R}^2$ .

Assume furthermore that  $\psi(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Then, by Proposition 3.1,  $\psi$  must be radially symmetric about the origin. Hence  $\psi = \psi(r)$ ,  $r = |x|$ , satisfies the ordinary differential equation

$$\psi_{rr} + \left(\frac{1}{r} + \frac{\tau}{2}r\right)\psi_r + \sigma e^{-r^2/4}e^\psi = 0, \quad \text{or} \quad (re^{\tau r^2/4}\psi_r)_r + \sigma r e^{(\tau-1)r^2/4}e^\psi = 0 \quad \text{for } r > 0.$$

From  $\psi_r(0) = 0$ , we have

$$re^{\tau r^2/4}\psi_r = -\sigma \int_0^r se^{(\tau-1)s^2/4}e^\psi ds < 0 \quad \text{for } r > 0.$$

This implies that  $\psi_r(r) < 0$  for  $r > 0$ . From Lemma 3.1 we obtain  $\psi(r) = O(e^{-\min\{\tau, 1\}r^2/4})$  as  $r \rightarrow \infty$ . This completes the proof of Theorem 3.  $\square$

#### 4. Structure of the solutions set to (1.8) with (1.9)

From Theorem 3 the solution  $\psi$  of (1.8) with (1.9) must be radially symmetric about the origin. Then the study of the solutions is reduced to the problem:

$$\begin{cases} \psi_{rr} + \left(\frac{1}{r} + \frac{\tau}{2}r\right)\psi_r + \sigma e^{-r^2/4}e^\psi = 0, & r > 0, \\ \psi_r(0) = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} \psi(r) = 0, \end{cases} \quad (4.1)_\sigma$$

where  $\sigma > 0$ . In this section we investigate the structure of the pair  $(\sigma, \psi)$  of a parameter and a solution. Define the set  $\mathcal{C}$  as

$$\mathcal{C} = \{(\sigma, \psi) : \sigma > 0 \text{ and } \psi \in C^2(0, \infty) \cap C^1[0, \infty) \text{ is a solution of } (4.1)_\sigma\}. \quad (4.2)$$

For  $(\sigma, \psi) \in \mathcal{C}$  we have  $\psi \in C^2[0, \infty)$  by Lemma 4.1 below.

**Proposition 4.1.** *The set  $\mathcal{C}$  is written by one parameter families  $(\sigma(s), \psi(r; s))$  on  $s \in \mathbb{R}$ , that is,  $\mathcal{C} = \{(\sigma(s), \psi(r; s)) : s \in \mathbb{R}\}$ . The pairs  $(\sigma(s), \psi(r; s))$  satisfy the following properties:*

- (i)  $s \mapsto (\sigma(s), \psi(\cdot; s)) \in (0, \infty) \times C^2[0, \infty)$  is continuous;
- (ii)  $\lim_{s \rightarrow -\infty} \sigma(s) = 0$  and  $\lim_{s \rightarrow -\infty} \psi(\cdot; s) = 0$  in  $C^2[0, \infty)$ ;
- (iii)  $\lim_{s \rightarrow \infty} \|\psi(\cdot; s)\|_{L^\infty[0, \infty)} = \lim_{s \rightarrow \infty} \psi(0; s) = \infty$ .

First we show the following:

**Lemma 4.1.** *Let  $\psi \in C^2(0, \infty) \cap C^1[0, \infty)$  be a solution to  $(4.1)_\sigma$ . Then  $\psi \in C^2[0, \infty)$  and  $\sup_{r \geq 0} \psi(r) = \psi(0)$ . Moreover we have*

$$\sup_{r \geq 0} |\psi_r(r)| \leq \pi^{1/2} \sigma e^{\psi(0)} \quad \text{and} \quad \sup_{r \geq 0} |\psi_{rr}(r)| \leq \frac{3 + 2\tau}{2} \sigma e^{\psi(0)}. \quad (4.3)$$

*Proof.* From  $(4.1)_\sigma$  we have  $(re^{\tau r^2/4} \psi_r)_r + \sigma r e^{(\tau-1)r^2/4} e^\psi = 0$  for  $r > 0$ . From  $\psi_r(0) = 0$ , it follows that

$$\psi_r(r) = -\frac{\sigma}{r} e^{-\tau r^2/4} \int_0^r \xi e^{(\tau-1)\xi^2/4} e^{\psi(\xi)} d\xi. \quad (4.4)$$

By using the L'Hospital's rule we obtain

$$\lim_{r \rightarrow 0} \frac{\psi_r(r)}{r} = \lim_{r \rightarrow 0} -\frac{\sigma}{r^2 e^{\tau r^2/4}} \int_0^r \xi e^{(\tau-1)\xi^2/4} e^{\psi(\xi)} d\xi = -\frac{\sigma e^{\psi(0)}}{2},$$

which implies  $\psi \in C^2[0, \infty)$ . Since  $\psi_r(r) < 0$  for  $r > 0$  from (4.4), we have  $\sup_{r \geq 0} \psi(r) = \psi(0)$ .

From (4.4) we have

$$|\psi_r(r)| \leq \left( \frac{1}{r} \int_0^r \xi e^{-\xi^2/4} d\xi \right) \sigma e^{\psi(0)}. \quad (4.5)$$

We see that  $(1/r) \int_0^r \xi e^{-\xi^2/4} d\xi \leq \int_0^\infty e^{-\xi^2/4} d\xi = \pi^{1/2}$ . Then the left hand side of (4.3) holds.

From the equation in  $(4.1)_\sigma$  we have

$$|\psi_{rr}(r)| \leq \left( \frac{1}{r} + \frac{\tau}{2} r \right) |\psi_r(r)| + \sigma e^{-r^2/4} e^{\psi(r)} \leq \left( \frac{1}{r} + \frac{\tau}{2} r \right) |\psi_r(r)| + \sigma e^{\psi(0)}.$$

We note here that

$$\left(\frac{1}{r} + \frac{\tau}{2}r\right) \frac{1}{r} \int_0^r \xi e^{-\xi^2/4} d\xi \leq \frac{1}{r^2} \int_0^r \xi d\xi + \frac{\tau}{2} \int_0^\infty \xi e^{-\xi^2/4} d\xi = \frac{1}{2} + \tau. \quad (4.6)$$

It follows from (4.5) and (4.6) that

$$\left(\frac{1}{r} + \frac{\tau}{2}r\right) |\psi_r(r)| \leq \frac{1+2\tau}{2} \sigma e^{\psi(0)}.$$

Therefore we obtain the right hand side of (4.3). This completes the proof of Lemma 4.1.

□

To prove Proposition 4.1 we consider the initial value problem

$$\begin{cases} w_{rr} + \left(\frac{1}{r} + \frac{\tau}{2}r\right) w_r + e^{-r^2/4} e^w = 0, & r > 0, \\ w_r(0) = 0 \quad \text{and} \quad w(0) = s, \end{cases} \quad (4.7)_s$$

where  $s \in \mathbb{R}$ . We denote by  $w(r; s)$  the solution of the problem  $(4.7)_s$ . We easily see that  $w(r; s)$  and  $w_r(r; s)$  satisfy, respectively,

$$w(r; s) = s - \int_0^r \frac{1}{\xi} e^{-\tau\xi^2/4} \left( \int_0^\xi \eta e^{(\tau-1)\eta^2/4} e^{w(\eta;s)} d\eta \right) d\xi \quad (4.8)$$

and

$$w_r(r; s) = -\frac{1}{r} e^{-\tau r^2/4} \int_0^r \xi e^{(\tau-1)\xi^2/4} e^{w(\xi;s)} d\xi. \quad (4.9)$$

Define  $I(\tau)$  as

$$I(\tau) = \int_0^\infty \frac{1}{\xi} e^{-\tau\xi^2/4} \left( \int_0^\xi \eta e^{(\tau-1)\eta^2/4} d\eta \right) d\xi.$$

From [21, Lemma 1] it follows that  $I(\tau) = (\log \tau)/(\tau - 1)$  if  $\tau \neq 1$ ,  $I(\tau) = 1$  if  $\tau = 1$ . We easily obtain  $w_r(r; s) < 0$  for  $r > 0$  and  $w(r; s) \geq s - e^s I(\tau)$  for  $r \geq 0$ . (See [21, Lemma 2].) Then  $\lim_{r \rightarrow \infty} w(r; s)$  exists and is a finite value. Put  $t(s) = \lim_{r \rightarrow \infty} w(r; s)$ .

