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(Citation)
Econometric Theory,18(5):1086-1098

(Issue Date)
2002-10

(Resource Type)
journal article

(Version)
Version of Record

(URL)
https://hdl.handle.net/20.500.14094/90000343

Econometric Theory, **18**, 2002, 1086–1098. Printed in the United States of America.
DOI: 10.1017.S0266466602185033

PMSE PERFORMANCE OF THE BIASED ESTIMATORS IN A LINEAR REGRESSION MODEL WHEN RELEVANT REGRESSORS ARE OMITTED

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In this paper, we consider a linear regression model when relevant regressors are omitted. We derive the explicit formulae for the predictive mean squared errors (PMSEs) of the Stein-rule (SR) estimator, the positive-part Stein-rule (PSR) estimator, the minimum mean squared error (MMSE) estimator, and the adjusted minimum mean squared error (AMMSE) estimator. It is shown analytically that the PSR estimator dominates the SR estimator in terms of PMSE even when there are omitted relevant regressors. Also, our numerical results show that the PSR estimator and the AMMSE estimator have much smaller PMSEs than the ordinary least squares estimator even when the relevant regressors are omitted.

1. INTRODUCTION

In the context of linear regression, the Stein-rule (SR) estimator proposed by Stein (1956) and James and Stein (1961) dominates the ordinary least squares (OLS) estimator in terms of predictive mean squared error (PMSE) if the model is specified correctly. Further, as is shown in Baranchik (1970), the SR estimator is further dominated by the positive-part Stein-rule (PSR) estimator when the specified model is correct.

As an improved estimator, Theil (1971) proposes the minimum mean squared error (MMSE) estimator. Because Theil's (1971) MMSE estimator includes unknown parameters, Farebrother (1975) suggests an operational variant of the MMSE estimator that is obtained by replacing the unknown parameters of the MMSE estimator by the OLS estimators. Hereafter, we use the term *MMSE estimator* to denote the operational variant of the MMSE estimator. Because the MMSE estimator satisfies Baranchik's (1970) condition, it dominates the OLS estimator if the number of regressors is larger than or equal to three and if

The author is grateful to Kazuhiro Ohtani for his helpful guidance and to Peter C.B. Phillips and anonymous referees for their useful comments and suggestions. Address correspondence to: Akio Namba, Graduate School of Economics, Kobe University, Rokko, Nada-ku, Kobe 657-8501, Japan; e-mail: namba@kobe-u.ac.jp.

  0266-4666/02 $9.50

the model is specified correctly. As an extension of the MMSE estimator, Ohtani (1996) considers the adjusted minimum mean squared error (AMMSE) estimator that is obtained by adjusting the degrees of the freedom of the component of the MMSE estimator.

These estimators are called biased estimators because they are not unbiased even if the model is specified correctly. However, in most of the practical situations, it is hard to determine which regressors should be included in the model. Thus, a researcher may exclude relevant regressors mistakenly. If relevant regressors are omitted, even the OLS estimator is not unbiased. In such situations, there may be a strong incentive to use the biased estimators if we consider that they are superior to the OLS estimator in some criterion. However, there is little research on the properties of biased estimators when relevant regressors are omitted in the specified model. Some exceptions are Mittelhammer (1984) and Ohtani (1993, 1998). Mittelhammer (1984) shows that the SR estimator no longer dominates the OLS estimator when relevant regressors are omitted. Ohtani (1993) derives the formulae for the PMSEs of the SR estimator and the PSR estimator in the misspecified model and derives a sufficient condition for the PSR estimator to dominate the SR estimator. Also, Ohtani (1998) derives the formulae for the PMSEs of the MMSE estimator and the AMMSE estimator. His numerical results show that the PMSE performance of the AMMSE estimator is better than that of the PSR estimator when the number of regressors included in the specified model is less than or equal to five and the model misspecification is severe. However, in Ohtani (1993, 1998), the formulae for the PMSEs of the SR, PSR, MMSE, and AMMSE estimators are not correct.

Thus, in this paper, we limit our attention to PMSE and compare the sampling performances of the OLS, SR, PSR, MMSE, and AMMSE estimators though there are several other criteria. (As to other criteria, see, e.g., Gourieroux and Monfort, 1995.) The plan of this paper is as follows. In Section 2 the model and the estimators are presented, and in Section 3 we derive the explicit formulae for the PMSEs of the SR, PSR, MMSE, and AMMSE estimators. It is shown analytically that the PSR estimator dominates the SR estimator in terms of PMSE even when relevant regressors are omitted. In Section 4, using the formulae derived in Section 3, we compare the PMSEs of the estimators by numerical evaluations.

