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Risk Comparison of the Stein-Rule Estimator in a Linear Regression Model with Omitted Relevant Regressors and Multivariate *t* Errors under the Pitman Nearness Criterion*

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Abstract

In this paper we consider a linear regression model with omitted relevant regressors and multivariate t error terms. The explicit formula for the Pitman nearness criterion of the Stein-rule (SR) estimator relative to the ordinary least squares (OLS) estimator is derived. It is shown numerically that the dominance of the SR estimator over the OLS estimator under the Pitman nearness criterion can be extended to the case of the multivariate t error distribution when the specification error is not severe. It is also shown that the dominance of the SR estimator over the OLS estimator cannot be extended to the case of the multivariate t error distribution when the specification error is severe.

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1 Introduction

In the context of linear regression, the ordinary least squares (OLS) estimator is known as the best linear unbiased estimator (BLUE). However, the Stein-rule (SR) estimator for regression coefficients proposed by Stein (1956) and James and Stein (1961) dominates the OLS estimator in terms of mean squared error (MSE) of prediction. Since the findings of Stein (1956) and James and Stein (1961), lots of shrinkage estimators have been proposed, and their sampling properties have been examined.

When shrinkage estimators are compared with the OLS estimator, the MSE has usually been used as a criterion of comparison. However, there are several studies in which another criterion is used to compare the sampling performances of these estimators. For example, using the Pitman nearness criterion, Keating and Czitrom (1989) showed by numerical evaluations that the SR estimator is uniformly preferred to the OLS estimator. This indicates that the dominance of the SR estimator over the OLS estimator holds even when the Pitman nearness criterion is used instead of MSE. The Pitman nearness criterion is also used in many studies. Some examples are Rao, Keating and Mason (1988), Sen, Kubokawa and Saleh (1989), Srivastava and Srivastava (1993), and Chaturvedi and Bhatti (1998).

Although it is assumed in these studies that the model is specified correctly, the model may be specified incorrectly. One example of such a specification error is to exclude some relevant independent variables in the specified model. It is also assumed in these studies that the error terms obey a normal distribution. However, as is stated in Fama (1965), and Blattberg and Gonedes (1974), many economic data may be generated by a distribution with fatter tails than a normal distribution. One example of such a distribution is a multivariate t distribution.

Mittelhammer (1984), Ohtani (1993b), and Namba (2002) examined the MSE performances of the Stein-rule (SR) estimator and its variants when relevant independent variables are omitted. They showed that although the SR estimator has smaller MSE than the OLS estimator in a wide region of the parameter space, the dominance of the SR estimator over the OLS estimator does not hold necessarily when there are omitted variables. On the other hand, there are many studies on the sampling performances of estimators when error terms in a linear regression model obey a multivariate t distribution. Some examples are Zellner (1976), Prucha and Kelejian (1984), Ullah and Zinde-Walsh (1984), Judge et al. (1985), Sutradhar and Ali (1986), Sutradhar (1988), Singh (1988, 1991), Giles (1991, 1992), Ohtani (1991, 1993a), Ohtani and Giles (1993), Ohtani and Hasegawa (1993), Namba (2001), and Namba and Ohtani (2002).

In this paper we consider a linear regression model when relevant independent variables are omitted in the specified model and error terms obey a multivariate t distribution. Using the above model, we compare the sampling performances of the SR estimator and the OLS estimator under the Pitman nearness criterion. Thus, our analysis is an extension of Keating and Czitrom (1989) to two directions: omitted relevant independent variables and multivariate t error terms. In the next section we introduce a model and estimators. In section 3 we derive the exact formula of the Pitman nearness criterion of the SR estimator relative to the OLS estimator. In section 4, using this exact formula, we compare the SR estimator and the OLS estimator numerically. Our numerical results show that the dominance of the SR estimator over the OLS estimator under the Pitman nearness criterion can be extended to the case of the multivariate t error distribution when the specification error is not severe. The numerical results also show that the dominance of the SR estimator over the OLS estimator cannot be extended to the case of the multivariate t error distribution when the specification error is severe.

