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# Non-uniqueness of solutions to the Cauchy problem for semilinear heat equations with singular initial data

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**Abstract.** The Cauchy problem for semilinear heat equations with singular initial data

$$w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0, \infty) \quad \text{and} \quad w(x, 0) = \lambda a(x/|x|) |x|^{-2/(p-1)} \quad \text{in } \mathbf{R}^N \setminus \{0\}$$

is studied, where  $N \geq 2$ ,  $\lambda > 0$  is a parameter, and  $a \geq 0$ ,  $a \not\equiv 0$ . We show that when  $p > (N+2)/N$  and  $(N-2)p < N+2$ , there exists a positive constant  $\bar{\lambda}$  such that the problem has two positive self-similar solutions  $\underline{w}_\lambda$  and  $\overline{w}_\lambda$  with  $\underline{w}_\lambda < \overline{w}_\lambda$  if  $\lambda \in (0, \bar{\lambda})$  and no positive self-similar solutions if  $\lambda > \bar{\lambda}$ . Furthermore, for each fixed  $t > 0$ ,  $\underline{w}_\lambda(\cdot, t) \rightarrow 0$  and  $\overline{w}_\lambda(\cdot, t) \rightarrow w_0(\cdot, t)$  in  $L^\infty(\mathbf{R}^N)$  as  $\lambda \rightarrow 0$ , where  $w_0$  is a non-unique solution to the problem with zero initial data, which is constructed by Haraux and Weissler.

## 1. Introduction

We consider the Cauchy problem for semilinear heat equations with singular initial data:

$$(1.1) \quad w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0, \infty),$$

$$(1.2)_\lambda \quad w(x, 0) = \lambda a(x/|x|) |x|^{-2/(p-1)} \quad \text{in } \mathbf{R}^N \setminus \{0\},$$

where  $N \geq 2$ ,  $p > (N+2)/N$ ,  $a : S^{N-1} \rightarrow \mathbf{R}$ , and  $\lambda > 0$  is a parameter. We assume that  $a \in L^\infty(S^{N-1})$  and  $a \geq 0$ ,  $a \not\equiv 0$ . A typical case is  $a \equiv 1$ .

The equation (1.1) is invariant under the similarity transformation

$$w(x, t) \mapsto w_\mu(x, t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$

A solution  $w$  is said to be self-similar, when  $w = w_\mu$  for all  $\mu > 0$ , that is,

$$(1.3) \quad w(x, t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$

Such self-similar solutions are global in time and often used to describe the large time behavior of global solutions to (1.1), see, e.g., [20, 21, 4, 28, and 29].

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If  $w(x, t)$  is a self-similar solution of (1.1) and has an initial value  $A(x)$ , then we easily see that  $A$  has the form  $A(x) = A(x/|x|)|x|^{-2/(p-1)}$ . Then the problem of existence of self-similar solutions is essentially depend on the solvability of the Cauchy problem (1.1)-(1.2) $_{\lambda}$ .

It is well known by [10, 34, and 20] that if  $1 < p \leq (N + 2)/N$  then (1.1) has no time global solution  $w$  such that  $w \geq 0$  and  $w \not\equiv 0$ . Therefore, the condition  $p > (N + 2)/N$  is necessary for the existence of positive self-similar solutions of (1.1).

We briefly review some results concerning the Cauchy problem for (1.1) with initial data in  $L^q(\mathbf{R}^N)$ . Weissler [32, 33] showed that the IVP (1.1) with  $w(x, 0) = A \in L^q(\mathbf{R}^N)$  admits a unique time-local solution if  $q \geq N(p - 1)/2$ . He also showed in [34] that the solution exists time-globally if  $q = N(p - 1)/2$  and if  $\|A\|_{L^q(\mathbf{R}^N)}$  is sufficiently small. Giga [12] has constructed a unique local regular solution in  $L^\alpha(0, T : L^\beta)$ , where  $\alpha$  and  $\beta$  are chosen so that the norm of  $L^\alpha(0, T : L^\beta)$  is invariant under scaling. On the other hand, for  $1 \leq q < N(p - 1)/2$ , Haraux and Weissler [16] constructed a solution  $w_0 \in C([0, \infty); L^q(\mathbf{R}^N))$  of (1.1) satisfying  $w_0(x, t) > 0$  for  $t > 0$  and  $\|w_0(\cdot, t)\|_{L^q(\mathbf{R}^N)} \rightarrow 0$  as  $t \rightarrow 0$  when  $p > (N + 2)/N$  and  $(N - 2)p < N + 2$  by seeking solutions of self-similar form. Therefore, if  $p > (N + 2)/N$  and  $(N - 2)p < N + 2$ , the Cauchy problem

$$(1.4) \quad w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0, \infty) \quad \text{and} \quad w(x, 0) = 0 \quad \text{in } \mathbf{R}^N$$

admits a non-unique solution in  $C([0, \infty); L^q(\mathbf{R}^N))$  for  $1 \leq q < N(p - 1)/2$ .

Kozono and Yamazaki [22] constructed Besov-type function spaces based on the Morrey spaces, and then obtained global existence results for the equation (1.1) and the Navier-Stokes system with small initial data in these spaces. By [22] the problem (1.1)-(1.2) $_{\lambda}$  admits a time-global solution for sufficiently small  $\lambda > 0$ . Cazenave and Weissler [4] proved the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equations and the equations (1.1) with small initial data by using the weighted norms.

Galaktionov and Vazquez [11] have investigated the uniqueness of the solutions to the problem (1.1)-(1.2) $_{\lambda}$  with  $a \equiv 1$ . In [11, p. 41] they have conjectured that the problem (1.1)-(1.2) $_{\lambda}$  has exactly two solutions (the minimal and maximal) when  $N \geq 3$  and  $N/(N - 2) < p \leq (N + 2)/(N - 2)$ .

Letting  $\mu = t^{-1/2}$  in (1.3), we see that the self-similar solution  $w$  has the form

$$(1.5) \quad w(x, t) = t^{-1/(p-1)} u(x/\sqrt{t}),$$

where  $u$  satisfies the elliptic equation

$$(1.6) \quad \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + u^p = 0 \quad \text{in } \mathbf{R}^N.$$

By Lemma B.1 in Appendix below we find that if  $w$  satisfies  $(1.2)_\lambda$  in the sense of  $L^1_{\text{loc}}(\mathbf{R}^N)$ , that is,

$$\int_K |w(x, t) - \lambda a(x/|x|)|x|^{-2/(p-1)}|dx \rightarrow 0 \quad \text{as } t \rightarrow 0$$

for any compact subset  $K$  of  $\mathbf{R}^N$ , then  $u$  satisfies

$$(1.7)_\lambda \quad \lim_{r \rightarrow \infty} r^{2/(p-1)}u(r\omega) = \lambda a(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

Conversely, if  $u \in C^2(\mathbf{R}^N)$  is a solution of (1.6) satisfying  $(1.7)_\lambda$ , then the function  $w$  defined by (1.5) satisfies (1.1) and  $(1.2)_\lambda$  in the sense of  $L^1_{\text{loc}}(\mathbf{R}^N)$ .

In this paper we investigate the problem (1.6)-(1.7) $_\lambda$  by making use of the methods for semilinear elliptic equations, and then derive the results for the Cauchy problem (1.1)-(1.2) $_\lambda$  to give a partially affirmative answer to the conjecture by [11]. First we will state the results concerning the problem (1.6)-(1.7) $_\lambda$ , and then apply these results to the problem (1.1)-(1.2) $_\lambda$ .

Before stating our results, we introduce some notations. Set  $\rho(x) = e^{|x|^2/4}$ . We define the weighted Sobolev space

$$(1.8) \quad H^1_\rho(\mathbf{R}^N) = \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)\rho dx < \infty \right\}$$

equipped with the norms

$$\|u\|_{H^1_\rho(\mathbf{R}^N)} = \left( \int_{\mathbf{R}^N} (|\nabla u|^2 + u^2)\rho dx \right)^{1/2}.$$

It has been shown by Weissler [36, Theorem 1] and Escobedo and Kavian [8, Theorem 0.14] independently that there exists a solution  $u_0$  of the problem

$$(1.9) \quad \begin{cases} \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + u^p = 0 & \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^N, \end{cases}$$

with  $p > (N+2)/N$  and  $(N-2)p < (N+2)$ . The uniqueness of the solution to the problem (1.9) has been obtained by [25, Corollary 2].

We refer to  $u$  as a solution of (1.6) if  $u \in C^2(\mathbf{R}^N)$  is a classical solution of (1.6). Our main results are stated in the following theorems:

**Theorem 1.** Assume that  $p > (N + 2)/N$ . Then there exists a constant  $\bar{\lambda} > 0$  such that

- (i) for  $0 < \lambda < \bar{\lambda}$ , (1.6)-(1.7) $_{\lambda}$  has a minimal positive solution  $\underline{u}_{\lambda}$ ;  $\underline{u}_{\lambda}$  is increase with respect to  $\lambda$  and satisfies  $\|\underline{u}_{\lambda}\|_{L^{\infty}(\mathbf{R}^N)} = O(\lambda)$  as  $\lambda \rightarrow 0$ ;
- (ii) for  $\lambda > \bar{\lambda}$ , there are no positive solutions of (1.6)-(1.7) $_{\lambda}$ ;

**Theorem 2.** Assume that  $p > (N + 2)/N$  and  $(N - 2)p < N + 2$ . Let  $\bar{\lambda} > 0$  be the constant in Theorem 1. Then, for  $0 < \lambda < \bar{\lambda}$ , (1.6)-(1.7) $_{\lambda}$  has a positive solution  $\bar{u}_{\lambda}$  satisfying  $\bar{u}_{\lambda} > \underline{u}_{\lambda}$  and

$$\bar{u}_{\lambda} - \underline{u}_{\lambda} \in H_{\rho}^1(\mathbf{R}^N) \quad \text{and} \quad \bar{u}_{\lambda}(x) - \underline{u}_{\lambda}(x) = O(e^{-|x|^2/4}) \quad \text{as } |x| \rightarrow \infty.$$

Furthermore

$$\bar{u}_{\lambda} - \underline{u}_{\lambda} \rightarrow u_0 \quad \text{in } H_{\rho}^1(\mathbf{R}^N) \cap L^{\infty}(\mathbf{R}^N) \quad \text{as } \lambda \rightarrow 0,$$

where  $u_0$  is the solution of the problem (1.9). In particular, we have  $\bar{u}_{\lambda} \rightarrow u_0$  in  $L^{\infty}(\mathbf{R}^N)$  as  $\lambda \rightarrow 0$ .

*Remark 1.* (i) We now restrict our attention to radial solutions of (1.6), i.e., a solution of the form  $u = u(r)$ ,  $r = |x|$ . Then  $u(r)$  must satisfies the initial value problem

$$\begin{cases} u_{rr} + \left( \frac{N-1}{r} + \frac{r}{2} \right) u_r + \frac{1}{p-1} u + |u|^{p-1} u = 0, & r > 0, \\ u(0) = \alpha > 0 \quad \text{and} \quad u_r(0) = 0. \end{cases}$$

We denote by  $u(r; \alpha)$  the unique solution of this problem. It has been shown by Haraux-Weissler [16, Proposition 3.4 and Theorem 5] that  $u(r; \alpha)$  has the following properties: the limit  $L(\alpha) = \lim_{r \rightarrow \infty} r^{2/(p-1)} u(r; \alpha)$  exists and is a locally Lipschitz function of  $\alpha \in \mathbf{R}$ ; if  $p > (N + 2)/N$  then  $u(r; \alpha) > 0$  for all  $r > 0$  and  $L(\alpha) > 0$  for sufficiently small  $\alpha > 0$ ; if in addition  $(N - 2)p < (N + 2)$  then there exists some  $\alpha_0 > 0$  such that  $L(\alpha_0) = 0$  and  $u(r; \alpha_0) > 0$  for  $r > 0$ ;  $u(r; \alpha)$  is positive for  $r > 0$  if  $0 < \alpha < \alpha_0$ . (See the proof of Proposition 3.7 in [16].) Moreover, it has been shown by Yanagida [38, Theorem 1] and Dohmen-Hirose [7, Theorem 1.2 and Corollary 1.3] that  $u(r; \alpha)$  cannot be positive for all  $r > 0$  if  $\alpha > \alpha_0$ . (In fact, they in [38, 7] showed the uniqueness of the  $\alpha_0$  such that  $L(\alpha_0) = 0$  and  $u(r; \alpha_0) > 0$  for  $r > 0$ .) From these facts we find that, when  $p > (N + 2)/N$  and  $(N - 2)p < (N + 2)$ ,  $u(r; \alpha) > 0$  for  $r > 0$  if and only if  $\alpha \in (0, \alpha_0]$ , and that  $L$  is a nonnegative continuous function on  $[0, \alpha_0]$  with

$L(0) = L(\alpha_0) = 0$  and  $L \not\equiv 0$ . Let  $\bar{\lambda} = \max\{L(\alpha) : 0 \leq \alpha \leq \alpha_0\}$ . Then  $\bar{\lambda} > 0$ , and it is clear that there are at least two different values of  $\alpha$  satisfying  $L(\alpha) = \lambda$  if  $\lambda \in (0, \bar{\lambda})$ , and there exists at least one value of  $\alpha$  satisfying  $L(\alpha) = \bar{\lambda}$ . Thus, when  $p > (N+2)/N$  and  $(N-2)p < (N+2)$ , the problem (1.6)-(1.7) $_{\lambda}$  with  $a \equiv 1$  has at least two radial solutions if  $0 < \lambda < \bar{\lambda}$ , and at least one radial solution if  $\lambda = \bar{\lambda}$ . Moreover, there is no radial positive solution of (1.6)-(1.7) $_{\lambda}$  with  $a \equiv 1$  if  $\lambda > \bar{\lambda}$ .

(ii) In the case  $N = 1$ , Weissler [35] has shown that  $u(r; \alpha) > 0$  for  $r > 0$  if and only if  $\alpha \in (0, \alpha_0]$  for some  $\alpha_0 > 0$ , and that  $L$  is a positive concave function on  $(0, \alpha_0)$  with  $L(0) = L(\alpha_0) = 0$ . Thus there are precisely two different value of  $\alpha$  satisfying  $L(\alpha) = \lambda$  if  $0 < \lambda < \bar{\lambda}$ , and so the problem has precisely two solutions if  $0 < \lambda < \bar{\lambda}$ .

(iii) At this time we do not know whether (1.6)-(1.7) $_{\lambda}$  with  $\lambda = \bar{\lambda}$  has a positive solution when  $a$  is not constant. The exact multiplicity of positive solutions of (1.6)-(1.7) $_{\lambda}$  for  $\lambda \in (0, \bar{\lambda}]$  is an open question.

Now we consider the Cauchy problem (1.1)-(1.2) $_{\lambda}$ . We refer to  $w$  as a solution of (1.1) if  $w \in C^2(\mathbf{R}^N \times (0, \infty))$  is a classical solution of (1.1). If  $u$  is a solution of (1.6)-(1.7) $_{\lambda}$ , then the function  $w$  defined by (1.5) is a solution of (1.1) satisfying (1.2) $_{\lambda}$  in the sense of  $L^1_{\text{loc}}(\mathbf{R}^N)$  by Lemma B.1 below. Put

$$(1.10) \quad w_0(x, t) = t^{-1/(p-1)} u_0(x/\sqrt{t}),$$

where  $u_0$  is the solution of the problem (1.9). It has been shown by [8, Proposition 3.5] that  $u_0 \in C^2(\mathbf{R}^N)$  and  $u_0(x) = O(e^{-|x|^2/8})$  as  $|x| \rightarrow \infty$ . (See, also, [26, Theorem 1].) Then we have  $u_0 \in L^q(\mathbf{R}^N)$  for all  $q \geq 1$  and

$$\|w_0(\cdot, t)\|_{L^q(\mathbf{R}^N)} = t^{-1/(p-1)+N/2q} \|u_0\|_{L^q(\mathbf{R}^N)}.$$

Consequently,  $w_0$  solves the the Cauchy problem (1.4) in  $C([0, \infty); L^q(\mathbf{R}^N))$  for  $1 \leq q < N(p-1)/2$ . We note that the positive solution  $u$  of (1.6) satisfying

$$u(x) = o(|x|^{-2/(p-1)}) \quad \text{as } |x| \rightarrow \infty$$

is unique and radially symmetric by [25, Corollary 1]. Therefore,  $w_0$  defined by (1.10) coincides with the non-unique solution constructed by Haraux and Weissler [16].