**Lemma 4.2.** *For  $s \in \mathbb{R}$ , let  $\psi(r; s) = w(r; s) - t(s)$ . Then  $\psi(r; s)$  is a solution to  $(4.1)_\sigma$  with  $\sigma = e^{t(s)}$ . Conversely, let  $\psi(r)$  be a solution of  $(4.1)_\sigma$ . Then, for some  $s \in \mathbb{R}$ ,  $\psi(r) = \psi(r; s)$  and  $\sigma = e^{t(s)}$ .*

*Proof.* It is clear that  $\psi(r; s)$  is a solution to  $(4.1)_\sigma$  with  $\sigma = e^{t(s)}$ . Conversely, let  $\psi(r)$  be a solution of  $(4.1)_\sigma$ , and let  $w(r) = \psi(r) + \log \sigma$ . Then  $w(r)$  satisfies  $(4.7)_s$  with  $s = \psi(0) + \log \sigma$ . By the uniqueness we obtain  $w(r) = w(r; s)$  with  $s = \psi(0) + \log \sigma$ . We

have  $\lim_{r \rightarrow \infty} w(r; s) = \lim_{r \rightarrow \infty} w(r) = \log \sigma$ . Then  $t(s) = \log \sigma$ , that is,  $\sigma = e^{t(s)}$ . Hence we obtain  $\psi(r) = w(r) - \log \sigma = w(r; s) - t(s)$ , which implies  $\psi(r) = \psi(r; s)$ .  $\square$

From [21, (ii) of Lemma 5] it follows that, for  $s_1, s_2 \in \mathbb{R}$ ,

$$\sup_{r \geq 0} |w(r; s_1) - w(r; s_2)| \leq C_1 |s_1 - s_2|, \quad (4.10)$$

where  $C_1 = \exp(e^m I(\tau))$  and  $m = \max\{s_1, s_2\}$ . Moreover we have the following:

**Lemma 4.3.** *Let  $s_1, s_2 \in \mathbb{R}$ , and let  $m = \max\{s_1, s_2\}$ . Then we have*

- (i)  $\sup_{r \geq 0} |w_r(r; s_1) - w_r(r; s_2)| \leq C_2 |s_1 - s_2|$ , where  $C_2 = \pi^{1/2} e^m C_1$ ;
- (ii)  $\sup_{r \geq 0} |w_{rr}(r; s_1) - w_{rr}(r; s_2)| \leq C_3 |s_1 - s_2|$ , where  $C_3 = (3 + 2\tau) e^m C_1/2$ .

*Proof.* From (4.9) we have

$$|w_r(r; s_1) - w_r(r; s_2)| \leq \frac{1}{r} e^{-\tau r^2/4} \int_0^r \xi e^{(\tau-1)\xi^2/4} |e^{w(\xi; s_1)} - e^{w(\xi; s_2)}| d\xi.$$

Note that  $|e^{w(t; s_1)} - e^{w(t; s_2)}| \leq e^m |w(t; s_1) - w(t; s_2)|$  with  $m = \max\{s_1, s_2\}$ . Then from (4.10) we have  $|e^{w(t; s_1)} - e^{w(t; s_2)}| \leq C_1 e^m |s_1 - s_2|$ . Then it follows that

$$|w_r(r; s_1) - w_r(r; s_2)| \leq C_1 e^m |s_1 - s_2| \left( \frac{1}{r} \int_0^r \xi e^{-\xi^2/4} d\xi \right). \quad (4.11)$$

From  $(1/r) \int_0^r \xi e^{-\xi^2/4} d\xi \leq \int_0^\infty e^{-\xi^2/4} d\xi = \pi^{1/2}$ , we obtain (i).

From (4.7)<sub>s</sub> we see that  $w_{rr}(r; s) = -(1/r + \tau r/2) w_r(r; s) - e^{-r^2/4} e^{w(r; s)}$ . Then we have

$$|w_{rr}(r; s_1) - w_{rr}(r; s_2)| \leq \left( \frac{1}{r} + \frac{\tau}{2} r \right) |w_r(r; s_1) - w_r(r; s_2)| + e^m |w(r; s_1) - w(r; s_2)|.$$

Then from (4.11) and (4.6) we obtain

$$\left( \frac{1}{r} + \frac{\tau}{2} r \right) |w_r(r; s_1) - w_r(r; s_2)| \leq \frac{1 + 2\tau}{2} C_1 e^m |s_1 - s_2|.$$

Therefore we obtain (ii).  $\square$

**Lemma 4.4.** *Let  $s_1, s_2 \in \mathbb{R}$ , and let  $m = \max\{s_1, s_2\}$ . Then we have*

- (i)  $|t(s_1) - t(s_2)| \leq C_1 |s_1 - s_2|$ , where  $C_1 = \exp(e^m I(\tau))$ ;
- (ii)  $\lim_{s \rightarrow -\infty} (s - t(s)) = 0$ ;
- (iii)  $\sup_{s \in \mathbb{R}} t(s) \leq -\log I(\tau)$ .

*Proof.* Letting  $r \rightarrow \infty$  in (4.10), we have (i). Since  $w(r; s) < s$  for  $r > 0$ , it follows from (4.8) that

$$0 < s - w(r; s) \leq e^s \int_0^r \frac{1}{\xi} e^{-\tau \xi^2/4} \left( \int_0^\xi \eta e^{(\tau-1)\eta^2/4} d\eta \right) d\xi.$$

Letting  $r \rightarrow \infty$  we have  $0 < s - t(s) \leq e^s I(\tau)$  for  $s \in \mathbb{R}$ . This implies that (ii) holds.

Since  $w(r; s)$  is decreasing in  $r > 0$ , it follows from (4.9) that

$$w_r(r; s) \leq -\frac{1}{r} e^{w(r; s)} e^{-\tau r^2/4} \int_0^r \xi e^{(\tau-1)\xi^2/4} d\xi.$$

Then we obtain

$$\frac{d}{dr} \left( -e^{-w(r; s)} \right) \leq -\frac{1}{r} e^{-\tau r^2/4} \int_0^r \xi e^{(\tau-1)\xi^2/4} d\xi.$$

Integrating the above on  $[0, \infty)$  we have  $e^{-t(s)} - e^{-s} \geq I(\tau)$  or  $e^{-t(s)} \geq I(\tau)$ . This implies that (iii) holds.  $\square$

*Proof of Proposition 4.1.* By Lemma 4.2 we have  $\mathcal{C} = \{(\sigma(s), \psi(\cdot; s)) : s \in \mathbb{R}\}$ , where  $\sigma(s) = e^{t(s)}$  and  $\psi(r; s) = w(r; s) - t(s)$ . We see that  $w(\cdot; s) \in C^2[0, \infty)$  and  $t(s) \in \mathbb{R}$  are continuous for  $s \in \mathbb{R}$  by Lemma 4.3 and (i) of Lemma 4.4, respectively. Thus (i) holds.

By (ii) of Lemma 4.4 we have  $\sigma(s) = e^{t(s)} \rightarrow 0$  and  $\psi(0; s) = s - t(s) \rightarrow 0$  as  $s \rightarrow -\infty$ . Then, by Lemma 4.1 we conclude that  $\psi(\cdot; s) \rightarrow 0$  in  $C^2[0, \infty)$  as  $s \rightarrow -\infty$ . Thus (ii) holds.

