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(Citation)
Communications in Statistics: Simulation and Computation,33(1):99-108

(Issue Date)
2004

(Resource Type)
journal article

(Version)
Accepted Manuscript

(URL)
https://hdl.handle.net/20.500.14094/90000340

SIMULATION STUDIES ON BOOTSTRAP EMPIRICAL LIKELIHOOD TESTS

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Abstract

In this paper we consider to test the hypothesis using the empirical likelihood. To calculate the critical value of the test, two bootstrap methods are applied. Our simulation results indicate that the bootstrap methods improve the small sample property of the test.

1 Introduction

Likelihood methods are very important in parametric models since they are very effective. However, we suppose that the distribution of data has a known form when parametric likelihood methods are used. Empirical likelihood is a nonparametric method of statistical inference which has lots of similarities to parametric likelihood methods.

As shown by Owen (1988, 1990), Qin and Lawless (1994), El Barmi (1996), we can test hypotheses or construct confidence regions using the empirical likelihood ratio. These inferences are asymptotically valid and have coverage error or size distortion in small sample size. Several authors proposed ways to improve the accuracy of inference based on the empirical likelihood ratio. Some examples are DiCiccio and Romano (1989), Hall (1990), DiCiccio et al. (1991) and Tsao (2001). Owen (1988, 2001) proposed to use the bootstrap method in calculating the critical value of the empirical likelihood ratio. Also, Hall and Presnell (1999), Brown and Newey (2001) proposed to utilize the probabilities obtained by the empirical likelihood in the bootstrap resampling. However, the validity of these procedure have not been examined except for some simulations in Owen (1988), Hall and Presnell (1999), and Brown and Newey (2001). Though they examined the size of the test or the coverage probability of the confidence region, the power of the test has not been examined yet.

Thus, in this paper we apply two bootstrap methods to obtain the critical values of the empirical likelihood ratio. The first is the usual bootstrap method where the resampling is executed based on the empirical distribution function. And the second is the method where the resampling is executed by utilizing the probabilities obtained by empirical likelihood. In sections 2 and 3, we briefly explain the empirical likelihood and the bootstrap methods. In section 4, we execute the Monte Carlo simulations to examine the validity of the procedure.

2 Empirical Likelihood

Let $X_1, X_2, \ldots, X_n$ be $d \times 1$ independent random vectors with the common distribution F_0. Suppose that we have a set of estimating equations

$$\mathrm{E}[g(X, \theta)] = 0, \tag{1}$$

where θ is a $l \times 1$ vector of unknown parameters, $g(X, \theta)$ is a $s \times 1$ vector-valued function, and X is a $d \times 1$ random vector distributed as F_0. Then the empirical likelihood ratio for θ is defined as

$$\mathcal{R}(\theta) = \max\left\{\prod_{i=1}^{n} np_i \Big| \sum_{i=1}^{n} p_i g(X_i, \theta) = 0, p_i \geq 0, \sum_{i=1}^{n} p_i = 1\right\}. \tag{2}$$

Assume that $\mathrm{E}[g(X, \theta_0)] = 0$, and $\mathrm{Var}[g(X, \theta_0)]$ is finite and has rank $q > 0$. Then, as shown by Qin and Lawless (1994) and Owen (2001), $-2 \log \mathcal{R}(\theta_0) \to \chi^2_q$ in distribution as $n \to \infty$.

Using this fact, we can test the hypotheses about θ or construct confidence regions for θ. However, these inferences are valid only asymptotically, and the test has size distortion if the critical value is calculated from the chi-square distribution. Several authors proposed ways to improve the accuracy of inference based on the empirical likelihood ratio. Some examples are DiCiccio and Romano (1989), Hall (1990), DiCiccio et al. (1991) and Tsao (2001). In particular, Owen (1988, 2001) suggested to apply the bootstrap method. In the next section, we introduce the way to apply the bootstrap methods to the empirical likelihood ratio.

3 Bootstrap Method

Though the empirical likelihood ratio given in (2) is distributed as chi-square distribution as $n \to \infty$ under the null hypothesis $\mathrm{H}_0 : \mathrm{E}[g(X, \theta_0)] = 0$, its distribution is unknown in small sample size. When the distribution of a statistic is unknown, the bootstrap method proposed by Efron (1979) is often valid. Owen (1988) used the bootstrap method to construct a confidence interval based on empirical likelihood ratio, and executed some simulations. Also, Hall and Presnell (1999), and Brown and Newey (2001) proposed to use the probabilities obtained by the empirical likelihood as the resampling probabilities of the bootstrap. We call this method the EL bootstrap hereafter. In this section, we briefly summarize the usual bootstrap and the EL bootstrap.

