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On the existence of multiple solutions of the boundary value problem for nonlinear second order differential equations

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1. Introduction

In this paper we consider the second order ordinary differential equation

$$u'' + a(x)f(u) = 0, \quad 0 < x < 1 \tag{1.1}$$

with the boundary condition

$$u(0) = u(1) = 0. \tag{1.2}$$

In equation (1.1) we assume that a satisfies

$$a \in C^1[0,1], \quad a(x) > 0 \quad \text{for } 0 \le x \le 1, \tag{1.3}$$

and that f satisfies the following conditions (H1)–(H3):

(H1) $f \in C(\mathbf{R})$, $f(s) > 0$ for $s > 0$, $f(-s) = -f(s)$ for $s > 0$, and f is locally Lipschitz continuous on $(0, \infty)$;

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(H2) There exist limits f_0 and f_∞ such that $0 \le f_0,\ f_\infty \le \infty$,

$$f_0 = \lim_{s \to +0} \frac{f(s)}{s} \quad \text{and} \quad f_\infty = \lim_{s \to \infty} \frac{f(s)}{s};$$

(H3) In the case where $f_0 = \infty$ in (H2), $f(s)$ is nondecreasing and $f(s)/s$ is nonincreasing on $(0, s_0]$ for some $s_0 > 0$.

From (H1) we see that $f(0) = 0$. The case where $f(s) = |s|^{p-1}s$ with $p > 0$ is a typical case satisfying (H1)–(H3). Thus, $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. While, if $0 < f_0 < \infty$ and $0 < f_\infty < \infty$, then f is asymptotically linear at 0 and ∞, respectively.

In this paper we investigate the existence of multiple solutions of the problem (1.1) and (1.2). This kind of problem has been studied by many authors with various methods and techniques. We refer for instance to the papers [1–10, 12–17, 19] and the references cited therein.

In this paper we establish the precise conditions concerning the behavior of the ratio $f(s)/s$ for the existence and non-existence of the solutions. In particular, we will show that the problem (1.1) and (1.2) has at least k solutions, where k represents the number of eigenvalues crossed by the ratio $f(s)/s$. (See Theorem 2 below.) For the autonomous case $a(t) \equiv$ const, this result can be shown by applying a time-mapping method, see, e.g. [10]. However, it seems that very little is known about the result for the nonautonomous case $a(t) \not\equiv$ const. We will obtain here the result for the nonautonomous case by employing the shooting method together with the Sturm's comparison theorem. Moreover our results will help us to develop the previous arguments, and to treat the known results from a unified point of view.

Let λ_k be the k-th eigenvalue of

$$\text{(1.4)} \qquad \begin{cases} \varphi'' + \lambda a(x)\varphi = 0, \quad 0 < x < 1, \\ \varphi(0) = \varphi(1) = 0, \end{cases}$$

and let φ_k be an eigenfunction corresponding to λ_k. It is known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \qquad \lim_{k \to \infty} \lambda_k = \infty,$$

and that φ_k has exactly $k-1$ zeros in $(0,1)$. (See, e.g., [18, Chap. VI, Sec. 27].) For convenience, we put $\lambda_0 = 0$.

First we consider the case where the range of $f(s)/s$ contains no eigenvalue of the problem (1.4).

Theorem 1. *Assume that there exists an integer $k \in \mathbf{N} = \{1, 2, \ldots\}$ such that*

$$\lambda_{k-1} < \frac{f(s)}{s} < \lambda_k \quad \text{for } s \in (0, \infty). \tag{1.5}$$

Then the problem (1.1) *and* (1.2) *has no solution $u \in C^2[0, 1]$.*

Next we consider the case where the range of $f(s)/s$ contains at least one eigenvalue of the problem (1.4). Note that if u is a solution of (1.1), so is $-u$, because of $f(-s) = -f(s)$. Hence we consider solutions u of the problem (1.1) and (1.2) with $u'(0) > 0$ only.

Theorem 2. *Assume that either $f_0 < \lambda_k < f_\infty$ or $f_\infty < \lambda_k < f_0$ for some $k \in \mathbf{N}$. Then the problem* (1.1) *and* (1.2) *has a solution u_k which have exactly $k - 1$ zeros in $(0, 1)$.*

Theorem 3. *Assume that either the following* (i) *or* (ii) *holds for some $k \in \mathbf{N}$:*

$$\text{(i)} \quad f_0 < \lambda_k < \lambda_{k+1} < f_\infty; \qquad \text{(ii)} \quad f_\infty < \lambda_k < \lambda_{k+1} < f_0.$$

Then the problem (1.1) *and* (1.2) *has solutions u_k and u_{k+1} such that u_k and u_{k+1} have exactly $k - 1$ and k zeros in $(0, 1)$, respectively, and satisfy $0 < u_k'(0) < u_{k+1}'(0)$ if* (i) *holds, and $u_k'(0) > u_{k+1}'(0) > 0$ if* (ii) *holds.*

Let us consider the cases where either f is superlinear or sublinear. As a consequence of Theorem 3 we obtain the following:

Corollary 1. *Assume that either the following* (i) *or* (ii) *holds:*

$$\text{(i)} \quad f_0 = 0, \ f_\infty = \infty; \qquad \text{(ii)} \quad f_0 = \infty, \ f_\infty = 0.$$

Then there exist solutions u_k $(k = 1, 2, \ldots)$ of the problem (1.1) *and* (1.2) *such that u_k has exactly $k - 1$ zeros in $(0, 1)$ for each $k \in \mathbf{N}$, and that*

$$0 < u_1'(0) < u_2'(0) < \cdots < u_k'(0) < u_{k+1}'(0) < \cdots$$

if (i) *holds, and*

$$u_1'(0) > u_2'(0) > \cdots > u_k'(0) > u_{k+1}'(0) > \cdots > 0$$

if (ii) *holds.*

Remark. (i) For the superlinear and sublinear cases, the existence of positive solutions of (1.1) and (1.2) has been obtained by Erbe and Wang [7] by using fixed point techniques.

(ii) The existence of an infinite sequence of solutions of (1.1) and (1.2) has been studied by Hartman [12] and Hooker [13] for the superlinear case, and Capietto and Dambrosio [3] for the sublinear case. In [12] and [13], Corollary 1 has been given under a weaker condition on a and f. For the superlinear and sublinear cases, we refer to [17].

