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On the Use of the Stein Variance Estimator in the Double k-class Estimator when Each Individual Regression Coefficient is Estimated

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Abstract

In this paper we consider the double k-class estimator which incorporates the Stein variance estimator. This estimator is called the SVKK estimator. We derive the explicit formula for the MSE of the SVKK estimator for each individual regression coefficient. It is shown analytically that the MSE performance of the Stein-rule estimator for each individual regression coefficient can be improved by utilizing the Stein variance estimator. Also, MSE's of several estimators included in a family of the SVKK estimators are compared by numerical evaluations.

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1 Introduction

In the context of a linear regression model, the Stein-rule (SR) estimator proposed by Stein (1956) and James and Stein (1961) dominates the ordinary least squares (OLS) estimator in terms of predictive mean squared error (PMSE) if the number of the regression coefficient is larger than or equal to three. Following George's (1990) suggestion, Berry (1994) proved that the modified Stein-rule estimator can be improved by incorporating the Stein variance estimator proposed by Stein (1964). Also, Ohtani (1996a) showed that Berry's estimator is a kind of pre-test estimator, and its PMSE performance can be further improved if an appropriate critical value of the pre-test is used.

As an improved estimator, Theil (1971) proposed the minimum mean squared error (MMSE) estimator. However, Theil's (1971) MMSE estimator is not operational since it includes unknown parameters. Thus, Farebrother (1975) suggested the operational variant of the MMSE estimator which can be obtained by replacing the unknown parameters in Theil's (1971) MMSE estimator by the OLS estimators. Hereafter, we call the operational variant of the MMSE estimator the MMSE estimator simply. The MMSE estimator dominates the OLS estimator in terms of PMSE since it satisfies Baranchik's (1970) condition. As an extension of the MMSE estimator, Ohtani (1996b) considered the adjusted minimum mean squared error (AMMSE) estimator which is obtained by adjusting the degrees of freedom of the MMSE estimator.

Ullah and Ullah (1978) considered the double k-class (KK) estimator. The KK estimator includes the SR, MMSE and AMMSE estimators as special cases. Several authors have investigated the sampling performance and the choice of the parameters of the KK estimator. Some examples are Vinod (1980), Carter (1981), Menjoge (1984), Carter et al. (1993), and Vinod and Srivastava (1995).

Though most of the studies on the properties of the SR estimator and its variants have assumed that all the regression coefficients are estimated simultaneously, there are several studies which examined the properties of estimators for each individual regression coefficient. Some examples are Ullah (1974), Rao and Shinozaki (1978), Ohtani and Kozumi (1996), and Ohtani

(1997). The estimation of each individual regression coefficient is important, for example, in the following case. Suppose that we estimate an import demand equation expressed as

$$y_t = \beta_0 + x_{1t}\beta_1 + x_{2t}\beta_2 + \epsilon_t,$$

where y_t is the quantity of imports of a country at time t, x_{1t} is the real income (real GNP), and x_{2t} is the ratio of the import price to the domestic price (relative price). [See, for example, Houthakker and Magee (1969) and Goldstein and Khan (1985) for specification of import demand equations.] If our concern is to predict the quantity of imports which may lead to a surplus or deficit of the trade balance when the real income changes (ceteris paribus), then it is important to estimate β_1 as accurately as possible since the change in the quantity of imports depends on only the change in the real income (under the assumption of "ceteris paribus").

Recently, Ohtani and Wan (1999) suggested the Stein variance double k-class (SVKK) estimator and examined its PMSE performance. However, the MSE performance of the SVKK estimator for each individual regression coefficient have not been examined. Thus, in this paper we examined the MSE performance of the SVKK estimator for the each individual regression coefficient. In Section 2 we present the model and the estimators. We derive the MSE of the SVKK estimator for each individual regression coefficient in Section 3. Also, we show that the SR estimator for each individual regression coefficient can be improved by incorporating the Stein variance estimator. In Section 4 we compare the MSE's of several estimators included in a family of SVKK estimators by numerical evaluations.

2 Model and the estimators

Consider a linear regression model,

$$y = X\beta + \epsilon, \quad \epsilon \sim N(0, \sigma^2 I_n), \tag{1}$$

where y is an $n \times 1$ vector of observations on a dependent variable, X is an $n \times k$ matrix of full column rank of observations on nonstochastic independent variables, β is a $k \times 1$ vector of regression coefficients, and ϵ is an $n \times 1$ vector of normal error terms.