As a consequence of Theorems 1 and 2, we obtain the following results.

**Corollary 1.** Assume that  $p > (N + 2)/N$ . Then there exists a constant  $\bar{\lambda} > 0$  such that

(i) for  $0 < \lambda < \bar{\lambda}$ , (1.1)-(1.2) $_{\lambda}$  has a positive self-similar solution  $\underline{w}_{\lambda}$ ; for each fixed  $t > 0$ , the solution  $\underline{w}_{\lambda}(\cdot, t)$  is increasing with respect to  $\lambda$  and satisfies

$$\|\underline{w}_{\lambda}(\cdot, t)\|_{L^{\infty}(\mathbf{R}^N)} = O(\lambda) \quad \text{as } \lambda \rightarrow 0;$$

(ii) for  $\lambda > \bar{\lambda}$ , (1.1)-(1.2) $_{\lambda}$  has no positive self-similar solutions;

**Corollary 2.** Assume that  $p > (N + 2)/N$  and  $(N - 2)p < N + 2$ . Let  $\bar{\lambda} > 0$  be the constant in Corollary 1. Then, for  $0 < \lambda < \bar{\lambda}$ , (1.1)-(1.2) $_{\lambda}$  has a positive self-similar solution  $\bar{w}_{\lambda}$  satisfying  $\bar{w}_{\lambda}(x, t) > \underline{w}_{\lambda}(x, t)$  for  $(x, t) \in (\mathbf{R}^N \times (0, \infty))$ ; the solution  $\bar{w}_{\lambda}$  satisfies, for each fixed  $t > 0$ ,

$$\|\bar{w}_{\lambda}(\cdot, t) - w_0(\cdot, t)\|_{L^{\infty}(\mathbf{R}^N)} \rightarrow 0 \quad \text{as } \lambda \rightarrow 0,$$

where  $w_0$  is the non-unique solution of (1.4) in  $C([0, \infty); L^q(\mathbf{R}^N))$  for  $1 \leq q < N(p - 1)/2$ , which is constructed by [16].

*Remark 2.* (i) The existence of a positive self-similar solution of (1.1)-(1.2) $_{\lambda}$  has been shown by [4] under a weaker condition on  $a$ .

(ii) It is already known that there is no solutions of (1.1)-(1.2) $_{\lambda}$  if  $\lambda$  is large enough, see, e.g., [33, Corollary 5.1], [37, Corollary 1.1], and [24, Remark 3.7]. These results, however, do not quite apply to self-similar solutions in stated, we easily see that the proofs easily apply to self-similar solutions, or any positive measurable solutions.

(iii) From (i) of Remark 1 we see that, when  $p > (N + 2)/N$  and  $(N - 2)p < (N + 2)$ , the problem (1.1)-(1.2) $_{\lambda}$  with  $\lambda = \bar{\lambda}$  has a radial positive self-similar solution if  $a \equiv 1$ . It is an open question whether (1.1)-(1.2) $_{\lambda}$  with  $\lambda = \bar{\lambda}$  has a positive self-similar solution if  $a$  is not constant. The exact multiplicity of positive self-similar solutions of (1.1)-(1.2) $_{\lambda}$  for  $\lambda \in (0, \bar{\lambda}]$  is still an open question.

We prove Theorem 1 by using of the explicit supersolution and comparison arguments based on the maximum principle. We prove Theorem 2 by variational approach essentially due to Ambrosetti-Rabinowitz [1] and Crandall-Rabinowitz [5].

As far as we are aware, the idea of constructing self-similar solutions by solving the initial value problem for homogeneous initial data was first used by Giga and

Miyakawa [13], for the Navier-Stokes equation in vorticity form. The idea of [13] was used later by several authors for various problems. Concerning the equation

$$(1.11) \quad u_t - \Delta u + u^p = 0 \quad \text{in } \mathbf{R}^N,$$

we refer to Kwak [23] and Cazenave et. al. [3]. They also obtained the asymptotically self-similar behavior for a class of general solutions. See, also [15, 19, 9, and 18].

After the paper was completed, we learned the work by Souplet and Weissler [30] where the existence of radial self-similar solutions of (1.6) were studied precisely in the subcritical, supercritical, and critical cases by using a shooting argument.

This paper is organized as follows: in Section 2 we show the maximum principle and comparison results for the operator related to the equation (1.6). In Section 3 we consider the linearized eigenvalue problems. Sections 4 and 5 devoted to the proofs of Theorems 1 and 2, respectively. For completeness, we show the regularity and some properties of the solutions in the appendixes.

In the remaining part of the paper, we assume that  $p > (N + 2)/N$ .

## 2. Preliminaries

In this section we show the following two propositions which are crucial for the proofs of the theorems. For simplicity, we define  $Lu$  by

$$Lu = -\Delta u - \frac{1}{2}x \cdot \nabla u - \frac{1}{p-1}u$$

for  $u \in C^2(\mathbf{R}^N)$ .

**Proposition 2.1.** *Assume that  $Lu \geq 0$  in  $\mathbf{R}^N$ , and that*

$$\liminf_{|x| \rightarrow \infty} |x|^{2/(p-1)} u(x) \geq 0.$$

*Then  $u > 0$  or  $u \equiv 0$  in  $\mathbf{R}^N$ . In particular, if  $Lu \geq 0$  and  $u \geq 0$  in  $\mathbf{R}^N$  then  $u > 0$  or  $u \equiv 0$  in  $\mathbf{R}^N$ .*

**Proposition 2.2.** *Assume that  $\alpha, \beta : S^{N-1} \rightarrow \mathbf{R}$  satisfy  $\alpha, \beta \in L^\infty(S^{N-1})$  and*

$$0 \leq \alpha(\omega) \leq \beta(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

*Suppose that there exists a positive function  $v$  satisfying  $Lv \geq v^p$  in  $\mathbf{R}^N$  and*

$$\lim_{r \rightarrow \infty} r^{2/(p-1)} v(r\omega) = \beta(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

Then there exists a positive solution  $u$  of  $Lu = u^p$  in  $\mathbf{R}^N$  satisfying  $u \leq v$  in  $\mathbf{R}^N$  and

$$(2.1) \quad \lim_{r \rightarrow \infty} r^{2/(p-1)} u(r\omega) = \alpha(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

Moreover, for any positive function  $w$  satisfying  $Lw \geq w^p$  in  $\mathbf{R}^N$  and

$$\liminf_{r \rightarrow \infty} r^{2/(p-1)} w(r\omega) \geq \alpha(\omega) \quad \text{for a.e. } \omega \in S^{N-1},$$

we have  $u \leq w$  in  $\mathbf{R}^N$ .

First we show the following lemma.

**Lemma 2.1.** *Assume that  $\alpha : S^{N-1} \rightarrow \mathbf{R}$  satisfies  $\alpha \in L^\infty(S^{N-1})$  and  $\alpha \geq 0$ ,  $\alpha \not\equiv 0$  for a.e.  $\omega \in S^{N-1}$ . Then there exists a positive function  $\phi_\alpha \in C^2(\mathbf{R}^N)$  satisfying  $L\phi_\alpha = 0$  in  $\mathbf{R}^N$  and*

$$(2.2) \quad \lim_{r \rightarrow \infty} r^{2/(p-1)} \phi_\alpha(r\omega) = \alpha(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

*Proof.* Put

$$(2.3) \quad w(x, t) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbf{R}^N} e^{-|x-y|^2/(4t)} \alpha(y/|y|) |y|^{-2/(p-1)} dy.$$

We note that  $\alpha(y/|y|) |y|^{-2/(p-1)} \in L^1_{\text{loc}}(\mathbf{R}^N)$  from  $p > (N+2)/N$ . By [6, Chapter 5, Theorem 6.1],  $w$  defined by (2.3) satisfies  $w_t = \Delta w$  in  $\mathbf{R}^N \times (0, \infty)$  and

$$w(x, t) \rightarrow \alpha(x/|x|) |x|^{-2/(p-1)} \quad \text{in } L^1_{\text{loc}}(\mathbf{R}^N) \quad \text{as } t \rightarrow 0.$$

Define the rescaled functions  $w_\mu$  by  $w_\mu(x, t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t)$  for  $\mu > 0$ . From (2.3) we obtain  $w(x, t) = w_\mu(x, t)$  for all  $\mu > 0$ . Putting  $\mu = 1/\sqrt{t}$ , we find that

$$(2.5) \quad w(x, t) = t^{-1/(p-1)} \phi_\alpha(x/\sqrt{t}),$$

where  $\phi_\alpha(x) = w(x, 1)$ . It can be easily checked that  $\phi_\alpha$  satisfies  $L\phi_\alpha = 0$  in  $\mathbf{R}^N$ . By Lemma B.1 in Appendix B below, we obtain (2.2).  $\square$

**Lemma 2.2.** *Let  $\Omega \subset \mathbf{R}^N$  be a bounded domain with smooth boundary  $\partial\Omega$ .*

(i) *Assume that  $Lu \geq 0$  in  $\Omega$  and  $u \geq 0$  on  $\partial\Omega$ . Then  $u > 0$  or  $u \equiv 0$  in  $\Omega$ .*

(ii) *Assume that  $f \in C^\theta(\Omega)$  for some  $\theta \in (0, 1)$  and  $g \in C(\partial\Omega)$ . Then there exists a solution  $u$  of  $Lu = f$  in  $\Omega$  and  $u = g$  on  $\partial\Omega$ .*

*Proof.* (i) From Lemma 2.1 there exists a positive function  $\phi$  satisfying

$$(2.6) \quad L\phi = 0 \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{r \rightarrow \infty} r^{2/(p-1)} \phi(r\omega) = 1 \quad \text{for a.e. } \omega \in S^{N-1}.$$

Let  $v(x) = u(x)/\phi(x)$ . Then  $v$  satisfies

$$-\Delta v - \left( \frac{2}{\phi} \nabla \phi + \frac{1}{2} x \right) \cdot \nabla v \geq 0 \quad \text{in } \Omega \quad \text{and} \quad v \geq 0 \quad \text{on } \partial\Omega.$$

By the maximum principle [27] we have  $v > 0$  or  $v \equiv 0$  in  $\Omega$ , which implies that  $u > 0$  or  $u \equiv 0$  in  $\Omega$ .

(ii) Let  $\phi$  be a positive function satisfying (2.6). We have a solution  $v \in C^{2,\theta}(\Omega)$  of

$$\begin{cases} -\Delta v - \left( \frac{2}{\phi} \nabla \phi + \frac{1}{2} x \right) \cdot \nabla v = \frac{f}{\phi} & \text{in } \Omega, \text{ and} \\ v = \frac{g}{\phi} & \text{on } \partial\Omega. \end{cases}$$

(See, e.g., [14].) Then  $u(x) = v(x)\phi(x)$  satisfies  $Lu = f$  in  $\Omega$  and  $u = g$  on  $\partial\Omega$ .

□

*Proof of Proposition 2.1.* Let  $v(x) = u(x)/\phi(x)$ , where  $\phi$  is a positive function satisfying (2.6). Then  $v$  satisfies

$$-\Delta v - \left( \frac{2}{\phi} \nabla \phi + \frac{1}{2} x \right) \cdot \nabla v \geq 0 \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \liminf_{|x| \rightarrow \infty} v(x) \geq 0.$$

First we show  $v \geq 0$  in  $\mathbf{R}^N$ . Assume to the contrary that  $v(x_0) < 0$  for some  $x_0 \in \mathbf{R}^N$ . Choose  $\varepsilon > 0$  so small that  $\varepsilon < -v(x_0)$ , and take  $R > 0$  so large that  $R > |x_0|$  and  $v(x) \geq -\varepsilon$  on  $|x| = R$ . By the maximum principle [27] we have  $v \geq -\varepsilon$  in  $|x| \leq R$ . This contradicts  $v(x_0) < -\varepsilon$ . Hence,  $v \geq 0$  in  $\mathbf{R}^N$ . As a consequence of (i) of Lemma 2.2 we have  $v > 0$  or  $v \equiv 0$  in  $\mathbf{R}^N$ , which implies that  $u > 0$  or  $u \equiv 0$  in  $\mathbf{R}^N$ . □

**Lemma 2.3.** *Let  $f \in C_{\text{loc}}^\theta(\mathbf{R}^N)$  for some  $\theta \in (0, 1)$ , and let  $f \geq 0$ . Assume that there exists a positive function  $v$  such that  $Lv \geq f$  in  $\mathbf{R}^N$ . Then there exists a solution  $u$  of  $Lu = f$  in  $\mathbf{R}^N$  such that  $0 \leq u \leq v$  in  $\mathbf{R}^N$ .*

*Proof.* Define  $B_r = \{x \in \mathbf{R}^N : |x| < r\}$  for  $r > 0$ . From (ii) of Lemma 2.2, there exists a solution  $u_k$  of

$$Lu_k = f \quad \text{in } B_k \quad \text{and} \quad u_k = v \quad \text{on } \partial B_k$$

for each  $k = 1, 2, \dots$ . From (i) of Lemma 2.2 we have  $u_k > 0$  in  $B_k$ . Put  $w_k(x) = v(x) - u_k(x)$ . Then  $w_k$  satisfies  $Lw_k \geq 0$  in  $B_k$  and  $w_k = 0$  on  $\partial B_k$ . From (i) of Lemma 2.2 again we have  $w_k \geq 0$ . Thus we have  $0 < u_k \leq v$  in  $B_k$ .

Take  $R > 0$ . Since  $u_k$  satisfies  $Lu_k = f$  in  $B_R$  for  $k \geq R$ , by the Schauder estimates  $\{u_k\}$  is bounded in  $C_{\text{loc}}^{2,\theta}(B_R)$  for some  $0 < \theta < 1$ . Then, by the Ascoli-Arzelà, a subsequence in  $\{u_k\}$  converges in  $C_{\text{loc}}^2(B_R)$ . We may do the same arguments for a sequence  $\{R_n\}$  such that  $R_n \rightarrow \infty$  as  $n \rightarrow \infty$ . By the diagonal method there exists a function  $u \in C^2(\mathbf{R}^N)$  such that a subsequence converges to  $u$  in  $C_{\text{loc}}^2(\mathbf{R}^N)$ . Thus  $u$  satisfies  $Lu = f$  in  $\mathbf{R}^N$  with  $0 \leq u \leq v$  in  $\mathbf{R}^N$ . This concludes the proof.  $\square$

**Lemma 2.4.** *Let  $\phi_\alpha$  be a positive function satisfying  $L\phi_\alpha = 0$  in  $\mathbf{R}^N$  and (2.2). Assume that there exists a positive function  $\hat{v}$  satisfying*

$$L\hat{v} \geq (\hat{v} + \phi_\alpha)^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} \hat{v}(x) = 0.$$

*Then there exists a solution  $\hat{u}$  of  $L\hat{u} = (\hat{u} + \phi_\alpha)^p$  in  $\mathbf{R}^N$  satisfying  $0 \leq \hat{u} \leq \hat{v}$ . Moreover, for any positive function  $\hat{w}$  satisfying*

$$(2.7) \quad L\hat{w} \geq (\hat{w} + \phi_\alpha)^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \liminf_{|x| \rightarrow \infty} |x|^{2/(p-1)} \hat{w}(x) \geq 0,$$

*we have  $\hat{u} \leq \hat{w}$  in  $\mathbf{R}^N$ .*

*Proof.* For each  $u \in C^2(\mathbf{R}^N)$ , we define the mapping  $Tu$  as follows:  $v = Tu$  if

$$(2.8) \quad Lv = (u + \phi_\alpha)^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad 0 \leq v \leq \hat{v} \quad \text{in } \mathbf{R}^N.$$

Assume that  $0 \leq u \leq \hat{v}$ . Since  $\hat{v}$  satisfies  $L\hat{v} \geq (u + \phi_\alpha)^p$ , from Lemma 2.3 there exists a function  $v$  satisfying (2.8). Then the mapping  $T$  is well defined for each  $u \in C^2(\mathbf{R}^N)$  satisfying  $0 \leq u \leq \hat{v}$ . We also find that

$$(2.9) \quad \lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} Tu(x) = 0$$

from  $0 \leq Tu \leq \hat{v}$  and  $\lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} \hat{v}(x) = 0$ .