From Lemma 4.1 we have  $\|\psi(\cdot; s)\|_{L^\infty[0, \infty)} = \psi(0; s)$ . From (iii) of Lemma 4.4 we have  $\lim_{s \rightarrow \infty} \psi(0; s) = \lim_{s \rightarrow \infty} (s - t(s)) \geq \lim_{s \rightarrow \infty} (s + \log I(\tau)) = \infty$ . Thus (iii) holds. This completes the proof of Proposition 4.1.  $\square$

## 5. Blow-up analysis to self-similar solutions

This section is concerned with the case (iii) of Theorem 2. We study the asymptotic behavior of sequences  $\{(\phi_k, \psi_k)\} \subset \mathcal{S}$  satisfying  $\|\psi_k\|_{L^\infty(\mathbb{R}^2)} \rightarrow \infty$  as  $k \rightarrow \infty$ . We show the following:

**Proposition 5.1.** *Let  $(\phi_k, \psi_k) \in \mathcal{S}$ , and let  $\lambda_k = \|\phi_k\|_{L^1(\mathbb{R}^2)}$ . Assume that*

$$\|\psi_k\|_{L^\infty(\mathbb{R}^2)} \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (5.1)$$

*and that  $\{\lambda_k\}$  is bounded. Then there exists a subsequence, which we call again  $(\psi_k, \phi_k)$  and  $\lambda_k$ , satisfying  $\lambda_k \rightarrow 8\pi$  as  $k \rightarrow \infty$  and*

$$\phi_k(x) dx \rightarrow 8\pi \delta_0(dx) \quad \text{as } k \rightarrow \infty \quad (5.2)$$

*in the sense of measure, where  $\delta_0(dx)$  is Dirac's delta function with the support in origin.*

In order to prove Proposition 5.1 we make use of Theorems A and Theorem 4 in Section 1. We also need the following result by Brezis and Merle [2].

**Theorem B [2].** *Assume  $\{u_k\}$  is a sequence of solutions of (1.11) such that*

$$\|V_k\|_{L^\infty(\Omega)} \leq C, \quad \|u_k^+\|_{L^1(\Omega)} \leq C, \quad \text{and} \quad \int_{\Omega} V_k e^{u_k} dx < 4\pi,$$

*for some constant  $C > 0$ , where  $u^+ = \max\{u, 0\}$ . Then  $\{u_k^+\}$  is bounded in  $L^\infty_{\text{loc}}(\Omega)$ .*

Now we prepare several lemmas.

**Lemma 5.1.** *Assume that  $f \in C(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ . Let  $w \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$  be a solution of*

$$-\Delta w - \frac{\tau}{2} x \cdot \nabla w = f \quad \text{for } x \in \mathbb{R}^2. \quad (5.3)$$

*Then we have  $\|w\|_{L^1(\mathbb{R}^2)} + \|\nabla w\|_{L^1(\mathbb{R}^2)} \leq C\|f\|_{L^1(\mathbb{R}^2)}$  for some positive constant  $C$ .*

*Proof.* Define  $W$  and  $F$  respectively as

$$W(x, t) = w\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad F(x, t) = \frac{1}{t} f\left(\frac{x}{\sqrt{t}}\right).$$

Then  $W$  and  $F$  satisfy

$$\|W(\cdot, t)\|_{L^1(\mathbb{R}^2)} = t\|w\|_{L^1(\mathbb{R}^2)} \quad \text{and} \quad \|F(\cdot, t)\|_{L^1(\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2)} \quad (5.4)$$

for  $t > 0$ . Furthermore, from (5.3) we have  $\tau W_t = \Delta W + F$  in  $\mathbb{R}^2 \times (0, \infty)$ . Since  $W \rightarrow 0$  in  $L^1(\mathbb{R}^2)$  as  $t \rightarrow 0$  from (5.4), we obtain

$$W(x, t) = \frac{1}{\tau} \int_0^t e^{((t-s)/\tau)\Delta} F(\cdot, s) ds.$$

Then it follows from (5.4) that

$$t\|w\|_{L^1(\mathbb{R}^2)} = \|W(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq \frac{1}{\tau} \int_0^t \|F(\cdot, s)\|_{L^1(\mathbb{R}^2)} ds \leq \frac{t}{\tau} \|f\|_{L^1(\mathbb{R}^2)}.$$

Therefore we obtain  $\|w\|_{L^1(\mathbb{R}^2)} \leq \tau^{-1} \|f\|_{L^1(\mathbb{R}^2)}$ .

Next we show  $\|\nabla w\|_{L^1(\mathbb{R}^2)} \leq C\|f\|_{L^1(\mathbb{R}^2)}$ . By the  $L^p$ - $L^q$  estimates (2.5) with  $p = q = 1$  we have

$$\|\nabla e^{((t-s)/\tau)\Delta} F(\cdot, s)\|_{L^1(\mathbb{R}^2)} \leq C(t-s)^{-1/2} \|F(\cdot, s)\|_{L^1(\mathbb{R}^2)} = C(t-s)^{-1/2} \|f\|_{L^1(\mathbb{R}^2)}.$$

Then we obtain

$$\|\nabla W(\cdot, t)\|_{L^1(\mathbb{R}^2)} \leq \frac{1}{\tau} \int_0^t \|\nabla e^{((t-s)/\tau)\Delta} F(\cdot, s)\|_{L^1(\mathbb{R}^2)} ds \leq Ct^{1/2} \|f\|_{L^1(\mathbb{R}^2)}.$$

By the definition of  $W$  it follows that  $\|\nabla W(\cdot, t)\|_{L^1(\mathbb{R}^2)} = t^{1/2} \|\nabla w\|_{L^1(\mathbb{R}^2)}$ . Therefore we conclude that  $\|\nabla w\|_{L^1(\mathbb{R}^2)} \leq C \|f\|_{L^1(\mathbb{R}^2)}$ . This completes the proof of Lemma 5.1.  $\square$

Let  $(\phi_k, \psi_k) \in \mathcal{S}$ , and let  $\lambda_k = \|\phi_k\|_{L^1(\mathbb{R}^2)}$ . Then  $(\lambda_k, \psi_k)$  solves (1.10), that is,

$$\Delta \psi_k + \frac{\tau}{2} x \cdot \nabla \psi_k + \lambda_k e^{-|x|^2/4} e^{\psi_k} \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy = 0 \quad \text{for } x \in \mathbb{R}^2. \quad (5.5)$$

From Theorem 3 we have  $\psi_k \in L^1(\mathbb{R}^2)$ ,  $\psi_k = \psi_k(r)$ ,  $r = |x|$ , and  $\partial \psi_k / \partial r < 0$  for  $r > 0$ . Assume that (5.1) holds. Then  $\|\psi_k\|_{L^\infty(\mathbb{R}^2)} = \psi_k(0) \rightarrow \infty$  as  $k \rightarrow \infty$ . We always use  $B_r$  to denote a ball of radius  $r$  centered at origin, that is,  $B_r = \{x \in \mathbb{R}^2 : |x| < r\}$ .

**Lemma 5.2.** (i) We have  $\|\psi_k\|_{L^1(\mathbb{R}^2)} + \|\nabla \psi_k\|_{L^1(\mathbb{R}^2)} = O(1)$  as  $k \rightarrow \infty$ .

(ii) For all  $r > 0$  we have  $\sup_k \|\psi_k\|_{L^\infty(\mathbb{R}^2 \setminus B_r)} < \infty$ .

*Proof.* (i) Put

$$f_k(x) = \lambda_k e^{-|x|^2/4} e^{\psi_k(x)} \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy.$$

Then  $f_k \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ . We have  $\psi_k \in L^1(\mathbb{R}^2)$  and

$$-\Delta \psi_k - \frac{\tau}{2} x \cdot \nabla \psi_k = f_k \quad \text{for } x \in \mathbb{R}^2.$$

By Lemma 5.1 we obtain  $\|\psi_k\|_{L^1(\mathbb{R}^2)} + \|\nabla \psi_k\|_{L^1(\mathbb{R}^2)} \leq C \|f_k\|_{L^1(\mathbb{R}^2)}$  for some constant  $C > 0$ . Since  $\|f_k\|_{L^1(\mathbb{R}^2)} = \lambda_k = O(1)$  as  $k \rightarrow \infty$ , the assertion of (i) holds.

(ii) Assume to the contrary that  $\sup_k \|\psi_k\|_{L^\infty(\mathbb{R}^2 \setminus B_{r_0})} = \infty$  for some  $r_0 > 0$ . Since  $\psi_k(r)$  is decreasing in  $r > 0$ , there exists a subsequence, which we call again  $\{\psi_k\}$ , such that  $\inf_{y \in B_{r_0}} \psi_k(y) \rightarrow \infty$  as  $k \rightarrow \infty$ . Then  $\|\psi_k\|_{L^1(\mathbb{R}^2)} \rightarrow \infty$  as  $k \rightarrow \infty$ , which contradicts the assertion (i).  $\square$

Take  $R > 0$ . Let  $g_k$  be a unique solution of the problem

$$-\Delta g_k = \frac{\tau}{2} x \cdot \nabla \psi_k \quad \text{in } B_R, \quad g_k = 0 \quad \text{on } \partial B_R.$$

**Lemma 5.3.** We have  $\|g_k\|_{L^\infty(B_R)} = O(1)$  and  $\|\nabla g_k\|_{L^\infty(B_R)} = O(1)$  as  $k \rightarrow \infty$ .