2 Model and Estimators

Consider a linear regression model,

$$\begin{aligned} y &= X_1\beta_1 + X_2\beta_2 + u \\ &= X\beta + u, \qquad (1) \end{aligned}$$

where y is an $n \times 1$ vector of observations on a dependent variable, X_1 and X_2 are $n \times k_1$ and $n \times k_2$ matrices of observations on nonstochastic independent variables and $X = [X_1, X_2]$, β_1 and β_2 are $k_1 \times 1$ and $k_2 \times 1$ vectors of regression coefficients and $\beta' = [\beta_1', \beta_2']$. We assume that X_1 and $X = [X_1, X_2]$ are of full column rank. Also, in this paper, we assume that u has a multivariate t distribution with the probability density function (p.d.f) given by

$$p(u|\nu,\sigma) = \frac{\nu^{\nu/2}\Gamma((\nu+n)/2)}{\pi^{n/2}\Gamma(\nu/2)\sigma^n} \frac{1}{\{\nu + u'u/\sigma^2\}^{(n+\nu)/2}}, \qquad (2)$$

where σ is a scale parameter and ν is a degrees of freedom parameter. It is easy to show that $E[u] = 0$ and $E[uu'] = [\nu\sigma^2/(\nu-2)]I_n$ for $\nu > 2$. When $\nu \to \infty$, the p.d.f of u approaches that of a normal distribution with mean 0 and covariance matrix $\sigma^2 I_n$. As is shown in Zellner (1976), the multivariate t distribution can be viewed as a mixture of multivariate normal and inverted gamma distributions:

$$p(u|\nu,\sigma) = \int_0^\infty P_N(u|\tau)P_{IG}(\tau|\nu,\sigma)d\tau, \tag{3}$$

where

$$P_N(u|\tau) = (2\pi\tau^2)^{-n/2}\exp[-u'u/2\tau^2], \tag{4}$$

$$P_{IG}(\tau|\nu,\sigma) = \frac{2(\nu\sigma^2/2)^{\nu/2}}{\Gamma(\nu/2)}\tau^{-(\nu+1)}\exp[-\nu\sigma^2/2\tau^2]. \tag{5}$$

Suppose that the matrix of regressors X_2 is omitted mistakenly and the model is specified as

$$y = X_1\beta_1 + u^*, \quad u^* = X_2\beta_2 + u. \tag{6}$$

Then, based on the misspecified model, the ordinary least squares (OLS) estimator of β_1 is

$$b_1 = S_{11}^{-1}X_1'y, \tag{7}$$

where $S_{11} = X_1'X_1$. Also, the Stein-rule (SR) estimator proposed by Stein (1956) and James and Stein (1961) is defined as

$$b_{SR1} = \left(1 - \frac{ae_1'e_1}{b_1'S_{11}b_1}\right)b_1, \tag{8}$$

where $e_1 = y - X_1b_1$, and a is a constant such that $0 \leqq a \leqq 2(k_1-2)/(n-k_1+2)$. If we use the loss function

$$L(\bar{\beta}_1) = (X_1\bar{\beta}_1 - X\beta)'(X_1\bar{\beta}_1 - X\beta), \tag{9}$$

where $\bar{\beta}_1$ is any estimator of β_1, and no relevant regressors are omitted, then Stein (1956) showed that the SR estimator dominates the OLS estimator. Moreover, as is shown in James and Stein (1961), if there are no omitted regressors, the MSE of the SR estimator is minimized when $a = (k_1-2)/(n-k_1+2)$. Thus, we use this value of a hereafter. Also, Ohtani (1993a) and Namba (2002) showed by numerical evaluations that although the SR estimator has smaller MSE than the OLS estimator in a wide region of the parameter space, the dominance of the SR estimator over the OLS estimator does not hold necessarily when there are omitted variables. In the next section, we derive the explicit formula for the Pitman nearness criterion of the SR estimator relative to the OLS estimator.