In order to obtain the critical value of the test for $\mathrm{H}_0 : \mathrm{E}[g(X, \theta_0)] = 0$, the bootstrap methods are applied to the empirical likelihood ratio in the following way.

1. I. In the case of usual bootstrap, draw a random sample of size n, $X_1^*, X_2^*, \ldots, X_n^*$ from the original sample $X_1, X_2, \ldots, X_n$ with replacement. This is equivalent to drawing n i.i.d. observations from the distribution with $P(X = X_i) = 1/n$.

   II. In the case of EL bootstrap, draw a random sample of size n, $X_1^*, X_2^*, \ldots, X_n^*$ from the distribution with $P(X = X_i) = \widehat{p}_i$, where $\widehat{p}_i$ is obtained by maximizing the empirical likelihood ratio given in (2) under the null hypothesis $\mathrm{H}_0 : \mathrm{E}[g(X, \theta_0)] = 0$.

2. Using $X_1^*, X_2^*, \ldots, X_n^*$, calculate

$$\mathcal{R}_b(\theta_0) = \max\left\{ \prod_{i=1}^{n} np_i^* \middle| \sum_{i=1}^{n} p_i^* g^*(X_i^*, \theta_0) = 0, p_i^* \geq 0, \sum_{i=1}^{n} p_i^* = 1 \right\}$$

   where $g^*(X, \theta)$ is defined as follows.

I. In the case usual bootstrap, as discussed by Hall and Horowitz (1996), recentering is required since $\frac{1}{n}\sum_{i=1}^{n} g(X_i, \theta_0) \neq 0$ in general. Thus, recentering $g(X, \theta)$, we define $g^*(X, \theta) = g(X, \theta) - \frac{1}{n}\sum_{i=1}^{n} g(X_i, \theta)$.

II. In the case of EL bootstrap, we define $g^*(X, \theta) = g(X, \theta)$. Since $\sum_{i=1}^{n} \widehat{p}_i g(X_i, \theta_0) = 0$, recentering is not required in the EL bootstrap.

3. Repeating steps 1–2 B times, we obtain B values of $\mathcal{R}_b(\theta_0)$. Sorting $-2\log\mathcal{R}_b(\theta_0)$ into the ascending numerical order, and letting $-2\log\mathcal{R}_\alpha$ be the $B \times (1-\alpha)$th value, we obtain the $100 \times \alpha\%$ critical value of $-2\log\mathcal{R}(\theta_0)$ as $-2\log\mathcal{R}_\alpha$. We call $-2\log\mathcal{R}_\alpha$ the bootstrap critical value.

The differences between the usual and the EL bootstrap methods are steps 1 and 2. In step 1, while X_i is drawn with probability $1/n$ in the usual bootstrap, it is drawn with probability $\widehat{p}_i$ in the EL bootstrap. Also, in step 2, though the usual bootstrap requires the recentering, it is not needed in the EL bootstrap.

To examine the validity of these procedures, we execute some simulations in the next section.

4 Monte Carlo Results

In this section, we examine the efficiency of the procedures introduced in the previous section by Monte Carlo simulations. For simplicity, we assume $d = l = s = 1$ and let $g(X, \theta) = X - \theta$. In this setup, we can test the hypothesis about the mean θ of a random scalar X. In the following simulations, we consider to test $\mathrm{H}_0 : \theta = \theta_0 = 0$ against $\mathrm{H}_1 : \theta = \theta_1$. In the usual bootstrap, $g^*(X, \theta_1) = X - \overline{X}$.[1]

[1]Efron and Tibshirani (1993) stated that for the test for the mean $\mathrm{H}_0 : \theta = \theta_0$, $Z_i^* = X_i^* - \overline{X} + \theta_0$ should be used in place of X_i^*. If Z_i^* is used in place of X_i, $g(Z_i^*, \theta) = X_i^* - \overline{X}$. Thus, in this case, recentering and the method proposed by Efron and Tibshirani (1993) yield the same results. Also, if we consider $g(X, \theta)$ as a random vector, and test for $\mathrm{H}_0 : E[g(X, \theta_0)] = 0$, recentering and the method of Efron and Tibshirani

The design of the simulation is as follows.