*Proof.* We have  $g_k = g_k(r)$ ,  $r = |x|$ , since  $\psi_k = \psi_k(r)$ . We see that  $g_k(r)$  satisfies

$$-(rg'_k)' = \frac{\tau}{2}r^2\psi_k, \quad 0 < r < R, \quad g'_k(0) = g_k(R) = 0,$$

where  $' = d/dr$ . We will show that

$$\|g_k\|_{L^\infty[0,R]} = O(1), \quad \|g'_k\|_{L^\infty[0,R]} = O(1) \quad \text{as } k \rightarrow \infty. \quad (5.6)$$

By integrating the equation above, we obtain

$$-rg'_k(r) = \frac{\tau}{2} \int_0^r s^2 \psi'_k(s) ds.$$

Then it follows that

$$|g'_k(r)| \leq \frac{\tau}{2r} \int_0^r s^2 |\psi'_k(s)| ds \leq \frac{\tau}{2} \int_0^r s |\psi'_k(s)| ds \quad \text{for } 0 \leq r \leq R.$$

Thus we obtain

$$\|g'_k\|_{L^\infty[0,R]} \leq \frac{\tau}{2} \int_0^R s |\psi'_k(s)| ds. \quad (5.7)$$

We note that  $\int_r^R g'_k(s) ds = g_k(R) - g_k(r) = -g_k(r)$ . Then

$$|g_k(r)| \leq \int_0^R |g'_k(s)| ds \leq R \|g'_k\|_{L^\infty[0,R]} \quad \text{for } 0 \leq r \leq R.$$

From (5.7) we obtain

$$\|g_k\|_{L^\infty[0,R]} \leq \frac{\tau R}{2} \int_0^R s |\psi'_k(s)| ds. \quad (5.8)$$

By (i) of Lemma 5.2 we have

$$2\pi \int_0^R s |\psi'_k(s)| ds = \|\nabla \psi_k\|_{L^1(B_R)} \leq \|\nabla \psi_k\|_{L^1(\mathbb{R}^2)} = O(1) \quad \text{as } k \rightarrow \infty.$$

From (5.7) and (5.8) we obtain (5.6). This completes the proof of Lemma 5.3.  $\square$

Now define  $v_k$  as

$$v_k(x) = \psi_k(x) - g_k(x) - \log \left( \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy \right). \quad (5.9)$$

It follows from (5.5) that

$$-\Delta v_k = -\Delta \psi_k - \frac{\tau}{2} x \cdot \nabla \psi_k = \lambda_k e^{-|x|^2/4} e^{g_k} e^{v_k} \quad \text{for } x \in B_R. \quad (5.10)$$

Then we have

$$-\Delta v_k = V_k(x) e^{v_k} \quad \text{in } B_R, \quad (5.11)$$

where  $V_k(x) = \lambda_k e^{-|x|^2/4} e^{g_k}$ . Since  $\{\lambda_k\}$  is bounded and by Lemma 5.3, we have  $0 \leq V_k(x) \leq C_0$  and  $\|\nabla V_k\|_{L^\infty(B_R)} \leq C_1$  for some constants  $C_0$  and  $C_1$ . Since  $v_k$  is radial symmetry and satisfies  $-\Delta v_k \geq 0$  in  $B_R$ ,  $v_k(r)$  is nonincreasing in  $r \in (0, R)$  by the maximum principle.

**Lemma 5.4.** *There exists a subsequence, which we call again  $\{v_k\}$ , such that  $v_k(0) \rightarrow \infty$  and  $v_k(x) \rightarrow -\infty$  uniformly on compact subset of  $B_R \setminus \{0\}$  as  $k \rightarrow \infty$ . Moreover,*

$$\int_{B_R} V_k e^{v_k} dx \rightarrow 8\pi \quad \text{as } k \rightarrow \infty \quad (5.12)$$

and

$$\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (5.13)$$

*Proof.* We see that

$$\int_{B_R} e^{v_k(y)} dy \leq e^{\|g_k\|_{L^\infty(B_R)}} \int_{B_R} e^{\psi_k(y)} dy \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4 + \psi_k(y)} dy \leq C$$

for some constant  $C > 0$ . Hence, by applying Theorem A, there exists a subsequence (still denoted by  $\{v_k\}$ ) satisfying one of the alternatives (i), (ii), and (iii) in Theorem A.

Assume that the first alternative (i) holds. Since  $\{v_k\}$  and  $\{g_k\}$  are bounded in  $L_{\text{loc}}^\infty(B_R)$  and  $\psi_k(0) \rightarrow \infty$  as  $k \rightarrow \infty$ , it follows from (5.9) that

$$\log \left( \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy \right) = \psi_k(0) - g_k(0) - v_k(0) \rightarrow \infty \quad \text{as } k \rightarrow \infty.$$

Let  $y_0 \in B_R \setminus \{0\}$ . Then from (5.9) we have  $\psi_k(y_0) \rightarrow \infty$  as  $k \rightarrow \infty$ . This contradicts (ii) of Lemma 5.2.

Assume that the second alternative (ii) holds. Since  $v_k(r)$  is nonincreasing in  $r$ , we have  $v_k \rightarrow -\infty$  uniformly on  $B_R$ . Then

$$\int_{B_R} e^{v_k} dx \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (5.14)$$

Put

$$w_k = \psi_k - g_k \quad \text{and} \quad W_k(x) = V_k(x) \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy.$$

Then we have  $-\Delta w_k = W_k e^{w_k}$  in  $B_R$ . Because  $\psi_k \geq 0$ , we have

$$W_k(x) \leq V_k(x) \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} dy \leq C$$

for some constant  $C > 0$ . We find that  $\|w_k\|_{L^1(B_R)} \leq \|\psi_k\|_{L^1(B_R)} + \|g_k\|_{L^1(B_R)} = O(1)$  as  $k \rightarrow \infty$  by Lemmas 5.2 and 5.3. It follows from (5.14) that

$$\int_{B_R} W_k(y) e^{w_k(y)} dy = \int_{B_R} V_k(y) e^{v_k(y)} dy \leq C_0 \int_{B_R} e^{v_k(y)} dy \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence, by applying Theorem B we obtain  $\|w_k^+\|_{L^\infty(B_R)} = O(1)$  as  $k \rightarrow \infty$ . This contradicts  $w_k(0) = \psi_k(0) - g_k(0) \rightarrow \infty$  as  $k \rightarrow \infty$ .

Therefore, the third alternative (iii) must hold. By (ii) of Lemma 5.2 we have the blow-up set  $\mathcal{B} = \{0\}$ . Then  $v_k(0) \rightarrow \infty$  and  $v_k(x) \rightarrow -\infty$  uniformly on compact subset of  $B_R \setminus \{0\}$ . Moreover

$$\int_{B_R} V_k e^{v_k} dx \rightarrow \alpha \quad \text{as } k \rightarrow \infty \quad (5.15)$$

for some  $\alpha \geq 4\pi$ . Since  $v_k$  is radial symmetry, we have  $\max_{\partial B_R} v_k - \min_{\partial B_R} v_k = 0$ . By applying Theorem 4, we obtain  $\alpha = 8\pi$  in (5.15).

Let  $x_0 \in B_R \setminus \{0\}$ . From  $v_k(x_0) \rightarrow -\infty$  as  $k \rightarrow \infty$  we have

$$\log \left( \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy \right) = \psi_k(x_0) - g_k(x_0) - v_k(x_0) \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

which implies that (5.13) holds.  $\square$

*Proof of Proposition 5.1.* Let  $\{v_k\}$  be a subsequence obtained in Lemma 5.4. First we verify that, for all  $r > 0$ ,

$$\int_{\mathbb{R}^2 \setminus B_r} V_k e^{v_k} dy \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (5.16)$$

From (ii) of Lemma 5.2 there exists a constant  $M = M(r) > 0$  such that  $|\psi_k(x)| \leq M$  for  $|x| \geq r$ . Since

$$\int_{\mathbb{R}^2 \setminus B_r} V_k(y) e^{v_k(y)} dy = \frac{\lambda_k \int_{\mathbb{R}^2 \setminus B_r} e^{-|y|^2/4} e^{\psi_k(y)} dy}{\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy} \leq \frac{\lambda_k e^M \int_{\mathbb{R}^2 \setminus B_r} e^{-|y|^2/4} dy}{\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi_k(y)} dy},$$

it follows from (5.13) that (5.16) holds.