Let us consider the nonlinear eigenvalue problem of the form

$$\begin{cases} u'' + \lambda a(x) f(u) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.6}$$

where $\lambda > 0$ is a real parameter. We assume in (1.6) that a satisfies (1.3) and f satisfies (H1)–(H3). By virtue of Theorem 2 we obtain the following corollary, which is motivated by the results of Kolodner [14] and Dinca and Sanchez [6].

Corollary 2. *Assume that either the following* (i) *or* (ii) *holds*:

$$\text{(i)} \quad f_0 = 0,\ f_\infty = 1; \qquad \text{(ii)} \quad f_0 = 1,\ f_\infty = 0.$$

Assume, in addition, that $\lambda_k < \lambda < \lambda_{k+1}$ *for some* $k \in \mathbf{N}$, *where* λ_k *is the* k*-th eigenvalue of the problem* (1.4). *Then the problem* (1.6) *possesses* k *solutions* u_j $(j = 1, 2, \ldots, k)$ *such that* u_j *has exactly* $j - 1$ *zeros in* $(0, 1)$.

By a change of variable, it can be shown that the existence of solution of the problem (1.1) and (1.2) is equivalent to the existence of radial solutions of the following Dirichlet problem for semilinear elliptic equations in annular domains:

$$\Delta u + a(|x|) f(u) = 0 \quad \text{in } \Omega, \tag{1.7}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

where $\Omega = \{x \in \mathbf{R}^N : R_1 < |x| < R_2\}$, $R_1 > 0$ and $N \geq 2$. We assume in (1.7) that $a \in C^1[R_1, R_2]$, $a(r) > 0$ for $R_1 \leq r \leq R_2$, and that f satisfies conditions (H1)–(H3).

Let μ_k be the k-th eigenvalue of

$$\begin{cases} (r^{N-1}\phi')' + \mu r^{N-1} a(r)\phi = 0, \quad R_1 \leq r \leq R_2, \\ \phi(R_1) = \phi(R_2) = 0. \end{cases} \tag{1.9}$$

It is known (see, e.g., [18, Chap. IV, Sec. 27]) that

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_k < \mu_{k+1} < \cdots, \quad \lim_{k\to\infty} \mu_k = \infty.$$

From Theorems 1 and 2 and Corollary 1, we obtain the following result, which will be proved in Section 3.

Corollary 3. (i) *Assume that there exists an integer $k \in \mathbf{N}$ such that*

$$\mu_{k-1} < \frac{f(s)}{s} < \mu_k \quad \textit{for } s \in (0, \infty).$$

Then the problem (1.7) *and* (1.8) *has no radial solution $u(r)$ in $C^2[R_1, R_2]$, where $r = |x|$.*

(ii) *Assume that either $f_0 < \mu_k < f_\infty$ or $f_\infty < \mu_k < f_0$ for some $k \in \mathbf{N}$. Then there exists a radial solution $u_k(r)$ of the problem* (1.7) *and* (1.8)*, which has exactly $k-1$ zeros in (R_1, R_2). In particular, if either* (i) *or* (ii) *in Corollary* 1 *holds, then there exist radial solutions $u_k(r)$ $(k = 1, 2, \ldots)$ of* (1.7) *and* (1.8) *such that $u_k(r)$ has exactly $k - 1$ zeros in (R_1, R_2) for each $k \in \mathbf{N}$.*

Remark. (i) The existence of radial positive solutions of (1.7) and (1.8) has been studied by many authors. For example we refer to [1, 2, 4, 5, 8, 9, 16] for the superlinear case, and to [19] for the sublinear case.

(ii) The existence of solutions with prescribed numbers of zeros is discussed by Coffman and Marcus [4] for the superlinear case.

(iii) Recently, Ercole and Zumpano [8] have established the existence of radial positive solutions with no assumptions on the behavior of the nonlinearity f either at zero or at infinity. They have used the fixed point theorem.

(iv) In order to treat radial solutions of the Dirichlet problem on a ball, we need to study the singular boundary value problem. See, e.g., Lemmert and Walter [15].

Theorem 1 follows immediately from the Sturm–Picone theorem. The proofs of Theorems 2 and 3 depends on the shooting method combined with the Sturm's comparison theorem. Namely we consider the solution $u(x;\mu)$ of (1.1) satisfying the initial condition

$$u(0) = 0 \quad \text{and} \quad u'(0) = \mu,$$

and observe the number of zeros of $u(x;\mu)$ in $(0,1]$ when $\mu \to 0$ and $\mu \to \infty$, by using the Sturm's comparison theorem. Here $\mu \in \mathbf{R}$ is a parameter.

This paper is organized as follows. In Section 2 we give a global existence and uniqueness of the solution for the initial value problem. In Section 3 we study the properties of the solution $u(x;\mu)$, and then prove Theorems 1–3 by employing the Prüfer transformation. In Section 4 we prove the lemmas stated in Section 3.

2. Preliminaries

In this section we consider the solution u of (1.1) with the initial condition

$$u(0) = 0 \quad \text{and} \quad u'(0) = \mu, \tag{2.1}$$

where $\mu \in \mathbf{R}$ is a parameter. We denote by $u(x;\mu)$ the solution of the problem (1.1) and (2.1).

Proposition 2.1. *For each $\mu \in \mathbf{R}$, the solution $u(x;\mu)$ exists on $[0,1]$ and is unique. Furthermore, the solution $u(x;\mu)$ satisfies the following properties* (i) *and* (ii):

(i) *$u(x;\mu)$ and $u'(x;\mu)$ are continuous functions of $(x,\mu) \in [0,1] \times \mathbf{R}$;*

(ii) *For each $\mu \in \mathbf{R} \backslash \{0\}$, the number of zeros of $u(x;\mu)$ in $[0,1]$ is finite.*

We will prove Proposition 2.1 by using the arguments introduced by Wong [20] and Coffman and Wong [5]. First we give some lemmas.