1. Draw a random sample of size n, $X_1, X_2, \ldots, X_n$, from the

   I. normal distribution

   II. uniform distribution

   III. chi-square distribution with 2 degrees of freedom

   IV. t-distribution with 3 degrees of freedom

   where $n = 20,\ 50,\ 100,\ 200$. Each sample is transformed so as to have mean θ_1 and variance 1, where $\theta_1 = 0,\ 0.1,\ 0.2,\ 0.4$.

2. Using the bootstrap methods explained in the previous section, calculate the $100 \times \alpha\%$ critical values, where $\alpha = 0.1,\ 0.05,\ 0.01$ and the number of iteration $B = 10000$.

3. Calculate the value of $\mathcal{R}(0)$. If $-2 \log \mathcal{R}(0)$ is larger than the critical values obtained in step 2, $\mathrm{H}_0 : \theta = 0$ is rejected. In a similar way, we can test H_0 using the critical values of the chi-square distribution.

4. Repeating steps 1–3 10000 times, and calculating the rate such that $\mathrm{H}_0 : \theta = 0$ is rejected, we obtain the empirical power of the test $\widehat{p}$. To examine the validity of the procedures, we also test the null hypothesis using the critical values of the chi-square distribution.

Empirical powers obtained by the simulations are shown in Table 1. Let p the true power of the test. When $\theta_1 = 0$, we test $\mathrm{H}_0 : p = \alpha$ against $\mathrm{H}_1 : p \neq \alpha$ using the normal approximation of a binomial distribution. In Table 1, $*$, † and ‡ represent that $\mathrm{H}_0 : p = \alpha$ is rejected at 10%, 5%, and 1% significance levels respectively.

When the sample is drawn from the normal distribution, $\mathrm{H}_0 : p = \alpha$ can not be rejected even for $n = 20$ if the bootstrap critical values are used. On the other hand, if the critical

(1993) mean the same thing. The author is grateful to an anonymous referee for indicating the discussion of Efron and Tibshirani (1993).

values of the chi-square distribution are used, $\mathrm{H}_0 : p = \alpha$ is rejected even for $n = 50$. When the sample is drawn from the uniform distribution, the empirical sizes obtained from the usual bootstrap critical values and critical values of the chi-square distribution are almost comparable. The EL bootstrap yields the preferable results than the other two methods. When the sample is distributed as the chi-square distribution and $n = 20$ and 50, $\mathrm{H}_0 : p = \alpha$ is rejected at 1% significance level for all critical values. However, the usual bootstrap critical values yield empirical sizes closer to α. Also, when $n = 200$, both usual and EL bootstrap critical values yield the preferable results. When the sample has the t distribution, $\mathrm{H}_0 : p = \alpha$ is rejected at 1% significance level even for $n = 200$ if the critical values of the chi-square distribution are used. However, $\mathrm{H}_0 : p = \alpha$ can not be rejected even when $n = 20$, if the usual bootstrap critical values are used. As a whole, we see that the bootstrap critical values yield the preferable empirical sizes compared with those of the chi-square distribution. Also, critical values obtained by the usual and EL bootstrap methods are almost comparable though the usual bootstrap yields the preferable results in some cases.

In most cases, the powers of the test are slightly smaller when the bootstrap critical values are used than when those of the chi-square distribution are used. However, the differences of powers are very small and the critical values of the chi-square distribution yield large size distortions. Also, large powers obtained by the critical values of chi-square distribution are caused by these size distortions.

Thus, our simulation results show that the test based on the empirical likelihood ratio is improved by the bootstrap methods introduced in this paper.

Acknowledgment

The author is grateful to an anonymous referee for valuable comments and suggestions.

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Table 1: Empirical sizes obtained by Monte Carlo simulations.