From (5.10), (5.11), and the second equation of (1.4) we have

$$V_k e^{v_k} = -\Delta v_k = -\Delta \psi_k - \frac{\tau}{2} x \cdot \nabla \psi_k = \phi_k.$$

From (5.12) and (5.16) we have

$$\lambda_k = \|\phi_k\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} V_k e^{v_k} dy = \int_{B_R} V_k e^{v_k} dy + \int_{\mathbb{R}^2 \setminus B_R} V_k e^{v_k} dy \rightarrow 8\pi \quad \text{as } k \rightarrow \infty.$$

Thus  $\lambda_k \rightarrow 8\pi$  as  $k \rightarrow \infty$ . Since  $\{\phi_k\}$  is bounded in  $L^1(\mathbb{R}^2)$ , we may extract a subsequence, which we call again  $\{\phi_k\}$ , such that  $\phi_k$  converges in the sense of measures on  $\mathbb{R}^2$  to some nonnegative bounded measure  $\mu$ , i.e.,

$$\int_{\mathbb{R}^2} \phi_k(x) \eta dx \rightarrow \int_{\mathbb{R}^2} \eta d\mu$$

for every  $\eta \in C(\mathbb{R}^2)$  with compact support. From (5.16) we have  $\int_{\mathbb{R}^2 \setminus B_r} \phi_k(x) dx \rightarrow 0$  as  $k \rightarrow \infty$  for every  $r > 0$ . Then  $\phi_k \rightarrow 0$  in  $L^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\})$  and hence  $\mu$  is supported on  $\{0\}$ . Thus we obtain  $d\mu = \alpha \delta_0(dx)$  with  $\alpha = 8\pi$ , which implies that (5.2) holds. This completes the proof of Proposition 5.1.  $\square$

## 6. $L^1$ -norms of self-similar solutions

This section is concerned with the case (iv) of Theorem 2 and we investigate the upper bounds of  $\|\phi\|_{L^1(\mathbb{R}^2)}$  for  $(\phi, \psi) \in \mathcal{S}$ .

**Proposition 6.1.** *Let  $(\phi, \psi) \in \mathcal{S}$ . Then*

$$\|\phi\|_{L^1(\mathbb{R}^2)} \leq \max \left\{ \frac{4}{3}\pi^3, \frac{4}{3}\pi^3\tau^2 \right\}.$$

Moreover, if  $0 < \tau \leq 1/2$  then  $\|\phi\|_{L^1(\mathbb{R}^2)} < 8\pi$ .

We prove Proposition 6.1, following the idea of Biler[1]. By Theorem 1 the solution  $(\phi, \psi) \in \mathcal{S}$  must be radially symmetric about the origin. Define  $\Phi$  and  $\Psi$ , respectively, as

$$\Phi(s) = \frac{1}{2} \int_0^s \phi(\sqrt{t}) dt \quad \text{and} \quad \Psi(s) = \frac{1}{2} \int_0^s \psi(\sqrt{t}) dt.$$

First we show the following:

**Lemma 6.1.** *We have  $\|\phi\|_{L^1(\mathbb{R}^2)} = 2\pi \lim_{s \rightarrow \infty} \Phi(s)$ . Moreover,  $(\Phi, \Psi)$  solves*

$$\begin{cases} \Phi'' + \frac{1}{4}\Phi' - 2\Phi'\Psi'' = 0 \\ 4s\Psi'' + \tau s\Psi' - \tau\Psi + \Phi = 0 \end{cases} \quad (6.1)$$

for  $s > 0$ , where  $' = d/ds$ .

*Proof.* We see that

$$\int_{\mathbb{R}^2} \phi(|y|) dy = 2\pi \int_0^\infty r \phi(r) dr = 2\pi \left( \frac{1}{2} \int_0^\infty \phi(\sqrt{t}) dt \right),$$

which implies  $\|\phi\|_{L^1(\mathbb{R}^2)} = 2\pi \lim_{s \rightarrow \infty} \Phi(s)$ .

Define  $u$  and  $v$  as

$$u(r, t) = \frac{1}{t} \phi\left(\frac{r}{\sqrt{t}}\right) \quad \text{and} \quad v(r, t) = \psi\left(\frac{r}{\sqrt{t}}\right),$$

respectively. Put  $U$  and  $V$  as

$$U(r, t) = \int_0^r s u(s, t) ds \quad \text{and} \quad V(r, t) = \int_0^r s v(s, t) ds.$$

Then, by the change of variables, we obtain

$$U(r, t) = \frac{1}{2} \int_0^{r^2/t} \phi(\sqrt{s}) ds \quad \text{and} \quad V(r, t) = \frac{t}{2} \int_0^{r^2/t} \psi(\sqrt{s}) ds.$$

By the definition of  $\Psi$  and  $\Phi$  we have

$$U(r, t) = \Phi\left(\frac{r^2}{t}\right) \quad \text{and} \quad V(r, t) = t\Psi\left(\frac{r^2}{t}\right). \quad (6.2)$$

Now we verify that  $(U, V)$  satisfies

$$\begin{cases} U_t = r(r^{-1}U_r)_r - U_r(r^{-1}V_r)_r \\ \tau V_t = r(r^{-1}V_r)_r + U \end{cases} \quad (6.3)$$

for  $(r, t) \in [0, \infty) \times (0, \infty)$ . Since  $(u, v)$  solves (1.1), we see that

$$ru_t = (ru_r)_r - ru_r v_r - u(rv_r)_r \quad \text{and} \quad \tau r v_t = (rv_r)_r + ru.$$

Then we obtain

$$\int_0^r s u_t(s, t) ds = ru_r - ru_r v_r \quad \text{and} \quad \tau \int_0^r s v_t(s, t) ds = rv_r + \int_0^r s u(s, t) ds.$$

Thus we obtain (6.3). By virtue of (6.2) we have (6.1).  $\square$

**Lemma 6.2.** *We have*

$$-s\Psi''(s) = \frac{1}{4}e^{-\tau s/4} \int_0^s e^{\tau t/4} \Phi'(t) dt > 0 \quad \text{for } s > 0. \quad (6.4)$$

*Proof.* Put  $W(s) = -4s\Psi''(s)$ . From the second equation of (6.1) we have

$$\Phi' = (-4s\Psi'')' - \tau s\Psi'' = W' + \frac{\tau}{4}W.$$

Since  $s\Psi''(s) = \sqrt{s}\psi'(\sqrt{s})/4$ , we have  $W(0) = \lim_{s \rightarrow 0} W(s) = 0$ . Then we obtain

$$W(s) = e^{-\tau s/4} \int_0^s e^{\tau t/4} \Phi'(t) dt.$$

Since  $\Phi'(s) = \phi(\sqrt{s})/2 > 0$ , we obtain the assertion.  $\square$

**Lemma 6.3.** *We have  $s\Psi''(s) \rightarrow 0$  as  $s \rightarrow \infty$  and, for  $s > 0$ ,*

$$0 < \Psi(s) - s\Psi'(s) \leq \begin{cases} \frac{\tau}{4} \int_0^s \frac{t}{e^{\tau t/4} - 1} dt < s & \text{if } 0 < \tau \leq 1, \\ \frac{\tau}{4} \int_0^s \frac{t}{e^{t/4} - 1} dt & \text{if } \tau > 1. \end{cases}$$

*Proof.* From the first equation of (6.1) and (6.4) we have

$$\Phi'' + \frac{1}{4}\Phi' + \frac{1}{2s}e^{-\tau s/4}\Phi' \int_0^s e^{\tau t/4}\Phi'(t) dt = 0.$$

We note that  $\Phi'(s) = \phi(\sqrt{s})/2 > 0$ . Then, for the case  $0 < \tau \leq 1$ , we have

$$\Phi'' + \frac{\tau}{4}\Phi' + \frac{1}{2s}e^{-\tau s/4}\Phi' \int_0^s e^{\tau t/4}\Phi'(t) dt \leq 0,$$

that is,

$$(e^{\tau s/4}\Phi')' + \frac{1}{2s}\Phi' \int_0^s e^{\tau t/4}\Phi'(t) dt \leq 0. \quad (6.5)$$