Assume that  $u_1, u_2 \in C^2(\mathbf{R}^N)$ . We show that  $0 \leq u_1 < u_2 \leq \hat{v}$  implies  $Tu_1 < Tu_2$ . In fact, if  $u_1 < u_2$  then  $L(Tu_2 - Tu_1) > 0$  in  $\mathbf{R}^N$ . From (2.9) we have

$$\lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} (Tu_2(x) - Tu_1(x)) = 0.$$

Hence, from Proposition 2.1 we have  $Tu_1 < Tu_2$ .

Define  $\{\hat{u}_k\}$  inductively by

$$(2.10) \quad \hat{u}_0 \equiv 0 \quad \text{and} \quad \hat{u}_k = T\hat{u}_{k-1} \quad \text{for } k = 1, 2, \dots$$

Since we have  $L(T\hat{u}_0) = \phi_\alpha^p > 0$  in  $\mathbf{R}^N$  and  $\lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} T\hat{u}_0(x) = 0$ , we obtain  $T\hat{u}_0 > 0$  in  $\mathbf{R}^N$  by Proposition 2.1. Then, by induction,  $\hat{u}_k$  is well defined and satisfies

$$0 \equiv \hat{u}_0 < \hat{u}_1 < \dots < \hat{u}_k < \hat{u}_{k+1} < \dots < \hat{v} \quad \text{in } \mathbf{R}^N.$$

Define  $\hat{u}(x) = \lim_{k \rightarrow \infty} \hat{u}_k(x)$ . Take  $R > 0$  and define  $B_R = \{x \in \mathbf{R}^N : |x| < R\}$ . Since  $\{\hat{u}_k\}$  satisfies

$$L\hat{u}_k = (\hat{u}_{k-1} + \phi_\alpha)^p \leq (\hat{v} + \phi_\alpha)^p \quad \text{in } B_R,$$

it follows from elliptic interior estimates that  $\{\hat{u}_k\}$  is bounded in  $W_{\text{loc}}^{2,p}(B_R)$  for every  $p > 1$ . By the Sobolev embedding theorem and the Schauder estimates,  $\{\hat{u}_k\}$  is bounded in  $C_{\text{loc}}^{2,\theta}(B_R)$  for some  $\theta \in (0, 1)$ . Therefore,  $\{\hat{u}_k\}$  converges to  $\hat{u}$  in  $C_{\text{loc}}^2(B_R)$ . We may do the same arguments for a sequence  $\{R_n\}$  such that  $R_n \rightarrow \infty$  as  $n \rightarrow \infty$ . By the diagonal method  $\{\hat{u}_k\}$  converges to  $\hat{u}$  in  $C_{\text{loc}}^2(\mathbf{R}^N)$ , and thus we have  $L\hat{u} = (\hat{u} + \phi_\alpha)^p$  and  $0 < \hat{u} \leq \hat{v}$  in  $\mathbf{R}^N$ .

Let  $\hat{w}$  be a positive function satisfying (2.7). We claim that  $\hat{w} > u$  implies  $\hat{w} > Tu$  for  $u \in C^2(\mathbf{R}^N)$  satisfying  $0 \leq u \leq \hat{v}$ . In fact, if  $\hat{w} > u$  we have  $L(\hat{w} - Tu) > 0$  in  $\mathbf{R}^N$  and

$$\liminf_{|x| \rightarrow \infty} |x|^{2/(p-1)} (\hat{w} - Tu(x)) = \liminf_{|x| \rightarrow \infty} |x|^{2/(p-1)} \hat{w} \geq 0.$$

From Proposition 2.1 we obtain  $\hat{w} > Tu$ .

Let  $\{\hat{u}_n\}$  be the sequence defined by (2.10). Then we have  $\hat{w} > \hat{u}_0 \equiv 0$  and  $\hat{w} > \hat{u}_k$  for  $k = 1, 2, \dots$ , by induction. Therefore, we have  $\hat{w} \geq \hat{u}$ .  $\square$

*Proof of Proposition 2.2.* Let  $\phi_\beta$  be a positive function satisfying  $L\phi_\beta = 0$  in  $\mathbf{R}^N$  and

$$\lim_{r \rightarrow \infty} r^{2/(p-1)} \phi(r\omega) = \beta(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.$$

Then, from Proposition 2.1 we have  $\phi_\alpha \leq \phi_\beta$  in  $\mathbf{R}^N$ . Define  $\hat{v}(x) = v(x) - \phi_\beta(x)$ . From  $L\hat{v} = v^p > 0$  in  $\mathbf{R}^N$  and  $\lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} \hat{v}(x) = 0$  we have  $\hat{v} > 0$  by Proposition 2.1. We also find that  $\hat{v}$  satisfies  $L\hat{v} \geq (\hat{v} + \phi_\beta)^p \geq (\hat{v} + \phi_\alpha)^p$  in  $\mathbf{R}^N$ . Then it follows from Lemma 2.4 that there exists a solution  $\hat{u}$  of  $L\hat{u} = (\hat{u} + \phi_\alpha)^p$  in  $\mathbf{R}^N$  satisfying  $0 \leq \hat{u} \leq \hat{v}$  in  $\mathbf{R}^N$ . In particular, we have  $\lim_{|x| \rightarrow \infty} |x|^{2/(p-1)} \hat{u}(x) = 0$ . Put  $u = \hat{u} + \phi_\alpha$ . Then  $u$  satisfies  $Lu = u^p$  in  $\mathbf{R}^N$  with (2.1).

Define  $\hat{w} = w - \phi_\alpha$ . Then  $\hat{w}$  satisfies

$$L\hat{w} \geq w^p = (\hat{w} + \phi_\alpha)^p > 0 \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \liminf_{|x| \rightarrow \infty} |x|^{2/(p-1)} \hat{w}(x) \geq 0.$$

Proposition 2.1 implies that  $\hat{w} > 0$  in  $\mathbf{R}^N$ . From Lemma 2.4 we have  $\hat{u} \leq \hat{w}$ , which implies  $u \leq w$  in  $\mathbf{R}^N$ . This completes the proof of Proposition 2.2.  $\square$

### 3. Eigenvalue problems

We recall here some results about the weighted Sobolev space  $H_\rho^1(\mathbf{R}^N)$  defined by (1.8). For  $1 \leq p < \infty$ , we define

$$L_\rho^p(\mathbf{R}^N) = \left\{ u \in L^p(\mathbf{R}^N) : \int_{\mathbf{R}^N} u^p \rho dx < \infty \right\} \quad \text{and} \quad \|u\|_{L_\rho^p} = \left( \int_{\mathbf{R}^N} u^p \rho dx \right)^{1/p},$$

where  $\rho(x) = e^{|x|^2/4}$ .

**Lemma 3.1.** (i) *For every  $u \in H_\rho^1(\mathbf{R}^N)$ , we have*

$$\frac{N}{2} \int_{\mathbf{R}^N} u^2 \rho dx \leq \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx.$$

(ii) *The embedding  $H_\rho^1(\mathbf{R}^N) \subset L_\rho^2(\mathbf{R}^N)$  is compact.*

(iii) *If  $N \geq 3$ , then the embedding  $H_\rho^1(\mathbf{R}^N) \subset L_\rho^{p+1}(\mathbf{R}^N)$  is continuous for  $1 \leq p \leq (N+2)/(N-2)$ , and is compact for  $1 < p < (N+2)/(N-2)$ . If  $N = 2$  then the embedding  $H_\rho^1(\mathbf{R}^2) \subset L_\rho^{p+1}(\mathbf{R}^2)$  is continuous and compact for  $p > 1$ .*

For the proof, see Escobedo and Kavian [8] and Kavian [20]. From (i) of Lemma 3.1, for  $u \in H_\rho^1(\mathbf{R}^N)$  we have

$$(3.1) \quad \int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx \geq \left( \frac{N}{2} - \frac{1}{p-1} \right) \int_{\mathbf{R}^N} u^2 \rho dx, \quad \text{and}$$

$$(3.2) \quad \int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx \geq \left( 1 - \frac{2}{N(p-1)} \right) \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx.$$

Let us consider the eigenvalue problem

$$(3.3) \quad \begin{cases} -\Delta u - \frac{1}{2} x \cdot \nabla u - \frac{1}{p-1} u = \mu m(x) u & \text{in } \mathbf{R}^N, \\ u \in H_\rho^1(\mathbf{R}^N), \end{cases}$$

where  $m \in L^\infty(\mathbf{R}^N) \cap C^\theta(\mathbf{R}^N)$  for some  $\theta \in (0, 1)$  and  $m > 0$  in  $\mathbf{R}^N$ . First, we show the following:

**Lemma 3.2.** *The problem (3.3) has the first eigenvalue  $\mu_0 > 0$  and the corresponding eigenfunction  $u_0 > 0$  in  $\mathbf{R}^N$ . Furthermore, we have*

$$(3.4) \quad \mu_0 = \inf \left\{ \int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx : u \in H_\rho^1(\mathbf{R}^N), \int_{\mathbf{R}^N} m u^2 \rho dx = 1 \right\}.$$

*Proof.* We claim that  $\mu_0 > 0$  and the minimization problem (3.4) is achieved by some function  $u_0 > 0$ . First we show  $\mu_0 > 0$ . Indeed, we see that

$$1 = \int_{\mathbf{R}^N} m u^2 \rho dx \leq \|m\|_{L^\infty} \int_{\mathbf{R}^N} u^2 \rho dx.$$

Then it follows from (3.1) that

$$\int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx \geq \left( \frac{N}{2} - \frac{1}{p-1} \right) \frac{1}{\|m\|_{L^\infty}}$$

for  $u \in H_\rho^1(\mathbf{R}^N)$ , which implies  $\mu_0 > 0$ .

Let  $\{u_k\} \subset H_\rho^1(\mathbf{R}^N)$  be a minimizing sequence of  $\mu_0$ , that is,

$$\int_{\mathbf{R}^N} m u_k^2 \rho dx = 1 \quad \text{and} \quad \int_{\mathbf{R}^N} \left( |\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx \rightarrow \mu_0 \quad \text{as } k \rightarrow \infty.$$

From (3.2) and (i) of Lemma 3.1 we find that  $\{u_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ . Then, from (ii) of Lemma 3.1, there exist a subsequence that we still denoted  $\{u_k\}$  and a function  $u_0 \in H_\rho^1(\mathbf{R}^N)$  such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } H_\rho^1(\mathbf{R}^N) \quad \text{as } k \rightarrow \infty,$$

$$u_k \rightarrow u_0 \quad \text{strongly in } L_\rho^2(\mathbf{R}^N) \quad \text{as } k \rightarrow \infty.$$

Then we obtain

$$\int_{\mathbf{R}^N} \left( |\nabla u_0|^2 - \frac{1}{p-1} u_0^2 \right) \rho dx \leq \liminf_{k \rightarrow \infty} \int_{\mathbf{R}^N} \left( |\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx = \mu_0$$

and

$$1 = \lim_{k \rightarrow \infty} \int_{\mathbf{R}^N} m u_k^2 \rho dx = \int_{\mathbf{R}^N} m u_0^2 \rho dx.$$

Hence, we find that  $u_0$  achieves  $\mu_0$ . Clearly,  $|u_0|$  also achieves  $\mu_0$ . By the elliptic regularity theory and Proposition 2.1, we have  $u_0 \in C^2(\mathbf{R}^N)$  and  $u_0 > 0$  in  $\mathbf{R}^N$ .

□

In this section we show the following two propositions.

**Proposition 3.1.** *Assume that there is a positive function  $w \in C^2(\mathbf{R}^N)$  satisfying*

$$(3.5) \quad \Delta w + \frac{1}{2}x \cdot \nabla w + \frac{1}{p-1}w + \mu m(x)w \leq 0, \quad x \in \mathbf{R}^N,$$

*for some  $\mu \in \mathbf{R}$ . Then  $\mu \leq \mu_0$ , where  $\mu_0$  is the first eigenvalue of the problem (3.3).*

**Proposition 3.2.** *Assume that  $m_1, m_2 \in L^\infty(\mathbf{R}^N)$  satisfy  $0 < m_1(x) \leq m_2(x)$ ,  $m_1(x) \not\equiv m_2(x)$ . Let  $\mu_i$  be the first eigenvalue of the problem*

$$(3.6)_i \quad \begin{cases} -\Delta u - \frac{1}{2}x \cdot \nabla u - \frac{1}{p-1}u = \mu m_i(x)u & \text{in } \mathbf{R}^N, \\ u \in H_\rho^1(\mathbf{R}^N), \end{cases}$$

*for each  $i = 1, 2$ . Then  $\mu_1 > \mu_2$ .*

To prove Proposition 3.1, we consider the eigenvalue problems

$$(3.7)_k \quad \begin{cases} -\Delta v - \frac{1}{2}x \cdot \nabla v - \frac{1}{p-1}v = \mu m(x)v & \text{in } B_k, \\ v \in H_0^1(B_k), \end{cases}$$

where  $B_k = \{x \in \mathbf{R}^N : |x| < k\}$ , for  $k = 1, 2, \dots$ . We can prove that the problem (3.7)<sub>k</sub> has the first eigenvalue  $\mu_k > 0$  and the corresponding eigenfunction  $v_k > 0$  in  $B_k$ . Furthermore, we find that

$$(3.8)_k \quad \mu_k = \inf \left\{ \int_{B_k} \left( |\nabla v|^2 - \frac{1}{p-1}v^2 \right) \rho dx : v \in H_0^1(B_k), \int_{B_k} m v^2 \rho dx = 1 \right\},$$

and that  $v_k \in C^2(\overline{B_k})$  achieves the minimization (3.8)<sub>k</sub>.