2. THE MODEL AND THE ESTIMATORS

Consider a linear regression model,

$$y = X_1\beta_1 + X_2\beta_2 + \epsilon, \qquad \epsilon \sim N(0, \sigma^2 I_n), \tag{1}$$

where y is an $n \times 1$ vector of observations on a dependent variable, X_1 and X_2 are $n \times k_1$ and $n \times k_2$ matrices of observations on nonstochastic independent variables, β_1 and β_2 are $k_1 \times 1$ and $k_2 \times 1$ vectors of regression coefficients, and ϵ is an $n \times 1$ vector of normal error terms. We assume that X_1 and $[X_1, X_2]$ are of full column rank.

Suppose that the matrix of regressors X_2 is omitted mistakenly and the model is specified as

$$y = X_1\beta_1 + \eta, \qquad \eta = X_2\beta_2 + \epsilon. \tag{2}$$

Then, based on the misspecified model, the ordinary least squares (OLS) estimator of β_1 is

$$b_1 = S_{11}^{-1}X_1'y, \tag{3}$$

where $S_{11} = X_1'X_1$.

Also, the SR, PSR, MMSE, and AMMSE estimators based on the misspecified model are, respectively,

$$b_{SR1} = \left(1 - \frac{ae_1'e_1}{b_1'S_{11}b_1}\right)b_1, \tag{4}$$

$$b_{PS1} = \max\left[0, 1 - \frac{ae_1'e_1}{b_1'S_{11}b_1}\right]b_1, \tag{5}$$

$$b_{M1} = \left(\frac{b_1'S_{11}b_1}{b_1'S_{11}b_1 + e_1'e_1/\nu_1}\right)b_1, \tag{6}$$

$$b_{AM1} = \left(\frac{b_1'S_{11}b_1/k_1}{b_1'S_{11}b_1/k_1 + e_1'e_1/\nu_1}\right)b_1, \tag{7}$$

where $e_1 = y - X_1b_1$, $\nu_1 = n - k_1$, and a is a constant such that $0 \le a \le 2(k_1 - 2)/(\nu_1 + 2)$. If the model is specified correctly, the PMSE of the SR estimator is minimized when $a = (k_1 - 2)/(\nu_1 + 2)$. Thus, we use this value of a hereafter.

3. PMSE OF THE ESTIMATORS

To derive the explicit formulae for the PMSEs of the SR, PSR, MMSE, and AMMSE estimators, we consider the general pretest estimator defined as

$$\hat{\beta}_1 = I(F \ge c)\left(1 + \alpha\frac{e_1'e_1}{b_1'S_{11}b_1}\right)^r b_1, \tag{8}$$

where $I(A)$ is an indicator function such that $I(A) = 1$ if an event A occurs and $I(A) = 0$ otherwise, $F = (b_1'S_{11}b_1/k_1)/(e_1'e_1/\nu_1)$ is the test statistic for the null hypothesis $H_0\colon \beta_1 = 0$ against the alternative $H_1\colon \beta_1 \ne 0$ based on the misspecified model, c is the critical value of the pretest, and r is an arbitrary integer. The term $\hat{\beta}_1$ reduces to the SR estimator when $c = 0$, $\alpha = -a$, and $r = 1$, and it reduces to the PSR estimator when $c = a\nu_1/k_1$, $\alpha = -a$, and $r = 1$. Furthermore, $\hat{\beta}_1$ reduces to the MMSE estimator when $c = 0$, $\alpha = 1/\nu_1$, and $r = -1$, and it reduces to the AMMSE estimator when $c = 0$, $\alpha = k_1/\nu_1$, and $r = -1$, respectively.

The PMSE of $\hat{\beta}_1$ is defined as

$$\begin{aligned} PMSE[\hat{\beta}_1] &= E[(X_1\hat{\beta}_1 - X\beta)'(X_1\hat{\beta}_1 - X\beta)] \\ &= E[\hat{\beta}_1' S_{11}\hat{\beta}_1] - 2E[\beta' X' X_1 \hat{\beta}_1] + \beta' S \beta, \end{aligned} \tag{9}$$

where $X = [X_1, X_2]$, $\beta' = [\beta_1', \beta_2']$, and $S = X'X$. The meaning and some discussions of the PMSE are given in Mittelhammer (1984) and Ohtani (1993).