3 Pitman Nearness Criterion

Using (9), the Pitman nearness criterion of the SR estimator relative to the OLS estimator is given by

$$\mathrm{PN}(b_{SR1}, b_1) = \Pr[L(b_{SR1}) \leqq L(b_1)], \tag{10}$$

where $\Pr[\cdot]$ denotes the probability and $L(\cdot)$ is defined in (9). If $\mathrm{PN}(b_{SR1}, b_1) \geqq 1/2$, the SR estimator is preferable to the OLS estimator in the Pitman nearness sense.

If k_1 is an even integer, we can derive the explicit formula of $\mathrm{PN}(b_{SR1}, b_1)$ in a similar way to Keating and Czitrom (1989). As is shown in appendix, the explicit formula for $\mathrm{PN}(b_{SR1}, b_1)$ is

$$\begin{aligned}\mathrm{PN}(b_{SR1}, b_1) \;=\; & \nu^{\nu/2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{r=0}^{m+i-1} \frac{\theta_1^{m+2i-r-1} \theta_2^{j}}{(2\theta_1 + \theta_2 + \nu)^{\nu/2+m+2i+j-r-1}} \\ & \times \frac{\Gamma(\nu/2 + m + 2i + j - r - 1)}{\Gamma(\nu/2)\Gamma(m+i-r) i! j!} [1 - I_{\frac{a}{2+a}}(r+1, n/2 - m + j)],\end{aligned} \tag{11}$$

where $\theta_1 = \beta' S \beta / 4\sigma^2$, $\theta_2 = \beta' X' M_1 X \beta / \sigma^2$, $M_1 = I - X_1 S_{11}^{-1} X_1'$, $m = k_1/2$, k_1 is an even integer and $I_x(a, b)$ is the incomplete beta function ratio defined as

$$I_x(a, b) = \frac{1}{B(a, b)} \int_0^x t^{a-1} (1-t)^{b-1} dt. \tag{12}$$

Using this formula, we examine the Pitman nearness of the SR estimator relative to the OLS estimator in the next section.

4 Numerical Analysis

In this section, we compare the SR estimator with the OLS estimator based on the Pitman nearness criterion using (11). The noncentrality parameters are expressed as $\theta_1 = \theta/4$ and $\theta_2 = \theta(1 - R_1^2)$, where $\theta = \beta' S \beta / \sigma^2$, $R_1^2 = \beta' X' X_1 S_{11}^{-1} X_1' X \beta / \beta' S \beta$ and $S = X'X$. As is discussed in Ohtani (1993a), θ is the noncentrality parameter which appeared in the test for the null hypothesis that all the regression coefficients are zeros in the correctly specified model. Also, R_1^2 is interpreted as the coefficient of determination in regression of $X\beta$ on X_1. Thus, if R_1^2 is close to unity, the magnitude of model misspecification is regarded as small, and vice versa. Thus, in the numerical evaluations, we use the following parameter values; $\theta =$ various values, $R_1^2 = 0.1$, 0.3, 0.5, 0.7, 0.9, 1.0, $m = 2$, 3, 4, $n = 20$, 30, 40, and $\nu = 3$, 5, 7, 10, 20, 30, ∞. The numerical evaluations were executed on a personal computer using the FORTRAN code. The double infinite series in (11) were judged to converge when the increment of the series got smaller than 10^{-12}.

The results for $m = 2$, 4, $n = 20$, 30, and $\nu = 3$, 20 are shown in Tables 1 to 4. We see from Table 1 that the SR estimator is preferred to the OLS estimator for all values of θ and R_1^2 considered here when $m = 2$, $n = 20$, and $\nu = 3$. In particular, when θ is close to 0 and R_1^2 is close to 1.0, the SR estimator is much preferred to the OLS estimator. However, the efficiency gets small as θ gets large and R_1^2 gets small. When ν increases from 3 to 20, the OLS estimator is preferred to the SR estimator for $\theta = 50.0$ and $R_1^2 = 0.1$. This indicates that as the tails of the error distribution get flat, the preference for the SR estimator gets strong under the Pitman nearness criterion. We see from Table 2 that as n increases from 20 to 40, the SR estimator is preferred to the OLS estimator for all values of θ and R_1^2 considered here even when $\nu = 20$, though the performance of efficiency is similar to that for $n = 20$. We see from Table 3 that the OLS estimator is preferred to the SR estimator for large values of θ and small values of R_1^2 when $m = 4$, $n = 20$, and $\nu = 3$. In particular, when θ is close to 50.0 and R_1^2 is close to 0.1, the OLS estimator is much preferred to the SR estimator. This indicates that the region of θ and R_1^2 such that the SR estimator is preferred to the OLS estimator gets much narrow as m increases from 2 to 4. However, we see from Table 4 that as n increases from 20 to 40, the SR estimator is preferred to the OLS estimator for all values of θ and R_1^2 considered here, though the performance of efficiency is similar to that for $m = 2$, $n = 20$ and $\nu = 3$ given in Table 1.