Suppose that  $v \in H_0^1(B_k)$ , and extend  $v$  to be zero outside  $B_k$ . Then  $v \in H_0^1(B_{k+1})$ . From (3.8)<sub>k</sub> we have  $\mu_k \geq \mu_{k+1}$  for  $k = 1, 2, \dots$

**Lemma 3.3.** *We have  $\lim_{k \rightarrow \infty} \mu_k = \mu_0$ , where  $\mu_0$  is the first eigenvalue of the problem (3.3).*

*Proof.* Suppose that  $v_k \in H_0^1(B_k)$  is the first eigenfunction of the problem (3.7)<sub>k</sub>, and extend  $v_k$  to be zero outside  $B_k$ . Then  $v_k \in H_\rho^1(\mathbf{R}^N)$  and satisfies

$$\int_{\mathbf{R}^N} \left( \nabla v_k \cdot \nabla \phi - \frac{1}{p-1}v_k \phi - \mu_k m v_k \phi \right) \rho dx = 0$$

for any  $\phi \in C_0^\infty(B_k)$ . Since  $v_k$  achieves the the minimization (3.8)<sub>k</sub>, we have

$$\int_{\mathbf{R}^N} m v_k^2 \rho dx = 1 \quad \text{and} \quad \int_{\mathbf{R}^N} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx = \mu_k.$$

From (3.4) we have  $\mu_k \geq \mu_0$ . From (3.2) and  $\mu_k \geq \mu_{k+1}$ ,  $k = 1, 2, \dots$ , it follows that

$$\int_{\mathbf{R}^N} |\nabla v_k|^2 \rho dx \leq \left( 1 - \frac{2}{N(p-1)} \right)^{-1} \mu_k \leq \left( 1 - \frac{2}{N(p-1)} \right)^{-1} \mu_1.$$

Therefore, from (i) of Lemma 3.1,  $\{v_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ . Then, from (ii) of Lemma 3.1, there exist a subsequence that we still denote  $\{v_k\}$  and a function  $v_0 \in H_\rho^1(\mathbf{R}^N)$  such that

$$v_k \rightharpoonup v_0 \quad \text{weakly in } H_\rho^1(\mathbf{R}^N) \quad \text{as } k \rightarrow \infty,$$

$$v_k \rightarrow v_0 \quad \text{strongly in } L_\rho^2(\mathbf{R}^N) \quad \text{as } k \rightarrow \infty.$$

Then we obtain  $\int_{\mathbf{R}^N} m v_0^2 \rho dx = 1$ , which implies that  $v_0 \not\equiv 0$ . We also obtain

$$(3.9) \quad \int_{\mathbf{R}^N} \left( \nabla v_0 \cdot \nabla \phi - \frac{1}{p-1} v_0 \phi - \mu_\infty m v_0 \phi \right) \rho dx = 0$$

for any  $\phi \in C_0^\infty(\mathbf{R}^N)$ , where  $\mu_\infty = \lim_{k \rightarrow \infty} \mu_k$ . Since  $C_0^\infty(\mathbf{R}^N)$  is dense in  $H_\rho^1(\mathbf{R}^N)$ , we obtain (3.9) for any  $\phi \in H_\rho^1(\mathbf{R}^N)$ . Putting  $\phi = u_0 > 0$  in (3.9), where  $u_0$  is the eigenfunction of the problem (3.3), we obtain

$$\begin{aligned} \mu_\infty \int_{\mathbf{R}^N} m v_0 u_0 \rho dx &= \int_{\mathbf{R}^N} \left( \nabla v_0 \cdot \nabla u_0 - \frac{1}{p-1} v_0 u_0 \right) \rho dx \\ &= \mu_0 \int_{\mathbf{R}^N} m v_0 u_0 \rho dx, \end{aligned}$$

which implies  $\mu_\infty = \mu_0$ . □

*Proof of Proposition 3.1.* We claim that  $\mu < \mu_k$  for each  $k = 1, 2, \dots$ , where  $\mu_k$  is the first eigenvalue of the problem (3.7)<sub>k</sub>. Assume that  $v_k$  is the corresponding eigenfunction. We note that  $v_k \in C^2(\overline{B_k})$  and satisfies

$$(3.10) \quad \int_{B_k} m v_k^2 \rho dx = 1 \quad \text{and} \quad \int_{B_k} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx = \mu_k.$$

Let  $w \in C^2(\mathbf{R}^N)$  be a positive function satisfying (3.5). Then, by the straightforward calculation we have the following Picone's identity (cf. [17, 31]):

$$\rho w^2 \left| \nabla \left( \frac{v_k}{w} \right) \right|^2 + \nabla \cdot \left( \frac{v_k^2}{w} (\rho \nabla w) \right) = \rho |\nabla v_k|^2 + \frac{v_k^2}{w} \nabla \cdot (\rho \nabla w) \quad \text{in } B_k.$$

Since  $w$  satisfies

$$\nabla \cdot (\rho \nabla w) + \rho \left( \frac{1}{p-1} w + \mu m w \right) \leq 0,$$

we obtain

$$(3.11) \quad \rho w^2 \left| \nabla \left( \frac{v_k}{w} \right) \right|^2 + \nabla \cdot \left( \frac{v_k^2}{w} (\rho \nabla w) \right) \leq \left( |\nabla v_k|^2 - \left( \frac{1}{p-1} + \mu m \right) v_k^2 \right) \rho$$

in  $B_k$ . Note that  $v_k = 0$  on  $\partial B_k$ . Then, by using Green's formula, we have

$$\int_{B_k} \nabla \cdot \left( \frac{v_k^2}{w} (\rho \nabla w) \right) dx = 0.$$

Therefore, integrating (3.11) on  $B_k$  we obtain

$$0 < \int_{B_k} \rho w^2 \left| \nabla \left( \frac{v_k}{w} \right) \right|^2 dx \leq \int_{B_k} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx - \mu \int_{B_k} m v_k^2 \rho dx.$$

From (3.10) we have  $0 < \mu_k - \mu$ . Then  $\mu_k > \mu$  for each  $k = 1, 2, \dots$ . From Lemma 3.3 we obtain  $\mu \leq \mu_0$ .  $\square$

*Proof of Proposition 3.2.* Let  $u_i > 0$  be the first eigenfunction of the problem (3.6)<sub>i</sub> for each  $i = 1, 2$ . Then  $u_i$ ,  $i = 1, 2$ , satisfies

$$\int_{\mathbf{R}^N} \left( \nabla u_i \cdot \nabla \phi - \frac{1}{p-1} u_i \phi \right) \rho dx = \mu_i \int_{\mathbf{R}^N} m_i u_i \phi \rho dx$$

for any  $\phi \in H_\rho^1(\mathbf{R}^N)$ . Therefore, we have

$$\begin{aligned} \mu_1 \int_{\mathbf{R}^N} m_1 u_1 u_2 \rho dx &= \int_{\mathbf{R}^N} \left( \nabla u_1 \cdot \nabla u_2 - \frac{1}{p-1} u_1 u_2 \right) \rho dx \\ &= \mu_2 \int_{\mathbf{R}^N} m_2 u_1 u_2 \rho dx. \end{aligned}$$

Since  $m_1 \leq m_2$ ,  $m_1 \not\equiv m_2$ , we obtain  $\mu_1 > \mu_2$ .  $\square$

#### 4. Existence of the minimal solution: Proof of Theorem 1

For each  $\lambda > 0$  we introduce the solution set

$$S_\lambda = \{u \in C^2(\mathbf{R}^N) : u \text{ is a positive solution of (1.6)-(1.7)}_\lambda\}.$$

We call a minimal solution  $\underline{u}_\lambda \in S_\lambda$ , if  $\underline{u}_\lambda$  satisfies  $\underline{u}_\lambda \leq u$  for all  $u \in S_\lambda$ .

First we show the following results.

**Lemma 4.1.** (i) We have  $S_\lambda \neq \emptyset$  for some  $\lambda > 0$ . Moreover, if  $S_{\lambda_0} \neq \emptyset$  for some  $\lambda_0 > 0$ , then  $S_\lambda \neq \emptyset$  for all  $\lambda \in (0, \lambda_0)$ .

(ii) If  $S_\lambda \neq \emptyset$  then there exists a minimal solution  $\underline{u}_\lambda \in S_\lambda$ . Moreover, for any positive function  $w$  satisfying

$$(4.1) \quad \begin{cases} -\Delta w - \frac{1}{2}x \cdot \nabla w - \frac{1}{p-1}w \geq w^p & \text{in } \mathbf{R}^N \quad \text{and} \\ \liminf_{r \rightarrow \infty} r^{2/(p-1)}w(r\omega) \geq \lambda a(\omega) & \text{for a.e. } \omega \in S^{N-1}, \end{cases}$$

we have  $\underline{u}_\lambda \leq w$ .

*Proof.* (i) Let  $v = v(r)$ ,  $r = |x|$ , be a positive solution of (1.6) satisfying

$$\lim_{r \rightarrow \infty} r^{2/(p-1)}v(r) = \ell$$

for some  $\ell > 0$ . The existence of such  $v$  is obtained by [16, Theorem 5]. Take  $\lambda_* > 0$  so small that  $\lambda_* \leq \ell/\|a\|_{L^\infty(S^{N-1})}$ . By applying Proposition 2.2 with  $\alpha(\omega) = \lambda_* a(\omega)$  and  $\beta(\omega) \equiv \ell$ , we obtain a positive solution  $u$  of (1.6)-(1.7) $_{\lambda}$  with  $\lambda = \lambda_*$ , that is,  $S_{\lambda_*} \neq \emptyset$ .

Assume that  $S_{\lambda_0} \neq \emptyset$  for some  $\lambda_0 > 0$ . Let  $\lambda \in (0, \lambda_0)$ . Then, by applying Proposition 2.2 with  $\alpha(\omega) = \lambda a(\omega)$  and  $\beta(\omega) = \lambda_0 a(\omega)$ , we have a positive solution  $u$  of (1.6)-(1.7) $_{\lambda}$ . Therefore,  $S_\lambda \neq \emptyset$  for all  $\lambda \in (0, \lambda_0)$ .

(ii) Assume that  $u_\lambda \in S_\lambda$ . Applying Proposition 2.2 with  $v = u_\lambda$  and  $\alpha(\omega) = \beta(\omega) = \lambda a(\omega)$ , we have a positive solution  $\underline{u}_\lambda$  of (1.6)-(1.7) $_{\lambda}$  such that  $\underline{u}_\lambda \leq w$  for any  $w > 0$  satisfying (4.1). In particular, we obtain  $\underline{u}_\lambda \leq u$  for all  $u \in S_\lambda$ . This implies that  $\underline{u}_\lambda$  is the minimal solution of  $S_\lambda$ .  $\square$

**Lemma 4.2.** (i) Assume that  $\underline{u}_{\lambda_1} \in S_{\lambda_1}$  and  $\underline{u}_{\lambda_2} \in S_{\lambda_2}$  are minimal solutions with  $0 < \lambda_1 < \lambda_2$ . Then

$$(4.2) \quad \frac{\underline{u}_{\lambda_1}}{\lambda_1} \leq \frac{\underline{u}_{\lambda_2}}{\lambda_2} \quad \text{in } \mathbf{R}^N.$$

In particular,  $\underline{u}_{\lambda_1} < \underline{u}_{\lambda_2}$  in  $\mathbf{R}^N$ .

(ii) Let  $\underline{u}_\lambda \in S_\lambda$  be the minimal solution. Then  $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} = O(\lambda)$  as  $\lambda \rightarrow 0$ .

(iii) Let  $\bar{\lambda} = \sup\{\lambda > 0 : S_\lambda \neq \emptyset\}$ . Then  $\bar{\lambda} < \infty$ .

*Remark 4.1.* As already mentioned in (ii) of Remark 2, the result (iii) of this lemma is essentially obtained by [33, 37, and 24]. However, we give here a slight simple proof for convenience.

*Proof.* (i) Define  $v = \underline{u}_{\lambda_2}/\lambda_2$ . Then  $v$  satisfies

$$\begin{cases} -\Delta v - \frac{1}{2}x \cdot \nabla v - \frac{1}{p-1}v = \lambda_2^{p-1}v^p \geq \lambda_1^{p-1}v^p & \text{in } \mathbf{R}^N \quad \text{and} \\ \lim_{r \rightarrow \infty} r^{2/(p-1)}v(r\omega) = a(\omega) & \text{for a.e. } \omega \in S^{N-1}. \end{cases}$$

Put  $w = \lambda_1 v$ . Then  $w$  satisfies

$$\begin{cases} -\Delta w - \frac{1}{2}x \cdot \nabla w - \frac{1}{p-1}w \geq w^p & \text{in } \mathbf{R}^N \quad \text{and} \\ \lim_{r \rightarrow \infty} r^{2/(p-1)}w(r\omega) = \lambda_1 a(\omega) & \text{for a.e. } \omega \in S^{N-1}. \end{cases}$$

From (ii) of Lemma 4.1 we have  $\underline{u}_{\lambda_1} \leq w$ , which implies that (4.2) holds. In particular, we have  $\underline{u}_{\lambda_1} < \underline{u}_{\lambda_2}$  in  $\mathbf{R}^N$ .

(ii) Take  $\lambda_0 > 0$  so that  $S_{\lambda_0} \neq \emptyset$ . Let  $\lambda \in (0, \lambda_0)$ . From (i) of this lemma, we have

$$\frac{\underline{u}_\lambda}{\lambda} \leq \frac{\underline{u}_{\lambda_0}}{\lambda_0} \quad \text{in } \mathbf{R}^N.$$

Then we obtain  $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} \leq (\lambda/\lambda_0)\|\underline{u}_{\lambda_0}\|_{L^\infty(\mathbf{R}^N)}$  for  $\lambda \in (0, \lambda_0)$ . This implies that (ii) holds.

(iii) Assume that  $S_\lambda \neq \emptyset$  for some  $\lambda > 0$ . Let  $\underline{u}_\lambda \in S_\lambda$  be the minimal solution. Then  $v = \underline{u}_\lambda/\lambda$  satisfies

$$(4.3) \quad \Delta v + \frac{1}{2}x \cdot \nabla v + \frac{1}{p-1}v + \underline{u}_\lambda^{p-1}v = 0 \quad \text{in } \mathbf{R}^N.$$

Take  $\lambda_0 \in (0, \lambda)$ , and let  $\underline{u}_{\lambda_0} \in S_{\lambda_0}$  be the minimal solution. Then, from (i) of this lemma, we have  $\underline{u}_\lambda/\lambda \geq \underline{u}_{\lambda_0}/\lambda_0$ . Hence, from (4.3) we have

$$\Delta v + \frac{1}{2}x \cdot \nabla v + \frac{1}{p-1}v + \lambda^{p-1} \left( \frac{\underline{u}_{\lambda_0}}{\lambda_0} \right)^{p-1} v \leq 0 \quad \text{in } \mathbf{R}^N.$$

On the other hand, from Lemma 3.2 the eigenvalue problem

$$\begin{cases} -\Delta w - \frac{1}{2}x \cdot \nabla w - \frac{1}{p-1}w = \mu \left( \frac{\underline{u}_{\lambda_0}}{\lambda_0} \right)^{p-1} w & \text{in } \mathbf{R}^N, \\ w \in H_\rho^1(\mathbf{R}^N), \end{cases}$$

has the first eigenvalue  $\mu_0 > 0$ . By Proposition 3.1 we have  $\lambda^{p-1} \leq \mu_0$ . This implies that  $\sup\{\lambda > 0 : S_\lambda \neq \emptyset\} \leq \mu_0^{1/(p-1)}$ .  $\square$

*Proof of Theorem 1.* (i) Let  $\bar{\lambda} = \sup\{\lambda > 0 : S_\lambda \neq \emptyset\}$ . Then, from (i) of Lemma 4.1 and (iii) of Lemma 4.2, we have  $0 < \bar{\lambda} < \infty$ . By Lemma 4.1, for  $\lambda \in (0, \bar{\lambda})$ ,

$S_\lambda \neq \emptyset$  and there exists a minimal solution  $\underline{u}_\lambda \in S_\lambda$ . From (i) and (ii) of Lemma 4.2,  $\underline{u}_\lambda$  is increasing in  $\lambda$  and satisfies  $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} = O(\lambda)$  as  $\lambda \rightarrow 0$ .

(ii) By the definition of  $\bar{\lambda}$ , we can conclude that  $(1.6)-(1.7)_\lambda$  has no positive solution for  $\lambda > \bar{\lambda}$ .  $\square$

## 5. Existence of the second solution: Proof of Theorem 2

Let  $\underline{u}_\lambda$  be the minimal positive solution of  $(1.6)-(1.7)_\lambda$  for  $\lambda \in (0, \bar{\lambda})$  obtained in Theorem 1. In order to find a second solution of  $(1.6)-(1.7)_\lambda$  we introduce the following problem:

$$(5.1)_\lambda \quad \begin{cases} \Delta u + \frac{1}{2}x \cdot \nabla u + \frac{1}{p-1}u + (u + \underline{u}_\lambda)^p - \underline{u}_\lambda^p = 0 & \text{in } \mathbf{R}^N, \\ u \in H_\rho^1(\mathbf{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^N. \end{cases}$$

Clearly, we can get another positive solution  $\bar{u}_\lambda = \underline{u}_\lambda + u_\lambda$  of  $(1.6)-(1.7)_\lambda$ , if  $(5.1)_\lambda$  possesses a solution  $u_\lambda$  satisfying (5.2) below. In this section we show the following two propositions.