For the case  $\tau > 1$  we have

$$\Phi'' + \frac{1}{4}\Phi' + \frac{1}{2s}e^{-\tau s/4}\Phi' \int_0^s e^{t/4}\Phi'(t) dt \leq 0,$$

that is,

$$(e^{s/4}\Phi')' + \frac{1}{2s}e^{(1-\tau)s/4}\Phi' \int_0^s e^{t/4}\Phi'(t) dt \leq 0. \quad (6.6)$$

First we consider the case  $0 < \tau \leq 1$ . Define  $Z$  as

$$Z(s) = \int_0^s e^{\tau t/4}\Phi'(t) dt.$$

From (6.5) we have

$$sZ'' + \frac{1}{2}e^{-\tau s/4}Z'Z \leq 0. \quad (6.7)$$

By integrating the above on  $[0, s]$  we obtain

$$sZ' - Z + \frac{1}{4}e^{-\tau s/4}Z^2 + \frac{\tau}{16} \int_0^s e^{-\tau t/4}Z^2(t) dt \leq 0.$$

Then we have  $sZ' - Z + e^{-\tau s/4}Z^2/4 \leq 0$ . Dividing the inequality by  $Z^2$  it follows that  $(s/Z)' \geq e^{-\tau s/4}/4$ . Therefore we obtain

$$Z(s) \leq \frac{\tau s}{1 - e^{-\tau s/4}}. \quad (6.8)$$

From (6.4) we have  $-s\Psi'' = e^{-\tau s/4}Z(s)/4 > 0$ . Then

$$0 < -s\Psi''(s) \leq \frac{\tau s}{4(e^{\tau s/4} - 1)} < 1 \quad \text{for } s > 0.$$

This implies  $s\Psi''(s) \rightarrow 0$  as  $s \rightarrow \infty$ . By integrating the above on  $[0, s]$  we obtain the assertion.

Next we consider the case  $\tau > 1$ . Define  $Z$  as

$$Z(s) = \int_0^s e^{t/4} \Phi'(t) dt.$$

Then from (6.6) we have (6.7). By the similar argument above we obtain (6.8). We see that

$$e^{-\tau s/4} \int_0^s e^{\tau t/4} \Phi'(t) dt = \int_0^s e^{-\tau(s-t)/4} \Phi'(t) dt \leq \int_0^s e^{-(s-t)/4} \Phi'(t) dt = e^{-s/4} Z(s).$$

Then from (6.4) and (6.8) we have

$$0 < -s\Psi''(s) \leq \frac{1}{4} e^{-s/\tau} Z(s) \leq \frac{\tau s}{4(e^{s/4} - e^{(1-\tau)s/4})} \leq \frac{\tau s}{4(e^{s/4} - 1)}.$$

Therefore  $s\Psi''(s) \rightarrow 0$  as  $s \rightarrow \infty$ . By integrating the above we obtain the assertion.  $\square$

*Proof of Proposition 6.1.* First we consider the case  $0 < \tau \leq 1$ . From the second equation of (6.1) we have  $\Phi(s) = -4s\Psi''(s) + \tau(\Psi(s) - s\Psi'(s))$ . From Lemma 6.3 we obtain

$$\lim_{s \rightarrow \infty} \Phi(s) = \lim_{s \rightarrow \infty} \tau(\Psi(s) - s\Psi'(s)) \leq \frac{\tau^2}{4} \int_0^\infty \frac{s}{e^{\tau s/4} - 1} ds.$$

By the change of variable  $z = \tau s/4$  it follows that

$$\lim_{s \rightarrow \infty} \Phi(s) \leq 4 \int_0^\infty \frac{z}{e^z - 1} dz = \frac{2}{3} \pi^2.$$

Since  $\|\phi\|_{L^1(\mathbb{R}^2)} = 2\pi \lim_{s \rightarrow \infty} \Phi(s)$  from Lemma 6.1, we obtain the assertion.

Next we consider the case  $\tau > 1$ . By the similar argument we obtain

$$\lim_{s \rightarrow \infty} \Phi(s) \leq \frac{\tau^2}{4} \int_0^\infty \frac{s}{e^{s/4} - 1} ds = 4\tau^2 \int_0^\infty \frac{z}{e^z - 1} dz = \frac{2}{3} \pi^2 \tau^2,$$

which implies the assertion.

Finally we consider the case  $0 < \tau \leq 1/2$ . The change of variables

$$t = (\log s)/2, \quad k(t) = \Phi(s), \quad \ell(t) = 2s\Phi'(s), \quad m(t) = \Psi(s), \quad n(t) = 2s\Psi'(s)$$

transforms (6.1) into

$$\begin{cases} \dot{k} = \ell, & \dot{m} = n, \\ \dot{\ell} = \left(2 - k + \tau m - \frac{\tau n}{2} - \frac{e^{2t}}{2}\right) \ell, \\ \dot{n} = 2n + e^{2t} \left(\frac{\tau n}{2} + \tau m - k\right), \end{cases}$$

where  $\dot{\cdot} = d/dt$ . Hence we have

$$\frac{d}{dt} \left( (k-2)^2 + 2\ell \right) = 2\ell \left( \tau m - \frac{\tau n}{2} - \frac{e^{2t}}{2} \right) = 4s\Phi'(s) \left( \tau(\Psi(s) - s\Psi'(s)) - \frac{s}{2} \right) \leq 0$$

by Lemma 6.3. Then  $(k(t) - 2)^2 + 2\ell(t)$  is decreasing for  $t > -\infty$ . We note that  $\lim_{t \rightarrow -\infty} k(t) = \Phi(0) = 0$  and  $\lim_{t \rightarrow -\infty} \ell(t) = \lim_{s \rightarrow 0} 2s\Phi'(s) = \lim_{s \rightarrow 0} s\phi(\sqrt{s}) = 0$ . Then we have

$$(k(t) - 2)^2 + 2\ell(t) < 4 \quad \text{for } t > -\infty.$$

Since  $\ell(t) = 2s\Phi'(s) = s\phi(\sqrt{s}) > 0$  and  $\lim_{t \rightarrow \infty} ((k(t) - 2)^2 + 2\ell(t)) < 4$ , we obtain  $\lim_{t \rightarrow \infty} k(t) < 4$ . Thus  $\lim_{s \rightarrow \infty} \Phi(s) < 4$ , which implies  $\|\phi\|_{L^1(\mathbb{R}^2)} < 8\pi$ .  $\square$

## 7. Proof of Theorem 2

By Theorem 3 it is shown that  $(\phi, \psi) \in \mathcal{S}$  if and only if  $\psi = \psi(r)$ ,  $r = |y|$ , solves  $(4.1)_\sigma$  for some  $\sigma > 0$  and  $\phi$  is given by (1.7). By Proposition 4.1 the set  $\mathcal{C}$  defined by (4.2) is written by one parameter families  $(\sigma(s), \psi(r; s))$  on  $s \in \mathbb{R}$ . Let

$$\phi(r; s) = \sigma(s)e^{-r^2/4}e^{\psi(r; s)}. \quad (7.1)$$

Then  $\mathcal{S}$  is written by one parameter families  $(\phi(r, s), \psi(r, s))$  on  $s \in \mathbb{R}$ . From (i) and (ii) of Proposition 4.1 and (7.1) we have  $s \mapsto (\phi(\cdot; s), \psi(\cdot; s)) \in C^2[0, \infty) \times C^2[0, \infty)$  is continuous and  $(\phi(\cdot; s), \psi(\cdot; s)) \rightarrow (0, 0)$  in  $C^2[0, \infty) \times C^2[0, \infty)$  as  $s \rightarrow -\infty$ . We see that

$$\lambda(s) = 2\pi \int_0^\infty r\phi(r; s)dr. \quad (7.2)$$

Then  $\lambda(s)$  is continuous and satisfies  $\lambda(s) \rightarrow 0$  as  $s \rightarrow -\infty$ . Hence, (i) and (ii) holds. By Proposition 6.1 we obtain (iv).