Lemma 2.1. *Assume that $f(s)$ is nondecreasing and $f(s)/s$ is nonincreasing on $(0, s_0]$ for some $s_0 > 0$. If $s_1, s_2, s_3 \in (0, s_0]$ satisfy $s_2, s_3 \geq s_1$ and $s_2 \neq s_3$, then*

$$0 \leq \frac{f(s_3) - f(s_2)}{s_3 - s_2} \leq \frac{f(s_1)}{s_1}. \tag{2.2}$$

Proof. In view of the monotonicity of $f(s)$, we have $(f(s_3) - f(s_2))/(s_3 - s_2) \geq 0$. We may assume that $s_3 > s_2$. Since $f(s)/s$ is nonincreasing, we obtain $s_2 f(s_3) \leq s_3 f(s_2)$. It follows that $s_2 f(s_3) - s_2 f(s_2) \leq s_3 f(s_2) - s_2 f(s_2)$, which implies that

$$\frac{f(s_3) - f(s_2)}{s_3 - s_2} \leq \frac{f(s_2)}{s_2}.$$

By the monotonicity of $f(s)/s$, we conclude that (2.2) holds. □

Let u be a solution of (1.1), and let $I[u] \subset [0, 1]$ be the maximal interval of existence for u. We define the energy function $E[u]$ as follows:

$$E[u](x) = \frac{[u'(x)]^2}{2} + a(x)F(u(x)) \quad \text{for } x \in I[u], \tag{2.3}$$

where

$$F(s) = \int_0^s f(\sigma)\, d\sigma \quad \text{for } s \in \mathbf{R}.$$

It is easy to see that $F(s) = F(|s|) > 0$ for all $s \in \mathbf{R} \setminus \{0\}$, $F(0) = 0$, and $F(s)$ is strictly increasing in $s \in (0, \infty)$. Thus $E[u](x) \geq 0$ on $I[u]$, and we conclude that $E[u](x) = 0$ on $I[u]$ if and only if $u(x) = 0$ on $I[u]$. Furthermore we obtain the following properties.

Lemma 2.2. *Let $x_0, x \in I[u]$. Then*

$$E[u](x) \leq E[u](x_0) \exp\left(\int_{x_0}^{x} \frac{[a'(t)]_+}{a(t)} dt\right) \quad \textit{for } x_0 < x, \tag{2.4}$$

and

$$E[u](x) \leq E[u](x_0) \exp\left(\int_{x}^{x_0} \frac{[a'(t)]_-}{a(t)} dt\right) \quad \textit{for } x < x_0, \tag{2.5}$$

where $[s]_+ = \max\{s, 0\}$ and $[s]_- = \max\{-s, 0\}$.

Proof. In view of (1.1) and (2.3), we find that

$$\frac{d}{dx}E[u](x) = a'(x)F(u(x)) \le \frac{[a'(x)]_+}{a(x)}E[u](x).$$

Then, for x_0, $x \in I[u]$, $x_0 < x$, we have

$$\frac{d}{dx}\left(E[u](x)\exp\left(-\int_{x_0}^{x}\frac{[a'(t)]_+}{a(t)}dt\right)\right) \le 0.$$

An integration of the above over $[x_0, x]$ gives inequality (2.4). Inequality (2.5) can be obtained in a similar fashion. □

Let us consider the equation (1.1) with the general initial condition

$$u(x_0) = \alpha, \quad u'(x_0) = \beta, \tag{2.6}$$

where $x_0 \in [0,1]$ and α, $\beta \in \mathbf{R}$ are arbitrarily given.

Lemma 2.3. *Let $x_0 \in [0,1]$, and let α, $\beta \in \mathbf{R}$. Then the initial value problem* (1.1) *and* (2.6) *has a unique local solution.*

Proof. The existence of a local solution of (1.1) and (2.6) is guaranteed by the Peano existence theorem. Then it suffices to show the uniqueness of a local solution of the problem. If either $f_0 < \infty$ or $\alpha \ne 0$ in (2.6), then the uniqueness of a local solution of the problem is clear since f satisfies a Lipschitz condition on $[\alpha - \delta, \alpha + \delta]$, where $\delta > 0$ is taken sufficiently small. Then the question is the case where $f_0 = \infty$ and $\alpha = 0$. In this case we make a distinction between $\beta \ne 0$ and $\beta = 0$.

The case where $f_0 = \infty$, $\alpha = 0$ and $\beta \ne 0$ in (2.6). We may suppose that $\beta > 0$ without loss of generality. Let u_1 and u_2 be local solutions of (1.1) satisfying (2.6) with $\alpha = 0$ and $\beta > 0$. Then there exists a number $x_1 \in (x_0, 1)$ such that

$$\frac{\beta}{2}(x - x_0) \le u_i(x) \le s_0 \quad \text{for } x_0 \le x \le x_1,\ i = 1, 2, \tag{2.7}$$

where s_0 is the number in (H3). Since u_1 and u_2 satisfy

$$u_i(x) = \beta(x - x_0) - \int_{x_0}^{x}(x - t)a(t)f(u_i(t))dt$$

for $x_0 \le x \le x_1$, $i = 1, 2$, we have

$$|u_1(x) - u_2(x)| \le (x - x_0) \int_{x_0}^{x} a(t)|f(u_1(t)) - f(u_2(t))|dt$$

for $x_0 \le x \le x_1$. By (2.7), Lemma 2.1 implies that

$$|f(u_1(x)) - f(u_2(x))| \le \frac{f(\beta(x - x_0)/2)}{\beta(x - x_0)/2}|u_1(x) - u_2(x)|$$

for $x_0 \le x \le x_1$. Then we obtain

$$\frac{|u_1(x) - u_2(x)|}{x - x_0} \le \frac{2}{\beta} \int_{x_0}^{x} a(t) f\left(\frac{\beta}{2}(t - x_0)\right) \frac{|u_1(t) - u_2(t)|}{t - x_0} dt$$

for $x_0 \le x \le x_1$. We note here that the function $U(x) = |u_1(x)-u_2(x)|/(x-x_0)$ has the finite limit as $x \to x_0 + 0$, and hence $U(x)$ can be regarded as a continuous function on the closed interval $[x_0, x_1]$. Then by Gronwall's inequality we see that $U(x) \equiv 0$ for $x_0 \le x \le x_1$, which implies that $u_1(x) \equiv u_2(x)$ for $x_0 \le x \le x_1$. The uniqueness in a left-neighborhood of x_0 is similarly proved.

The case where $f_0 = \infty$, $\alpha = 0$ *and* $\beta = 0$ *in* (2.6). In this case we will show $u \equiv 0$ by using of the function $E[u]$ defined by (2.3). Suppose that u is a local solution of (1.1) and (2.6) with $\alpha = \beta = 0$ and that it exists on an interval $[x_0, x_1]$. Then, making use of (2.4) and nothing that $E[x](x_0) = 0$, we see that $E[u](x) \equiv 0$, which implies the uniqueness on a right-neighborhood of x_0. Similarly, the use of (2.5) yields the uniqueness on a left-neighborhood of x_0. This completes the proof of Lemma 2.3. □

Proof of Proposition 2.1. From Lemma 2.3 the initial value problem (1.1) and (2.1) has a unique local solution $u(x; \mu)$ for each $\mu \in \mathbf{R}$, and the solution $u(x; \mu)$ is unique as far as the solution exists. Let $I[u]$ be the maximal interval of existence for $u(x; \mu)$. From Lemma 2.2 the function $E[u(\cdot, \mu)]$ defined by (2.3) satisfies

$$E[u(\cdot, \mu)](x) \le \frac{\mu^2}{2} \exp\left(\int_0^1 \frac{|a'(t)|}{a(t)} dt\right) \quad \text{for } x \in I[u].$$

This means that both $u(x; \mu)$ and $u'(x; \mu)$ are bounded as far as the solution exists. Thus, by a standard argument, we conclude that $u(x; \mu)$ is continuable on $[0, 1]$, that is, $u(x; \mu)$ exists on $[0, 1]$ and is unique.