Denoting $u_1 = b_1' S_{11} b_1/\sigma^2$ and $u_2 = e_1' e_1/\sigma^2$, (9) reduces to

$$\begin{aligned} PMSE[\hat{\beta}_1] = \sigma^2 E\left[I\left(\frac{\nu_1}{k_1}\frac{u_1}{u_2} \geq c\right)\left(\frac{u_1 + \alpha u_2}{u_1}\right)^{2r} u_1\right] \\ - 2E\left[I\left(\frac{\nu_1}{k_1}\frac{u_1}{u_2} \geq c\right)\left(\frac{u_1 + \alpha u_2}{u_1}\right)^{r} \beta' X' X_1 b_1\right] + \beta' S \beta. \end{aligned} \tag{10}$$

We define the functions $H(p,q;\alpha,c)$ and $J(p,q;\alpha,c)$ as

$$H(p,q;\alpha,c) = E\left[I\left(\frac{\nu_1}{k_1}\frac{u_1}{u_2} \geq c\right)\left(\frac{u_1 + \alpha u_2}{u_1}\right)^{p} u_1^q\right], \quad , \tag{11}$$

$$J(p,q,;\alpha,c) = E\left[I\left(\frac{\nu_1}{k_1}\frac{u_1}{u_2} \geq c\right)\left(\frac{u_1 + \alpha u_2}{u_1}\right)^{p} u_1^q(\beta' X' X_1 b_1/\sigma^2)\right], \tag{12}$$

where p and q are arbitrary integers. Then, we obtain

$$PMSE[\hat{\beta}_1]/\sigma^2 = H(2r,1;\alpha,c) - 2J(r,0;\alpha,c) + \lambda_1 + \lambda_2, \tag{13}$$

where $\lambda_1 = \beta' X' X_1 S_{11}^{-1} X_1' X\beta/\sigma^2$, $\lambda_2 = \beta' X' M_1 X\beta/\sigma^2$, $M_1 = I_n - X_1 S_{11}^{-1} X_1'$, and $\lambda_1 + \lambda_2 = \beta' S\beta/\sigma^2$.

As is shown in the Appendix, the explicit formulae of $H(p,q;\alpha,c)$ and $J(p,q;\alpha,c)$ are

$$H(p,q;\alpha,c) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1) w_j(\lambda_2) G_{ij}(p,q;\alpha,c), \tag{14}$$

$$J(p,q;\alpha,c) = \lambda_1 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1) w_j(\lambda_2) G_{i+1,j}(p,q;\alpha,c), \tag{15}$$

where

$$\begin{aligned} G_{ij}(p,q;\alpha,c) = \frac{2^q \Gamma((k_1 + \nu_1)/2 + q + i + j)}{\Gamma(k_1/2 + i)\Gamma(\nu_1/2 + j)} \\ \times \int_{c^*}^{1} t^{k_1/2 - p + q + i - 1}(1-t)^{\nu_1/2 + j - 1}[\alpha + (1-\alpha)t]^p \, dt, \end{aligned} \tag{16}$$

$w_i(\lambda) = \exp(-\lambda/2)(\lambda/2)^i/i!$, and $c^* = k_1 c/(\nu_1 + k_1 c)$.

The formula for $J(p,q;\alpha,c)$ is substantially different from the formula derived in Theorem 1 of Ohtani (1993). In Ohtani (1993), there are several unnecessary terms in the formula for the PMSEs of the SR estimator and the PSR estimator.

When $c^* \in (0,1)$, the integral in (16) can be written as

$$\begin{aligned}
&\int_{c^*}^{1} t^{k_1/2-p+q+i-1}(1-t)^{\nu_1/2+j-1}[\alpha+(1-\alpha)t]^p\,dt \\
&\quad = \sum_{l=0}^{\infty}\binom{k_1/2-p+q+i-1}{l}(-1)^l \\
&\qquad \times \int_{c^*}^{1}(1-t)^{\nu_1/2+l+j-1}[\alpha+(1-\alpha)t]^p\,dt \\
&\quad = \sum_{l=0}^{\infty}\binom{k_1/2-p+q+i-1}{l}(-1)^l\,\frac{(1-c^*)^{\nu_1/2+j+l}}{\nu_1/2+j+l} \\
&\qquad \times {}_2F_1(-p,\nu_1/2+j+l;\nu_1/2+j+l+1;(1-\alpha)(1-c^*)),
\end{aligned} \tag{17}$$

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function. (For the definition and properties of the hypergeometric function, see, e.g., Luke, 1969; Abramowitz and Stegun, 1972; Abadir, 1999.) In the case of the MMSE and AMMSE estimators, the hypergeometric function in (17) converges because $|(1-\alpha)\times(1-c^*)|<1$. In the case of the SR and PSR estimators, it is a finite series with $p+1$ terms because p is a positive integer. Thus, (17) converges for all the estimators considered in this paper. Also, when $|(1-\alpha)(1-c^*)|>1$, analytic continuation formulae of the hypergeometric function make (17) convergent. (The author is deeply grateful to one of the referees for suggesting that (17) be derived and instructing the properties of the hypergeometric function.)