**Proposition 5.1.** *Let  $p > (N+2)/N$  and  $(N-2)p < N+2$ . For  $\lambda \in (0, \bar{\lambda})$ , there exists a solution  $u_\lambda \in C^2(\mathbf{R}^N)$  of  $(5.1)_\lambda$  satisfying*

$$(5.2) \quad u_\lambda(x) = O(e^{-|x|^2/4}) \quad \text{as } |x| \rightarrow \infty.$$

**Proposition 5.2.** *Assume that  $p > (N+2)/N$  and  $(N-2)p < N+2$ . Let  $u_\lambda$  be the solution of  $(5.1)_\lambda$  obtained in Proposition 5.1. Then  $u_\lambda \rightarrow u_0$  in  $H_\rho^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$  as  $\lambda \rightarrow 0$ , where  $u_0$  is the solution of the problem (1.9).*

As a consequence of Propositions 5.1 and 5.2 we obtain Theorem 2.

We show the existence of the solution of  $(5.1)_\lambda$  by using a variational method. To this end we define the corresponding variational functional of  $(5.1)_\lambda$  by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \int_{\mathbf{R}^N} G(u, \underline{u}_\lambda) \rho dx$$

with  $u \in H_\rho^1(\mathbf{R}^N)$ , where

$$G(t, s) = \frac{1}{p+1} (t+s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t.$$

We know that the nontrivial critical point  $u \in H_\rho^1(\mathbf{R}^N)$  of the functional  $I_\lambda$  is a weak solution of the equation in  $(5.1)_\lambda$ , that is,  $u$  satisfies

$$\int_{\mathbf{R}^N} \left( \nabla u \cdot \nabla \phi - \frac{1}{p-1} u \phi \right) \rho dx - \int_{\mathbf{R}^N} g(u, \underline{u}_\lambda) \phi \rho dx = 0$$

for any  $\phi \in H_\rho^1(\mathbf{R}^N)$ , where

$$g(t, s) = (t + s)^p - s^p.$$

We easily see that  $u_\lambda \in C^2(\mathbf{R}^N)$  and  $u_\lambda > 0$  in  $\mathbf{R}^N$  from Proposition A.1 in Appendix A and Proposition 2.1.

First we investigate the properties of the functions  $g(t, s)$  and  $G(t, s)$ .

**Lemma 5.1.** (i) *For  $s_0 > 0$ , there is a constant  $C = C(s_0) > 0$  such that*

$$0 \leq g(t, s) \leq C(t + t^p), \quad t \geq 0, \quad 0 \leq s \leq s_0.$$

(ii) *For  $\delta > 0$ , there is a constant  $C = C(\delta) > 0$  such that*

$$0 \leq g(t, s) \leq Ct, \quad 0 \leq s, t \leq \delta.$$

Furthermore,  $C(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

(iii) *We have*

$$G(t, s) \geq \frac{1}{p+1} t^{p+1}, \quad s, t \geq 0.$$

(iv) *For any  $\varepsilon > 0$  and  $s_0 > 0$ , there is a constant  $C = C(\varepsilon, s_0) > 0$  such that*

$$G(t, s) - \frac{p}{2} s^{p-1} t^2 \leq \varepsilon t^2 + C t^{p+1}, \quad t \geq 0, \quad 0 \leq s \leq s_0.$$

(v) *Put  $c_p = \min\{1, p-1\}$ . Then*

$$g(t, s)t - (2 + c_p)G(t, s) \geq -\frac{c_p p}{2} s^{p-1} t^2, \quad s, t \geq 0.$$

*Proof.* (i) For  $0 \leq s \leq s_0$  we have

$$\lim_{t \rightarrow \infty} \frac{g(t, s)}{t^p} = 1 \quad \text{and} \quad \lim_{t \rightarrow 0} \frac{g(t, s)}{t} = p s^{p-1}$$

by using l'Hospital's rule. Hence we obtain (i).

(ii) For  $0 \leq s, t \leq \delta$  we have

$$g_t(t, s) = p(t + s)^{p-1} \leq p(2\delta)^{p-1}.$$

Integrating the above on  $[0, t]$  with respect  $t$ , we obtain  $g(t, s) \leq C(\delta)t$ , where  $C(\delta) = p(2\delta)^{p-1}$ . Thus,  $C(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ .

(iii) We have  $G(0, s) = G_t(0, s) = 0$  and  $G_{tt}(t, s) = p(t + s)^{p-1} \geq pt^{p-1}$  for  $t, s \geq 0$ . By integrating on  $[0, t]$  twice with respect  $t$ , we obtain (iii).

(iv) Put  $h(t, s) = G(t, s) - (p/2)s^{p-1}t^2$ . We have  $h(0, s) = h_t(0, s) = h_{tt}(0, s) = 0$ . Then, by using l'Hospital's rule, we obtain

$$\lim_{t \rightarrow 0} \frac{h(t, s)}{t^2} = 0.$$

By virtue of

$$\lim_{t \rightarrow \infty} \frac{h(t, s)}{t^{p+1}} = \frac{1}{p+1},$$

we obtain (iv).

(v) Define

$$H(t, s) = g(t, s)t - (2 + c_p)G(t, s) + \frac{c_p p}{2}s^{p-1}t^2.$$

Then we have  $H(0, s) = H_t(0, s) = H_{tt}(0, s) = 0$  and

$$H_{ttt}(t, s) = \begin{cases} p(p-1)(2-p)(t+s)^{p-3}s & \text{if } 1 < p < 2, \\ p(p-1)(p-2)(t+s)^{p-3}t & \text{if } p \geq 2. \end{cases}$$

Thus  $H_{ttt}(t, s) \geq 0$  for  $s, t \geq 0$ . By integrating on  $[0, t]$  three times with respect  $t$ , we obtain  $H(t, s) \geq 0$  for  $s, t \geq 0$ . Thus (v) holds.  $\square$

Let  $\underline{u}_\lambda$  be the minimal positive solution of (1.6)-(1.7) $_\lambda$  for  $\lambda \in (0, \bar{\lambda})$ . By Lemma 3.2 the corresponding eigenvalue problem

$$\begin{cases} -\Delta w - \frac{1}{2}x \cdot \nabla w - \frac{1}{p-1}w = \mu p \underline{u}_\lambda^{p-1} w & \text{in } \mathbf{R}^N, \\ w \in H_\rho^1(\mathbf{R}^N), \end{cases}$$

has the first eigenvalue  $\mu(\lambda) > 0$ . Furthermore, we have

$$\mu(\lambda) = \inf \left\{ \int_{\mathbf{R}^N} \left( |\nabla w|^2 - \frac{1}{p-1}w^2 \right) \rho dx : w \in H_\rho^1(\mathbf{R}^N), p \int_{\mathbf{R}^N} \underline{u}_\lambda^{p-1} w^2 \rho dx = 1 \right\}.$$

Then it follows that

$$(5.3) \quad \int_{\mathbf{R}^N} \left( |\nabla w|^2 - \frac{1}{p-1}w^2 \right) \rho dx \geq \mu(\lambda) p \int_{\mathbf{R}^N} \underline{u}_\lambda^{p-1} w^2 \rho dx$$

for any  $w \in H_\rho^1(\mathbf{R}^N)$ .

**Lemma 5.2.** *For  $0 < \lambda < \bar{\lambda}$ , we have  $\mu(\lambda) > 1$ . Moreover,  $\mu(\lambda)$  is strictly decreasing in  $\lambda \in (0, \bar{\lambda})$ .*

*Proof.* Take  $\lambda_1, \lambda_2 \in (0, \bar{\lambda})$  with  $\lambda_1 < \lambda_2$ . From (i) of Theorem 1 we have  $\underline{u}_{\lambda_2} > \underline{u}_{\lambda_1}$  in  $\mathbf{R}^N$ , and hence  $\underline{u}_{\lambda_2}^{p-1} > \underline{u}_{\lambda_1}^{p-1}$ . By Proposition 3.2, we have  $\mu(\lambda_2) < \mu(\lambda_1)$ . Therefore,  $\mu(\lambda)$  is strictly decreasing in  $\lambda$ .

Let  $\lambda \in (0, \bar{\lambda})$ , and let  $\lambda_0 \in (\lambda, \bar{\lambda})$ . Put  $w = \underline{u}_{\lambda_0} - \underline{u}_\lambda$ . Then  $w > 0$  and  $w$  satisfies

$$\Delta w + \frac{1}{2}x \cdot \nabla w + \frac{1}{p-1}w + p\underline{u}_\lambda^{p-1}w \leq 0, \quad x \in \mathbf{R}^N.$$

By Proposition 3.1 we have  $\mu(\lambda) \geq 1$ . Then  $\mu(\lambda) \geq 1$  for  $\lambda \in (0, \bar{\lambda})$ . Since  $\mu(\lambda)$  is strictly decreasing, we have  $\mu(\lambda) > 1$  for all  $\lambda \in (0, \bar{\lambda})$ .  $\square$

In the following we verify the existence of nontrivial solution of  $(5.1)_\lambda$  by means of the Mountain Pass lemma.

**Lemma 5.3.** *Assume that  $\{u_k\}$  is the Palais-Smale sequence for  $I_\lambda(u)$ , that is,*

$$(5.4) \quad u_k \in H_\rho^1(\mathbf{R}^N), \quad \{I_\lambda(u_k)\} \text{ is bounded, and } I'_\lambda(u_k) \rightarrow 0 \text{ as } k \rightarrow \infty$$

*in the dual space of  $H_\rho^1(\mathbf{R}^N)$ . Then  $\{u_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ .*

*Proof.* Since  $\{I_\lambda(u_k)\}$  is bounded, we have

$$(5.5) \quad \frac{1}{2} \int_{\mathbf{R}^N} \left( |\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx - \int_{\mathbf{R}^N} G(u_k, \underline{u}_\lambda) \rho dx \leq M$$

for some  $M > 0$ . Let  $\varepsilon > 0$ . From  $I'_\lambda(u_k) \rightarrow 0$  as  $k \rightarrow \infty$ , we have, for sufficient large  $k$ ,

$$\left| \int_{\mathbf{R}^N} \left( \nabla u_k \cdot \nabla \phi - \frac{1}{p-1} u_k \phi \right) \rho dx - \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda) \phi \rho dx \right| \leq \varepsilon \|\phi\|_{H_\rho^1}$$

for any  $\phi \in H_\rho^1(\mathbf{R}^N)$ . Putting  $\phi = u_k / \|u_k\|_{H_\rho^1}$ , we have

$$\left| \int_{\mathbf{R}^N} \left( |\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx - \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda) u_k \rho dx \right| \leq \varepsilon \|u_k\|_{H_\rho^1}.$$

Then we obtain

$$(5.6) \quad \int_{\mathbf{R}^N} \left( |\nabla u_k|^2 - \frac{1}{p-1} u_k^2 \right) \rho dx \geq \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda) u_k \rho dx - \varepsilon \|u_k\|_{H_\rho^1}.$$

Put  $c_p = \min\{1, p-1\}$ . From (5.5) and (5.6) we have

$$\begin{aligned}
(2 + c_p)M &\geq \left(1 + \frac{c_p}{2}\right) \int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2\right) \rho dx \\
&\quad - (2 + c_p) \int_{\mathbf{R}^N} G(u_k, \underline{u}_\lambda) \rho dx \\
&\geq \frac{c_p}{2} \int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2\right) \rho dx \\
&\quad + \int_{\mathbf{R}^N} (g(u_k, \underline{u}_\lambda) u_k - (2 + c_p) G(u_k, \underline{u}_\lambda)) \rho dx - \varepsilon \|u_k\|_{H_\rho^1}.
\end{aligned}$$

From (v) of Lemma 5.1, (5.3), and (3.2), it follows that

$$\begin{aligned}
(2 + c_p)M &\geq \frac{c_p}{2} \left( \int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2\right) \rho dx - p \int_{\mathbf{R}^N} \underline{u}_\lambda^{p-1} u_k^2 \rho dx \right) \\
&\quad - \varepsilon \|u_k\|_{H_\rho^1} \\
&\geq \frac{c_p}{2} \left(1 - \frac{1}{\mu(\lambda)}\right) \int_{\mathbf{R}^N} \left(|\nabla u_k|^2 - \frac{1}{p-1} u_k^2\right) \rho dx - \varepsilon \|u_k\|_{H_\rho^1} \\
&\geq \frac{c_p}{2} \left(1 - \frac{1}{\mu(\lambda)}\right) \left(1 - \frac{2}{N(p-1)}\right) \|\nabla u_k\|_{L_\rho^2}^2 - \varepsilon \|u_k\|_{H_\rho^1}.
\end{aligned}$$

We note here that  $\mu(\lambda) > 1$  from Lemma 5.2. Therefore,  $\{\|\nabla u_k\|_{L_\rho^2}\}$  is bounded, and hence, from (i) of Lemma 3.1,  $\{u_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ .  $\square$

**Lemma 5.4.** *The functional  $I_\lambda$  satisfies the Palais-Smale condition, that is, any Palais-Smale sequence contains a subsequence which converges in  $H_\rho^1(\mathbf{R}^N)$ .*

*Proof.* We show the case where  $N \geq 3$ . We can verify the case where  $N = 2$  with a slight modification. Let  $\{u_k\}$  be a Palais-Smale sequence, that is, (5.4) holds. By Lemma 5.3 we have  $\{u_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ . Then, from (ii) and (iii) of Lemma 3.1, there exist a subsequence that we still denote  $\{u_k\}$  and a function  $u \in H_\rho^1(\mathbf{R}^N)$  such that

$$(5.7) \quad u_k \rightharpoonup u \quad \text{weakly in } H_\rho^1(\mathbf{R}^N) \text{ as } k \rightarrow \infty,$$

$$(5.8) \quad u_k \rightarrow u \quad \text{strongly in } L_\rho^2(\mathbf{R}^N) \cap L_\rho^{p+1}(\mathbf{R}^N) \text{ as } k \rightarrow \infty.$$

We claim that  $\|\nabla(u_k - u)\|_{L_\rho^2} \rightarrow 0$  as  $k \rightarrow \infty$ . We see that

$$\|\nabla(u_k - u)\|_{L_\rho^2}^2 = \int_{\mathbf{R}^N} \nabla u_k \cdot (\nabla u_k - \nabla u) \rho dx - \int_{\mathbf{R}^N} \nabla u \cdot (\nabla u_k - \nabla u) \rho dx.$$

It follows from (5.7) that

$$\int_{\mathbf{R}^N} \nabla u \cdot (\nabla u_k - \nabla u) \rho dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We observe that

$$\begin{aligned} \int_{\mathbf{R}^N} \nabla u_k \cdot (\nabla u_k - \nabla u) \rho dx &= I'_\lambda(u_k)(u_k - u) \\ &+ \frac{1}{p-1} \int_{\mathbf{R}^N} u_k(u_k - u) \rho dx + \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda)(u_k - u) \rho dx. \end{aligned}$$

Since  $I'_\lambda(u_k) \rightarrow 0$  as  $k \rightarrow \infty$ , we have

$$|I'_\lambda(u_k)(u_k - u)| \leq |I'_\lambda(u_k)| \|u_k - u\|_{H_\rho^1} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

From (i) of Lemma 5.1 we obtain

$$\left| \int_{\mathbf{R}^N} g(u_k, \underline{u}_\lambda)(u_k - u) \rho dx \right| \leq C \left( \int_{\mathbf{R}^N} u_k(u_k - u) \rho dx + \int_{\mathbf{R}^N} u_k^p(u_k - u) \rho dx \right)$$

for some constant  $C > 0$ . By using Hölder inequality and (5.8), we obtain

$$\left| \int_{\mathbf{R}^N} u_k(u_k - u) \rho dx \right| \leq \|u_k\|_{L_\rho^2}^2 \|u_k - u\|_{L_\rho^2}^2 \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

and

$$\left| \int_{\mathbf{R}^N} u_k^p(u_k - u) \rho dx \right| \leq \|u_k\|_{L_\rho^{p+1}}^{p+1} \|u_k - u\|_{L_\rho^{p+1}} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Therefore, we have

$$\int_{\mathbf{R}^N} \nabla u_k \cdot (\nabla u_k - \nabla u) \rho dx \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

and conclude that  $\|\nabla(u_k - u)\|_{L_\rho^2} \rightarrow 0$  as  $k \rightarrow \infty$ . From (i) of Lemma 3.1 we have  $u_k \rightarrow u$  in  $H_\rho^1(\mathbf{R}^N)$ .  $\square$

**Lemma 5.5.** *There exist some constants  $\delta = \delta(\lambda) > 0$  and  $\eta = \eta(\lambda) > 0$  such that*

$$(5.9) \quad I_\lambda(u) \geq \eta > 0$$

for all  $u \in H_\rho^1(\mathbf{R}^N)$  satisfying  $\|\nabla u\|_{L_\rho^2} = \delta$ .