From the above results, we see that when the specification error is not severe (R_1^2 is close to 1.0), the SR estimator is preferred to the OLS estimator under the Pitman nearness criterion even when the error terms obey the multivariate t distribution. Thus, the dominance of the SR estimator over the OLS estimator under the Pitman nearness criterion can be extended to the case of the multivariate t error distribution when the specification error is not severe. The results also show that when R_1^2 is close to 0.1, the OLS estimator is preferred to the SR estimator for some values of θ, though the region of θ and R_1^2 such that the OLS estimator is preferred to the SR estimator gets narrow as n increases. Thus, the dominance of the SR estimator over the OLS estimator under the Pitman nearness criterion cannot be extended to the case of the multivariate t error distribution when the specification error is severe.

Appendix

Using (7), (8) and (9), we have

$$\Pr[L(b_{SR1}) \leqq L(b)] = \Pr\left[\left(X_1b_1 - \frac{X\beta}{2}\right)'\left(X_1b_1 - \frac{X\beta}{2}\right) \geqq \frac{\beta' S\beta}{4} + \frac{ae_1'e_1}{2}\right]. \tag{13}$$

First, we derive the formula for $\Pr[L(b_{SR1}) \leqq L(b_1)|\tau]$, assuming that τ is given. If we define $v_1 = (X_1b_1 - X\beta/2)'(X_1b_1 - X\beta/2)/\tau^2$ and $v_2 = e_1'e_1/\tau^2$, then $v_1 \sim \chi'^2_{k_1}(\lambda_1)$ and $v_2 \sim \chi'^2_{n-k_1}(\lambda_2)$ for given τ, where $\chi'^2_f(\lambda)$ is the noncentral chi-square distribution with f degrees of freedom and noncentrality parameter λ, $\lambda_1 = \beta' S\beta/4\tau^2$, and $\lambda_2 = \beta' X' M_1 X\beta/\tau^2$. Thus, using v_1 and v_2, we have

$$\Pr[L(b_{SR1}) \leqq L(b_1)|\tau] = \Pr\left[\frac{v_1 - \lambda_1}{v_2} \geqq \frac{a}{2}\middle|\tau\right]. \tag{14}$$

Since v_1 and v_2 are mutually independent for given τ, we have

$$\Pr\left[\frac{v_1 - \lambda_1}{v_2} \geqq \frac{a}{2}\middle|\tau\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} \iint_R v_1^{k_1/2+i-1} v_2^{(n-k_1)/2+j-1} \exp\left[-\frac{v_1+v_2}{2}\right] dv_1\, dv_2, \tag{15}$$

where R is the region such that $(v_1 - \lambda_2)/v_2 \geqq a/2$,

$$K_{ij} = \frac{w_i(\lambda_1)w_j(\lambda_2)}{2^{n/2+i+j}\Gamma(k_1/2+i)\Gamma((n-k_1)/2+j)}, \tag{16}$$

and $w_i(\lambda) = \exp(-\lambda/2)(\lambda/2)^i/i!$.