Let $\alpha=-a$ and $r=1$. We have

$$\begin{aligned}
PMSE(\hat\beta_1)/\sigma^2 &= H(2,1;-a,c)-2J(1,0;-a,c)+\lambda_1+\lambda_2 \\
&= 2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}w_i(\lambda_1)w_j(\lambda_2)\,\frac{\Gamma((k_1+\nu_1)/2+i+j+1)}{\Gamma(k_1/2+i)\Gamma(\nu_1/2+j)} \\
&\quad \times \int_{c^*}^{1} t^{k_1/2+i-2}(1-t)^{\nu_1/2+j-1}[-a+(1+a)t]^2\,dt \\
&\quad - 2\lambda_1\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}w_i(\lambda_1)w_j(\lambda_2)\,\frac{\Gamma((k_1+\nu_1)/2+i+j+1)}{\Gamma(k_1/2+i+1)\Gamma(\nu_1+j)} \\
&\quad \times \int_{c^*}^{1} t^{k_1/2+i-1}(1-t)^{\nu_1/2+j-1}[-a+(1+a)t]\,dt+\lambda_1+\lambda_2.
\end{aligned} \tag{18}$$

Differentiating (18) with respect to c and performing some manipulations, we obtain

$$\begin{aligned}\frac{\partial PMSE(\hat{\beta}_1)/\sigma^2}{\partial c} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)\,\frac{2\Gamma((k_1+\nu_1)/2+i+j+1)}{\Gamma(k_1/2+i)\Gamma(\nu_1/2+j)} \\ &\quad\times \frac{k_1^{k_1/2+i-1}\nu_1^{\nu_1/2+j}c^{k_1/2+i-2}}{(\nu_1+k_1c)^{(\nu_1+k_1)/2+i+j-1}} \\ &\quad\times\left[-a+(1+a)\,\frac{k_1c}{\nu_1+k_1c}\right] \\ &\quad\times\left\{-\left[-a+(1+a)\,\frac{k_1c}{\nu_1+k_1c}\right]+\lambda_1\,\frac{1}{k_1/2+i}\,\frac{k_1c}{\nu_1+k_1c}\right\}. \end{aligned} \tag{19}$$

From (19), when $\alpha = -a$ and $r = 1$, a condition for $PMSE[\hat{\beta}_1]$ to be monotonically decreasing is

$$-a+(1+a)\,\frac{k_1c}{\nu_1+k_1c} \leq 0. \tag{20}$$

Thus, $PMSE[\hat{\beta}_1]$ is monotonically decreasing on $c \in [0, a\nu_1/k_1]$ if $\alpha = -a$ and $r = 1$. Because $\hat{\beta}_1$ reduces to the SR estimator when $\alpha = -a$, $r = 1$, and $c = 0$, and it reduces to the PSR estimator when $\alpha = -a$, $r = 1$, and $c = a\nu_1/k_1$, we obtain the following theorem.

THEOREM 1. *The* PSR *estimator dominates the* SR *estimator in terms of* PMSE *even when relevant regressors are omitted in the specified model.*

Theorem 2 in Ohtani (1993) requires the condition $\lambda_1 \geq \lambda_2$ for the PMSE of the PSR estimator to be smaller than that of the SR estimator. This condition is caused by the fact that his formula of PMSE is mistakenly derived. However, our result shows that such a condition is not required when the formula is derived correctly.

Because further theoretical analysis of the PMSEs of the SR, PSR, MMSE, and AMMSE estimators is difficult, we compare them numerically in the next section.

4. NUMERICAL ANALYSIS

In this section, we compare the PMSE performances of the SR, PSR, MMSE, and AMMSE estimators by numerical evaluations. As is shown in Ohtani (1993), the noncentrality parameters are expressed as $\lambda_1 = F_\beta R_1^2$ and $\lambda_2 = F_\beta(1-R_1^2)$, where $F_\beta = \beta'S\beta/\sigma^2$ and $R_1^2 = \beta'X'X_1S_{11}^{-1}X_1'X\beta/\beta'S\beta$. Here F_β is the noncentrality parameter that appeared in the test for the null hypothesis that all the regression coefficients are zeros in the correctly specified model. Also, R_1^2 is

Table 1. Relative PMSEs of the SR, PSR, MMSE, and AMMSE estimators for $k_1 = 5$ and $n = 20$