Following Judge and Yancey (1986, p. 11), we reparameterize the model (1) and work with the following orthonormal counterpart:

$$y = Z\gamma + \epsilon, \tag{2}$$

where $Z = XS^{-1/2}$, $\gamma = S^{1/2}\beta$ and $S^{1/2}$ is the symmetric matrix such that $S^{-1/2}SS^{-1/2} = I_k$, where $S = X'X$.

Then the ordinary least squares (OLS) estimator is

$$\widehat{\gamma} = Z'y. \tag{3}$$

In the context of reparameterized model, the Stein-rule (SR) estimator proposed by Stein (1956) is defined as

$$\widehat{\gamma}_{SR} = \left(1 - \frac{k-2}{\nu+2}\frac{e'e}{\widehat{\gamma}'\widehat{\gamma}}\right)\widehat{\gamma}, \tag{4}$$

where $e = y - Z\widehat{\gamma}$, $\nu = n - k$ and $e'e/(\nu+2)$ is an estimator of σ^2. The SR estimator dominates the OLS estimator under predictive mean squared error (PMSE) for $k \geq 3$.

The minimum mean squared error (MMSE) estimator proposed by Farebrother (1975) is

$$\widehat{\gamma}_M = \left(\frac{\widehat{\gamma}'\widehat{\gamma}}{\widehat{\gamma}'\widehat{\gamma} + e'e/(\nu+2)}\right)\widehat{\gamma}. \tag{5}$$

As an extension of the MMSE estimator, Ohtani (1996b) considered the adjusted minimum mean squared error (AMMSE) estimator which is obtained by the degrees of freedom of $\widehat{\gamma}'\widehat{\gamma}$ (i.e., k):

$$\widehat{\gamma}_{AM} = \left(\frac{\widehat{\gamma}'\widehat{\gamma}/k}{\widehat{\gamma}'\widehat{\gamma}/k + e'e/(\nu+2)}\right)\widehat{\gamma}. \tag{6}$$

Also, Ullah and Ullah (1978) proposed the double k-class (KK) estimator:

$$\begin{aligned}\widehat{\gamma}_{KK} &= \left(1 - \frac{k_1 e'e}{y'y - k_2 e'e}\right)\widehat{\gamma} \\ &= \left(\frac{\widehat{\gamma}'\widehat{\gamma} + \alpha_1 e'e}{\widehat{\gamma}'\widehat{\gamma} + \alpha_2 e'e}\right)\widehat{\gamma},\end{aligned} \tag{7}$$

where k_1 and k_2 are constants chosen appropriately, $\alpha_1 = 1 - k_1 - k_2$ and $\alpha_2 = 1 - k_2$. The double k-class estimator includes the SR, MMSE and AMMSE estimators as special cases.

The Stein variance estimator is defined as

$$\widehat{\sigma}_s^2 = \min[y'y/(n+2),\ e'e/(\nu+2)]. \tag{8}$$

The Stein variance estimator can be regarded as the following pre-test estimator:

$$\widehat{\sigma}_s^2 = I(F \geq c_1)\frac{e'e}{\nu+2} + I(F < c_1)\frac{y'y}{n+2}, \tag{9}$$

where $c_1 = \nu/(\nu+2)$, $F = (\widehat{\gamma}'\widehat{\gamma}/k)/(e'e/\nu)$ is the test statistic for testing the null hypothesis $H_0 : \gamma = 0$ against the alternative $H_1 : \gamma \neq 0$, and $I(A)$ is an indicator function such that $I(A) = 1$ if an event A occurs and $I(A) = 0$ otherwise. Thus, as to the Stein variance estimator, the error variance is estimated by $y'y/(n+2)$ if the null hypothesis is not rejected, and it is estimated by $e'e/(\nu+2)$ if the null hypothesis is rejected. Utilizing the Stein variance estimator, Berry (1994) and Ohtani (1996a) considered the following Stein variance Stein-rule (SVSR) estimator:

$$\widehat{\gamma}_{SR}(c) = I(F \geq c)\left(1 - \frac{k-2}{\nu+2}\frac{e'e}{\widehat{\gamma}'\widehat{\gamma}}\right)\widehat{\gamma} + I(F < c)\left(1 - \frac{k-2}{n+2}\frac{y'y}{\widehat{\gamma}'\widehat{\gamma}}\right)\widehat{\gamma}. \tag{10}$$

Berry (1994) showed that the SVSR estimator dominates the SR estimator in terms of PMSE when $c_0 = k/(\nu+2)$ is used as the critical value of the pre-test. Ohtani (1996a) showed that the PMSE performance of the SVSR estimator can be further improved if $c_2 = (k-2)\nu(2n-k+4)/[k(\nu+2)(2n-k+6)]$ is used instead of c_0.