Making use of the change of variables, $w = (v_1 - \lambda_1)/v_2$ and $z = v_2$, the integral in (15) reduces to

$$\int_0^{\infty}\int_{a/2}^{\infty} (wz + \lambda_1)^{k_1/2+i-1} z^{(n-k_1)/2+j} \exp\left[-\frac{wz + z + \lambda_1}{2}\right] dw\, dz. \tag{17}$$

When $k_1 = 2m$, where m is a positive integer, (17) reduces to

$$\sum_{r=0}^{m+i-1} {}_{m+i-1}C_r \lambda_1^{m+i-1-r} \exp(-\lambda_1/2) \int_0^{\infty}\int_{a/2}^{\infty} w^r z^{n/2+r-m+j} \exp\left[-\frac{(1+w)z}{2}\right] dw\, dz, \tag{18}$$

where ${}_pC_m = \frac{p!}{m!(p-m)!}$.

Again, making use of the change of a variable, $z_1 = (1+w)z/2$, the integral in (18) reduces to

$$2^{n/2+r-m+j+1}\Gamma(n/2+r-m+j+1)\int_{a/2}^{\infty}\frac{w^r}{(1+w)^{n/2+r-m+j+1}}dw. \tag{19}$$

Further, making use of the change of a variable, $t = w/(1+w)$, we obtain

$$\begin{aligned}\Pr\left[\frac{v_1-\lambda_1}{v_2} \geqq \frac{a}{2}\Big|\tau\right] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{r=0}^{m+i-1}\frac{(\lambda_1/2)^{m+2i-r-1}(\lambda_2/2)^j\exp[-(\lambda_1+\lambda_2/2)]}{\Gamma(m+i-r)i!j!}\\ &\qquad\times[1-I_{\frac{a}{2+a}}(r+1,n/2-m+j)].\end{aligned} \tag{20}$$

Utilizing the relation,

$$\begin{aligned}\Pr[L(b_{SR1}) \leqq L(b)] &= \int_0^{\infty}\Pr[L(b_{SR1}) \leqq L(b)|\tau]P_{IG}(\tau|\nu,\sigma)d\tau\\ &= \int_0^{\infty}\Pr\left[\frac{v_1-\lambda_1}{v_2} \geqq \frac{a}{2}\Big|\tau\right]P_{IG}(\tau|\nu,\sigma)d\tau,\end{aligned} \tag{21}$$

and making use of the change of variable, $t_1 = (2\eta_1+\eta_2+\nu\sigma^2)/2\tau^2$, where $\eta_1 = \beta' S\beta/4$ and $\eta_2 = \beta' X' M_1 X\beta$, we obtain (11) in the text.

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Table 1: Results for $m = 2$ and $n = 20$.

		R_1^2					
ν	θ	0.1	0.3	0.5	0.7	0.9	1.0
3	0.0	0.9221	0.9221	0.9221	0.9221	0.9221	0.9221
	1.0	0.8893	0.8910	0.8926	0.8943	0.8959	0.8967
	2.0	0.8600	0.8632	0.8664	0.8695	0.8727	0.8743
	4.0	0.8125	0.8182	0.8239	0.8297	0.8354	0.8382
	6.0	0.7759	0.7837	0.7914	0.7992	0.8069	0.8108
	8.0	0.7466	0.7561	0.7655	0.7750	0.7845	0.7892
	10.0	0.7223	0.7333	0.7442	0.7552	0.7662	0.7717
	15.0	0.6757	0.6896	0.7037	0.7179	0.7320	0.7391
	20.0	0.6412	0.6576	0.6743	0.6910	0.7078	0.7162
	25.0	0.6139	0.6325	0.6513	0.6703	0.6894	0.6990
	30.0	0.5914	0.6119	0.6326	0.6536	0.6747	0.6853
	40.0	0.5556	0.5792	0.6033	0.6278	0.6525	0.6649
	50.0	0.5275	0.5538	0.5808	0.6083	0.6362	0.6501
20	0.0	0.9221	0.9221	0.9221	0.9221	0.9221	0.9221
	1.0	0.8888	0.8905	0.8921	0.8938	0.8955	0.8963
	2.0	0.8572	0.8605	0.8638	0.8671	0.8704	0.8720
	4.0	0.8025	0.8087	0.8149	0.8211	0.8273	0.8303
	6.0	0.7589	0.7674	0.7760	0.7845	0.7930	0.7972
	8.0	0.7239	0.7344	0.7449	0.7554	0.7658	0.7710
	10.0	0.6954	0.7075	0.7196	0.7318	0.7439	0.7500
	15.0	0.6422	0.6577	0.6732	0.6888	0.7044	0.7122
	20.0	0.6047	0.6228	0.6411	0.6595	0.6779	0.6871
	25.0	0.5759	0.5964	0.6170	0.6377	0.6586	0.6690
	30.0	0.5527	0.5751	0.5979	0.6207	0.6438	0.6553
	40.0	0.5164	0.5423	0.5686	0.5953	0.6221	0.6355
	50.0	0.4883	0.5172	0.5466	0.5765	0.6066	0.6218