	R_1^2					
F_β	0.1	0.3	0.5	0.7	0.9	1.0
			SR			
.0	.4706	.4706	.4706	.4706	.4706	.4706
1.0	.5616	.5632	.5640	.5641	.5635	.5629
2.0	.6306	.6349	.6372	.6374	.6352	.6332
4.0	.7299	.7381	.7421	.7415	.7355	.7300
6.0	.7995	.8080	.8116	.8093	.7997	.7909
8.0	.8519	.8579	.8594	.8552	.8427	.8314
10.0	.8935	.8948	.8935	.8873	.8727	.8596
15.0	.9692	.9538	.9447	.9347	.9176	.9018
20.0	1.0216	.9873	.9715	.9592	.9416	.9248
25.0	1.0605	1.0079	.9872	.9734	.9562	.9391
30.0	1.0908	1.0215	.9971	.9825	.9659	.9489
40.0	1.1344	1.0377	1.0086	.9930	.9777	.9613
50.0	1.1640	1.0467	1.0147	.9987	.9844	.9689
100.0	1.2274	1.0621	1.0249	1.0081	.9965	.9843
150.0	1.2471	1.0662	1.0275	1.0104	.9997	.9895
			PSR			
.0	.3533	.3533	.3533	.3533	.3533	.3533
1.0	.4457	.4522	.4579	.4628	.4666	.4682
2.0	.5148	.5294	.5414	.5506	.5566	.5584
4.0	.6121	.6419	.6643	.6792	.6858	.6856
6.0	.6783	.7197	.7486	.7653	.7693	.7658
8.0	.7267	.7764	.8084	.8244	.8245	.8175
10.0	.7640	.8196	.8522	.8659	.8621	.8521
15.0	.8292	.8920	.9204	.9262	.9149	.9003
20.0	.8720	.9363	.9571	.9558	.9410	.9245
25.0	.9028	.9658	.9785	.9721	.9561	.9391
30.0	.9262	.9866	.9919	.9820	.9659	.9489
40.0	.9598	1.0133	1.0066	.9929	.9777	.9613
50.0	.9829	1.0294	1.0139	.9987	.9844	.9689
100.0	1.0390	1.0582	1.0249	1.0081	.9965	.9843
150.0	1.0618	1.0651	1.0275	1.0104	.9997	.9895
			MMSE			
.0	.7254	.7254	.7254	.7254	.7254	.7254
1.0	.7614	.7631	.7644	.7654	.7660	.7661
2.0	.7883	.7925	.7957	.7977	.7985	.7984
4.0	.8260	.8354	.8421	.8460	.8466	.8456
6.0	.8514	.8651	.8743	.8792	.8794	.8772
8.0	.8699	.8868	.8976	.9029	.9025	.8993
10.0	.8842	.9033	.9149	.9203	.9192	.9152
15.0	.9089	.9310	.9427	.9475	.9454	.9400
20.0	.9251	.9479	.9586	.9625	.9599	.9539
25.0	.9367	.9591	.9685	.9716	.9690	.9626
30.0	.9455	.9671	.9752	.9777	.9750	.9686
40.0	.9582	.9774	.9834	.9849	.9825	.9762
50.0	.9669	.9838	.9882	.9891	.9869	.9809
100.0	.9880	.9963	.9969	.9964	.9949	.9904
150.0	.9963	1.0003	.9995	.9985	.9972	.9935

continued

Table 1. *continued*

F_β	R_1^2 0.1	0.3	0.5	0.7	0.9	1.0
			AMMSE			
.0	.3357	.3357	.3357	.3357	.3357	.3357
1.0	.4328	.4316	.4295	.4267	.4229	.4207
2.0	.5050	.5064	.5056	.5023	.4963	.4921
4.0	.6057	.6154	.6202	.6193	.6110	.6033
6.0	.6731	.6910	.7016	.7037	.6946	.6841
8.0	.7216	.7463	.7615	.7661	.7566	.7439
10.0	.7585	.7885	.8070	.8133	.8036	.7891
15.0	.8215	.8600	.8824	.8902	.8797	.8620
20.0	.8616	.9044	.9272	.9342	.9228	.9032
25.0	.8897	.9346	.9562	.9614	.9490	.9284
30.0	.9107	.9563	.9761	.9793	.9659	.9448
40.0	.9399	.9853	1.0011	1.0003	.9855	.9641
50.0	.9595	1.0037	1.0157	1.0117	.9957	.9745
100.0	1.0049	1.0424	1.0426	1.0295	1.0103	.9913
150.0	1.0224	1.0556	1.0503	1.0333	1.0127	.9953

interpreted as the coefficient of determination in regression of $X\beta$ on X_1. Thus, if R_1^2 is close to unity, the magnitude of model misspecification is regarded as small and vice versa.