Recently, Ohtani and Wan (1999) proposed the Stein variance double k-class (SVKK) estimator:

$$\widehat{\gamma}_{KK}(c) = I(F \geq c)\left(\frac{\widehat{\gamma}'\widehat{\gamma} + \alpha_1 e'e}{\widehat{\gamma}'\widehat{\gamma} + \alpha_2 e'e}\right)\widehat{\gamma} + I(F < c)\left(\frac{\widehat{\gamma}'\widehat{\gamma} + \alpha_3 y'y}{\widehat{\gamma}'\widehat{\gamma} + \alpha_4 y'y}\right)\widehat{\gamma}. \tag{11}$$

The SVKK estimator reduces to the SVSR estimator when $\alpha_1 = -(k-2)/(\nu+2)$, $\alpha_2 = 0$, $\alpha_3 = -(k-2)/(n+2)$ and $\alpha_4 = 0$. When $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = 1/(\nu+2)$ and $\alpha_4 = 1/(n+2)$, the SVKK estimator reduces to SVMMSE estimator. Also, when $\alpha_1 = \alpha_3 = 0$, $\alpha_2 = k/(\nu+2)$ and $\alpha_4 = k/(n+2)$, SVKK estimator reduces to the SVAMMSE estimator. Ohtani and Wan (1999) examined the PMSE performances of a family of SVKK estimators. However, the MSE performance of the SVKK estimator for each individual regression coefficient has not been examined so far. Thus, we derive the explicit formula for the MSE of the SVKK estimator for each individual regression coefficient and examine its performance in the next section.

3 MSE

Let h be a $k \times 1$ vector with known elements. If h' is the ith row vector of $S^{-1/2}$, the estimator $h'\widehat{\gamma}_{KK}(c)$ is the ith element of the SVKK estimator for β in the original model. Since the elements of h are known, we assume that $h'h = 1$ without loss of generality. Then the MSE of $h'\widehat{\gamma}$ is

$$\begin{aligned}
MSE[h'\widehat{\gamma}_{KK}(c)] &= E[(h'\widehat{\gamma}_{KK}(c) - h'\gamma)^2] \\
&= E\left[I(F \geq c)\left(\frac{\widehat{\gamma}'\widehat{\gamma} + \alpha_1 e'e}{\widehat{\gamma}'\widehat{\gamma} + \alpha_2 e'e}\right)^2 (h'\widehat{\gamma})^2\right] \\
&\quad + E\left[I(F < c)\left(\frac{(1+\alpha_3)\widehat{\gamma}'\widehat{\gamma} + \alpha_3 e'e}{(1+\alpha_4)\widehat{\gamma}'\widehat{\gamma} + \alpha_4 e'e}\right)^2 (h'\widehat{\gamma})^2\right] \\
&\quad - 2h'\gamma E\left[I(F \geq c)\left(\frac{\widehat{\gamma}'\widehat{\gamma} + \alpha_1 e'e}{\widehat{\gamma}'\widehat{\gamma} + \alpha_2 e'e}\right) h'\widehat{\gamma}\right] \\
&\quad - 2h'\gamma E\left[I(F < c)\left(\frac{(1+\alpha_3)\widehat{\gamma}'\widehat{\gamma} + \alpha_3 e'e}{(1+\alpha_4)\widehat{\gamma}'\widehat{\gamma} + \alpha_4 e'e}\right) h'\widehat{\gamma}\right] + (h'\gamma)^2,
\end{aligned} \tag{12}$$

since $y'y = \widehat{\gamma}'\widehat{\gamma} + e'e$.

We define the functions, $H(p, q; a_1, a_2, a_3, a_4; c)$ and $J(p, q; a_1, a_2, a_3, a_4; c)$, as

$$H(p, q; a_1, a_2, a_3, a_4; c) = E\left[I(F \geq c)\left(\frac{a_1\widehat{\gamma}'\widehat{\gamma} + a_2 e'e}{a_3\widehat{\gamma}'\widehat{\gamma} + a_4 e'e}\right)^p (h'\widehat{\gamma})^{2q}\right], \tag{13}$$

$$J(p, q; a_1, a_2, a_3, a_4; c) = E\left[I(F \geq c)\left(\frac{a_1\widehat{\gamma}'\widehat{\gamma} + a_2 e'e}{a_3\widehat{\gamma}'\widehat{\gamma} + a_4 e'e}\right)^p (h'\widehat{\gamma})^{2q}(h'\widehat{\gamma})\right]. \tag{14}$$

