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Large gauge hierarchy in gauge-Higgs unification

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We study a five dimensional $SU(3)$ nonsupersymmetric gauge theory compactified on $M^4 \times S^1/Z_2$ and discuss the gauge hierarchy in the scenario of the gauge-Higgs unification. Making use of the calculability of the Higgs potential and a curious feature that coefficients in the potential are given by discrete values, we find two models, in which the large gauge hierarchy is realized, that is, the weak scale is naturally obtained from a unique large scale such as a grand unified theory scale or the Planck scale. The size of the Higgs mass is also discussed in each model. One of the models we find realizes both large gauge hierarchy and consistent Higgs mass, and shows that the Higgs mass becomes heavier as the compactified scale becomes smaller.

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I. INTRODUCTION

The standard model has had great success in the last three decades, and at the moment, there is no discrepancy of the prediction from precision measurements. The standard model, however, has potential shortcomings, one of which is the stability of the Higgs sector against radiative corrections. Namely, the Higgs mass is sensitive to ultraviolet effects because of quadratic dependence on the cutoff. Two tremendously separated energy scales cannot be stabilized without fine-tuning of parameters, and there is the gauge hierarchy problem in the standard model.

The problem entirely comes from lack of symmetry to control the Higgs sector. Any attempts to overcome the problem always lead us to physics beyond the standard model. Supersymmetry (SUSY) has been considered as one of the promising candidates. It controls the Higgs sector to suppress the ultraviolet effect on the Higgs mass. The gauge hierarchy problem is (technically) solved by introducing SUSY. However, SUSY does not possess the mechanism to yield the large gauge hierarchy naturally at tree level. The solution to the gauge hierarchy problem requires the mechanism to generate the large gauge hierarchy at tree level, and at the same time, it must be stable against radiative corrections. SUSY guarantees only the stability against radiative corrections. Moreover, the superpartners, which are the prediction of supersymmetry, have not yet been discovered. Hence, it is important to seek a different approach to the gauge hierarchy problem.

Recently, higher dimensional gauge symmetry has received much attention as a new approach to the gauge hierarchy problem without resorting to supersymmetry. The higher dimensional gauge symmetry plays the role of controlling the Higgs sector. In particular, the gauge-Higgs unification, originally proposed by Manton [1] and Fairlie [2], is a very attractive idea of physics beyond the standard model [3]. Four dimensional gauge and Higgs fields are unified into a higher dimensional gauge field, and the theory is completely controlled by the higher dimensional gauge symmetry. Hence, the mass term for the Higgs field is forbidden by the gauge invariance. The gauge-Higgs unification has been studied extensively from various points of view [4,5].

In the gauge-Higgs unification, the Higgs field corresponds to the Wilson line phase, which is a nonlocal quantity. The Higgs potential is generated at quantum level after the compactification and, reflecting the nonlocality of the Higgs field, the potential never suffers from the ultraviolet effect [6]. As a result, the Higgs mass calculated from the potential is finite as well. In other words, the Higgs mass and the potential are calculable. This is a very remarkable fact, which rarely happens in the usual quantum field theory. The specific feature is entirely due to shift symmetry manifest through the Wilson line phase (Higgs field), which is a remnant of the higher dimensional gauge symmetry. Thanks to the finite Higgs mass, two tremendously separated energy scales can be stable in the gauge-Higgs unification.

The compactification scale is set by the magnitude of the vacuum expectation values (VEV) of the Wilson line phase in the gauge-Higgs unification when the extra dimension is flat.[1] In the usual scenario, the scale is at around a few TeV.[2] In this paper, we discuss whether or not the compactification scale can be an enormously large scale such as a grand unified theory (GUT) scale or the Planck scale.[3] In discussing the gauge hierarchy, we make use of a curious feature of the Higgs potential in the gauge-Higgs unifica-

[1]The gauge-Higgs unification with a warped extra dimension has been studied in [7].

[2]This is the reason that the gauge-Higgs unification can be used to solve the little hierarchy problem.

[3]A different attempt to solve the large gauge hierarchy problem has been made in the Higgsless gauge symmetry breaking scenario [8].

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tion. The dynamics of the potential is mostly governed by massless bulk matter introduced into the theory. For small VEV of the Higgs field, in which we are really interested, the Higgs potential is approximated in terms of the logarithm and polynomials with their coefficients being a discrete value given by the flavor number. This curious feature is hardly observed in the usual quantum field theory, in which the coefficients are continuous, scale-dependent parameters.

The point for obtaining the large gauge hierarchy is that the coefficient for the mass term can be set to zero. We introduce massless bulk matter satisfying not only the periodic boundary condition for the S^1 direction but also the antiperiodic one. The massless bulk field satisfying the antiperiodic boundary condition has an opposite sign for the coefficient of the mass term from that satisfying the periodic one. This is why the coefficient can be set to zero even though we do not introduce supersymmetry. We also notice that this is not the fine-tuning of the parameter, but just a choice of the flavor set because the coefficient is the discrete value given by the flavor number.