The parameter values used in the numerical evaluations were $k_1 = 3, 4, 5, 8$, $n = 20, 30, 40$, $R_1^2 = 0.1, 0.3, 0.5, 0.7, 0.9, 1.0$, and $F_\beta =$ various values. To compare the PMSEs of the estimators, we evaluated the values of relative PMSE defined as $PMSE[\bar{\beta}_1]/PMSE[b_1]$, where $\bar{\beta}_1$ is any estimator of β_1. Thus, the estimator $\bar{\beta}_1$ has smaller PMSE than the OLS estimator when the value of relative PMSE is smaller than unity. The numerical evaluations were executed on a personal computer using the FORTRAN code. In evaluating the integral in $G_{ij}(p, q; \alpha, c)$ given in (16), we used Simpson's rule with 200 equal subdivisions. The double infinite series in $H(p, q; \alpha, c)$ and $J(p, q; \alpha, c)$ were judged to converge when the increment of series became smaller than 10^{-12}. Because the results for the cases of $k_1 = 5, 8$, and $n = 20$ are qualitatively typical, we do not show the results for the other cases.

Table 1 shows the relative PMSEs of the SR, PSR, MMSE, and AMMSE estimators for $k_1 = 5$ and $n = 20$. From Table 1, we can make sure of Theorem 1; that is, the PMSE of the PSR estimator is uniformly smaller than or equal to that of the SR estimator even if relevant regressors are omitted. Also, we can see that the AMMSE estimator has the smallest PMSE over a wide region of the parameter space (e.g., $F_\beta \leq 25.0$). In particular, the PMSE of the AMMSE estimator is smaller than the other estimators when the model misspecification is severe (e.g., $R_1^2 = 0.1$ and $F_\beta \leq 50.0$). Though the maximum of the relative PMSE of the AMMSE estimator is slightly larger than unity, the minimum is much smaller than unity. This indicates that the gain in PMSE

TABLE 2. Relative PMSEs of the SR, PSR, MMSE, and AMMSE estimators for $k_1 = 8$ and $n = 20$

	R_1^2					
F_β	0.1	0.3	0.5	0.7	0.9	1.0
			SR			
.0	.3571	.3571	.3571	.3571	.3571	.3571
1.0	.4321	.4320	.4318	.4313	.4306	.4301
2.0	.4980	.4975	.4964	.4946	.4920	.4904
4.0	.6105	.6067	.6020	.5960	.5880	.5831
6.0	.7053	.6944	.6839	.6724	.6584	.6500
8.0	.7877	.7663	.7485	.7312	.7116	.6998
10.0	.8611	.8265	.8005	.7774	.7526	.7379
15.0	1.0170	.9408	.8931	.8571	.8225	.8023
20.0	1.1456	1.0212	.9529	.9066	.8657	.8419
25.0	1.2553	1.0804	.9938	.9396	.8946	.8684
30.0	1.3506	1.1255	1.0233	.9630	.9152	.8874
40.0	1.5089	1.1891	1.0624	.9932	.9422	.9127
50.0	1.6355	1.2314	1.0868	1.0116	.9589	.9288
100.0	2.0101	1.3252	1.1364	1.0477	.9926	.9629
150.0	2.1888	1.3589	1.1527	1.0590	1.0033	.9750
			PSR			
.0	.2606	.2606	.2606	.2606	.2606	.2606
1.0	.3218	.3279	.3336	.3389	.3437	.3460
2.0	.3739	.3864	.3976	.4072	.4153	.4188
4.0	.4585	.4834	.5040	.5198	.5304	.5335
6.0	.5251	.5609	.5888	.6076	.6165	.6171
8.0	.5792	.6245	.6576	.6769	.6818	.6786
10.0	.6244	.6778	.7144	.7322	.7318	.7247
15.0	.7106	.7801	.8198	.8293	.8144	.7984
20.0	.7722	.8539	.8917	.8898	.8626	.8408
25.0	.8184	.9099	.9432	.9295	.8935	.8682
30.0	.8542	.9542	.9815	.9568	.9147	.8874
40.0	.9057	1.0200	1.0337	.9909	.9421	.9127
50.0	.9407	1.0670	1.0669	1.0108	.9589	.9288
100.0	1.0200	1.1872	1.1326	1.0477	.9926	.9629
150.0	1.0489	1.2400	1.1518	1.0590	1.0033	.9750
			MMSE			
.0	.8030	.8030	.8030	.8030	.8030	.8030
1.0	.8145	.8165	.8184	.8203	.8220	.8228
2.0	.8242	.8283	.8320	.8353	.8382	.8395
4.0	.8400	.8479	.8546	.8600	.8643	.8660
6.0	.8525	.8636	.8725	.8793	.8841	.8857
8.0	.8627	.8766	.8870	.8945	.8994	.9008
10.0	.8714	.8875	.8990	.9069	.9116	.9126
15.0	.8885	.9083	.9211	.9290	.9329	.9331
20.0	.9014	.9232	.9361	.9435	.9465	.9460
25.0	.9117	.9345	.9468	.9535	.9559	.9549
30.0	.9202	.9432	.9549	.9608	.9626	.9612
40.0	.9334	.9559	.9659	.9706	.9716	.9698
50.0	.9435	.9647	.9731	.9768	.9774	.9753
100.0	.9713	.9855	.9889	.9898	.9893	.9871
150.0	.9844	.9935	.9944	.9942	.9934	.9912