n	Dist	θ_1	χ^2 10%	χ^2 5%	χ^2 1%	Usual bootstrap 10%	Usual bootstrap 5%	Usual bootstrap 1%	EL bootstrap 10%	EL bootstrap 5%	EL bootstrap 1%
20	I	0.00	$0.1179^{\ddagger}$	$0.0648^{\ddagger}$	$0.0192^{\ddagger}$	0.1000	0.0495	0.0092	0.1011	0.0504	0.0095
		0.10	0.1565	0.0966	0.0313	0.1351	0.0745	0.0168	0.1377	0.0777	0.0187
		0.20	0.2536	0.1675	0.0640	0.2226	0.1344	0.0393	0.2284	0.1405	0.0401
		0.40	0.5688	0.4416	0.2440	0.5257	0.3903	0.1636	0.5343	0.4007	0.1727
	II	0.00	$0.1076^{\dagger}$	$0.0545^{\dagger}$	$0.0128^{\ddagger}$	0.0959	$0.0444^{\dagger}$	$0.0064^{\ddagger}$	0.1000	0.0475	0.0095
		0.10	0.1390	0.0789	0.0211	0.1247	0.0646	0.0113	0.1273	0.0678	0.0157
		0.20	0.2347	0.1492	0.0486	0.2188	0.1307	0.0295	0.2201	0.1338	0.0350
		0.40	0.5702	0.4528	0.2298	0.5503	0.4289	0.1762	0.5475	0.4224	0.1723
	III	0.00	$0.1616^{\ddagger}$	$0.1040^{\ddagger}$	$0.0461^{\ddagger}$	$0.1176^{\ddagger}$	$00665^{\ddagger}$	$0.0217^{\ddagger}$	$0.1328^{\ddagger}$	$0.0832^{\ddagger}$	$0.0386^{\ddagger}$
		0.10	0.1783	0.1107	0.0356	0.1287	0.0667	0.0128	0.1190	0.0575	0.0154
		0.20	0.3098	0.2138	0.0814	0.2330	0.1282	0.0258	0.1947	0.0874	0.0088
		0.40	0.8211	0.7464	0.5453	0.7176	0.5749	0.2482	0.6537	0.4299	0.0757
	IV	0.00	$0.1500^{\ddagger}$	$0.0900^{\ddagger}$	$0.0308^{\ddagger}$	0.1017	0.497	0.0098	$0.1111^{\ddagger}$	$0.0589^{\ddagger}$	0.0117^{*}
		0.10	0.2023	0.1333	0.0544	0.1493	0.0826	0.0194	0.1622	0.0943	0.0243
		0.20	0.3377	0.2461	0.1235	0.2690	0.1701	0.0550	0.2951	0.1975	0.0723
		0.40	0.6599	0.5595	0.3786	0.5819	0.4569	0.2266	0.6259	0.5137	0.2963
50	I	0.00	$0.1073^{\dagger}$	$0.0552^{\dagger}$	0.0116	0.1015	0.0498	0.0101	0.1029	0.0500	0.0098
		0.10	0.1949	0.1141	0.0350	0.1872	0.1081	0.0303	0.1872	0.1075	0.0318
		0.20	0.4187	0.2991	0.1272	0.4087	0.2831	0.1154	0.4091	0.2848	0.1162
		0.40	0.8806	0.8059	0.5942	0.8750	0.7953	0.5674	0.8758	0.7960	0.5707
	II	0.00	0.1053^{*}	0.0537^{*}	$0.0120^{\dagger}$	0.1018	0.515	0.0106	0.1020	0.0520	0.0112
		0.10	0.1888	0.1136	0.0350	0.1847	0.1095	0.0330	0.1839	0.1105	0.0337
		0.20	0.4236	0.3019	0.1279	0.4178	0.2955	0.1212	0.4182	0.2946	0.1221
		0.40	0.8891	0.8210	0.6226	0.8868	0.8159	0.6119	0.8859	0.8154	0.6082
	III	0.00	$0.1302^{\ddagger}$	$0.0741^{\ddagger}$	$0.0246^{\ddagger}$	$0.1132^{\ddagger}$	$00610^{\ddagger}$	$0.0146^{\ddagger}$	$0.1165^{\ddagger}$	$0.0639^{\ddagger}$	$0.0184^{\ddagger}$
		0.10	0.2111	0.1303	0.0405	0.1833	0.1038	0.0209	0.1769	0.0946	0.0159
		0.20	0.5121	0.3933	0.1982	0.4693	0.3376	0.1305	0.4530	0.3150	0.0987
		0.40	0.9906	0.9779	0.9331	0.9847	0.9636	0.8591	0.9834	0.9594	0.8115
	IV	0.00	$0.1270^{\ddagger}$	$0.0689^{\ddagger}$	$0.0192^{\ddagger}$	0.0999	0.480	0.0097	$0.1081^{\ddagger}$	$0.0554^{\dagger}$	$0.0125^{\dagger}$
		0.10	0.2347	0.1542	0.0590	0.2021	0.1237	0.0367	0.2130	0.1352	0.0450
		0.20	0.4890	0.3748	0.1967	0.4497	0.3255	0.1454	0.4645	0.3478	0.1686
		0.40	0.8716	0.8068	0.6360	0.8434	0.7604	0.5549	0.8625	0.7907	0.6078