It is well known that the scalar equation (1.1) can be written by the system of the first order differential equation $\boldsymbol{u}' = \boldsymbol{F}(x, \boldsymbol{u})$, where

$$\boldsymbol{u} = \begin{pmatrix} u \\ u' \end{pmatrix} \quad \text{and} \quad \boldsymbol{F}(x, \boldsymbol{u}) = \begin{pmatrix} u' \\ -a(x)f(u) \end{pmatrix}.$$

By a general theory on the continuous dependence of solutions on parameters and initial conditions (see, for example, [11, Chap. 1, Theorem 2.4]), it follows that $u(x;\mu)$ and $u'(x;\mu)$ are continuous in (x,μ) on the set $[0,1] \times \mathbf{R}$. Then (i) of Proposition 2.1 holds.

By the uniqueness we see that if $u(x;\mu)$ has infinitely many zeros in the finite interval $[0,1]$ then $u(x;\mu) \equiv 0$ on $[0,1]$. Thus, if $u(x;\mu) \not\equiv 0$ on $[0,1]$, then the number of zeros of $u(x;\mu)$ in $[0,1]$ is finite, that is (ii) of Proposition 2.1 holds. The proof is complete. □

3. Proofs of Theorems 1-3

In this section we give proofs of Theorems 1-3.

Proof of Theorem 1. Assume to the contrary that the problem (1.1) and (1.2) has a solution $u \in C^2[0,1]$. We see that u satisfies $u'' + b(x)u = 0$ for $0 < x < 1$, where

$$b(x) = a(x)\frac{f(u(x))}{u(x)}.$$

Note that $f_0 \le \lambda_k < \infty$ by (1.5), and hence $f(s)/s$ can be regarded as a continuous function on $\mathbf{R}$. Thus we have $b \in C[0,1]$.

From (1.5) it follows that

$$\lambda_{k-1}a(x) < b(x) < \lambda_k a(x) \quad \text{for } u(x) \neq 0.$$

First assume $k \ge 2$. Recall that an eigenfunction φ_{k-1} corresponding to λ_{k-1} has exactly $k-2$ zeros in $(0,1)$. By applying the Sturm–Picone theorem to φ_{k-1} and u (see, e.g., [18, Chap. IV, Sec. 27]), we see that u has at least $k-1$ zeros in $(0,1)$. Now, by applying the Sturm–Picone theorem again to u and φ_k, we find that φ_k has at least k zeros in $(0,1)$. This is a contradiction. If $k=1$, then by applying the Sturm–Picone theorem to u and φ_1, we conclude that φ_1 has at least one zero in $(0,1)$. This contradicts the positivity of φ_1. Consequently the problem (1.1) and (1.2) has no solution. □

To prove Theorems 2 and 3 we employ the Prüfer transformation for the solution $u(x;\mu)$ of the problem (1.1) and (2.1). For the solution $u(x;\mu)$ with $\mu \in \mathbf{R}$, we define the functions $r(x;\mu)$ and $\theta(x;\mu)$ by

$$\begin{cases} u(x;\mu) = r(x;\mu)\sin\theta(x;\mu), \\ u'(x;\mu) = r(x;\mu)\cos\theta(x;\mu), \end{cases}$$

where $' = d/dx$. Since $u(x;\mu)$ and $u'(x;\mu)$ cannot vanish simultaneously, $r(x;\mu)$ and $\theta(x;\mu)$ are written in the forms

$$r(x;\mu) = \Big([u(x;\mu)]^2 + [u'(x;\mu)]^2\Big)^{1/2} > 0$$

and

$$\theta(x;\mu) = \arctan\frac{u(x;\mu)}{u'(x;\mu)},$$

respectively. Therefore $r(x;\mu)$ and $\theta(x;\mu)$ are determined as continuously differentiable function with respect to $x \in [0,1]$. By a simple calculation we see that

$$\theta'(x;\mu) = \cos^2\theta(x;\mu) + a(x)\frac{\sin\theta(x;\mu) f(r(x;\mu)\sin\theta(x;\mu))}{r(x;\mu)} > 0$$

for $x \in [0,1]$, which shows that $\theta(x;\mu)$ is strictly increasing in $x \in [0,1]$ for each fixed $\mu \neq 0$. From the initial condition (2.1) it follows that $r(0;\mu) = \mu$ and $\theta(0;\mu) \equiv 0 \pmod{2\pi}$. For simplicity we take $\theta(0;\mu) = 0$. By (i) of Proposition 2.1, the function $\theta(x;\mu)$ is continuous in $(x,\mu) \in [0,1] \times \mathbf{R}$. It is easy to see that $u(x;\mu)$ has exactly k zeros in $(0,1)$ if and only if

$$k\pi < \theta(1;\mu) \leq (k+1)\pi.$$

To show Theorems 2 and 3 we need Lemmas 3.1–3.4 below. Recall that λ_k is the k-th eigenvalue of the problem (1.4). Since $u(x;-\mu) = -u(x;\mu)$ for $x \in [0,1]$, we consider only the case where $\mu > 0$.

Lemma 3.1. *Assume that $f_0 < \lambda_k$ for some $k \in \mathbf{N}$. Then there exists $\mu_* \in (0,1)$ such that, for each $\mu \in (0,\mu_*]$, the solution $u(x;\mu)$ has at most $k-1$ zeros in $(0,1)$.*

Lemma 3.2. *Assume that* $f_0 > \lambda_k$ *for some* $k \in \mathbf{N}$*. Then there exists* $\mu_* \in (0,1)$ *such that, for each* $\mu \in (0,\mu_*]$*, the solution* $u(x;\mu)$ *has at least* k *zeros in* $(0,1)$*.*

Lemma 3.3. *Assume that* $f_\infty > \lambda_k$ *for some* $k \in \mathbf{N}$*. Then there exists* $\mu^* > 1$ *such that, for each* $\mu \geq \mu^*$*, the solution* $u(x;\mu)$ *has at least* k *zeros in* $(0,1)$*.*

Lemma 3.4. *Assume that* $f_\infty < \lambda_k$ *for some* $k \in \mathbf{N}$*. Then there exists* $\mu^* > 1$ *such that, for each* $\mu \geq \mu^*$*, the solution* $u(x;\mu)$ *has at most* $k-1$ *zeros in* $(0,1)$*.*

The proofs of Lemmas 3.1–3.4 will be given in Section 4.