continued

TABLE 2. *continued*

	R_1^2					
F_β	0.1	0.3	0.5	0.7	0.9	1.0
			AMMSE			
.0	.3201	.3201	.3201	.3201	.3201	.3201
1.0	.3783	.3783	.3779	.3772	.3762	.3755
2.0	.4271	.4284	.4287	.4279	.4260	.4245
4.0	.5047	.5107	.5139	.5139	.5103	.5068
6.0	.5642	.5756	.5822	.5835	.5782	.5727
8.0	.6116	.6282	.6382	.6406	.6338	.6259
10.0	.6504	.6718	.6847	.6882	.6797	.6696
15.0	.7229	.7542	.7725	.7772	.7648	.7493
20.0	.7737	.8123	.8339	.8384	.8222	.8021
25.0	.8115	.8556	.8789	.8825	.8628	.8388
30.0	.8409	.8892	.9133	.9154	.8926	.8653
40.0	.8837	.9380	.9622	.9609	.9327	.9005
50.0	.9135	.9717	.9951	.9904	.9578	.9223
100.0	.9855	1.0526	1.0705	1.0534	1.0077	.9650
150.0	1.0143	1.0847	1.0988	1.0747	1.0226	.9781

from using the AMMSE estimator instead of the OLS estimator is much larger than the loss even when there are omitted regressors. Also, the loss from using the MMSE estimator instead of the OLS estimator is very small, though the gain is not so large.

The relative PMSEs of the SR, PSR, MMSE, and AMMSE estimators for $k_1 = 8$ and $n = 20$ are shown in Table 2. Similar to the case of $k_1 = 5$, we can make sure of Theorem 1. When $k_1 = 8$, the PSR estimator has the smallest PMSE over a wide region of the parameter space. Also, the AMMSE estimator has much smaller PMSE than the OLS estimator over a wide region of the parameter space. Though the maximums of the relative PMSEs of the PSR and the AMMSE estimators are slightly larger than unity, their minimums are much smaller than unity. Also, the MMSE estimator has smaller PMSE than the OLS estimator for all the values of F_β and R_1^2 considered here, though the minimum of its PMSE is not so small.

The results stated previously are almost similar to the numerical results in Ohtani (1998). In other words, the effect of his calculative error on the numerical results is not so large.

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APPENDIX

In this Appendix, we derive the formulae for $H(p,q;\alpha,c)$ and $J(p,q;\alpha,c)$. First, we derive the formula for $H(p,q;\alpha,c)$. The OLS estimator b_1 is distributed as $N(\beta_1 + S_{11}^{-1}S_{12}\beta_2, \sigma^2 S_{11}^{-1}) = N(S_{11}^{-1}X_1'X\beta, \sigma^2 S_{11}^{-1})$, where $X = [X_1, X_2]$, $\beta' = [\beta_1', \beta_2']$, and $S_{12} = X_1'X_2$. Thus, $u_1 = b_1'S_{11}b_1/\sigma^2$ is distributed as the noncentral chi-square distribution with k_1 degrees of freedom and noncentrality parameter $\lambda_1 = \beta'X'X_1S_{11}^{-1}X_1'X\beta/\sigma^2$. Also, the residual vector e_1 is distributed as $N(M_1X\beta, \sigma^2 M_1)$, where $M_1 = (I_n - X_1S_{11}^{-1}X_1')$. Thus, $u_2 = e_1'e_1/\sigma^2$ is distributed as the noncentral chi-square distribution with ν_1 degrees of freedom and noncentrality parameter $\lambda_2 = \beta'X'M_1X\beta/\sigma^2$. Furthermore, u_1 and u_2 are mutually independent.

Using u_1 and u_2, $H(p,q;\alpha,c)$ is expressed as

$$H(p,q;\alpha,c) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} \iint_R u_1^{k_1/2-p+q+i-1} u_2^{\nu_1/2+j-1}(u_1+\alpha u_2)^p \times \exp[-(u_1+u_2)/2]\, du_2\, du_1, \tag{A.1}$$

where

$$K_{ij} = \frac{w_i(\lambda_1)w_j(\lambda_2)}{2^{(k_1+\nu_1)/2+i+j}\Gamma(k_1/2+i)\Gamma(\nu_1/2+j)}, \tag{A.2}$$

$w_i(\lambda) = \exp(-\lambda/2)(\lambda/2)^i/i!$, and R is the region such that $\{u_1, u_2 | u_1/u_2 \geq k_1c/\nu_1\}$.