Then, the MSE of $h'\widehat{\gamma}_{KK}(c)$ is written as

$$\begin{aligned}
&MSE[h'\widehat{\gamma}_{KK}(c)] \\
&= H(2, 1; 1, \alpha_1, 1, \alpha_2; c) \\
&\quad + H(2, 1; 1+\alpha_3, \alpha_3, 1+\alpha_4, \alpha_4; 0) - H(2, 1; 1+\alpha_3, \alpha_3, 1+\alpha_4, \alpha_4; c) \\
&\quad - 2h'\gamma J(1, 0; 1, \alpha_1, 1, \alpha_2; c) \\
&\quad - 2h'\gamma J(1, 0; 1+\alpha_3, \alpha_3, 1+\alpha_4, \alpha_4; 0) + 2h'\gamma J(1, 0; 1+\alpha_3, \alpha_3, 1+\alpha_4, \alpha_4; c) \\
&\quad + (h'\gamma)^2.
\end{aligned} \tag{15}$$

As shown in Appendix, the explicit formulae of $H(p,q;a_1,a_2,a_3,a_4;c)$ and $J(p,q;a_1,a_2,a_3,a_4;c)$ are

$$H(p,q;a_1,a_2,a_3,a_4;c) = (2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)G_{ij}(p,q;a_1,a_2,a_3,a_4;c), \tag{16}$$

$$J(p,q;a_1,a_2,a_3,a_4;c) = h'\gamma(2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)G_{i+1,j}(p,q;a_1,a_2,a_3,a_4;c), \tag{17}$$

where

$$\begin{aligned} G_{ij}(p,q;a_1,a_2,a_3,a_4;c) = {} & \frac{\Gamma((\nu+k)/2+q+i+j)\Gamma(1/2+q+i)}{\Gamma(k/2+q+i+j)\Gamma(1/2+i)\Gamma(\nu/2)} \\ & \times \int_{c^*}^{1}\left(\frac{a_2+(a_1-a_2)t}{a_4+(a_3-a_4)t}\right)^p t^{k/2+q+i+j-1}(1-t)^{\nu/2-1}dt, \end{aligned} \tag{18}$$

$w_i(\lambda) = \exp(-\lambda/2)(\lambda/2)^i/i!$, $\lambda_1 = (h'\gamma)^2/\sigma^2$, $\lambda_2 = \gamma'(I_k - hh')\gamma/\sigma^2$ and $c^* = kc/(\nu+kc)$.

Differentiating (18) with respect to c, we have:

$$\begin{aligned} & \frac{\partial G_{ij}(p,q;a_1,a_2,a_3,a_4;c)}{\partial c} \\ & \quad = -\frac{\Gamma((\nu+k)/2+q+i+j)\Gamma(1/2+q+i)}{\Gamma(k/2+q+i+j)\Gamma(1/2+i)\Gamma(\nu/2)}\left(\frac{a_2\nu+a_1kc}{a_4\nu+a_3kc}\right)^p \\ & \qquad \times \frac{k^{k/2+q+i+j}\nu^{\nu/2}c^{k/2+q+i+j-1}}{(\nu+kc)^{(\nu+k)/2+q+i+j}}. \end{aligned} \tag{19}$$

Using (19) and performing some manipulations, we obtain:

$$\begin{aligned} & \frac{1}{2\sigma^2}\frac{\partial MSE[h'\widehat{\gamma}_{KK}(c)]}{\partial c} \\ & \quad = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)\frac{\Gamma((\nu+k)/2+i+j+1)\Gamma(1/2+1+i)}{\Gamma(k/2+i+j+1)\Gamma(1/2+i)\Gamma(\nu/2)} \\ & \qquad \times \frac{k^{k/2+i+j+1}\nu^{\nu/2}c^{k/2+i+j}}{(\nu+kc)^{(\nu+k)/2+i+j+1}}D_1(c)\left(D_2(c)-\frac{\lambda_1}{1/2+i}\right), \end{aligned} \tag{20}$$

where

$$D_1(c) = \frac{\alpha_3\nu+(1+\alpha_3)kc}{\alpha_4\nu+(1+\alpha_4)kc} - \frac{\alpha_1\nu+kc}{\alpha_2\nu+kc}, \tag{21}$$

$$D_2(c) = \frac{\alpha_3\nu+(1+\alpha_3)kc}{\alpha_4\nu+(1+\alpha_4)kc} + \frac{\alpha_1\nu+kc}{\alpha_2\nu+kc}. \tag{22}$$