We have  $\|\psi(\cdot, s)\|_{L^\infty[0, \infty)} = \psi(0, s) \rightarrow \infty$  as  $s \rightarrow \infty$  from (iii) of Proposition 4.1. Let  $\{s_k\}$  be a sequence satisfying  $s_k \rightarrow \infty$  as  $k \rightarrow \infty$ . We note that  $\{\lambda_k\}$  is bounded by Proposition 6.1. By applying Proposition 5.1, there exists a subsequence (still denoted by  $\{s_k\}$ ) such that  $\lambda(s_k) \rightarrow 8\pi$  and  $\phi_k(|x|, s_k)dx \rightarrow 8\pi\delta_0(dx)$  as  $k \rightarrow \infty$ . Therefore, (iii) holds. This completes the proof of Theorem 2.  $\square$

## Appendix A. Existence of solutions to (1.8) with (1.9)

The following theorem refines the previous results [20, Theorem 1], [21, Theorems 1 and 2], and [22, Theorem 1.1].

**Theorem A.1.** *For any  $\tau > 0$  there exists  $\sigma^* > 0$  such that*

- (i) *if  $\sigma > \sigma^*$ , then (1.8) with (1.9) has no solution;*
- (ii) *if  $\sigma = \sigma^*$ , then (1.8) with (1.9) has at least one solution;*
- (iii) *if  $0 < \sigma < \sigma^*$ , then (1.8) with (1.9) has at least two distinct solutions  $\underline{\psi}_\sigma, \overline{\psi}_\sigma$  satisfying  $\lim_{\sigma \rightarrow 0} \underline{\psi}_\sigma(0) = 0$  and  $\lim_{\sigma \rightarrow 0} \overline{\psi}_\sigma(0) = \infty$ .*

*Proof.* By Theorem 1 the problem (1.8) with (1.9) is reduced to the problem  $(4.1)_\sigma$ . By Proposition 4.1 the set  $\mathcal{C}$  defined by (4.2) is written by one parameter families  $(\sigma(s), \psi(r; s))$  on  $s \in \mathbb{R}$ . From (7.1) and (7.2) we find that

$$\sigma(s) = \lambda(s) \Big/ \left( 2\pi \int_0^\infty r e^{-r^2/4} e^{\psi(r;s)} dr \right) = \lambda(s) \Big/ \int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(|y|;s)} dy$$

From (5.13) in Lemma 5.4 we have

$$\int_{\mathbb{R}^2} e^{-|y|^2/4} e^{\psi(|y|;s)} dy \rightarrow \infty \quad \text{as } s \rightarrow \infty.$$

Then  $\sigma(s) \rightarrow 0$  as  $s \rightarrow \infty$ . Therefore, from (ii) of Proposition 4.1,  $\sigma(s)$  satisfies

$$\lim_{s \rightarrow \pm\infty} \sigma(s) = 0.$$

Let  $\sigma^* = \sup_{s \in \mathbb{R}} \sigma(s)$ . Then there exists  $s^* \in \mathbb{R}$  such that  $\sigma^* = \sigma(s^*)$ .

By Proposition 4.1 it is shown that  $(4.1)_\sigma$  has a solution if and only if  $\sigma = \sigma(s)$  for some  $s \in \mathbb{R}$ . Therefore,  $(4.1)_\sigma$  has no solution, if  $\sigma > \sigma^*$ , and  $(4.1)_\sigma$  has at least one solution, if  $\sigma = \sigma^*$ . If  $\sigma \in (0, \sigma^*)$ , by the mean value theorem, there exists  $s_1, s_2 \in \mathbb{R}$ ,  $s_1 < s^* < s_2$  such that  $\sigma = \sigma(s_1) = \sigma(s_2)$ . Then  $(4.1)_\sigma$  has at least two solutions  $\psi_{\sigma(s_1)}$  and  $\psi_{\sigma(s_2)}$ . We note that  $\lim_{s \rightarrow -\infty} \psi_{\sigma(s)}(0) = 0$  and  $\lim_{s \rightarrow \infty} \psi_{\sigma(s)}(0) = \infty$  by (ii) and (iii) of Proposition 4.1.

Since  $\lim_{s \rightarrow \pm\infty} \sigma(s) = 0$ , we can choose solutions  $\overline{\psi}_\sigma$  and  $\underline{\psi}_\sigma$  satisfying  $\lim_{\sigma \rightarrow 0} \underline{\psi}_\sigma(0) = 0$  and  $\lim_{\sigma \rightarrow 0} \overline{\psi}_\sigma(0) = \infty$ . This completes the proof of Theorem A.1.  $\square$

## Appendix B. Proof of Theorem 4

Define  $h_k \in C^2(\Omega) \cap C(\overline{\Omega})$  by

$$\Delta h_k = 0 \quad \text{in } \Omega \quad \text{and} \quad h_k = u_k \quad \text{on } \partial\Omega.$$

We may assume that  $\{0\} \in \Omega$  with no loss of generality.

**Lemma B.1.** *Let  $r > 0$  satisfying  $\overline{B}_r \subset \Omega$ . Then  $\|\nabla h_k\|_{L^\infty(B_r)} = O(1)$  as  $k \rightarrow \infty$ .*

*Proof.* By the maximum principle, we have  $\max_{\overline{\Omega}} h_k - \min_{\overline{\Omega}} h_k \leq \max_{\partial\Omega} h_k - \min_{\partial\Omega} h_k$ . Then from (1.15) we obtain

$$\max_{\overline{\Omega}} h_k - \min_{\overline{\Omega}} h_k \leq C_2$$

with a positive constant  $C_2$ . Let  $\tilde{h}_k(x) = h_k(x) - \min_{\overline{\Omega}} h_k$ . Then  $\tilde{h}_k$  satisfies

$$\Delta \tilde{h}_k = 0 \quad \text{in } \Omega, \quad 0 \leq \tilde{h}_k \leq C_2.$$

Since  $\partial \tilde{h}_k / \partial x_i$ ,  $i = 1, 2$ , is harmonic, by the mean value theorem and Gauss-Green Theorem, we obtain

$$\frac{\partial \tilde{h}_k}{\partial x_i} = \frac{1}{\pi r^2} \int_{B_r} \frac{\partial \tilde{h}_k}{\partial x_i} dx = \frac{1}{\pi r^2} \int_{\partial B_r} \tilde{h}_k n_i ds$$

for  $i = 1, 2$ , where  $n = (n_1, n_2)$  is the outer normal unit vector on  $\partial B_r$ . Then it follows that

$$\left| \frac{\partial \tilde{h}_k}{\partial x_i} \right| \leq \frac{1}{\pi r^2} \int_{\partial B_r} |\tilde{h}_k| ds \leq \frac{2C_1}{r}, \quad i = 1, 2.$$

Since  $|\nabla h_k| = |\nabla \tilde{h}_k|$ , we conclude that  $\|\nabla h_k\|_{L^\infty(B_r)} = O(1)$  as  $k \rightarrow \infty$ .  $\square$

Let  $w_k(x) = u_k(x) - h_k(x)$  in  $\Omega$ . Then

$$-\Delta w_k = W_k(x) e^{w_k} \quad \text{in } \Omega, \quad w_k = 0 \quad \text{on } \partial\Omega,$$

where  $W_k(x) = e^{h_k(x)} V_k(x)$ . Let  $G(x, y)$  be the Green's function of  $-\Delta$  in  $\Omega$  with respect to the zero boundary conditions:

$$-\Delta_x G(x, y) = \delta_y, \quad x \in \Omega, \quad G(x, y) = 0, \quad x \in \partial\Omega.$$

Then we have

$$\nabla w_k(x) = \int_{\Omega} \nabla_x G(x, y) W_k(y) e^{w_k(y)} dy, \quad x \in \Omega. \quad (\text{B.1})$$

Put  $z_k(x) = W_k(x) e^{w_k(x)}$ .

**Lemma B.2.** *For  $\psi \in C_0^2(\Omega)$  we have*

$$-\int_{\Omega} (\Delta \psi) z_k dx = \int_{\Omega} (\nabla(\log W_k) \cdot \nabla \psi) z_k dx + \frac{1}{2} \iint_{\Omega \times \Omega} \rho(x, y) z_k(x) z_k(y) dx dy, \quad (\text{B.2})$$

where  $\rho(x, y) = \nabla_x G(x, y) \cdot \nabla \psi(x) + \nabla_y G(x, y) \cdot \nabla \psi(y)$ .