Making use of the change of variables, $v_1 = u_1/u_2$ and $v_2 = u_2$, (A.1) reduces to

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} \int_{k_1 c/\nu_1}^{\infty}\int_0^{\infty} v_1^{k_1/2-p+q+i-1} v_2^{(k_1+\nu_1)/2+q+i+j-1} (\alpha + v_1)^p \times \exp[-v_2(1+v_1)/2]\, dv_2\, dv_1. \qquad \textbf{(A.3)}$$

Again, making use of the change of variable, $z = (1+v_1)v_2/2$, (A.3) reduces to

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} K_{ij} 2^{(k_1+\nu_1)/2+q+i+j} \Gamma((k_1+\nu_1)/2+q+i+j) \times \int_{k_1 c/\nu_1}^{\infty} \frac{v_1^{k_1/2-p+q+i-1}(\alpha+v_1)^p}{(1+v_1)^{(k_1+\nu_1)/2+q+i+j}}\, dv_1. \qquad \textbf{(A.4)}$$

Finally, making use of the change of variable, $t = v_1/(1+v_1)$, and performing some manipulations, we obtain (14) in the text.

Next, we derive the formula for $J(p,q;\alpha,c)$. Noting that $\partial\lambda_1/\partial\beta_1 = 2(S_{11}\beta_1 + S_{12}\beta_2)/\sigma^2$ and $\partial\lambda_2/\partial\beta_1 = 0$, and differentiating $H(p,q;\alpha,c)$ given in (14) with respect to β_1, we obtain

$$\begin{aligned}\frac{\partial H(p,q;\alpha,c)}{\partial\beta_1} &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{\partial w_i(\lambda_1)}{\partial\beta_1}\right] w_j(\lambda_2) G_{ij}(p,q;\alpha,c)\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{S_{11}\beta_1 + S_{12}\beta_2}{\sigma^2}\right][-w_i(\lambda_1) + w_{i-1}(\lambda_1)] w_j(\lambda_2) G_{ij}(p,q;\alpha,c)\\ &= -\left[\frac{S_{11}\beta_1 + S_{12}\beta_2}{\sigma^2}\right] H(p,q;\alpha,c)\\ &\quad + \left[\frac{S_{11}\beta_1 + S_{12}\beta_2}{\sigma^2}\right]\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i(\lambda_1) w_j(\lambda_2) G_{i+1,j}(p,q;\alpha,c). \qquad \textbf{(A.5)}\end{aligned}$$

Because $u_1 = b_1' X_{11} b_1/\sigma^2$ and $b_1 \sim N(S_{11}^{-1}X_1'X\beta, \sigma^2 S_{11}^{-1})$, $H(p,q;\alpha.c)$ can be expressed as

$$H(p,q;a,c) = \iint_R \left(\frac{u_1 + \alpha u_2}{u_1}\right)^p u_1^q f_N(b_1) f_2(u_2)\, du_2\, db_1, \qquad \textbf{(A.6)}$$

where $f_2(u_2)$ is the density function of u_2 and

$$f_N(b_1) = \frac{1}{(2\pi)^{k_1/2}|\sigma^2 S_{11}^{-1}|^{1/2}} \exp\left[-\frac{(b_1 - S_{11}^{-1}X_1'X\beta)' S_{11}(b_1 - S_{11}^{-1}X_1'X\beta)}{2\sigma^2}\right]. \qquad \textbf{(A.7)}$$

Differentiating (A.6) with respect to β_1, we obtain

$$\begin{aligned}\frac{\partial H(p,q;\alpha,c)}{\partial \beta_1} &= \iint_R \left(\frac{u_1+\alpha u_2}{u_1}\right)^p u_1^q f_N(b_1) f_2(u_2)\left[\frac{S_{11}b_1 - S_{11}\beta_1 - S_{12}\beta_2}{\sigma^2}\right] du_2\, db_1 \\ &= \frac{1}{\sigma^2} E\left[I\left(\frac{\nu_1}{k_1}\frac{u_1}{u_2} \geq c\right)\left(\frac{u_1+\alpha u_2}{u_1}\right)^p u_1^q S_{11} b_1\right] \\ &\quad - \left[\frac{S_{11}\beta_1 + S_{12}\beta_2}{\sigma^2}\right] H(p,q;\alpha,c). \qquad \textbf{(A.8)}\end{aligned}$$

Equating (A.5) and (A.8), and multiplying $\beta' X' X_1 S_{11}^{-1}$ from the left, we obtain (15) in the text.