Proof of Theorem 2. First we suppose that $f_0 < \lambda_k < f_\infty$. Lemma 3.1 implies that there exists $\mu_* \in (0,1)$ such that $u(x;\mu)$ has at most $k-1$ zeros in $(0,1)$ for $\mu \in (0,\mu_*]$, that is, $\theta(1;\mu) \leq k\pi$ for $\mu \in (0,\mu_*]$. By Lemma 3.3, there exists $\mu^* > 1$ such that $\theta(1;\mu) > k\pi$ for $\mu \geq \mu^*$. Since $\theta(1;\mu)$ is continuous in $\mu \in [0,\infty)$, there exists a number $\mu_k \in [\mu_*,\mu^*)$ such that $\theta(1;\mu_k) = k\pi$. This implies that $u(x;\mu_k)$ is a solution of the problem (1.1) and (1.2) and has exactly $k-1$ zeros in $(0,1)$.

In the same way, from Lemmas 3.2 and 3.4, we can prove Theorem 2 for the case $f_\infty < \lambda_k < f_0$. The proof is complete. □

Proof of Theorem 3. Suppose that (i) holds. From the proof of Theorem 2 there exist $\mu_k > 0$ such that $\theta(1,\mu_k) = k\pi$, that is, the problem (1.1) and (1.2) possesses a solution $u(x;\mu_k)$, which has exactly $k-1$ zeros in $(0,1)$ and satisfies $u'(0;\mu_k) = \mu_k$. By Lemma 3.3, there exist $\mu^* > 1$ such that $\theta(1;\mu) > (k+1)\pi$ for $\mu \geq \mu^*$. Hence we see that $\mu_k < \mu^*$. By the continuity of $\theta(1,\mu)$ there exists $\mu_{k+1} \in (\mu_k,\mu^*)$ such that $\theta(1;\mu_{k+1}) = (k+1)\pi$. Thus the problem (1.1) and (1.2) possesses a solution $u(x;\mu_{k+1})$, which has exactly k zeros in $(0,1)$. It follows from $\mu_k < \mu_{k+1}$ that $u'(0;\mu_k) < u'(0;\mu_{k+1})$.

By using Lemmas 3.2 and 3.4, it can be shown Theorem 3 for the case (ii). This completes the proof of Theorem 3. □

Now we give the proof of Corollary 3.

Proof of Corollary 3. For a radial solution $u(r)$, equation (1.7) is rewritten in the form

$$\frac{d^2u}{dr^2} + \frac{N-1}{r}\frac{du}{dr} + a(r)f(u) = 0 \quad \text{for } R_1 < r < R_2,$$

where $r = |x|$. Let $v(t) = u(e^t)$ for $N = 2$, and let $v(t) = u(t^{1/(2-N)})$ for $N \geq 3$. Then it follows that the problem (1.7) and (1.8) is transformed into the problem

$$v'' + b(t)f(v) = 0 \quad \text{for } T_1 < t < T_2, \tag{3.1}$$

$$v(T_1) = v(T_2) = 0, \tag{3.2}$$

where $' = d/dt$, and $b(t) = e^{2t}a(e^t)$, $T_1 = \log R_1$ and $T_2 = \log R_2$ for $N = 2$, and $b(t) = (2-N)^{-2}t^{-2(N-1)/(N-2)}a(t^{1/(2-N)})$, $T_1 = R_2^{2-N}$ and $T_2 = R_1^{2-N}$ for $N \geq 3$.

Similarly, define $\psi(t) = \phi(e^t)$ for $N = 2$, and $\psi(t) = \phi(t^{1/(2-N)})$ for $N \geq 3$. Then the eigenvalue problem (1.9) is transformed into the problem

$$\begin{cases} \psi'' + \mu b(t)\psi = 0, \quad T_1 < t < T_2, \\ \psi(T_1) = \psi(T_2) = 0, \end{cases}$$

where $' = d/dt$. Thus, by applying Theorems 1 and 2 and Corollary 1 to (3.1) and (3.2), we have Corollary 3. □

4. Proofs of Lemmas 3.1–3.4

In this section we prove Lemmas 3.1–3.4. The following notation will be used:

$$a_* = \min\{a(x) : 0 \leq x \leq 1\}, \quad a^* = \max\{a(x) : 0 \leq x \leq 1\},$$

$$A_* = \exp\left(-\int_0^1 \frac{[a'(t)]_-}{a(t)}dt\right), \quad A^* = \exp\left(\int_0^1 \frac{[a'(t)]_+}{a(t)}dt\right).$$

First we need the following results.

Lemma 4.1. (i) *Let* $\mu > 0$. *Then*

$$\frac{\mu^2}{2}A_* \le E[u(\cdot;\mu)](x) \le \frac{\mu^2}{2}A^* \quad \textit{for } 0 \le x \le 1, \tag{4.1}$$

where $E[u(\cdot;\mu)]$ *is the function defined by* (2.3).

(ii) *Let* $M > 0$. *If*

$$\mu^2 \le \frac{2a_* F(M)}{A^*}, \tag{4.2}$$

then $|u(x;\mu)| \le M$ *for* $0 \le x \le 1$.

Proof. (i) From Lemma 2.2 we have

$$E[u(\cdot;\mu)](x) \le E[u(\cdot;\mu)](0) \exp\left(\int_0^x \frac{[a'(t)]_+}{a(t)} dt\right) \quad \text{for } x > 0$$

and

$$E[u(\cdot;\mu)](0) \le E[u(\cdot;\mu)](x) \exp\left(\int_0^x \frac{[a'(t)]_-}{a(t)} dt\right) \quad \text{for } x > 0.$$

Then

$$E[u(\,\cdot\,;\mu)](x) \le \frac{\mu^2}{2} \exp\left(\int_0^1 \frac{[a(t)]_+}{a(t)} dt\right) \quad \text{for } 0 \le x \le 1$$

and

$$E[u(\,\cdot\,;\mu)](x) \ge \frac{\mu^2}{2} \exp\left(-\int_0^1 \frac{[a(t)]_-}{a(t)} dt\right) \quad \text{for } 0 \le x \le 1,$$

respectively, and hence (4.1) holds.