If we consider the case of the SVSR estimator (i.e., $\alpha_1 = -(k-2)/(\nu+2)$, $\alpha_3 = -(k-2)/(n+2)$ and $\alpha_2 = \alpha_4 = 0$), then, we have:

$$D_1(c) = \frac{(k-2)[\nu - (\nu+2)c]}{c(\nu+2)(n+2)}, \tag{23}$$

$$D_2(c) = \frac{-(k-2)(2n-k+4)\nu + (\nu+2)(2n-k+6)kc}{kc(n+2)(\nu+2)}. \tag{24}$$

Thus, $D_1(c) \geq 0$ when $0 < c \leq \nu/(\nu+2) = c_1$, and $D_2(c) \leq 0$ when $0 < c \leq [(k-2)(2n-k+4)\nu]/[k(\nu+2)(2n-k+6)] = c_2$. Since $c_1 > c_2$, the MSE of the SVSR estimator is monotonically decreasing on $c \in [0, c_2]$. Since the SVSR estimator reduces to the SR estimator when $c = 0$, we obtain the following theorem.

Theorem 1 *The SVSR estimator with $0 < c \leq c_2$ dominates the SR estimator in terms of MSE even when each individual regression coefficient is estimated. Particularly, the SVSR estimator with $c = c_2$ has the smallest MSE among the class of the SVSR estimators with $0 \leq c \leq c_2$.*

Since further theoretical analysis is difficult, we examine the MSE performances of a family of the SVKK estimators numerically in the next section.

4 Numerical analysis

In this section we compare the MSE performances of a family of the SVKK estimators for each individual regression coefficient by numerical evaluations. The parameter values used in the numerical evaluations are: $k = 3, 4, 5, 6, 8$, $n = 20, 30, 40$, $\lambda_1 + \lambda_2 =$ various values, and $\lambda_1 = w(\lambda_1 + \lambda_2)$, where $w = 0, 0.1, 0.2, \cdots, 1.0$. To compare the MSE's of the estimators, we evaluate the values of relative MSE defined as $MSE[h'\bar{\gamma}]/MSE[h'\hat{\gamma}]$, where $\bar{\gamma}$ is any estimator of γ. Thus, the estimator $h'\bar{\gamma}$ has smaller MSE than the OLS estimator when the value of relative MSE is smaller than unity.

The numerical evaluations were executed on a personal computer using the FORTRAN code. In evaluating the integral in $G_{ij}(p, q; a_1, a_2, a_3, a_4; c)$, we used Simpson's rule with 200 equal subdivisions. The double infinite series in $H(p, q; a_1, a_2, a_3, a_4; c)$ and $J(p, q; a_1, a_2, a_3, a_4; c)$ were judged to converge when the increment of the series got smaller than 10^{-12}.

As to the SVSR estimator, we used the critical value of the pre-test $c = c_2$ since Theorem 1 ensures that the SVSR estimator with $c = c_2$ has the smallest MSE in the class of the SVSR estimators with $0 \leq c \leq c_2$. However, no such critical values for the SVMMSE and SVAMMSE estimators could be obtained. Since some preliminary numerical results seem to show that the use of $c = 0$ yields better MSE performance than the use of $c > 0$, we use $c = 0$ for the SVMMSE and SVAMMSE estimators (i.e., the SVMMSE estimator and the SVAMMSE estimator reduce to the MMSE estimator and the AMMSE estimator respectively). Since the results for $k = 5$, 8 and $n = 30$ are qualitatively typical, we do not show the results for the other cases.

The results for $k = 5$ and $n = 20$ are shown in Table 1. We can make sure of Theorem 1 from Tables 1 and 2. We can see from Table 1 that the AMMSE estimator has the smallest MSE over a wide region of the parameter space when $k = 5$ (i.e. $\lambda_1 \leq 2.4$). Also, the SVSR estimator has the second smallest MSE when λ_1 is small (i.e., $\lambda_1 \leq 1.2$). Though the maximum of the MSE of the AMMSE estimator is larger than unity, the minimum is much smaller than unity. This indicates that the gain in MSE of using the AMMSE estimator instead of the OLS estimator is larger than loss. Also, the loss in MSE of using the MMSE estimator is very small though the gain is not so large.

Table 2 shows the results for $k = 8$ and $n = 20$. We see from Table 2 that the SVSR estimator has the smallest MSE when λ_1 is small. When λ is moderate, the MSE of the AMMSE estimator is smallest. Also, the loss in MSE of using the MMSE estimator is smaller than the loss of using the other estimators.