Table 2: Results for $m = 2$ and $n = 40$.

		R_1^2					
ν	θ	0.1	0.3	0.5	0.7	0.9	1.0
3	0.0	0.9157	0.9157	0.9157	0.9157	0.9157	0.9157
	1.0	0.8860	0.8868	0.8876	0.8884	0.8892	0.8896
	2.0	0.8601	0.8616	0.8631	0.8646	0.8661	0.8669
	4.0	0.8186	0.8213	0.8241	0.8268	0.8295	0.8308
	6.0	0.7871	0.7908	0.7945	0.7982	0.8019	0.8037
	8.0	0.7621	0.7666	0.7711	0.7757	0.7802	0.7824
	10.0	0.7416	0.7469	0.7521	0.7573	0.7625	0.7652
	15.0	0.7029	0.7096	0.7163	0.7231	0.7298	0.7332
	20.0	0.6748	0.6828	0.6908	0.6987	0.7067	0.7107
	25.0	0.6531	0.6621	0.6712	0.6802	0.6893	0.6938
	30.0	0.6354	0.6454	0.6554	0.6654	0.6755	0.6805
	40.0	0.6079	0.6195	0.6312	0.6430	0.6547	0.6606
	50.0	0.5868	0.5999	0.6131	0.6263	0.6395	0.6461
20	0.0	0.9157	0.9157	0.9157	0.9157	0.9157	0.9157
	1.0	0.8854	0.8862	0.8870	0.8878	0.8886	0.8890
	2.0	0.8573	0.8589	0.8605	0.8620	0.8636	0.8644
	4.0	0.8094	0.8124	0.8153	0.8183	0.8212	0.8227
	6.0	0.7717	0.7758	0.7798	0.7839	0.7879	0.7899
	8.0	0.7417	0.7467	0.7517	0.7567	0.7617	0.7641
	10.0	0.7175	0.7233	0.7291	0.7348	0.7406	0.7435
	15.0	0.6732	0.6806	0.6880	0.6955	0.7029	0.7066
	20.0	0.6427	0.6515	0.6602	0.6689	0.6777	0.6821
	25.0	0.6200	0.6298	0.6397	0.6496	0.6595	0.6645
	30.0	0.6020	0.6128	0.6237	0.6347	0.6456	0.6511
	40.0	0.5746	0.5873	0.6000	0.6127	0.6254	0.6318
	50.0	0.5542	0.5684	0.5826	0.5969	0.6112	0.6184

Table 3: Results for $m = 4$ and $n = 20$.