(ii) In view of (4.1) and (4.2) we obtain

$$E[u(\cdot;\mu)](x) \le \frac{\mu^2}{2}A^* \le a_* F(M) \quad \text{for } 0 \le x \le 1.$$

From $a_* F(u(x;\mu)) \le E[u(\cdot;\mu)](x)$, it follows that $a_* F(u(x;\mu)) \le a_* F(M)$. Therefore $|u(x;\mu)| \le M$ for $0 \le x \le 1$. □

Proof of Lemma 3.1. By $f_0 < \lambda_k$ there is a number $M > 0$ such that

$$\frac{f(s)}{s} < \lambda_k \quad \text{for } |s| \le M. \tag{4.3}$$

Take a number $\mu_* \in (0,1)$ so small that (4.2) is satisfied for all $\mu \in (0, \mu_*]$. Let $\mu \in (0, \mu_*]$. From (ii) of Lemma 4.1 we obtain $|u(x;\mu)| \le M$ for $x \in [0,1]$. Now we define the function $b(x)$ by

$$b(x) = a(x)\frac{f(u(x;\mu))}{u(x;\mu)}, \quad 0 \le x \le 1.$$

Since $f(s)/s$ is continuous on $\mathbf{R}$ by $f_0 < \infty$, we conclude that $b \in C[0,1]$. In view of (4.3) and the fact that $|u(x;\mu)| \le M$ on $[0,1]$, we have

$$b(x) < \lambda_k a(x) \quad \text{for } 0 \le x \le 1. \tag{4.4}$$

Recall that an eigenfunction φ_k corresponding to λ_k has exactly $k-1$ zeros in $(0,1)$. Since $u(x;\mu)$ satisfies the equation $u'' + b(x)u = 0$ and has a zero $x = 0$, by (4.4) and the Sturm–Picone theorem, we see that $u(x;\mu)$ has at most $k-1$ zeros in $(0,1)$. This completes the proof. □

Proof of Lemma 3.2. From $\lambda_k < f_0$ we have

$$\lambda_k < \frac{f(s)}{s} \quad \text{for } 0 < |s| \le M \tag{4.5}$$

for some $M > 0$. There is a number $\mu_* \in (0,1)$ so that (4.2) holds for all $\mu \in (0, \mu_*]$. Let $\mu \in (0, \mu_*]$. From (ii) of Lemma 4.1 it follows that $|u(x;\mu)| \le M$ for $x \in [0,1]$. Let x_i $(i = 0, 1, 2, \ldots, k)$ be zeros of an eigenfunction φ_k corresponding to λ_k such that

$$0 = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = 1. \tag{4.6}$$

Assume that there is an integer $i \in \{1, 2, \ldots, k\}$ such that $u(x;\mu)$ has no zero in (x_{i-1}, x_i). Without loss of generality we may suppose that $u(x) \equiv u(x;\mu) > 0$ and $\varphi_k(x) > 0$ for $x \in (x_{i-1}, x_i)$. Then we note that $\varphi_k'(x_{i-1}) > 0$ and $\varphi_k'(x_i) < 0$. Hence

$$u(x_i)\varphi_k'(x_i) - u(x_{i-1})\varphi_k'(x_{i-1}) \le 0. \tag{4.7}$$

On the other hand, by $|u(x)| \le M$ for $0 \le x \le 1$ and (4.5) we obtain

$$\int_{x_{i-1}}^{x_i} (-u''\varphi_k + u\varphi_k'')\,dx = \int_{x_{i-1}}^{x_i} a(x)\left[\frac{f(u)}{u} - \lambda_k\right] u\varphi_k\,dx > 0.$$

Hence from $\varphi_k(x_{i-1}) = \varphi(x_i) = 0$ it follows that

$$-u'\varphi_k + u\varphi_k'\Big|_{x_{i-1}}^{x_i} = u(x_i)\varphi_k'(x_i) - u(x_{i-1})\varphi_k'(x_{i-1}) > 0.$$

This contradicts (4.7). Consequently, $u(x;\mu)$ has at least one zero in (x_{i-1}, x_i) for each $i \in \{1, 2, \ldots, k\}$, which means that $u(x;\mu)$ has at least k zeros in $(0, 1)$. The proof is complete. □

To prove Lemmas 3.3 and 3.4 we require the following lemmas.

Lemma 4.2. *Let x_i $(i = 0, 1, 2, \ldots, k)$ be zeros of an eigenfunction φ_k corresponding to λ_k satisfying* (4.6).

(i) *Let $\lambda > \lambda_k$. Then, for each $i \in \{1, 2, \ldots, k\}$, there is a solution w_i of*

$$w'' + \lambda a(x)w = 0, \tag{4.8}$$

which has at least two zeros in (x_{i-1}, x_i).

(ii) *Let $\lambda < \lambda_k$. Then, for each $i \in \{1, 2, \ldots, k\}$, there is a solution w_i of* (4.8) *such that $w_i(x) > 0$ on $[x_{i-1}, x_i]$.*

Proof. (i) Let $i \in \{1, 2, \ldots, k\}$ be fixed. Consider the initial condition

$$w(x_{i-1} + \varepsilon) = 0 \quad \text{and} \quad w'(x_{i-1} + \varepsilon) = 1 \tag{4.9}$$

with $\varepsilon \geq 0$. Since $\lambda a(x) > \lambda_k a(x)$ on $[x_{i-1}, x_i]$ and $\varphi_k(x_{i-1}) = \varphi_k(x_i) = 0$, by the Sturm–Picone theorem, we see that the solution of (4.8) and (4.9) with $\varepsilon = 0$ has a zero z_0 in (x_{i-1}, x_i). By the continuous dependence of solutions on initial conditions, the solution of (4.8) and (4.9) with small $\varepsilon > 0$ has a zero z_ε near z_0. Hence, if $\varepsilon > 0$ is sufficiently small, the solution of (4.8) and (4.9) has two zeros $x_{i-1} + \varepsilon$ and z_ε in (x_{i-1}, x_i).