Appendix

In this appendix, we derive the formulae for $H(p, q; a_1, a_2, a_3, a_4; c)$ and $J(p, q; a_1, a_2, a_3, a_4; c)$. First, we derive the formula for $H(p, q; a_1, a_2, a_3, a_4; c)$. Let $u_1 = (h'\widehat{\gamma})^2/\sigma^2$, $u_2 = \widehat{\gamma}'(I_k - hh')\widehat{\gamma}/\sigma^2$ and $u_3 = e'e/\sigma^2$. Then, $u_1 \sim \chi_1'^2(\lambda_1)$ and $u_2 \sim \chi_{k-1}'^2(\lambda_2)$, where $\chi_f'^2(\lambda)$ is the noncentral chi-square distribution with f degrees of freedom and noncentrality parameter λ, $\lambda_1 = (h'\gamma)^2/\sigma^2$ and $\lambda_2 = \gamma'(I_k - hh')\gamma/\sigma^2$. Further, u_3 is distributed as the chi-square distribution with $\nu = n-k$ degrees of freedom, and u_1, u_2 and u_3 are mutually independent.

Using u_1, u_2 and u_3, $H(p,q;a_1,a_2,a_3,a_4;c)$ is expressed as

$$\begin{aligned} &H(p,q;a_1,a_2,a_3,a_4;c) \\ &\quad = (\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} \iiint_R \left(\frac{a_1(u_1+u_2)+a_2u_3}{a_3(u_1+u_2)+a_4u_3}\right)^q \\ &\qquad \times u_1^{1/2+q+i-1} u_2^{(k-1)/2+j-1} u_3^{\nu/2-1} \exp[-(u_1+u_2+u_3)/2]du_1\,du_2\,du_3, \end{aligned} \tag{25}$$

where

$$K_{ij} = \frac{w_i(\lambda_1)w_j(\lambda_2)}{2^{(\nu+k)/2+i+j}\Gamma(1/2+i)\Gamma((k-1)/2+j)\Gamma(\nu/2)}, \tag{26}$$

$w_i(\lambda) = \exp(-\lambda/2)(\lambda/2)^i/i!$, and R is the region such that $(u_1+u_2)/u_3 \geq kc/\nu = c^{**}$.

Making use of the change of variables, $v_1 = (u_1+u_2)/u_3$, $v_2 = u_1u_3/(u_1+u_2)$ and $v_3 = u_3$, (25) reduces to

$$\begin{aligned} &(\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} \int_0^{\infty}\int_0^{v_3}\int_{c^{**}}^{\infty} \left(\frac{a_1v_1+a_2}{a_3v_1+a_4}\right)^p v_1^{k/2+q+i+j-1} v_2^{1/2+q+i-1} v_3^{\nu/2} \\ &\quad \times (v_3-v_2)^{(k-1)/2+j-1} \exp[-v_3(v_1+1)/2]dv_1\,dv_2\,dv_3. \end{aligned} \tag{27}$$

Again, making use of the change of variable, $z_1 = v_2/v_3$, (27) reduces to

$$\begin{aligned} &(\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} \frac{\Gamma(1/2+q+i)\Gamma((k-1)/2+j)}{\Gamma(k/2+q+i+j)} \\ &\quad \times \int_0^{\infty}\int_{c^{**}}^{\infty} \left(\frac{a_1v_1+a_2}{a_3v_1+a_4}\right)^p v_1^{k/2+q+i+j-1} v_3^{(\nu+k)/2+q+i+j-1} \exp[-v_3(v_1+1)/2]dv_1\,dv_3. \end{aligned} \tag{28}$$

Further, making use of the change of variable, $z_2 = v_3(v_1+1)/2$, (28) reduces to

$$\begin{aligned} &(\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} 2^{(\nu+k)/2+q+i+j} \frac{\Gamma(1/2+q+i)\Gamma((k-1)/2+j)\Gamma((\nu+k)/2+q+i+j)}{\Gamma(k/2+q+i+j)} \\ &\quad \times \int_{c^{**}}^{\infty} \left(\frac{a_1v_1+a_2}{a_3v_1+a_4}\right)^p v_1^{k/2+q+i+j-1} \left(\frac{1}{1+v_1}\right)^{(\nu+k)/2+q+i+j} dv_1. \end{aligned} \tag{29}$$

Finally, making use of the change of variable, $t = v_1/(1+v_1)$. we obtain (16) in the text.