*Proof.* For any  $u \in H_\rho^1(\mathbf{R}^N)$  we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 - p \underline{u}_\lambda^{p-1} u^2 \right) \rho dx \\ &\quad - \int_{\mathbf{R}^N} \left( G(u, \underline{u}_\lambda) - \frac{p}{2} \underline{u}_\lambda^{p-1} u^2 \right) \rho dx \equiv J_1 - J_2. \end{aligned}$$

From (5.3) and Lemma 5.2 we obtain

$$J_1 \geq \frac{1}{2} \left( 1 - \frac{1}{\mu(\lambda)} \right) \int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx$$

with  $\mu(\lambda) > 1$ . Then, from (3.2), we have

$$J_1 \geq C_0 \|\nabla u\|_{L_\rho^2}^2, \quad \text{where } C_0 = \frac{1}{2} \left( 1 - \frac{1}{\mu(\lambda)} \right) \left( 1 - \frac{2}{N(p-1)} \right) > 0.$$

From (iv) of Lemma 5.1, for any  $\varepsilon > 0$  there is a constant  $C_1 = C_1(\varepsilon, \|\underline{u}_\lambda\|_{L^\infty}) > 0$  such that

$$J_2 \leq \varepsilon \int_{\mathbf{R}^N} u^2 \rho dx + C_1 \int_{\mathbf{R}^N} u^{p+1} \rho dx.$$

From (i) and (iii) of Lemma 3.1 we have

$$J_2 \leq \frac{2}{N} \varepsilon \|\nabla u\|_{L_\rho^2}^2 + C_1 C_2 \|\nabla u\|_{L_\rho^2}^{p+1}$$

for some constant  $C_2 > 0$ . Take  $\varepsilon > 0$  so small that  $\varepsilon < NC_0/2$ . Then we have

$$I_\lambda(u) \geq C_3 \|\nabla u\|_{L_\rho^2}^2 - C_1 C_2 \|\nabla u\|_{L_\rho^2}^{p+1}, \quad \text{where } C_3 = C_0 - \frac{2}{N} \varepsilon > 0,$$

which implies that (5.9) holds for some  $\delta > 0$  and  $\eta > 0$ .  $\square$

Define the corresponding functional of (1.9) by

$$I_0(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u^{p+1} \rho dx$$

with  $u \in H_\rho^1(\mathbf{R}^N)$ . Let  $u_0$  be the solution of the problem (1.9). Then  $u_0$  satisfies

$$(5.10) \quad \int_{\mathbf{R}^N} \left( |\nabla u_0|^2 - \frac{1}{p-1} u_0^2 \right) \rho dx = \int_{\mathbf{R}^N} u_0^{p+1} \rho dx.$$

Therefore, we have

$$(5.11) \quad I_0(u_0) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbf{R}^N} u_0^{p+1} \rho dx.$$

**Lemma 5.6.** *Let  $u_0$  be the solution of the problem (1.9), and let  $0 < \lambda < \bar{\lambda}$ . Then*

- (i)  $I_\lambda(tu_0) < 0$  for  $t > ((p+1)/2)^{1/(p-1)}$ ;
- (ii)  $\sup_{t>0} I_\lambda(tu_0) \leq I_0(u_0)$ .

*Proof.* From (5.10) we have

$$\begin{aligned} I_\lambda(tu_0) &= \frac{t^2}{2} \int_{\mathbf{R}^N} \left( |\nabla u_0|^2 - \frac{1}{p-1} u_0^2 \right) \rho dx - \int_{\mathbf{R}^N} G(tu_0, \underline{u}_\lambda) \rho dx \\ &= \frac{t^2}{2} \int_{\mathbf{R}^N} u_0^{p+1} \rho dx - \int_{\mathbf{R}^N} G(tu_0, \underline{u}_\lambda) \rho dx. \end{aligned}$$

From (iii) of Lemma 5.1 we have

$$G(tu_0, \underline{u}_\lambda) \geq \frac{t^{p+1}}{p+1} u_0^{p+1}.$$

Then it follows that

$$(5.12) \quad I_\lambda(tu_0) \leq \left( \frac{t^2}{2} - \frac{t^{p+1}}{p+1} \right) \int_{\mathbf{R}^N} u_0^{p+1} \rho dx.$$

Since  $(t^2/2 - t^{p+1}/(p+1)) < 0$  for  $t > ((p+1)/2)^{1/(p-1)}$ , we obtain (i). From (5.11) and (5.12) we obtain

$$\begin{aligned} \sup_{t>0} I_\lambda(tu_0) &\leq \sup_{t>0} \left( \frac{t^2}{2} - \frac{t^{p+1}}{p+1} \right) \int_{\mathbf{R}^N} u_0^{p+1} \rho dx \\ &= \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbf{R}^N} u_0^{p+1} \rho dx = I_0(u_0), \end{aligned}$$

which implies that (ii) holds.  $\square$

**Lemma 5.7.** *For  $0 < \lambda < \bar{\lambda}$ , there exists a critical point  $u_\lambda \in H_\rho^1(\mathbf{R}^N)$  of  $I_\lambda(u)$  such that  $I_\lambda(u_\lambda) \leq I_0(u_0)$ . Moreover,  $u_\lambda \in C^2(\mathbf{R}^N)$  and  $u_\lambda(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .*

*Proof.* From Lemma 5.4,  $I_\lambda(u)$  satisfies the Palais-Smale condition. From (i) of Lemma 5.6, there exists a constant  $T_1 > 0$  such that  $e = T_1 u_0$  satisfies  $\|\nabla e\|_{L_\rho^2} > \delta$  and  $I_\lambda(e) \leq 0$ , where  $\delta$  is the constant appearing in Lemma 5.5. Denote

$$c = \inf_{v \in \Gamma} \max_{s \in [0,1]} I_\lambda(v(s)),$$

where  $\Gamma = \{v \in C([0,1]; H_\rho^1(\mathbf{R}^N)) : v(0) = 0, v(1) = e\}$ . Then, from Lemma 5.5 and (ii) of Lemma 5.6, it follows that

$$0 < \eta \leq c \leq I_0(u_0).$$

The Mountain Pass Lemma [1, 5] enables us to find a critical point  $u_\lambda \in H_\rho^1(\mathbf{R}^N)$  of  $I_\lambda(u)$ . Hence,  $u_\lambda$  is a weak solution of the equation in  $(5.1)_\lambda$  and satisfies

$I_\lambda(u_\lambda) \leq I_0(u_0)$ . By Proposition A.1 in Appendix A, we have  $u_\lambda \in C^2(\mathbf{R}^N)$  and  $u_\lambda(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .  $\square$

*Proof of Proposition 5.1.* The existence of solution  $u_\lambda$  of the problem  $(5.1)_\lambda$  has been obtained by Lemma 5.7. Therefore it suffices to show (5.2). Take a constant  $c_0$  so that  $0 < c_0 < (N/2) - 1/(p-1)$ . Recall that both  $\underline{u}_\lambda(x)$  and  $u_\lambda(x)$  tend to 0 as  $|x| \rightarrow \infty$ . Then, from (ii) of Lemma 5.1, there is a constant  $R > 0$  such that

$$(5.13) \quad 0 \leq g(u_\lambda(x), \underline{u}_\lambda(x)) \leq c_0 u_\lambda(x), \quad |x| \geq R.$$

Put  $w(x) = C_1 e^{-|x|^2/4}$ , where  $C_1 = \max_{|x| \leq R} u_\lambda(x) e^{|x|^2/4}$ . Clearly,  $w(x) \geq u_\lambda(x)$  for  $|x| \leq R$ . Since  $w \in H_\rho^1(\mathbf{R}^N)$  and  $w$  satisfies  $-\nabla \cdot (\rho \nabla w) = (N/2)\rho w$  in  $\mathbf{R}^N$ , we have

$$(5.14) \quad \int_{\mathbf{R}^N} \nabla w \cdot \nabla \phi \rho dx = \frac{N}{2} \int_{\mathbf{R}^N} w \phi \rho dx$$

for any  $\phi \in H_\rho^1(\mathbf{R}^N)$ . Let  $\phi(x) = (u_\lambda(x) - w(x))^+$ , where  $a^+ = \max\{a, 0\}$ . Then  $\phi \in H_\rho^1(\mathbf{R}^N)$ ,  $\phi \equiv 0$  for  $|x| \leq R$ , and

$$(5.15) \quad \nabla \phi = \begin{cases} \nabla u_\lambda - \nabla w & \text{if } u_\lambda \geq w, \\ 0 & \text{if } u_\lambda < w. \end{cases}$$

Now we claim that  $\phi \equiv 0$  in  $\mathbf{R}^N$ . We observe that

$$\int_{\mathbf{R}^N} \left( \nabla u_\lambda \cdot \nabla \phi - \left( \frac{1}{p-1} + c_0 \right) u_\lambda \phi \right) \rho dx = \int_{\mathbf{R}^N} (g(u_\lambda, \underline{u}_\lambda) - c_0 u_\lambda) \phi \rho dx.$$

From (5.13) and  $\phi \equiv 0$  for  $|x| \leq R$  it follows that

$$(5.16) \quad \begin{aligned} & \int_{\mathbf{R}^N} \left( \nabla u_\lambda \cdot \nabla \phi - \left( \frac{1}{p-1} + c_0 \right) u_\lambda \phi \right) \rho dx \\ &= \int_{|x| \geq R} (g(u_\lambda, \underline{u}_\lambda) - c_0 u_\lambda) \phi \rho dx \leq 0. \end{aligned}$$

From (5.14) and  $c_0 < (N/2) - 1/(p-1)$ , we obtain

$$(5.17) \quad \begin{aligned} & \int_{\mathbf{R}^N} \left( \nabla w \cdot \nabla \phi - \left( \frac{1}{p-1} + c_0 \right) w \phi \right) \rho dx \\ &= \left( \frac{N}{2} - \frac{1}{p-1} - c_0 \right) \int_{\mathbf{R}^N} w \phi \rho dx \geq 0. \end{aligned}$$

Then, from (5.16) and (5.17) we obtain

$$\int_{\mathbf{R}^N} \left( (\nabla u_\lambda - \nabla w) \cdot \nabla \phi - \left( \frac{1}{p-1} + c_0 \right) (u_\lambda - w) \phi \right) \rho dx \leq 0.$$

By virtue of (5.15) it follows that

$$\int_{\mathbf{R}^N} \left( |\nabla \phi|^2 - \left( \frac{1}{p-1} + c_0 \right) \phi^2 \right) \rho dx \leq 0.$$

From (i) of Lemma 3.1 we have

$$\left( \frac{N}{2} - \frac{1}{p-1} - c_0 \right) \int_{\mathbf{R}^N} \phi^2 \rho dx \leq 0.$$

This implies that  $\phi \equiv 0$  in  $\mathbf{R}^N$ , and hence,  $u_\lambda(x) \leq w(x) = C_1 e^{-|x|^2/4}$  for  $x \in \mathbf{R}^N$ .  
□

The next result is fundamental to the proof of Proposition 5.2.

**Lemma 5.8.** *Let  $M_\lambda = \sup_{x \in \mathbf{R}^N} u_\lambda(x)$  for  $0 < \lambda < \bar{\lambda}$ . Then  $\liminf_{\lambda \rightarrow 0+} M_\lambda > 0$ .*

*Proof.* Assume to the contrary that  $\liminf_{\lambda \rightarrow 0+} M_\lambda = 0$ . Take a constant  $c_0$  so that  $0 < c_0 < (N/2) - 1/(p-1)$ . Recall that  $\|\underline{u}_\lambda\|_{L^\infty} \rightarrow 0$  as  $\lambda \rightarrow 0$ . From (ii) of Lemma 5.1, we can take a  $\lambda > 0$  so that  $g(u_\lambda(x), \underline{u}_\lambda(x)) \leq c_0 u_\lambda(x)$  for  $x \in \mathbf{R}^N$ . Then we have

$$\int_{\mathbf{R}^N} \left( |\nabla u_\lambda|^2 - \frac{1}{p-1} u_\lambda^2 \right) \rho dx = \int_{\mathbf{R}^N} g(u_\lambda, \underline{u}_\lambda) u_\lambda \rho dx \leq c_0 \int_{\mathbf{R}^N} u_\lambda^2 \rho dx.$$

It follows that

$$\int_{\mathbf{R}^N} |\nabla u_\lambda|^2 \rho dx \leq \left( c_0 + \frac{1}{p-1} \right) \int_{\mathbf{R}^N} u_\lambda^2 \rho dx < \frac{N}{2} \int_{\mathbf{R}^N} u_\lambda^2 \rho dx$$

with  $u_\lambda \in H_\rho^1(\mathbf{R}^N)$ . This contradicts (i) of Lemma 3.1. Hence, we obtain  $\liminf_{\lambda \rightarrow 0+} M_\lambda > 0$ . □

*Proof of Proposition 5.2.* Let  $\{\lambda_k\}$  be a sequence such that  $\lambda_k > \lambda_{k+1}$  and  $\lambda_k \rightarrow 0$  as  $k \rightarrow \infty$ . For simplicity, one sets  $v_k = u_{\lambda_k}$  and  $\underline{v}_k = \underline{u}_{\lambda_k}$ . The proof is divided into several steps.

**Step 1.** We claim that  $\{v_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ .

From Lemma 5.7 we have  $I_{\lambda_k}(v_k) \leq I_0(u_0)$ , that is,

$$\frac{1}{2} \int_{\mathbf{R}^N} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx - \int_{\mathbf{R}^N} G(v_k, \underline{v}_k) \rho dx \leq I_0(u_0).$$

Since  $v_k$  satisfies

$$\int_{\mathbf{R}^N} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx = \int_{\mathbf{R}^N} g(v_k, \underline{v}_k) v_k \rho dx,$$

we obtain

$$\begin{aligned} (2 + c_p) I_0(u_0) &\geq \left( 1 + \frac{c_p}{2} \right) \int_{\mathbf{R}^N} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx \\ &\quad - (2 + c_p) \int_{\mathbf{R}^N} G(v_k, \underline{v}_k) \rho dx \\ &\geq \frac{c_p}{2} \int_{\mathbf{R}^N} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx \\ &\quad + \int_{\mathbf{R}^N} (g(v_k, \underline{v}_k) v_k - (2 + c_p) G(v_k, \underline{v}_k)) \rho dx, \end{aligned}$$

where  $c_p = \min\{1, p-1\}$ . From (v) of Lemma 5.1 and (5.3), it follows that

$$\begin{aligned} (2 + c_p) I_0(u_0) &\geq \frac{c_p}{2} \left( \int_{\mathbf{R}^N} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx - p \int_{\mathbf{R}^N} \underline{v}_k^{p-1} v_k^2 \rho dx \right) \\ &\geq \frac{c_p}{2} \left( 1 - \frac{1}{\mu(\lambda_k)} \right) \int_{\mathbf{R}^N} \left( |\nabla v_k|^2 - \frac{1}{p-1} v_k^2 \right) \rho dx. \end{aligned}$$

Since  $\mu(\lambda)$  is strictly decreasing and  $\mu(\lambda) > 1$  by Lemma 5.2, we have  $\mu(\lambda_k) > \mu(\lambda_1) > 1$ . From (3.2) we obtain

$$(2 + c_p) I_0(u_0) \geq \frac{c_p}{2} \left( 1 - \frac{1}{\mu(\lambda_1)} \right) \left( 1 - \frac{2}{N(p-1)} \right) \|\nabla v_k\|_{L_\rho^2}^2,$$

which implies that  $\{\|\nabla v_k\|_{L_\rho^2}\}$  is bounded. Hence,  $\{v_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ .

**Step 2.** We show that there exist a subsequence that we still denote  $\{v_k\}$  and a function  $v_0 \in H_\rho^1(\mathbf{R}^N)$  such that  $v_k \rightarrow v_0$  in  $H_\rho^1(\mathbf{R}^N)$  as  $k \rightarrow \infty$ .

Since  $\{v_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ , from (ii) and (iii) of Lemma 3.1, there exist a subsequence (still denoted by  $\{v_k\}$ ) and some  $v_0 \in H_\rho^1(\mathbf{R}^N)$  such that

$$v_k \rightharpoonup v_0 \quad \text{weakly in } H_\rho^1(\mathbf{R}^N) \text{ as } k \rightarrow \infty,$$

$$v_k \rightarrow v_0 \quad \text{strongly in } L_\rho^2(\mathbf{R}^N) \cap L_\rho^{p+1}(\mathbf{R}^N) \text{ as } k \rightarrow \infty.$$

We claim that  $\|\nabla(v_k - v_0)\|_{L_\rho^2} \rightarrow 0$  as  $k \rightarrow \infty$ . We observe that

$$\|\nabla(v_k - v_0)\|_{L_\rho^2}^2 = \int_{\mathbf{R}^N} \nabla v_k \cdot (\nabla v_k - \nabla v_0) \rho dx - \int_{\mathbf{R}^N} \nabla v_0 \cdot (\nabla v_k - \nabla v_0) \rho dx$$

and

$$\begin{aligned} \int_{\mathbf{R}^N} \nabla v_k \cdot (\nabla v_k - \nabla v_0) \rho dx &= \frac{1}{p-1} \int_{\mathbf{R}^N} v_k (v_k - u) \rho dx \\ &\quad + \int_{\mathbf{R}^N} g(v_k, \underline{v}_k) (v_k - v_0) \rho dx. \end{aligned}$$

By the similar argument as in the proof of Lemma 5.4, we obtain  $\|\nabla(v_k - v_0)\|_{L_\rho^2} \rightarrow 0$  as  $k \rightarrow \infty$ , and hence,  $v_k \rightarrow v_0$  in  $H_\rho^1(\mathbf{R}^N)$  as  $k \rightarrow \infty$ .

**Step 3.** We show that  $v_0 = u_0$ , where  $u_0$  is the solution of the problem (1.9). Furthermore, we have  $v_k \rightarrow u_0$  in  $H_\rho^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$  as  $k \rightarrow \infty$ .

First we show that  $v_0$  satisfies the equation in (1.9). Since  $v_k \rightarrow v_0$  in  $H_\rho^1(\mathbf{R}^N)$  by Step 2, it suffices to prove that

$$(5.18) \quad \int_{\mathbf{R}^N} g(v_k, \underline{v}_k) \phi \rho dx \rightarrow \int_{\mathbf{R}^N} v_0^p \phi \rho dx \quad \text{as } k \rightarrow \infty$$

for any  $\phi \in H_\rho^1(\mathbf{R}^N)$ . From  $v_k \rightarrow v_0$  in  $L_\rho^2(\mathbf{R}^N) \cap L_\rho^{p+1}(\mathbf{R}^N)$ , there exist a subsequence (still denoted by  $\{v_k\}$ ) and a function  $h \in L_\rho^2(\mathbf{R}^N) \cap L_\rho^{p+1}(\mathbf{R}^N)$  such that

$$(5.19) \quad v_k(x) \leq h(x) \quad \text{a.e. } x \in \mathbf{R}^N$$

for  $k = 1, 2, \dots$ , and  $v_k \rightarrow v_0$  a.e.  $x \in \mathbf{R}^N$ . (See, e.g., [2].) By virtue of  $\|\underline{v}_k\|_{L^\infty} \rightarrow 0$  as  $k \rightarrow \infty$ , we have

$$g(v_k, \underline{v}_k) = (v_k - \underline{v}_k)^p - \underline{v}_k^p \rightarrow v_0^p \quad \text{a.e. } x \in \mathbf{R}^N.$$

From (i) of Lemma 5.1 and (5.19) it follows that

$$g(v_k, \underline{v}_k) \leq C(v_k + v_k^p) \leq C(h + h^p) \quad \text{a.e. } x \in \mathbf{R}^N.$$

By the Hölder inequality we have

$$\int_{\mathbf{R}^N} (h + h^p) \phi \rho dx \leq \|h\|_{L_\rho^2} \|\phi\|_{L_\rho^2} + \|h\|_{L_\rho^{p+1}}^p \|\phi\|_{L_\rho^{p+1}} < \infty.$$

Therefore, by the Lebesgue convergence theorem, we obtain (5.18). Hence,  $v_0$  satisfies the equation in (1.9).

Next we show  $v_0 > 0$ . From Proposition A.1 in Appendix A,  $v_0 \in C^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$  and  $|\nabla v_0| \in L^\infty(\mathbf{R}^N)$ . By (i) of Proposition A.2,  $\{v_k\}$  is bounded in  $C^1(\mathbf{R}^N)$ . Thus  $\{v_k - v_0\}$  is bounded in  $C^1(\mathbf{R}^N)$ . Recall that  $v_k - v_0 \rightarrow 0$  in  $H_\rho^1(\mathbf{R}^N)$  by Step 2. Then, by (ii) of Proposition A.2 we have  $v_k \rightarrow v_0$  in  $L^\infty(\mathbf{R}^N)$ , and hence  $v_0 \geq 0$ . Lemma 5.8 yields  $v_0 \not\equiv 0$ . Thus  $v_0 > 0$  by Proposition 2.1.

Therefore,  $v_0$  solves the problem (1.9). Since the solution of the problem (1.9) is unique by [25, Corollary 2], we conclude that  $v_0 = u_0$ . In particular, we obtain  $v_k \rightarrow u_0$  in  $H_\rho^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$  as  $k \rightarrow \infty$ .

Let  $\lambda_k$  be a sequence satisfying  $\lambda_k \rightarrow 0$  as  $k \rightarrow 0$ . Then, by Steps 1-3, there exists a subsequence (still denoted by  $\{\lambda_k\}$ ) such that  $u_{\lambda_k} \rightarrow u_0$  in  $H_\rho^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$  as  $k \rightarrow 0$ , which implies that  $u_\lambda \rightarrow u_0$  in  $H_\rho^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$  as  $\lambda \rightarrow 0$ . This completes the proof of Proposition 5.2.  $\square$

## Appendix A

**Proposition A.1.** *Let  $u \in H_\rho^1(\mathbf{R}^N)$  be a solution of*

$$(A.1) \quad \Delta u + \frac{1}{2}x \cdot \nabla u + f(x, u) = 0 \quad \text{in } \mathbf{R}^N,$$

where  $f$  is Hölder continuous and satisfies

$$(A.2) \quad |f(x, u)| \leq C(u + u^p), \quad x \in \mathbf{R}^N, \quad u \in [0, \infty),$$

for some constants  $C > 0$  and  $p > 1$ ,  $(N - 2)p < N + 2$ . Then  $u \in C^2(\mathbf{R}^N)$ , and both  $u(x)$  and  $|\nabla u(x)|$  tend to 0 as  $|x| \rightarrow \infty$ .

We prove Proposition A.1 by following the idea of Escobedo and Kavian [8]. First we prepare the following lemma.

**Lemma A.1.** (i) *Let  $u \in H_\rho^1(\mathbf{R}^N)$ . Then  $u \in L^r(\mathbf{R}^N)$  and  $|x||\nabla u| \in L^r(\mathbf{R}^N)$  for all  $r \in [1, 2]$ .*

(ii) *Assume that  $u \in L_\rho^2(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$  for some  $q > 2$ . Then  $u \in L^r(\mathbf{R}^N)$  for all  $r \in [2, q]$ .*

(iii) *Assume that  $|\nabla u| \in L_\rho^2(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$  for some  $q > 2$ . Then  $|x||\nabla u| \in L^r(\mathbf{R}^N)$  for all  $r \in [2, q]$ .*

(iv) *Let  $u \in L_\rho^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ . Then  $u \in L^q(\mathbf{R}^N)$  for all  $q > 2$ .*

(v) *Let  $|\nabla u| \in L_\rho^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ . Then  $|x||\nabla u| \in L^q(\mathbf{R}^N)$  for all  $q > 2$ .*

*Proof.* (i) It is clear if  $r = 2$ . For  $1 \leq r < 2$  we have

$$\begin{aligned} \int_{\mathbf{R}^N} u^r dx &= \int_{\mathbf{R}^N} u^r \rho^{r/2} \rho^{-r/2} dx \\ &\leq \left( \int_{\mathbf{R}^N} u^2 \rho dx \right)^{r/2} \left( \int_{\mathbf{R}^N} \rho^{-r/(2-r)} dx \right)^{(2-r)/2} < \infty \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbf{R}^N} |x|^r |\nabla u|^r dx &= \int_{\mathbf{R}^N} |x|^r |\nabla u|^r \rho^{r/2} \rho^{-r/2} dx \\ &\leq \left( \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx \right)^{r/2} \left( \int_{\mathbf{R}^N} |x|^{2r/(2-r)} \rho^{-r/(2-r)} dx \right)^{(2-r)/2} < \infty. \end{aligned}$$

(ii) Let  $r \in (2, q)$ . Put  $s = (q-2)/(r-2) > 1$ . Then we have

$$\begin{aligned} \int_{\mathbf{R}^N} |u|^r dx &\leq \int_{\mathbf{R}^N} u^{q/s} u^{(2s-2)/s} \rho^{(s-1)/s} dx \\ &\leq \left( \int_{\mathbf{R}^N} u^q dx \right)^{1/s} \left( \int_{\mathbf{R}^N} u^2 \rho dx \right)^{(s-1)/s} < \infty. \end{aligned}$$

(iii) Let  $r \in (2, q)$ . Put  $s = (q-2)/(r-2) > 1$ . Then we have

$$\begin{aligned} \int_{\mathbf{R}^N} |x|^r |\nabla u|^r dx &\leq \sup_{x \in \mathbf{R}^N} (|x|^r \rho^{-(s-1)/s}) \int_{\mathbf{R}^N} |\nabla u|^{q/s} |\nabla u|^{(2s-2)/s} \rho^{(s-1)/s} dx \\ &\leq \sup_{x \in \mathbf{R}^N} (|x|^r \rho^{-(s-1)/s}) \left( \int_{\mathbf{R}^N} |\nabla u|^q dx \right)^{1/s} \left( \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx \right)^{(s-1)/s} < \infty. \end{aligned}$$

(iv) For  $q > 2$ , we have

$$\int_{\mathbf{R}^N} |u|^q dx \leq \int_{\mathbf{R}^N} |u|^q \rho dx \leq \|u\|_{L^\infty}^{q-2} \int_{\mathbf{R}^N} u^2 \rho dx < \infty.$$

(v) For  $q > 2$ , we have

$$\begin{aligned} \int_{\mathbf{R}^N} |x|^q |\nabla u|^q dx &\leq \sup_{x \in \mathbf{R}^N} (|x|^q \rho^{-1}) \int_{\mathbf{R}^N} |\nabla u|^q \rho dx \\ &\leq \sup_{x \in \mathbf{R}^N} (|x|^q \rho^{-1}) \|\nabla u\|_{L^\infty}^{q-2} \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx < \infty. \end{aligned}$$

□

Set

$$(A.3) \quad h(x, u) = \frac{1}{2} x \cdot \nabla u + u + f(x, u).$$

Then the solution  $u$  of (A.1) satisfies

$$(A.4) \quad -\Delta u + u = h(x, u) \quad \text{in } \mathbf{R}^N.$$

We show the following:

**Lemma A.2.** *Let  $u \in H_\rho^1(\mathbf{R}^N)$  be a solution of (A.1) such that  $u \in L^q(\mathbf{R}^N)$  for some  $q > 2$ . Then  $h(x, u)$  defined by (A.3) satisfies  $h \in L^{q/p}(\mathbf{R}^N)$ .*

*Proof.* From (i) and (ii) of Lemma A.1,  $u \in L^r(\mathbf{R}^N)$  for all  $r \in [1, q]$ . Since  $f$  satisfies (A.2), we have  $f(x, u) \in L^r(\mathbf{R}^N)$  for  $1 \leq r \leq q/p$ . Then it suffices to show that  $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$ . If  $q/p \leq 2$ , from (i) of Lemma A.1, the result is established. So we assume that  $q/p > 2$ . From  $u \in H_\rho^1(\mathbf{R}^N)$ , we have  $h \in L^2(\mathbf{R}^N)$ . Then, by using the equation (A.4) we obtain  $u \in W^{2,2}(\mathbf{R}^N)$ . By the Sobolev embeddings, we have

$$|\nabla u| \in L^{r_1}(\mathbf{R}^N), \quad \frac{1}{r_1} = \frac{1}{2} - \frac{1}{N} \quad \text{if } N > 2,$$

$$|\nabla u| \in L^r(\mathbf{R}^N) \quad \text{for all } r > 2 \quad \text{if } N = 2.$$

In the cases where  $N = 2$  or  $r_1 > q/p$ , from (iii) of Lemma A.1, we have  $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$ , and the result is established. In the cases where  $N > 2$  and  $r_1 \leq q/p$ , from (iii) of Lemma A.1, we have  $|x||\nabla u| \in L^r(\mathbf{R}^N)$  for all  $r \in [1, r_1]$ . Then  $h \in L^r(\mathbf{R}^N)$  for  $r \in [1, r_1]$ , and so  $u \in W^{2,r}(\mathbf{R}^N)$  for  $r \in [1, r_1]$ . The Sobolev embeddings now yield

$$|\nabla u| \in L^r(\mathbf{R}^N) \quad \text{for all } r \in [1, r_2], \quad \frac{1}{r_2} = \frac{1}{r_1} - \frac{1}{N}, \quad \text{if } r_1 < N,$$

$$|\nabla u| \in L^\infty(\mathbf{R}^N), \quad \text{if } r_1 > N.$$

In the cases where  $r_1 > N$  or  $r_2 > q/p$ , we have  $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$ . In the cases where  $r_2 \leq p/q$ , we have  $|x||\nabla u| \in L^r(\mathbf{R}^N)$  for all  $r \in [1, r_2]$ . Repeating the arguments in finite times, we obtain  $|x||\nabla u| \in L^{q/p}(\mathbf{R}^N)$ .  $\square$

*Proof of Proposition A.1.* We show the case where  $N \geq 3$ . We can verify the case where  $N = 2$  with a slight modification.

First we show  $u \in L^\infty(\mathbf{R}^N)$ . From (iii) of Lemma 3.1,  $u \in L^{q_0}(\mathbf{R}^N)$ , where  $q_0 = 2N/(N-2)$ . Then, from Lemma A.2, we have  $h \in L^{q_0/p}(\mathbf{R}^N)$ . By using the equation (A.4) we obtain  $u \in W^{2,q_0/p}(\mathbf{R}^N)$ . Then the Sobolev embedding implies that

$$u \in L^{q_1}(\mathbf{R}^N), \quad \frac{1}{q_1} = \frac{p}{q_0} - \frac{2}{N} \quad \text{if } q_0 < \frac{pN}{2},$$

$$u \in L^\infty(\mathbf{R}^N), \quad \text{if } q_0 > \frac{pN}{2}.$$

We note that  $q_1 > q_0$  from the assumption  $p < (N+2)/(N-2)$ . If  $q_0 < pN/2$ , from Lemma A.2, we have  $h \in L^{q_1/p}(\mathbf{R}^N)$ , and hence  $u \in W^{2,q_1/p}(\mathbf{R}^N)$ . Then the Sobolev embedding implies that

$$u \in L^{q_2}(\mathbf{R}^N), \quad \frac{1}{q_2} = \frac{p}{q_1} - \frac{2}{N} \quad \text{if } q_1 < \frac{pN}{2},$$

$$u \in L^\infty(\mathbf{R}^N), \quad \text{if } q_1 > \frac{pN}{2}.$$

Repeating above arguments in finite times, we obtain  $u \in L^\infty(\mathbf{R}^N)$ .