(ii) Fix $i \in \{1, 2, \ldots, k\}$, and consider the initial condition

$$w(x_{i-1}) = \varepsilon \quad \text{and} \quad w'(x_{i-1}) = 1 \tag{4.10}$$

with $\varepsilon \geq 0$. Since $\lambda a(x) < \lambda_k a(x)$ on $[x_{i-1}, x_i]$ and φ_k has no zero in (x_{i-1}, x_i), the Sturm–Picone theorem shows that the solution of (4.8) and (4.10) with $\varepsilon = 0$ satisfies $w(x) > 0$ on $(x_{i-1}, x_i]$. By the continuous dependence of solutions on initial conditions, if $\varepsilon > 0$ is sufficiently small, then the solution of (4.8) and (4.10) satisfies $w(x) > 0$ on $[x_{i-1}, x_i]$. □

Lemma 4.3. *Let $M > 0$. If $\mu > 0$ satisfies*

$$\mu^2 > \frac{2a^* F(M)}{A_*}, \tag{4.11}$$

then the solution $u(x; \mu)$ has the following properties (i)–(iii):

(i) *If $u'(x_0; \mu) = 0$ for some $x_0 \in (0, 1]$, then $|u(x_0; \mu)| > M$;*

(ii) *Assume that $u(x; \mu)$ has no zero in (x_1, x_2) and satisfies $|u(x; \mu)| \leq M$ on $[x_1, x_2]$ for some $x_1, x_2 \in [0, 1]$. Then we have $x_2 - x_1 \leq \delta$, where*

$$\delta = \frac{M}{\sqrt{\mu^2 A_* - 2a^* F(M)}}; \tag{4.12}$$

(iii) *Define $\delta > 0$ by (4.12). Assume that $u(x; \mu)$ has no zero in (α, β) for some $\alpha, \beta \in [0, 1]$ satisfying $\beta - \alpha > 2\delta$. Then $|u(x; \mu)| \geq M$ for $x \in [\alpha + \delta, \beta - \delta]$.*

Proof. (i) From $u'(x_0; \mu) = 0$ we see that

$$E[u(\cdot; \mu)](x_0) = a(x_0) F(u(x_0; \mu)) \leq a^* F(u(x_0; \mu)).$$

In view of (4.1) and (4.11) we have

$$E[u(\cdot; \mu)](x_0) \geq \frac{\mu^2}{2} A_* > a^* F(M).$$

Then it follows that $a^* F(M) < a^* F(u(x_0; \mu))$, which implies that $M < |u(x_0; \mu)|$.

(ii) We may assume that $u(x) \equiv u(x; \mu) > 0$ on (x_1, x_2). Then we note that

$$0 \leq u(x) \leq M \quad \text{for } x \in [x_1, x_2]. \tag{4.13}$$

Hence we see that

$$E[u(\cdot\,; \mu)](x) \leq \frac{|u'(x)|^2}{2} + a^* F(M) \quad \text{for } x_1 \leq x \leq x_2.$$

Since $E[u(\cdot\,; \mu)](x) \geq (\mu^2/2) A_*$ by (4.1), we have

$$\frac{\mu^2}{2} A_* \leq \frac{|u'(x)|^2}{2} + a^* F(M) \quad \text{for } x_1 \leq x \leq x_2.$$

This implies that

$$|u'(x)| \geq \sqrt{\mu^2 A_* - 2a^* F(M)} = \frac{M}{\delta} \quad \text{for } x_1 \leq x \leq x_2.$$

Therefore we obtain either $u'(x) \geq M/\delta$ on $[x_1, x_2]$ or $-u'(x) \geq M/\delta$ on $[x_1, x_2]$.

Assume to the contrary that $x_2 - x_1 > \delta$. If $u'(x) \geq M/\delta$ on $[x_1, x_2]$, then we have

$$u(x_2) = u(x_1) + \int_{x_1}^{x_2} u'(x)dx \geq \frac{M}{\delta}(x_2 - x_1) > M.$$

This contradicts (4.13). On the other hand, if $-u'(x) \geq M/\delta$ on $[x_1, x_2]$, then

$$u(x_1) = u(x_2) - \int_{x_1}^{x_2} u'(x)dx \geq \frac{M}{\delta}(x_2 - x_1) > M,$$

which contradicts (4.13). Hence $x_2 - x_1 \leq \delta$.

(iii) It is sufficient to prove that $|u(x;\mu)| > M$ for $x \in (\alpha + \delta, \beta - \delta)$. We may assume that $u(x) \equiv u(x;\mu) > 0$ on (α, β). In view of (1.1) we see that $u''(x) < 0$ on (α, β), so that u' is strictly decreasing on $[\alpha, \beta]$. Assume to the contrary that there is a number $\gamma \in (\alpha + \delta, \beta - \delta)$ such that $u(\gamma) \leq M$.

First suppose that $u'(\gamma) \geq 0$. Since u' is nonincreasing on $[\alpha, \beta]$, we find that $u'(x) > u'(\gamma) \geq 0$ for $x \in [\alpha, \gamma)$, so that u is strictly increasing on $[\alpha, \gamma]$. Then we have $0 \leq u(x) \leq u(\gamma) \leq M$ on $[\alpha, \gamma]$. By applying (ii) of Lemma 4.3 with $x_1 = \alpha$ and $x_2 = \gamma$, we have $\gamma - \alpha \leq \delta$. This contradicts $\gamma \in (\alpha + \delta, \beta - \delta)$.

Next we assume that $u'(\gamma) < 0$. Since u' is strictly decreasing on $[\alpha, \beta]$, we have $u'(x) \leq u'(\gamma) < 0$ for $x \in [\gamma, \beta]$. Then $0 \leq u(x) \leq u(\gamma) \leq M$ for $x \in [\gamma, \beta]$. From (ii) of Lemma 4.3 we obtain $\beta - \gamma \leq \delta$. This is a contradiction. Therefore, $|u(x;\mu)| \geq M$ for $x \in [\alpha + \delta, \beta - \delta]$. This completes the proof. □

Proof of Lemma 3.3. Let $\lambda > 0$ satisfy $\lambda_k < \lambda < f_\infty$. Take $M > 0$ so large that

$$\frac{f(s)}{s} > \lambda > \lambda_k \quad \text{for } |s| \geq M. \tag{4.14}$$

Let x_i $(i = 0, 1, 2, \ldots, k)$ be zeros of an eigenfunction φ_k satisfying (4.6). By (i) of Lemma 4.2, for each $i \in \{1, 2, \ldots, k\}$, there exists a solution w_i of (4.8) having at least two zeros in (x_{i-1}, x_i).