Next, we derive the formula for $J(p,q;a_1,a_2,a_3,a_4;c)$. Differentiating $H(p,q;a_1,a_2,a_3,a_4;c)$ given if (16) with respect to γ, we have

$$
\begin{aligned}
&\frac{\partial H(p,q;a_1,a_2,a_3,a_4;c)}{\partial\gamma} \\
&\quad = (2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\partial w_i(\lambda_1)}{\partial\gamma}w_j(\lambda_2) + w_i(\lambda_1)\frac{\partial w_j(\lambda_2)}{\partial\gamma}\right] G_{ij}(p,q;a_1,a_2,a_3,a_4;c) \\
&\quad = -\frac{hh'\gamma}{\sigma^2}(2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)G_{ij}(p,q;a_1,a_2,a_3,a_4;c) \\
&\qquad +\frac{hh'\gamma}{\sigma^2}(2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)G_{i+1,j}(p,q;a_1,a_2,a_3,a_4;c) \\
&\qquad -\frac{(I_k-hh')\gamma}{\sigma^2}(2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)G_{ij}(p,q;a_1,a_2,a_3,a_4;c) \\
&\qquad +\frac{(I_k-hh')\gamma}{\sigma^2}(2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)G_{i,j+1}(p,q;a_1,a_2,a_3,a_4;c),
\end{aligned}
\tag{30}
$$

where we define $w_{-1}(\lambda_1) = w_{-1}(\lambda_2) = 0$. Since $h'h = 1$, we obtain

$$
\begin{aligned}
&h'\frac{\partial H(p,q;a_1,a_2,a_3,a_4;c)}{\partial\gamma} \\
&\quad = -\frac{h'\gamma}{\sigma^2}H(p,q;a_1,a_2,a_3,a_4;c) \\
&\qquad +\frac{h'\gamma}{\sigma^2}(2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)G_{i+1,j}(p,q;a_1,a_2,a_3,a_4;c).
\end{aligned}
\tag{31}
$$

Expressing $H(p,q;a_1,a_2,a_3,a_4;c)$ by $\widehat{\gamma}$ and $e'e$, we have

$$
\begin{aligned}
&H(p,q;a_1,a_2,a_3,a_4;c) \\
&\quad = \iint_{F\geq c}\left(\frac{a_1\widehat{\gamma}'\widehat{\gamma}+a_2e'e}{a_3\widehat{\gamma}'\widehat{\gamma}+a_4e'e}\right)^p (h'\widehat{\gamma})^{2q} f_N(\widehat{\gamma})f_e(e'e)\,d\widehat{\gamma}\,d(e'e),
\end{aligned}
\tag{32}
$$

where $F = (\widehat{\gamma}'\widehat{\gamma}/k)/(e'e/\nu)$, $f(e'e)$ is the density function of $e'e$, and

$$
f_N(\widehat{\gamma}) = \frac{1}{(2\pi)^{k/2}\sigma^k}\exp\left[-\frac{(\widehat{\gamma}-\gamma)'(\widehat{\gamma}-\gamma)}{2\sigma^2}\right],
\tag{33}
$$

is the density function of $\widehat{\gamma}$.

Differentiating $H(p,q;a_1,a_2,a_3,a_4;c)$ given in (32) with respect to γ, and multiplying h' from the left, we obtain

$$
h'\frac{\partial H(p,q;a_1,a_2,a_3,a_4;c)}{\partial\gamma}
$$

$$
\begin{aligned}
&= -\frac{h'\gamma}{\sigma^2} H(p,q;a_1,a_2,a_3,a_4;c) \\
&\quad + \frac{1}{\sigma^2} E\left[I(F \geq c) \left(\frac{a_1\widehat{\gamma}'\widehat{\gamma} + a_2 e'e}{a_3\widehat{\gamma}'\widehat{\gamma} + a_4 e'e} \right)^p (h'\widehat{\gamma})^{2q} h'\widehat{\gamma} \right]. \qquad (34)
\end{aligned}
$$

Equating (31) and (34), we obtain (17) in the text.

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Table 1: Relative MSE's for $k = 5$ and $n = 30$.