We consider a five dimensional $SU(3)$ gauge theory, where one of the spatial coordinates is compactified on an orbifold S^1/Z_2. We find two models, in which the large gauge hierarchy is realized. In model I, the form of the Higgs potential is reduced to the massless scalar field theory in the Coleman-Weinberg paper [9]. The large gauge hierarchy is interpreted as the magnitude of the ratio between the logarithmic and quartic terms in the Higgs potential. The flavor number of the massless bulk matter directly affects the size of the gauge hierarchy in model I.

In model II, we introduce massive bulk fermions in addition to the massless bulk matter considered in model I. The massive bulk matter contributes to the Higgs potential in a manner similar to the Boltzmann-like suppression factor, which is similar to finite temperature field theory [10]. Under the condition that the contribution from the massless bulk matter to the mass term vanishes, the mass term is controlled only by the term generated from the massive bulk fermion with the Boltzmann suppression factor to yield the exponentially small VEV. Then one can have the large gauge hierarchy. The flavor number of the massless bulk matter has no effect on the size of the gauge hierarchy.

We also study the Higgs mass in each model. In model I, because of the fact that the Higgs potential has the same form as the massless scalar field theory studied by Coleman and Weinberg, the Higgs mass is inevitably smaller than massive gauge bosons [9]. The heavier Higgs mass tends to decrease the size of the gauge hierarchy. The large gauge hierarchy and the heavy Higgs mass are not compatible in model I. On the other hand, in model II, the Higgs mass becomes heavier as the gauge hierarchy becomes larger. It is possible to have consistent Higgs mass with the experimental lower bound for the small flavor number.

This paper is organized as follows. In the next section we briefly review the gauge-Higgs unification in the five dimensional $SU(3)$ gauge theory compactified on the orbifold S^1/Z_2. We discuss the gauge hierarchy in the scenario of the gauge-Higgs unification and present two models which can realize the large gauge hierarchy in Sec. III. The final section is devoted to conclusions and discussions.

II. $SU(3)$ GAUGE THEORY ON $M^4 \times S^1/Z_2$

In this section, we quickly review the relevant part of the gauge-Higgs unification for later convenience. Readers who are familiar with the gauge-Higgs unification can skip this section and go directly to the next section. We consider an $SU(3)$ nonsupersymmetric gauge theory on $M^4 \times S^1/Z_2$ as the simplest example of the gauge-Higgs unification [11]. Here, M^4 is the four dimensional Minkowski space-time and S^1/Z_2 is an orbifold. The orbifold has two fixed points at $y = 0, \pi R$, where R is the radius of S^1. One needs to specify boundary conditions of fields for the S^1 direction and the fixed point.

We define

$$A_{\hat{\mu}}(x, y + 2\pi R) = U A_{\hat{\mu}}(x, y) U^\dagger, \tag{1}$$

$$\begin{pmatrix} A_\mu \\ A_y \end{pmatrix}(x, y_i - y) = P_i \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix}(x, y_i + y) P_i^\dagger \qquad (i = 0, 1), \tag{2}$$

where $U^\dagger = U^{-1}$, $P_i^\dagger = P_i = P_i^{-1}$ and $y_0 = 0$, $y_1 = \pi R$ and $\hat{\mu}$ stands for $\hat{\mu} = (\mu, y)$. The minus sign for A_y is needed to preserve the invariance of the Lagrangian under these transformations. A transformation $\pi R + y \xrightarrow{P_1} \pi R - y$ must be the same as $\pi R + y \xrightarrow{P_0} -(\pi R + y) \xrightarrow{U} \pi R - y$, so we obtain $U = P_1 P_0$. Hereafter, we consider P_i to be a fundamental quantity.

For the given matrix P_i, the parity of $A_{\hat{\mu}}^a (a = 1, \ldots, 8)$ under the transformation is assigned. The fields with even parity have zero modes, while those with odd parity have no zero modes. We choose $P_0 = P_1 = e^{\pi i \sqrt{3} \lambda_8} = \mathrm{diag.}(-1, -1, 1)$, where λ_8 is the 8th Gell-Mann matrix. Then, we observe that the zero modes are

$$A_\mu^{(0)} = \frac{1}{2} \begin{pmatrix} A_\mu^{(0)3} + \frac{A_\mu^{(0)8}}{\sqrt{3}} & A_\mu^{(0)1} - i A_\mu^{(0)2} & 0 \\ A_\mu^{(0)1} + i A_\mu^{(0)2} & -A_\mu^{(0)3} + \frac{A_\mu^{(0)8}}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} A_\mu^{(0)8} \end{pmatrix}, \tag{3}$$

$$A_y^{(0)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & A_y^{(0)4} - i A_y^{(0)5} \\ 0 & 0 & A_y^{(0)6} - i A_y^{(0)7} \\ A_y^{(0)4} + i A_y^{(0)5} & A_y^{(0)6} + i A_y^{(0)7} & 0 \end{pmatrix}. \tag{4}$$