*Proof.* We see that

$$\nabla z_k = (\nabla W_k) e^{w_k} + W_k e^{w_k} \nabla w_k = z_k \nabla(\log W_k) + z_k \nabla w_k.$$

Then, for  $\psi \in C_0^2(\Omega)$ , we obtain

$$-\int_{\Omega} (\Delta \psi) z_k dx = \int_{\Omega} (\nabla(\log W_k) \cdot \nabla \psi) z_k dx + \int_{\Omega} (\nabla w_k \cdot \nabla \psi) z_k dx. \quad (\text{B.3})$$

From (B.1) and Fubini's Theorem, we find that

$$\int_{\Omega} (\nabla w_k(x) \cdot \nabla \psi(x)) z_k(x) dx = \iint_{\Omega \times \Omega} (\nabla_x G(x, y) \cdot \nabla \psi(x)) z_k(x) z_k(y) dx dy. \quad (\text{B.4})$$

By changing the role of  $x$  and  $y$  in (B.4) we obtain

$$\int_{\Omega} (\nabla w_k(y) \cdot \nabla \psi(y)) z_k(y) dy = \iint_{\Omega \times \Omega} (\nabla_y G(x, y) \cdot \nabla \psi(y)) z_k(x) z_k(y) dx dy.$$

Hence, we obtain

$$\int_{\Omega} (\nabla w_k \cdot \nabla \psi) z_k dx = \frac{1}{2} \iint_{\Omega \times \Omega} \rho(x, y) z_k(x) z_k(y) dx dy.$$

From (B.3) we obtain (B.2).  $\square$

Without loss of generality, we may assume that the blowup set  $\mathcal{B}$  contains  $\{0\}$ , and that there exists a  $R > 0$  satisfying  $\{x : 0 < |x| < R\} \cap \mathcal{B} = \emptyset$ . Therefore,  $\{u_k\}$  satisfies

$$\max_{\overline{B_R}} u_k \rightarrow \infty \quad \text{and} \quad \max_{\overline{B_R} \setminus B_r} u_k \rightarrow -\infty \quad \text{as } k \rightarrow \infty \quad (\text{B.5})$$

for all  $r \in (0, R)$ . Moreover,

$$V_k e^{u_k} dx \rightharpoonup \alpha \delta_0(dx) \quad (\text{B.6})$$

on  $B_R$  in the sense of measure for some  $\alpha \geq 4\pi$ .

**Lemma B.3.** *There exist constants  $r_0 \in (0, R)$  and  $a > 0$  such that  $V_k(x) \geq a$  for  $x \in B_{r_0}$ .*

*Proof.* First we show  $\liminf_{k \rightarrow \infty} V_k(0) > 0$ . Assume to the contrary that

$$\liminf_{k \rightarrow \infty} V_k(0) = 0.$$

From (1.12) and (1.14), by taking a subsequence in  $\{V_k\}$  (still denoted by  $\{V_k\}$ ), there exists  $V_0 \in C(\Omega)$  such that  $V_k \rightarrow V_0$  in  $C(\overline{B_R})$  and  $V_0(0) = 0$ .

Let  $x_k \in B_R$ ,  $u_k(x_k) = \max_{x \in \overline{B_R}} u_k(x)$ . It follows from (B.5) that

$$x_k \rightarrow 0 \quad \text{and} \quad u_k(x_k) \rightarrow \infty. \quad (\text{B.7})$$

Let  $\delta_k = e^{-u_k(x_k)/2}$ . It follows from (B.7) that  $\delta_k \rightarrow 0$ . For  $|x| \leq R/(2\delta_k)$ , we consider the sequence of functions  $v_k(x) = u_k(\delta_k x + x_k) + 2 \log \delta_k$ . Then  $v_k$  satisfies

$$-\Delta v_k(x) = V_k(\delta_k x + x_k) e^{v_k(x)} \quad \text{for } x \in B_{R/(2\delta_k)}.$$

Moreover, we have  $v_k(0) = 0$ ,  $v_k(x) \leq 0$  in  $B_{R/(2\delta_k)}$ , and

$$\int_{B_{R/(2\delta_k)}} e^{v_k(x)} dx \leq \int_{B_R} e^{u_k(x)} dx \leq C$$

for some positive constant  $C$ .

For each  $r > 0$  the sequence  $\{v_k\}$  is well defined in  $B_r$  for  $k$  large enough. It follows from Theorem A that only alternative (i) may occur, hence  $\{v_k\}$  is bounded in  $L_{\text{loc}}^\infty(B_r)$  and, by standard elliptic estimates, also in  $C_{\text{loc}}^{2,\alpha}(B_r)$ ,  $0 < \alpha < 1$ . Therefore, a subsequence in  $\{v_k\}$  converges in  $C_{\text{loc}}^2(B_r)$ . We may do the same arguments for a sequence  $r_k \rightarrow \infty$ , and pass to a diagonal subsequence (which we will still denote as  $\{v_k\}$ ) converging in  $C_{\text{loc}}^2(\mathbb{R}^2)$  to  $v$  which satisfies  $-\Delta v = V_0(0)e^v$  in  $\mathbb{R}^2$ . Moreover,  $v(0) = 0$ ,  $v \leq 0$  in  $\mathbb{R}^2$ , and

$$\int_{\mathbb{R}^2} e^v dx \leq C. \quad (\text{B.8})$$

Since  $V_0(0) = 0$ ,  $v$  is harmonic in  $\mathbb{R}^2$ . Then  $v$  is a constant. This contradicts (B.8). Thus we conclude that  $\liminf_{k \rightarrow \infty} V_k(0) > 0$ .

From (1.14) there exists constants  $r_0 \in (0, R)$  and  $a > 0$  satisfying  $V_k(x) \geq a$  for  $x \in B_{r_0}$ .

□

*Proof of Theorem 4.* We will show that  $\alpha = 8\pi$  in (B.6). Take  $\phi \in C_0^2(B_R)$  so that  $0 \leq \phi \leq 1$  and  $\phi \equiv 1$  for  $x \in B_{r_0}$ , where  $r_0$  is a constant in Lemma B.3. Let  $\psi(x) = |x|^2 \phi(x)$ .

Then we have  $\psi \in C_0^2(B_R)$ . Moreover, it follows that  $\Delta\psi(x) = 4$  and  $\nabla\psi(x) = 2x$  for  $x \in B_{r_0}$ .

We recall that  $W_k(x) = e^{h_k(x)}V_k(x)$ . Then we have

$$\nabla(\log W_k) = \frac{\nabla W_k}{W_k} = \nabla h_k + \frac{\nabla V_k}{V_k}.$$

From Lemmas B.1 and B.3 and (1.12) we obtain  $|\nabla \log W_k(x)| \leq C$  for  $x \in B_{r_0}$  with some constant  $C$ . Then we have

$$|\nabla\psi(x) \cdot \nabla(\log W_k(x))| \leq 2C|x| \quad \text{for } x \in B_{r_0}. \quad (\text{B.9})$$

We see that  $G(x, y) = -(1/2\pi) \log|x - y| + K(x, y)$ , where  $K(x, y)$  is a smooth function on  $\overline{\Omega} \times \Omega$ . Then  $\rho(x, y)$  defined in Lemma B.2 satisfies

$$\rho(x, y) = -\frac{1}{\pi} + 2x \cdot \nabla_x K(x, y) + 2y \cdot \nabla_y K(x, y) \quad \text{for } x \in B_{r_0}. \quad (\text{B.10})$$

We see that  $z_k(x) = W_k(x)e^{w_k(x)} = V_k(x)e^{v_k(x)}$ . From (B.6) we have  $z_k(x)dx \rightarrow \alpha\delta_0(dx)$  on  $B_R$  in the sense of measure. Furthermore, we have

$$z_k(x)z_k(y)dxdy \rightarrow \alpha^2\delta_{x=0}(dx) \otimes \delta_{y=0}(dy) = \alpha^2\delta_{(x,y)=(0,0)}(dxdy)$$

on  $B_R$  in the sense of measure. Letting  $k \rightarrow \infty$  in (B.2), from (B.9) and (B.10), we have  $-4\pi\alpha = -\alpha^2/(2\pi)$ . From  $\alpha \geq 4\pi$ , we obtain  $\alpha = 8\pi$ . This completes the proof of Theorem 4.  $\square$

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