Now fix $i \in \{1, 2, \ldots, k\}$. Let t_1 and t_2 be zeros of w_i such that $x_{i-1} < t_1 < t_2 < x_i$. We can take a number $\mu_i > 0$ so large that, for all $\mu \geq \mu_i$, inequality

(4.11) is satisfied, and that $x_i - x_{i-1} > 2\delta$ and $[t_1, t_2] \subset [x_{i-1} + \delta, x_i - \delta]$, where $\delta > 0$ is defined by (4.12). Let $\mu \geq \mu_i$. We will show that $u(x; \mu)$ has at least one zero in (x_{i-1}, x_i).

Assume to the contrary that $u(x; \mu)$ has no zero in (x_{i-1}, x_i). From (iii) of Lemma 4.3 we obtain $|u(x; \mu)| \geq M$ for $x \in [x_{i-1} + \delta, x_i - \delta]$. By (4.14) we have

$$\lambda a(x) < a(x)\frac{f(u(x;\mu))}{u(x;\mu)} \equiv b(x), \quad x \in [t_1, t_2] \subset [x_{i-1} + \delta, x_i - \delta].$$

Since $u(x; \mu)$ satisfies $u'' + b(x)u = 0$, the Sturm–Picone theorem shows that $u(x; \mu)$ has at least one zero in (t_1, t_2). This contradicts the assumption that $u(x; \mu)$ has no zeros in (x_{i-1}, x_i). Hence, $u(x; \mu)$ has at least one zero in (x_{i-1}, x_i) if $\mu \geq \mu_i$.

Put $\mu^* = \max\{\mu_i : i = 1, 2, \ldots, k\}$. If $\mu \geq \mu^*$ then, $u(x; \mu)$ has at least one zero in (x_{i-1}, x_i) for each $i \in \{1, 2, \ldots, k\}$, which means that $u(x; \mu)$ has at least k zeros in $(0, 1)$. This completes the proof. □

Proof of Lemma 3.4. Choose $\lambda > 0$ such that $f_\infty < \lambda < \lambda_k$. Take $M > 0$ so large that

$$\frac{f(s)}{s} < \lambda < \lambda_k \quad \text{for } |s| \geq M. \tag{4.15}$$

Let x_i $(i = 0, 1, 2, \ldots, k)$ be zeros of an eigenfunction φ_k satisfying (4.6). By (ii) of Lemma 4.2, for every $i \in \{1, 2, \ldots, k\}$, there exists a solution w_i of (4.8) such that $w_i(x) > 0$ on $[x_{i-1}, x_i]$. For each $i \in \{1, 2, \ldots, k\}$, we define W_i by

$$W_i = \max\left\{\frac{|w_i'(x)|}{w_i(x)} : x \in [x_{i-1}, x_i]\right\}.$$

There exists a number $\mu^* > 1$ so large that if $\mu \geq \mu^*$, then (4.11) holds and

$$\frac{\sqrt{\mu^2 A_* - 2a^* F(M)}}{M} \geq \max\{W_i : i = 1, 2, \ldots k\}. \tag{4.16}$$

Let $\mu \geq \mu^*$. We will show that $u(x) \equiv u(x; \mu)$ has at most one zero in $[x_{i-1}, x_i)$ for each $i \in \{1, 2, \ldots, k\}$. Suppose that $u(x)$ has at least two zeros in $[x_{i-1}, x_i)$ for some $i \in \{1, 2, \ldots, k\}$. Then there exist numbers α, $\beta \in [x_{i-1}, x_i)$ with $\alpha < \beta$ such that $u(\alpha) = u(\beta) = 0$. We may assume that $u(x) > 0$ on (α, β).

Set $\overline{u} = \max_{x\in[\alpha,\beta]} u(x) > 0$ and take $\gamma \in (\alpha, \beta)$ such that $u(\gamma) = \overline{u}$. By (4.11) and $u'(\gamma) = 0$, (i) of Lemma 4.3 implies $\overline{u} > M$. Then there are numbers t_1 and t_2 such that $\alpha < t_1 < t_2 < \beta$, $u(t_1) = u(t_2) = M$ and $u(x) > M$ for $x \in (t_1, t_2)$. We note that $u'(t_1) > 0$ and $u'(t_2) < 0$. By (4.1) we have

$$\frac{\mu^2}{2} A_* \le E[u(\cdot;\mu)](t_j) \le \frac{|u'(t_j)|^2}{2} + a^* F(M) \quad \text{for } j = 1, 2,$$

which implies that $|u'(t_j)|^2 \ge \mu^2 A_* - 2a^* F(M)$ for $j = 1, 2$. From $u(t_1) = u(t_2) = M$ and (4.16), we obtain

$$\frac{|u'(t_j)|}{u(t_j)} \ge \frac{\sqrt{\mu^2 A_* - 2a^* F(M)}}{M} \ge W_i \quad \text{for } j = 1, 2. \tag{4.17}$$

In view of $u'(t_1) > 0$ we have $u'(t_1)/u(t_1) \ge W_i \ge w_i'(t_1)/w_i(t_1)$, that is,

$$u'(t_1) w_i(t_1) - u(t_1) w_i'(t_1) \ge 0. \tag{4.18}$$

Using (4.15) and the fact that $u(x) \ge M$ for $t_1 \le x \le t_2$, we see that

$$\int_{t_1}^{t_2} (u'' w_i - u w_i'') dx = \int_{t_1}^{t_2} a(x) \left[\lambda - \frac{f(u)}{u} \right] u w_i \, dx > 0,$$

so that

$$(u' w_i - u w_i') \Big|_{t_1}^{t_2} > 0.$$

Therefore (4.18) implies $u'(t_2) w_i(t_2) - u(t_2) w_i'(t_2) > 0$. Then we obtain

$$\frac{u'(t_2)}{u(t_2)} > \frac{w_i'(t_2)}{w_i(t_2)},$$

and hence

$$\frac{|u'(t_2)|}{u(t_2)} = \frac{-u'(t_2)}{u(t_2)} < \frac{-w_i'(t_2)}{w_i(t_2)} \le \frac{|w_i'(t_2)|}{w_i(t_2)} \le W_i.$$

This contradicts (4.17). Consequently $u(x;\mu)$ has at most one zero in $[x_{i-1}, x_i)$ for each $i \in \{1, 2, \ldots, k\}$. Note that $u(x;\mu)$ has no zero in (x_0, x_1), since $x_0 = 0$ is a zero of $u(x;\mu)$ in $[x_0, x_1)$. Hence $u(x;\mu)$ has at most $k-1$ zeros in $(0, 1)$. This completes the proof. □

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