		R_1^2					
ν	θ	0.1	0.3	0.5	0.7	0.9	1.0
3	0.0	0.9422	0.9422	0.9422	0.9422	0.9422	0.9422
	1.0	0.9222	0.9251	0.9280	0.9308	0.9335	0.9348
	2.0	0.9006	0.9068	0.9129	0.9188	0.9245	0.9273
	4.0	0.8571	0.8701	0.8828	0.8951	0.9069	0.9127
	6.0	0.8159	0.8352	0.8542	0.8726	0.8904	0.8990
	8.0	0.7780	0.8029	0.8276	0.8518	0.8752	0.8865
	10.0	0.7433	0.7732	0.8031	0.8326	0.8613	0.8751
	15.0	0.6687	0.7085	0.7494	0.7906	0.8310	0.8506
	20.0	0.6077	0.6548	0.7044	0.7553	0.8060	0.8306
	25.0	0.5568	0.6094	0.6659	0.7250	0.7847	0.8139
	30.0	0.5136	0.5702	0.6323	0.6985	0.7664	0.7997
	40.0	0.4438	0.5056	0.5760	0.6539	0.7359	0.7766
	50.0	0.3898	0.4542	0.5303	0.6173	0.7112	0.7584
20	0.0	0.9422	0.9422	0.9422	0.9422	0.9422	0.9422
	1.0	0.9225	0.9254	0.9281	0.9309	0.9336	0.9349
	2.0	0.9013	0.9073	0.9132	0.9190	0.9246	0.9273
	4.0	0.8566	0.8697	0.8824	0.8946	0.9065	0.9122
	6.0	0.8118	0.8320	0.8516	0.8705	0.8886	0.8974
	8.0	0.7688	0.7956	0.8218	0.8472	0.8715	0.8832
	10.0	0.7283	0.7612	0.7937	0.8252	0.8554	0.8700
	15.0	0.6390	0.6848	0.7307	0.7759	0.8197	0.8407
	20.0	0.5651	0.6206	0.6773	0.7342	0.7898	0.8166
	25.0	0.5035	0.5663	0.6318	0.6986	0.7645	0.7965
	30.0	0.4515	0.5197	0.5924	0.6677	0.7429	0.7795
	40.0	0.3687	0.4438	0.5273	0.6165	0.7078	0.7525
	50.0	0.3057	0.3843	0.4750	0.5753	0.6801	0.7319

Table 4: Results for $m = 4$ and $n = 40$.

		R_1^2					
ν	θ	0.1	0.3	0.5	0.7	0.9	1.0
3	0.0	0.9374	0.9374	0.9374	0.9374	0.9374	0.9374
	1.0	0.9234	0.9246	0.9258	0.9269	0.9281	0.9287
	2.0	0.9091	0.9116	0.9140	0.9165	0.9189	0.9201
	4.0	0.8811	0.8863	0.8915	0.8965	0.9014	0.9038
	6.0	0.8552	0.8630	0.8706	0.8781	0.8855	0.8891
	8.0	0.8314	0.8416	0.8517	0.8615	0.8711	0.8758
	10.0	0.8096	0.8221	0.8344	0.8464	0.8582	0.8639
	15.0	0.7625	0.7798	0.7971	0.8140	0.8306	0.8388
	20.0	0.7232	0.7448	0.7662	0.7875	0.8083	0.8185
	25.0	0.6897	0.7149	0.7401	0.7651	0.7897	0.8018
	30.0	0.6606	0.6889	0.7174	0.7458	0.7739	0.7877
	40.0	0.6117	0.6452	0.6794	0.7138	0.7480	0.7648
	50.0	0.5717	0.6095	0.6485	0.6880	0.7275	0.7469
20	0.0	0.9374	0.9374	0.9374	0.9374	0.9374	0.9374
	1.0	0.9235	0.9247	0.9258	0.9270	0.9281	0.9287
	2.0	0.9092	0.9116	0.9140	0.9164	0.9188	0.9199
	4.0	0.8802	0.8854	0.8905	0.8955	0.9004	0.9028
	6.0	0.8519	0.8599	0.8678	0.8754	0.8829	0.8866
	8.0	0.8252	0.8358	0.8463	0.8566	0.8666	0.8715
	10.0	0.8001	0.8133	0.8263	0.8391	0.8515	0.8576
	15.0	0.7448	0.7638	0.7825	0.8009	0.8188	0.8275
	20.0	0.6985	0.7225	0.7461	0.7693	0.7921	0.8032
	25.0	0.6593	0.6875	0.7154	0.7430	0.7700	0.7832
	30.0	0.6255	0.6574	0.6891	0.7206	0.7514	0.7665
	40.0	0.5697	0.6077	0.6460	0.6841	0.7217	0.7402
	50.0	0.5248	0.5678	0.6115	0.6554	0.6989	0.7203