Counting the zero modes for A_μ, we see that the gauge symmetry is broken down to $SU(2) \times U(1)$ [11]. On the other hand, the zero modes for A_y transform as the $SU(2)$ doublet, so we can regard it as the Higgs doublet,

$$\Phi \equiv \sqrt{2\pi R}\frac{1}{\sqrt{2}}\begin{pmatrix} A_y^{(0)4} - iA_y^{(0)5} \\ A_y^{(0)6} - iA_y^{(0)7} \end{pmatrix}. \tag{5}$$

In fact, Φ has the $SU(2) \times U(1)$ invariant kinetic term[4] arising from $\mathrm{tr}(F_{\mu y})^2$,

$$\int_0^{2\pi R} dy\{-\mathrm{tr}(F_{\mu y}F^{\mu y})\} = \left|\left(\partial_\mu - ig_4 A_\mu^{(0)a}\frac{\tau^a}{2} - i\frac{\sqrt{3}g_4}{2}A_\mu^{(0)8}\right)\Phi\right|^2, \tag{6}$$

where we have rescaled the zero modes of the gauge fields by $\sqrt{2\pi R}$ in order to have the correct canonical dimension. The VEV of the Higgs field is parametrized, by utilizing the $SU(2) \times U(1)$, as

$$\langle A_y^{(0)} \rangle \equiv \frac{a}{g_4 R}\frac{\lambda^6}{2} = A_y^{(0)6}\frac{\lambda^6}{2}, \tag{7}$$

where a is a dimensionless real parameter, and g_4 is the four dimensional gauge coupling defined from the original five dimensional gauge coupling by $g_4 \equiv g_5/\sqrt{2\pi R}$.

One usually evaluates the effective potential for the parameter a in order to determine it [13]. Let us note that a is closely related to the Wilson line phase,

$$W = \mathcal{P}\exp\left(ig_5 \oint_{S^1} dy\langle A_y\rangle\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi a_0) & i\sin(\pi a_0) \\ 0 & i\sin(\pi a_0) & \cos(\pi a_0) \end{pmatrix} \quad (a_0 \bmod 2), \tag{8}$$

where a_0 is determined as the minimum of the effective potential. The gauge symmetry breaking depends on a_0,

$$SU(2)\times U(1) \to \begin{cases} SU(2)\times U(1) & \text{for } a_0 = 0, \\ U(1)'\times U(1) & \text{for } a_0 = 1, \\ U(1)_{\mathrm{em}} & \text{otherwise.} \end{cases} \tag{9}$$

It has been known that, in order to realize the desirable gauge symmetry breaking, $SU(2) \times U(1) \to U(1)_{\mathrm{em}}$, the matter content in the bulk is crucial. For our purpose, let us consider the matter content studied in [14]. Following the standard prescription to calculate the effective potential for a [13,15], we obtain

$$\begin{aligned} V_{\mathrm{eff}}(a) = \frac{\Gamma(5/2)}{\pi^{5/2}(2\pi R)^5}\sum_{n=1}^{\infty}\frac{1}{n^5}\Bigg[&(-3 + 4N_{\mathrm{adj}}^{(+)} - dN_{\mathrm{adj}}^{(+)s})(\cos[2\pi n a] + 2\cos[\pi n a]) \\ &+ (4N_{\mathrm{adj}}^{(-)} - dN_{\mathrm{adj}}^{(-)s})\left(\cos\left[2\pi n\left(a - \frac{1}{2}\right)\right] + 2\cos[\pi n(a-1)]\right) + (4N_{fd}^{(+)} - 2N_{fd}^{(+)s})\cos[\pi n a] \\ &+ (4N_{fd}^{(-)} - 2N_{fd}^{(-)s})\cos[\pi n(a-1)]\Bigg], \end{aligned} \tag{10}$$

where the factor d coming from the adjoint scalar takes 1 (2) for the real (complex) field. $N_{\mathrm{adj}}^{(\pm)}(N_{fd}^{(\pm)})$ denotes the flavor number for fermions belonging to the adjoint (fundamental) representation under $SU(3)$, where the sign $(\pm)$ stands for the intrinsic parity defined in [16]. Similarly, $N_{\mathrm{adj}}^{(\pm)s}(N_{fd}^{(\pm)s})$ means the flavor number for bosons in corresponding representations. Since $V_{\mathrm{eff}}(a) = V_{\mathrm{eff}}(-a)$ and $V_{\mathrm{eff}}(a) = V_{\mathrm{eff}}(2 - a)$, the physical region of a is given by $0 \le a \le 1$.

We have obtained the effective potential for a nonlocal quantity, the Wilson line phase a. Nonlocal terms in the effective potential are expected not to suffer from ultraviolet effects [6]. That is why the divergence associated with the phase a does not appear in the potential (10). The effective potential is calculable in the gauge-Higgs unification. Accordingly, the Higgs mass, which is obtained from the second derivative of the effective potential evaluated at the minimum, is also calculable. Hence, the gauge-Higgs unification can provide a natural framework to address the gauge hierarchy problem.