From (iv) of Lemma A.1 we have  $u \in L^{pq}(\mathbf{R}^N)$  for all  $q > N$ . Then from Lemma A.2 we have  $h \in L^q(\mathbf{R}^N)$ , and hence  $u \in W^{2,q}(\mathbf{R}^N)$  for all  $q > N$ . By the Sobolev embedding theorem,  $u \in C^{1,\theta}(\mathbf{R}^N)$  for some  $\theta \in (0, 1)$ . Then, since  $f$  is Hölder continuous, we obtain  $u \in C^2(\mathbf{R}^N)$ . We note that  $C_0^\infty(\mathbf{R}^N)$  is dense in  $W^{2,q}(\mathbf{R}^N)$ . Then, by using the Sobolev embedding theorem again, we obtain  $u(x) \rightarrow 0$  and  $|\nabla u(x)| \rightarrow 0$  as  $|x| \rightarrow \infty$ .  $\square$

**Proposition A.2.** (i) Assume that  $\{u_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ , and that  $u_k$  satisfies

$$(A.5)_k \quad \Delta u_k + \frac{1}{2}x \cdot \nabla u_k + f_k(x, u_k) = 0 \quad \text{in } \mathbf{R}^N$$

for  $k = 1, 2, \dots$ . We assume in  $(A.5)_k$  that  $f_k$  satisfies

$$|f_k(x, u)| \leq C(u + u^p), \quad x \in \mathbf{R}^N, \quad u \in [0, \infty),$$

for some constants  $C > 0$  and  $p > 1$ ,  $(N - 2)p < N + 2$ , where  $C$  and  $p$  are independent of  $k$ . Then  $\{u_k\}$  is bounded in  $C^1(\mathbf{R}^N)$ .

(ii) Assume that  $\{u_k\}$  is bounded in  $C^1(\mathbf{R}^N)$ , and that  $u_k \rightarrow 0$  in  $H_\rho^1(\mathbf{R}^N)$ . Then  $u_k \rightarrow 0$  in  $L^\infty(\mathbf{R}^N)$  as  $k \rightarrow \infty$ .

From the proof of Lemma A.1 we obtain the following results.

**Lemma A.3.** (i) Assume that  $\{u_k\}$  is bounded in  $H_\rho^1(\mathbf{R}^N)$ . Then  $\{u_k\}$  and  $\{|x||\nabla u_k|\}$  are bounded in  $L^r(\mathbf{R}^N)$  for all  $r \in [1, 2]$ .

(ii) Assume that  $\{u_k\}$  is bounded in  $L_\rho^2(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$  for some  $q > 2$ . Then  $\{u_k\}$  is bounded in  $L^r(\mathbf{R}^N)$  for all  $r \in [2, q]$ .

(iii) Assume that  $\{|\nabla u_k|\}$  is bounded in  $L_\rho^2(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$  for some  $q > 2$ . Then  $\{|x||\nabla u_k|\}$  is bounded in  $L^r(\mathbf{R}^N)$  for all  $r \in [2, q)$ .

(iv) Let  $\{u_k\}$  is bounded in  $L_\rho^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ . Then  $\{u_k\}$  is bounded in  $L^q(\mathbf{R}^N)$  for all  $q > 2$ .

(v) Let  $\{|\nabla u_k|\}$  is bounded in  $L_\rho^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ . Then  $\{|x||\nabla u_k|\}$  is bounded in  $L^q(\mathbf{R}^N)$  for all  $q > 2$ .

Set

$$h_k(x, u) = \frac{1}{2}x \cdot \nabla u + u + f_k(x, u).$$

Then the solution  $u_k$  of  $(A.5)_k$  satisfies

$$-\Delta u_k + u_k = h_k(x, u_k) \quad \text{in } \mathbf{R}^N.$$

By the similar arguments in the proof of Lemma A.2, we obtain the following results.

**Lemma A.4.** *Assume that  $u_k$  is a solution of  $(A.5)_k$  such that  $\{u_k\}$  is bounded in  $L^q(\mathbf{R}^N)$  for some  $q > 2$ . Then  $\{h_k(\cdot, u_k)\}$  is bounded in  $L^{q/p}(\mathbf{R}^N)$ .*

*Proof of Proposition A.2.* (i) Following the arguments in the proof of Proposition A.1, we obtain  $\{u_k\}$  is bounded in  $L^\infty(\mathbf{R}^N)$ . From (iv) of Lemma A.3,  $\{u_k\}$  is bounded in  $L^{pq}(\mathbf{R}^N)$  for all  $q > N$ . Then  $\{h_k\}$  is bounded in  $L^q(\mathbf{R}^N)$ , and hence,  $\{u_k\}$  is bounded in  $W^{2,q}(\mathbf{R}^N)$  for all  $q > N$ . By the Sobolev embedding theorem,  $\{u_k\}$  is bounded in  $C^1(\mathbf{R}^N)$ .

(ii) Let  $q > N$ . Since  $\{u_k\}$  is bounded in  $C^1(\mathbf{R}^N)$  and  $u_k \rightarrow 0$  in  $H_\rho^1(\mathbf{R}^N)$ , we have

$$\|u_k\|_{L^q}^q \leq \int_{\mathbf{R}^N} u_k^q \rho dx \leq \|u_k\|_{L^\infty}^{q-2} \int_{\mathbf{R}^N} u_k^2 \rho dx \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

and

$$\|\nabla u_k\|_{L^q}^q \leq \int_{\mathbf{R}^N} |\nabla u_k|^q \rho dx \leq \|\nabla u_k\|_{L^\infty}^{q-2} \int_{\mathbf{R}^N} |\nabla u_k|^2 \rho dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence,  $u_k \rightarrow 0$  in  $W^{1,q}(\mathbf{R}^N)$  as  $k \rightarrow \infty$  for  $q > N$ . Then by the Sobolev embedding theorem, we have  $u_k \rightarrow 0$  in  $L^\infty(\mathbf{R}^N)$ .  $\square$

## Appendix B

**Lemma B.1.** *Let  $u$  be a positive function on  $\mathbf{R}^N$ , and let  $w$  be a function defined by (1.5) on  $\mathbf{R}^N \times (0, \infty)$ . Then  $w$  satisfies  $(1.2)_\lambda$  in the sense of  $L_{\text{loc}}^1(\mathbf{R}^N)$ , if and only if  $u$  satisfies  $(1.7)_\lambda$ .*

In order to prove Lemma B.1 we need the following

**Lemma B.2.** *Let  $w$  be the function in Lemma B.1. Put  $B_R = \{x \in \mathbf{R}^N : |x| < R\}$ , where  $R > 0$ . Then*

$$(B.1) \quad |x|^{2/(p-1)} w(x, t) \rightarrow \lambda a(x/|x|) \quad \text{as } t \rightarrow 0 \quad \text{for a.e. } x \in B_R$$

*if and only if*

$$(B.2) \quad w(\omega, t) \rightarrow \lambda a(\omega) \quad \text{as } t \rightarrow 0 \quad \text{for a.e. } \omega \in S^{N-1}.$$

*Proof.* Define  $E \subset \mathbf{R}^N$  such that if  $x \in \mathbf{R}^N \setminus E$  then

$$|x|^{2/(p-1)}w(x, t) \rightarrow \lambda a(x/|x|) \quad \text{as } t \rightarrow 0.$$

It follows from (1.5) that if  $x \in \mathbf{R}^N \setminus E$  and  $\mu > 0$  then

$$|\mu x|^{2/(p-1)}w(\mu x, t) = |x|^{2/(p-1)}w(x, t/\mu^2) \rightarrow \lambda a(x/|x|) \quad \text{as } t \rightarrow 0.$$

This implies that  $x \in \mathbf{R}^N \setminus E$  if and only if  $\mu x \in \mathbf{R}^N \setminus E$  for all  $\mu > 0$ . Thus we have

$$(B.3) \quad x \in E \quad \text{if and only if} \quad \mu x \in E \quad \text{for all } \mu > 0.$$

Put  $E_S = E \cap S^{N-1}$  and  $E_B = E \cap B_R$ . Then it follows from (B.3) that

$$\int_{E_B} dx = \int_0^R N\omega_N r^{N-1} \left( \int_{E_S} dS \right) dr = \omega_N R^N \int_{E_S} dS,$$

where  $\omega_N$  is the volume of unit ball in  $\mathbf{R}^N$  and  $dS$  denotes the surface measure on  $E_S$ . This implies that (B.1) holds if and only if (B.2) holds.  $\square$

*Proof of Lemma B.1.* From (1.5) we see that  $|x|^{2/(p-1)}w(x, t) = |y|^{2/(p-1)}u(y)$ , where  $y = x/\sqrt{t}$ . In particular, we have

$$(B.4) \quad w(\omega, t) = r^{2/(p-1)}u(r\omega), \quad \text{where } r = 1/\sqrt{t}.$$

Assume that  $u$  satisfies  $(1.7)_\lambda$ . Then, from (B.4), we obtain (B.2). Lemma B.2 implies that (B.1) holds for any  $R > 0$ . Now, fix a compact set  $K \subset \mathbf{R}^N$ . Then, by the Lebesgue dominated convergence theorem, we have

$$\int_K |w(x, t) - \lambda a(x/|x|)| dx = \int_K |x|^{-2/(p-1)} \left| |x|^{2/(p-1)}w(x, t) - \lambda a(x/|x|) \right| dx \rightarrow 0$$

as  $t \rightarrow 0$ . Therefore,  $w$  satisfies  $(1.2)_\lambda$  in the sense of  $L^1_{\text{loc}}(\mathbf{R}^N)$ .

Conversely, assume that  $w$  satisfies  $(1.2)_\lambda$  in the sense of  $L^1_{\text{loc}}(\mathbf{R}^N)$ . Then (B.1) holds for  $R > 0$ , which implies (B.2) by Lemma B.2. From (B.4) we find that  $u$  satisfies  $(1.7)_\lambda$ . This completes the proof of Lemma B.1.  $\square$

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## REFERENCES

- [1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14** (1973), 349-381.
- [2] H. Brezis, *Analyse Fonctionnelle*, Masson, Paris, 1983.
- [3] T. Cazenave, F. Dickstein, M. Escobedo, and F. B. Weissler, Self-similar solutions of a nonlinear heat equation, *J. Math. Sci. Univ. Tokyo* **8** (2001), 501-540.
- [4] T. Cazenave and F. B. Weissler, Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations, *Math. Z.* **228** (1998), 83-120.
- [5] M. G. Crandall and P. H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, *Arch. Rational Mech. Anal.* **58** (1975), 207-218.
- [6] E. DiBenedetto, *An Introduction to Partial Differential Equations*, Birkhäuser, Boston, 1995.
- [7] C. Dohmen and M. Hirose, Structure of positive radial solutions to the Haraux-Weissler equation, *Nonlinear Anal. TMA* **33** (1998), 51-69.
- [8] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions for the heat equation, *Nonlinear Anal. TMA* **11** (1987), 1103-1133.
- [9] M. Escobedo, O. Kavian, and H. Matano, Large time behavior of solutions of a dissipative semilinear heat equation, *Comm. Partial Differential Equations*, **27** (1995), 1427-1452.
- [10] H. Fujita, On the blowing up of solutions of the Cauchy problem for  $u_t = \Delta u + u^{1+\alpha}$ , *J. Fac. Sci. Univ. Tokyo, Sect. I* **13** (1966), 109-124.
- [11] V. A. Galaktionov and J. L. Vazquez, Continuation of blowup solutions of nonlinear heat equations in several space dimensions, *Comm. Pure Appl. Math.* **50** (1997) 1-67.

- [12] Y. Giga, Solutions for semilinear parabolic equations in  $L^p$  and regularity of weak solutions of the Navier-Stokes system, *J. Differential Equations* **62** (1986), 186-212.
- [13] Y. Giga and T. Miyakawa, Navier-Stokes flow in  $\mathbf{R}^3$  with measures as initial vorticity and Morrey spaces, *Comm. Partial Differential Equations* **14** (1989), 577-618.
- [14] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of the Second Order*, Springer-Verlag, Berlin, 1983.
- [15] A. Gmira and L. Veron, Large time behaviour of the solutions of a semilinear parabolic equation in  $\mathbf{R}^N$ , *J. Differential Equations* **53** (1984), 258-276.
- [16] A. Haraux and F. B. Weissler, Non-uniqueness for a semilinear initial value problem, *Indiana Univ. Math. J.* **31** (1982), 167-189.
- [17] P. Hartman and A. Wintner, On a comparison theorem for self-adjoint partial differential equations of elliptic type, *Proc. Amer. Math. Soc.* **6** (1955), 862-865.
- [18] L. Herraiz, Asymptotic behaviour of solutions of some semilinear parabolic problems, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **16** (1999), 49-105.
- [19] S. Kamin and L. A. Peletier, Large time behaviour of solutions of the heat equation with absorption, *Ann. Scuola Norm. Sup. Pisa* **12** (1985), 393-408.
- [20] O. Kavian, Remarks on the large time behavior of a nonlinear diffusion equation, *Annal. Institut Henri Poincaré-Analyse nonlinéaire* **4** (1987), 423-452.
- [21] T. Kawanago, Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **13** (1996), 1-15.
- [22] H. Kozono and M. Yamazaki Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, *Comm. Partial Differential Equations* **19** (1994), 959-1014.

- [23] M. Kwak, A semilinear heat equation with singular initial data, Proc. Royal Soc. Edinburgh Sect. A **128** (1998), 745-758.
- [24] T.-Y. Lee and W.-M. Ni, Global existence, large time behavior and life span of solutions of semilinear parabolic Cauchy problem, Trans. Amer. Math. Soc. **333** (1992), 365-371.
- [25] Y. Naito and T. Suzuki, Radial symmetry of self-similar solutions for semilinear heat equations, J. Differential Equations **163** (2000), 407-428.
- [26] L. A. Peletier, D. Terman, and F. B. Weissler, On the equation  $\Delta u + \frac{1}{2}x \cdot \nabla u + f(u) = 0$ , Arch. Rational Mech. Anal. **94** (1986), 83-99.
- [27] M. Protter and H. Weinberger, "Maximal Principles in Differential Equations", Prentice-Hall, Englewood Cliffs, N.J. 1967.
- [28] S. Snoussi and S. Tayachi, and F. B. Weissler, Asymptotically self-similar global solutions of a semilinear parabolic equation with a nonlinear gradient term, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), 1291-1307.
- [29] S. Snoussi and S. Tayachi, and F. B. Weissler, Asymptotically self-similar global solutions of a general semilinear heat equation, Math. Ann. **321** (2001), 131-155.
- [30] P. Souplet and F. B. Weissler, Regular self-similar solutions of the nonlinear heat equation with initial data above the singular steady state, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), 213-235.
- [31] C. A. Swanson, Comparison and Oscillation Theory of Linear Differential Equations, Academic Press, New York, 1968.
- [32] F. B. Weissler, Semilinear evolution equations in Banach spaces, J. Funct. Anal. **32** (1979), 277-296.
- [33] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in  $L^p$ , Indiana Univ. Math. J. **29** (1980), 79-102.
- [34] F. B. Weissler, Existence and non-existence of global solutions for a semilinear heat equation, Israel J. Math. **38** (1981), 29-40.

- [35] F. B. Weissler, Asymptotic analysis of an ordinary differential equation and nonuniqueness for a semilinear partial differential equation, Arch. Rational Mech. Anal. **91** (1985), 231-245.
- [36] F. B. Weissler, Rapidly decaying solutions of an ordinary differential equation with applications to semilinear elliptic and parabolic partial differential equations, Arch. Rational Mech. Anal. **91** (1985), 247-266.
- [37] F. B. Weissler,  $L^p$ -energy and blow-up for a semilinear heat equation, Proc. Sympos. Pure Math. **45** Part 2, Amer. Math. Soc., Providence, 1986, 545-551.
- [38] E. Yanagida, Uniqueness of rapidly decaying solutions to the Haraux-Weissler equation, J. Differential Equations **127** (1996), 561-570.