$\lambda_1 + \lambda_2$	λ_1	SR	SVSR	MMSE	AMMSE
.5	.00	.4465	.4139	.7509	.3642
	.05	.4714	.4372	.7547	.3773
	.10	.4962	.4606	.7585	.3903
	.15	.5211	.4839	.7623	.4034
	.20	.5460	.5073	.7661	.4164
	.25	.5709	.5306	.7699	.4295
	.30	.5958	.5540	.7737	.4425
	.35	.6207	.5773	.7775	.4556
	.40	.6456	.6007	.7813	.4687
	.45	.6705	.6240	.7851	.4817
	.50	.6954	.6474	.7889	.4948
3.0	.00	.4922	.4810	.8031	.4419
	.30	.5821	.5673	.8181	.5005
	.60	.6720	.6536	.8331	.5592
	.90	.7619	.7398	.8480	.6179
	1.20	.8518	.8261	.8630	.6765
	1.50	.9417	.9124	.8780	.7352
	1.80	1.0316	.9986	.8929	.7938
	2.10	1.1215	1.0849	.9079	.8525
	2.40	1.2114	1.1712	.9229	.9112
	2.70	1.3013	1.2575	.9378	.9698
	3.00	1.3912	1.3437	.9528	1.0285
10.0	.00	.6542	.6536	.8810	.5916
	1.00	.7534	.7519	.9003	.6902
	2.00	.8526	.8503	.9196	.7889
	3.00	.9518	.9487	.9390	.8876
	4.00	1.0511	1.0471	.9583	.9862
	5.00	1.1503	1.1454	.9776	1.0849
	6.00	1.2495	1.2438	.9969	1.1835
	7.00	1.3488	1.3422	1.0162	1.2822
	8.00	1.4480	1.4405	1.0355	1.3809
	9.00	1.5472	1.5389	1.0548	1.4795
	10.00	1.6465	1.6373	1.0741	1.5782
50.0	.00	.8986	.8986	.9662	.8487
	5.00	.9330	.9330	.9740	.9087
	10.00	.9673	.9673	.9819	.9688
	15.00	1.0017	1.0017	.9897	1.0288
	20.00	1.0361	1.0361	.9975	1.0888
	25.00	1.0704	1.0704	1.0054	1.1488
	30.00	1.1048	1.1048	1.0132	1.2088
	35.00	1.1392	1.1392	1.0211	1.2689
	40.00	1.1735	1.1735	1.0289	1.3289
	45.00	1.2079	1.2079	1.0368	1.3889
	50.00	1.2423	1.2423	1.0446	1.4489

Table 2: Relative MSE's for $k = 8$ and $n = 30$.

$\lambda_1 + \lambda_2$	λ_1	SR	SVSR	MMSE	AMMSE
.50	.00	.3138	.2738	.8217	.3421
	.05	.3455	.3024	.8238	.3546
	.10	.3772	.3310	.8258	.3672
	.15	.4088	.3596	.8278	.3797
	.20	.4405	.3882	.8298	.3922
	.25	.4721	.4168	.8318	.4047
	.30	.5038	.4454	.8338	.4172
	.35	.5354	.4740	.8358	.4298
	.40	.5671	.5026	.8378	.4423
	.45	.5988	.5312	.8398	.4548
	.50	.6304	.5598	.8418	.4673
3.0	.00	.3470	.3306	.8520	.3993
	.30	.4765	.4515	.8605	.4599
	.60	.6059	.5724	.8691	.5206
	.90	.7354	.6934	.8777	.5813
	1.20	.8648	.8143	.8862	.6420
	1.50	.9942	.9352	.8948	.7027
	1.80	1.1237	1.0561	.9034	.7633
	2.10	1.2531	1.1771	.9119	.8240
	2.40	1.3826	1.2980	.9205	.8847
	2.70	1.5120	1.4189	.9290	.9454
	3.00	1.6415	1.5398	.9376	1.0060
10.0	.00	.4922	.4908	.9020	.5211
	1.00	.6742	.6696	.9150	.6414
	2.00	.8562	.8483	.9281	.7617
	3.00	1.0382	1.0271	.9412	.8819
	4.00	1.2202	1.2058	.9543	1.0022
	5.00	1.4022	1.3846	.9674	1.1224
	6.00	1.5842	1.5633	.9804	1.2427
	7.00	1.7662	1.7421	.9935	1.3630
	8.00	1.9482	1.9208	1.0066	1.4832
	9.00	2.1302	2.0996	1.0197	1.6035
	10.00	2.3122	2.2783	1.0328	1.7237
50.0	.00	.8156	.8156	.9683	.7876
	5.00	.9022	.9022	.9752	.8903
	10.00	.9888	.9888	.9822	.9930
	15.00	1.0754	1.0754	.9891	1.0958
	20.00	1.1620	1.1620	.9960	1.1985
	25.00	1.2486	1.2486	1.0030	1.3012
	30.00	1.3352	1.3352	1.0099	1.4039
	35.00	1.4218	1.4218	1.0169	1.5067
	40.00	1.5084	1.5084	1.0238	1.6094
	45.00	1.5950	1.5950	1.0307	1.7121
	50.00	1.6816	1.6816	1.0377	1.8148