III. LARGE GAUGE HIERARCHY IN THE GAUGE-HIGGS UNIFICATION

In the scenario of the gauge-Higgs unification, the mass of the W bosons is given, from Eq. (6), by

$$M_W = \frac{a_0}{2R}. \tag{11}$$

This relation gives us the ratio between the weak scale and the compactification scale $M_c \equiv (2\pi R)^{-1}$,

$$\frac{M_W}{M_c} = \pi a_0. \tag{12}$$

Once the value of a_0 is determined from the effective potential, the compactification scale M_c is fixed through Eq. (12). In the usual scenario of the gauge-Higgs unification, the order of a_0 is $O(10^{-2})$ for an appropriate choice of

[4] This simplest example of the gauge-Higgs unification predicts wrong values of the Weinberg angle. It is known that there are a few prescriptions to overcome the problem [12].

the flavor set for the massless bulk matter [14,17]. Hence, the scale M_c is about a few TeV.[5]

We would like to realize the large gauge hierarchy such as $M_c \sim M_{\rm GUT}, M_{\rm Planck}$. One needs very small values of a_0. For small values of a_0, the effective potential is approximated in terms of the logarithm and polynomials with respect to a by using the following formulas,[6]

$$\sum_{n=1}^{\infty} \frac{1}{n^5}\cos(nx) = \zeta(5) - \frac{\zeta(3)}{2}x^2 + \frac{1}{24}\left(-\ln x + \frac{25}{12}\right)x^4 + O(x^6), \quad (13)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^5}\cos(nx - n\pi) = -\frac{15}{16}\zeta(5) + \frac{3}{8}\zeta(3)x^2 - \frac{\ln 2}{24}x^4 + O(x^6) \quad (14)$$

for $x \ll 1$ [17]. Applying the formulas to the effective potential (10), one obtains

$$\bar{V}_{eff}(a) = -\frac{\zeta(3)}{2}C^{(2)}(\pi a)^2 + \frac{(\pi a)^4}{24}\left[C^{(3)}\left(-\ln(\pi a) + \frac{25}{12}\right) + C^{(4)}(\ln 2)\right] + \cdots, \quad (15)$$

where $V_{\rm eff}(a) \equiv C\bar{V}_{\rm eff}(a)$ with $C \equiv \Gamma(\frac{5}{2})/\pi^{5/2}(2\pi R)^5$, and the coefficient $C^{(i)}(i = 2, 3, 4)$ is defined by

$$C^{(2)} \equiv 24N_{\rm adj}^{(+)} + 4N_{fd}^{(+)} + \frac{9d}{2}N_{\rm adj}^{(-)s} + \frac{3}{2}N_{fd}^{(-)s} - (18 + 6dN_{\rm adj}^{(+)s} + 2N_{fd}^{(+)s} + 18N_{\rm adj}^{(-)} + 3N_{fd}^{(-)}), \quad (16)$$

$$C^{(3)} \equiv 72N_{\rm adj}^{(+)} + 4N_{fd}^{(+)} - (54 + 18dN_{\rm adj}^{(+)s} + 2N_{fd}^{(+)s}), \quad (17)$$

$$C^{(4)} \equiv 48 + 16dN_{\rm adj}^{(+)s} + 18dN_{\rm adj}^{(-)s} + 2N_{fd}^{(-)s} - (64N_{\rm adj}^{(+)} + 4N_{fd}^{(-)} + 72N_{\rm adj}^{(-)}). \quad (18)$$

It should be noticed that each coefficient in the effective potential is given by the discrete values, that is, the flavor number of the massless bulk matter. This is the very curious feature of the Higgs potential, which is hardly seen in the usual quantum field theory, and is a key point to discuss the large gauge hierarchy in the gauge-Higgs unification.

[5]This is the reason that the gauge-Higgs unification can provide us with a solution to the little hierarchy problem.

[6]It turns out that the $\ln x$ term in Eq. (13) is important for later analyses. The term arises from the one-loop diagrams in which massless modes propagate.

A. Large gauge hierarchy in model I

We would like to obtain a hierarchically small VEV of a as the minimum of the effective potential (15). It is, however, hard to realize such small values of a_0 with nonvanishing $C^{(2)}$ in the potential (15) [17].

Let us consider the case where the coefficient of the mass term in the Higgs potential vanishes,

$$C^{(2)} = 0. \quad (19)$$

It should be noticed that the vanishing mass term (19) is not the fine-tuning of the parameter usually done in the quantum field theory. In the present case, all the coefficients in the effective potential are given by the discrete values, so that the condition is fulfilled just by the choice of the flavor set. We will discuss the matter content which realizes the vanishing mass term later. For a moment, we study the physical consequence of it.