Table 1 (continued)

n	Dist	θ_1	χ^2 10%	χ^2 5%	χ^2 1%	Usual bootstrap 10%	Usual bootstrap 5%	Usual bootstrap 1%	EL bootstrap 10%	EL bootstrap 5%	EL bootstrap 1%
100	I	0.00	0.1035	0.0537*	0.0113	0.1003	0.0514	0.0107	0.1014	0.0512	0.0104
		0.10	0.2662	0.1736	0.0609	0.2610	0.1709	0.0585	0.2609	0.1707	0.0585
		0.20	0.6363	0.5120	0.2791	0.6316	0.5060	0.2727	0.6319	0.5075	0.2721
		0.40	0.9915	0.9786	0.9174	0.9907	0.9768	0.9131	0.9908	0.9773	0.9140
	II	0.00	0.1045	0.0507	0.0117*	0.1029	0.0501	0.0115	0.1037	0.0501	0.0113
		0.10	0.2700	0.1781	0.0618	0.2676	0.1765	0.0602	0.2668	0.1764	0.0605
		0.20	0.6511	0.5297	0.2915	0.6500	0.5264	0.2862	0.6491	0.5268	0.2878
		0.40	0.9932	0.9818	0.9337	0.9932	0.9815	0.9325	0.9928	0.9813	0.9316
	III	0.00	0.1161‡	0.0619‡	0.0143‡	0.1081‡	0.0545†	0.0114	0.1080‡	0.0561‡	0.0125†
		0.10	0.2941	0.1978	0.0726	0.2773	0.1803	0.0589	0.2726	0.1733	0.0535
		0.20	0.7538	0.6545	0.4316	0.7357	0.6242	0.3784	0.7312	0.6157	0.3596
		0.40	0.9998	0.9998	0.9986	0.9998	0.9995	0.9973	0.9998	0.9994	0.9963
	IV	0.00	0.1169‡	0.0608‡	0.0133‡	0.0989	0.0476	0.0083*	0.1041	0.0535	0.0102
		0.10	0.3008	0.2065	0.0816	0.2791	0.1809	0.0630	0.2859	0.1910	0.0725
		0.20	0.6643	0.5557	0.3390	0.6421	0.5238	0.2948	0.6504	0.5359	0.3173
		0.40	0.9679	0.9438	0.8604	0.9553	0.9230	0.8210	0.9652	0.9389	0.8477
200	I	0.00	0.1075†	0.0511	0.0095	0.1066†	0.0507	0.0093	0.1060†	0.0507	0.0096
		0.10	0.4102	0.2940	0.1309	0.4074	0.2916	0.1292	0.4069	0.2915	0.1293
		0.20	0.8737	0.7993	0.5898	0.8725	0.7962	0.5877	0.8721	0.7972	0.5863
		0.40	0.9999	0.9998	0.9991	0.9999	0.9997	0.9990	0.9999	0.9998	0.9989
	II	0.00	0.1005	0.0521	0.0118*	0.1004	0.0512	0.0118*	0.0996	0.0511	0.0116
		0.10	0.4062	0.2901	0.1254	0.4046	0.2885	0.1243	0.4051	0.2887	0.1247
		0.20	0.8870	0.8151	0.6062	0.8863	0.8142	0.6054	0.8859	0.8148	0.6027
		0.40	1.0000	1.0000	0.9997	1.0000	1.0000	0.9996	1.0000	1.0000	0.9997
	III	0.00	0.1055*	0.0553†	0.0122†	0.1010	0.0512	0.0111	0.1009	0.0517	0.0105
		0.10	0.4585	0.3351	0.1654	0.4480	0.3257	0.1516	0.4450	0.3242	0.1477
		0.20	0.9481	0.9119	0.7862	0.9459	0.9060	0.7685	0.9459	0.9044	0.7598
		0.40	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	IV	0.00	0.1127‡	0.0584‡	0.0150‡	0.1015	0.0508	0.0107	0.1050*	0.0535	0.0122†
		0.10	0.4462	0.3330	0.1566	0.4280	0.3112	0.1381	0.4345	0.3201	0.1456
		0.20	0.8724	0.8019	0.6079	0.8594	0.7813	0.5715	0.8643	0.7903	0.5923
		0.40	0.9941	0.9900	0.9700	0.9887	0.9807	0.9541	0.9938	0.9885	0.9672