When Eq. (19) is satisfied, the minimum of the effective potential is given by

$$\pi a_0 \simeq \exp\left(\frac{C^{(4)}}{C^{(3)}}\ln 2 + \frac{11}{6}\right) = \exp\left(-\frac{C^{(4)}}{|C^{(3)}|}\ln 2 + \frac{11}{6}\right). \quad (20)$$

As we will see later, the coefficient $C^{(3)}$ should be negative in order for the minimum a_0 to be, at least, a local minimum. Remembering Eq. (12), we have

$$\frac{M_W}{M_c} = \exp\left(-\frac{C^{(4)}}{|C^{(3)}|}\ln 2 + \frac{11}{6}\right). \quad (21)$$

If we set $M_W = 10^2$ (GeV) and $M_c = 10^p$ (GeV), one obtains

$$\frac{C^{(4)}}{|C^{(3)}|} = \frac{1}{\ln 2}\left(\frac{11}{6} - (2 - p)\ln 10\right). \quad (22)$$

The magnitude of $C^{(4)}/|C^{(3)}|$ for various values of p is listed in Table I. One requires $C^{(4)}/|C^{(3)}| \gg 1$ for the large gauge hierarchy (large values of p). If one has the vanishing mass term in such a way that $C^{(4)}/|C^{(3)}|$ is as large as that listed in Table I, the large gauge hierarchy is realized in the scenario of the gauge-Higgs unification.

It is instructive to point out the difference between the present model and the famous Coleman-Weinberg potential of the massless scalar field theory [9]. The effective

TABLE I. The magnitude of $C^{(4)}/|C^{(3)}|$ for various values of p. $p = 19$ (16) corresponds to the Planck (GUT) scale.

	$p = 11$	$p = 12$	$p = 13$	$p = 16$	$p = 19$
$\frac{C^{(4)}}{\vert C^{(3)}\vert}$	32.54	35.86	39.19	49.15	59.12

potential with $C^{(2)} = 0$ (called model I hereafter) is exactly the same form as the one in the paper by Coleman and Weinberg [9]. The potential is controlled by the logarithmic and quartic terms. There is, however, a big difference among them. In the Coleman-Weinberg case, the quartic coupling exists at tree level and only the logarithmic term is generated at the one-loop level. The logarithmic term becomes a dominant contribution against the quartic term at the nontrivial vacuum configuration, so the vacuum configuration is outside of the validity of perturbation theory. On the other hand, for the present case, both the logarithmic and quartic terms are generated at the one-loop level; even if the quartic and logarithmic terms are the same order as each other at the nontrivial vacuum configuration (20), the perturbative reliability for the vacuum configuration is not spoiled. Rather, what one has to be concerned with is the stability against the two (higher)-loop contributions.[7] We will discuss this point later.

Since we understand that it is possible to have the large gauge hierarchy in the scenario of the gauge-Higgs unification, let us next present explicit examples of the flavor set to realize it. To this end, we investigate the condition (19), which is a key ingredient for the large gauge hierarchy.

We recast Eq. (16) as

$$C^{(2)} = 6(4N_{\rm adj}^{(+)} - 3 - dN_{\rm adj}^{(+)s}) + 2(2N_{fd}^{(+)} - N_{fd}^{(+)s}) + \tfrac{9}{2}(dN_{\rm adj}^{(-)s} - 4N_{\rm adj}^{(-)}) + \tfrac{3}{2}(N_{fd}^{(-)s} - 2N_{fd}^{(-)}). \quad (23)$$

In order to fulfill $C^{(2)} = 0$, the second parenthesis $(2N_{fd}^{(+)} - N_{fd}^{(+)s})$ in Eq. (23) must be an integral multiple of 3. Accordingly,

$$C^{(3)} = 18(4N_{\rm adj}^{(+)} - 3 - dN_{\rm adj}^{(+)s}) + 2(2N_{fd}^{(+)} - N_{fd}^{(+)s}) \quad (24)$$

is an integral multiple of 6. Then, we write

$$C^{(3)} = -6k(= -|C^{(3)}|) \qquad (k = \text{positive integer}). \quad (25)$$

Let us introduce another integer m defined by

$$4N_{\rm adj}^{(+)} - 3 - dN_{\rm adj}^{(+)s} \equiv -m. \quad (26)$$

From the coefficient $C^{(3)}$, we have

$$2N_{fd}^{(+)} - N_{fd}^{(+)s} = -3k + 9m. \quad (27)$$

Imposing the condition $C^{(2)} = 0$ gives us a relation given by

$$N_{fd}^{(-)s} - 2N_{fd}^{(-)} = -3(dN_{\rm adj}^{(-)s} - 4N_{\rm adj}^{(-)}) + 4k - 8m, \quad (28)$$

where we have used Eqs. (26) and (27). Equipped with these equations, we obtain

$$C^{(4)} = 12(dN_{\rm adj}^{(-)s} - 4N_{\rm adj}^{(-)}) + 8k, \quad (29)$$

which is independent of the integer m. Hence, we finally have

$$\frac{C^{(4)}}{|C^{(3)}|} = \frac{2}{k}(dN_{\rm adj}^{(-)s} - 4N_{\rm adj}^{(-)}) + \frac{4}{3}. \quad (30)$$

This result tells us an important point that, in order to make $C^{(4)}/|C^{(3)}|$ larger, smaller values of $|C^{(3)}|$ and $N_{\rm adj}^{(-)}$ are favored. In fact, $k = 1$ is the most desirable choice for the large gauge hierarchy.

Let us now study the matter content in model I. For a given p, the values for $C^{(4)}/|C^{(3)}|$ are determined. Then, we obtain a flavor set $(N_{\rm adj}^{(-)}, dN_{\rm adj}^{(-)s})$ for fixed values of k through Eq. (30). One also fixes the integer m. Then, a flavor set $(N_{fd}^{(-)}, N_{fd}^{(-)s})$ is determined by Eq. (28) and also $(N_{fd}^{(+)}, N_{fd}^{(+)s})$ by Eq. (27). Finally, one obtains $(N_{\rm adj}^{(+)}, dN_{\rm adj}^{(+)s})$ from Eq. (26). We notice that the flavor number of the massless bulk matter with $(+)$ parity depends only on the values of k and m, and not on the values of p, as seen from Eqs. (26) and (27). The flavor number of the massless bulk matter with $(-)$ parity depends on the values of p, which sets the hierarchy between M_W and M_c.

Let us present a few examples of the flavor set in model I. We choose $(k, m) = (1, 0)$ as an example. Then, we find that

$$4N_{\rm adj}^{(+)} - 3 - dN_{\rm adj}^{(+)s} = 0 \to (N_{\rm adj}^{(+)}, dN_{\rm adj}^{(+)s}) = (1, 1), (2, 5), (3, 9), \ldots, \quad (31)$$

$$2N_{fd}^{(+)} - N_{fd}^{(+)s} = -3 \to (N_{fd}^{(+)}, N_{fd}^{(+)s}) = (0, 3), (1, 5), (2, 7), \ldots. \quad (32)$$

For $(k, m, p) = (1, 0, 19)$,

$$dN_{\rm adj}^{(-)s} - 4N_{\rm adj}^{(-)} \simeq 29 \to (N_{\rm adj}^{(-)}, dN_{\rm adj}^{(-)s}) = (0, 29), (1, 33), (2, 37), \ldots, \quad (33)$$

$$N_{fd}^{(-)s} - 2N_{fd}^{(-)} = -83 \to (N_{fd}^{(-)}, N_{fd}^{(-)s}) = (42, 1), (43, 3), (44, 5), \ldots. \quad (34)$$

For $(k, m, p) = (1, 0, 11)$,

$$dN_{\rm adj}^{(-)s} - 4N_{\rm adj}^{(-)} \simeq 16 \to (N_{\rm adj}^{(-)}, dN_{\rm adj}^{(-)s}) = (0, 16), (1, 20), (2, 24), \ldots, \quad (35)$$

$$N_{fd}^{(-)s} - 2N_{fd}^{(-)} = -43 \to (N_{fd}^{(-)}, N_{fd}^{(-)s}) = (22, 1), (23, 3), (24, 5), \ldots. \quad (36)$$

[7] One, of course, cares about the vanishing mass term (19) at the two (higher)-loop level as well. We will come back to this point later.

We observe that the flavor numbers $dN_{\rm adj}^{(-)s}, N_{fd}^{(-)}$ are of order of $O(10)$. One should worry about the reliability of perturbation theory for such a large flavor number. This is because an expansion parameter in the present case may be given by $(g_4^2/4\pi^2)N_{\rm flavor}$, and it must be $(g_4^2/4\pi^2)N_{\rm flavor} \ll 1$ for reliable perturbative expansion.

Now, let us discuss the Higgs mass in model I. The Higgs mass squared is obtained by the second derivative of the effective potential evaluated at the minimum of the potential. We have

$$m_H^2 = (g_5R)^2 C\frac{\partial^2 \bar{V}_{\rm eff}}{\partial a^2}\bigg|_{a=a_0} = \frac{3g_4^2}{16\pi^2}M_W^2\left(-\frac{C^{(3)}}{6}\right)\left(=\frac{3g_4^2}{16\pi^2}M_W^2 k < M_W^2\right), \qquad (37)$$

where we have used Eq. (20). This shows that, as we have stated before, the coefficient $C^{(3)}$ must be negative in order for the minimum a_0 to be, at least, a local minimum of the effective potential. The larger values of $C^{(3)}$ make the Higgs mass heavier, while, as we have discussed, the large gauge hierarchy favors smaller values of $C^{(3)}$. The choice $k = 1$ is the most desirable one for the large gauge hierarchy, so the Higgs mass is lighter than M_W, which is the same result as in the original Coleman-Weinberg paper [9]. Therefore, one concludes that the large gauge hierarchy and the sufficiently heavy Higgs mass are not compatible in model I.

B. Large gauge hierarchy in model II

In this subsection, we study another model called model II. In model II, we introduce massive bulk fermions [18–20], that is, fermions with a bulk mass term in addition to the massless bulk matter in model I. As is well known, the bulk mass term for fermions in five dimensions is odd under the parity transformation, $y \to -y(\pi R - y \to \pi R + y)$. One needs a parity-even mass term for consistency of the Z_2 orbifolding. We resort to one of the prescriptions known to realize such a mass term. Here, we introduce a pair of fermion fields, $\psi^{(+)}, \psi^{(-)}$. $\psi^{(+)}$ and $\psi^{(-)}$ have different Z_2 parity, so that the mass term $M\bar{\psi}^{(+)}\psi^{(-)}$ has even parity under Z_2. A detailed discussion is given in [20].

Then, the contribution to the effective potential from the massive fermions is given by

$$V_{\rm eff}^{\rm massive} = -2^{[5/2]}(1+1)N_{\rm pair}\frac{1}{2\pi R}\frac{1}{2}\sum_{n=-\infty}^{\infty}\int\frac{d^4p_E}{(2\pi)^4} \times \ln\left[p_E^2 + \left(\frac{n+\frac{a}{2}}{R}\right)^2 + M^2\right], \qquad (38)$$

where we have assumed the fermions belong to the fundamental representation under the gauge group $SU(3)$ and $N_{\rm pair}$ stands for the number of the pair $(\psi^{(+)}, \psi^{(-)})$. According to the usual prescription [15], we have

$$V_{\rm eff}^{\rm massive} = \frac{3}{4\pi^2(2\pi R)^5}4\times 2N_{\rm pair}\sum_{n=1}^{\infty}\frac{1}{n^5}\left(1+nz+\frac{n^2z^2}{3}\right) \times e^{-nz}\cos(\pi n a), \qquad (39)$$

where we have defined a dimensionless parameter $z \equiv 2\pi RM = M/M_c$.

We are interested in the very small values of a, so that we expand the cosine function to obtain

$$V_{\rm eff}^{\rm massive} = \frac{3}{4\pi^3(2\pi R)^5}4\times 2N_{\rm pair}\left[B^{(0)} - B^{(2)}\frac{\pi^2}{2}a^2 + B^{(4)}\frac{\pi^4}{4!}a^4 + \cdots\right], \qquad (40)$$

where

$$B^{(0)} = \sum_{n=1}^{\infty}\frac{1}{n^5}\left(1+nz+\frac{n^2z^2}{3}\right)e^{-nz}, \qquad (41)$$

$$B^{(2)} = \sum_{n=1}^{\infty}\frac{1}{n^3}\left(1+nz+\frac{n^2z^2}{3}\right)e^{-nz}, \qquad (42)$$

$$B^{(4)} = \sum_{n=1}^{\infty}\frac{1}{n}\left(1+nz+\frac{n^2z^2}{3}\right)e^{-nz}. \qquad (43)$$

We note that the coefficient $B^{(i)}(i = 0, 2, 4)$ is suppressed by the Boltzmann-like factor e^{-nz}, reflecting the fact that the effective potential (39) shares a similarity with that in finite temperature field theory [10].

The total effective potential to the Wilson line phase a (Higgs field) is given by Eqs. (15) and (39),

$$\bar{V}_{\rm eff}^{\rm total} \simeq -\frac{1}{2}[\zeta(3)C^{(2)} + 8N_{\rm pair}B^{(2)}](\pi a)^2 + \frac{(\pi a)^4}{24} \times\left[C^{(3)}\left(-\ln(\pi a)+\frac{25}{12}\right) + C^{(4)}(\ln 2)\right], \qquad (44)$$

where we have ignored the contribution to the quartic term from the massive bulk fermion because it is highly suppressed. We require, again, that the contribution to the mass term from the massless bulk matter vanishes, that is, $C^{(2)} = 0$. Then the mass term is controlled only by the term coming from the massive bulk fermion. The magnitude of the VEV is governed by the mass term in the Higgs potential, so that the VEV for the present case is exponentially suppressed for appropriate large values of $z(>1)$. The essential behavior of the VEV is governed by the factor $B^{(2)}$, i.e.

$$\pi a_0 \simeq \gamma B^{(2)} \qquad (45)$$

with some numerical constant γ of order 1. Since we assume $z = M/M_c > 1$, the size of the bulk mass parameter M may be determined by physics above the compactification scale M_c. If we take M to be the order of the cutoff scale, it is natural to assume that M_c is, at most, around 10^{17} GeV in model II.

If we write $\pi a_0 = e^{-Y}$, then one finds, remembering Eq. (12),

$$-Y = \ln(\pi a_0)\left(= \ln\left(\frac{M_W}{M_c}\right)\right) = (2-p)\ln 10 \simeq \begin{cases} -34.539 & \text{for } p = 17, \\ -32.236 & \text{for } p = 16, \\ -25.328 & \text{for } p = 13, \\ -23.026 & \text{for } p = 12, \\ -20.723 & \text{for } p = 11. \end{cases} \quad (46)$$

The gauge hierarchy is controlled by the magnitude of Y, in other words, the bulk mass parameter z, and the large gauge hierarchy is achieved by $|z| \simeq 30 \sim 40$. The large gauge hierarchy is realized by the presence of the massive bulk fermion under the condition (19). We notice that the flavor number of the massless bulk matter is not essential for the large gauge hierarchy in model II.

Now, let us next discuss the Higgs mass in model II. Again, the Higgs mass is obtained by the second derivative of the total effective potential at the vacuum configuration a_0,

$$m_H^2 = (g_5 R)^2 C \frac{\partial^2 \bar{V}_{\text{total}}}{\partial a^2}\Bigg|_{a=a_0} = \frac{g_4^2}{16\pi^2} M_W^2 \left[-C^{(3)} \ln(\pi a_0) + \frac{4}{3} C^{(3)} + C^{(4)} \ln 2 \right] = \frac{g_4^2}{16\pi^2} M_W^2 F, \quad (47)$$

where we have defined

$$F \equiv -C^{(3)} \ln(\pi a_0) + \tfrac{4}{3} C^{(3)} + C^{(4)} \ln 2. \quad (48)$$

At first glance, one may think that the Higgs mass is lighter than M_W as in the case of model I. This is, however, not the case in model II. The Higgs mass depends on the logarithmic factor. The larger the gauge hierarchy is, the heavier the Higgs mass is. An important point is that the coefficient $C^{(3)}$ is not related to the realization of the large gauge hierarchy, so that it is not constrained by the requirement of the large gauge hierarchy at all.

The vacuum configuration must be, at least, a local minimum. We require that $F \geq 0$. Defining

$$l \equiv d N_{\text{adj}}^{(-)s} - 4 N_{\text{adj}}^{(-)} \quad (49)$$

and recalling $\pi a_0 = e^{-Y}$, we have, from Eq. (48),

$$F = k(-6Y + 8(\ln 2 - 1)) + l \times 12 \ln 2, \quad (50)$$

where we have used $C^{(3)} = -6k (k \in \mathbf{Z})$ and Eq. (29). In model II the coefficient $C^{(3)}$ is not necessarily negative. We separately discuss the size of the Higgs mass, depending on the sign of $C^{(3)}$.

For $C^{(3)} < 0 (k > 0)$, in order for m_H^2 to be positive, one needs

$$l > k \times \frac{6Y + 8(1 - \ln 2)}{12 \ln 2} \simeq k \times \begin{cases} 25.21 & \text{for } p = 17, \\ 23.55 & \text{for } p = 16, \\ 18.57 & \text{for } p = 13, \\ 16.90 & \text{for } p = 12, \\ 15.24 & \text{for } p = 11. \end{cases} \quad (51)$$

This shows that we need $O(10)$ numbers of the flavor for $dN_{\text{adj}}^{(-)s}$. The reliability of perturbation theory may be lost for such a large flavor number. Hence, we exclude the case of $C^{(3)} < 0$ in model II. Hereafter, we restrict ourselves to $C^{(3)} > 0$, that is, $k < 0$.

In addition to the sign of $C^{(3)}$, the integer l can take both signs. Let us first consider $l < 0$. Writing $l(k) = -|l|(-|k|)$, the requirement of $m_H^2 > 0$ yields

$$|k| > \frac{12 \ln 2}{6Y + 8(1 - \ln 2)} \times |l| \simeq |l| \times \begin{cases} 0.040 & \text{for } p = 17, \\ 0.042 & \text{for } p = 16, \\ 0.054 & \text{for } p = 13, \\ 0.059 & \text{for } p = 12, \\ 0.066 & \text{for } p = 11. \end{cases} \quad (52)$$

Since the minimum values of $|k|$ are given by $|k| = 1$, i.e., $k = -1$, one obtains

$$0 < |l| \leq \begin{cases} 25 & \text{for } p = 17, \\ 23 & \text{for } p = 16, \\ 18 & \text{for } p = 13, \\ 16 & \text{for } p = 12, \\ 15 & \text{for } p = 11. \end{cases} \quad (53)$$

The upper bound of $|l|$ is larger if we choose larger values of $|k|$. In order to avoid a large flavor number, let us choose $l = -1$ as an example. And we impose the constraint on the Higgs mass from the experimental lower bound, $m_H \gtrsim 114$ GeV. Then, we find the possible values of k,

$$|k| \gtrsim \frac{1}{6Y + 8(1 - \ln 2)} \left(\frac{16\pi^2}{g_4^2 M_W^2} (114 \text{ GeV})^2 + 12|l| \ln 2 \right) \simeq \begin{cases} 3.64 & \text{for } p = 17, \\ 3.90 & \text{for } p = 16, \\ 4.95 & \text{for } p = 13, \\ 5.43 & \text{for } p = 12, \\ 6.02 & \text{for } p = 11, \end{cases} \quad (54)$$

where we have used $g_4^2 \simeq 0.42$. If we choose the larger values of $|l|$, the possible values of k become larger. We observe that the large p suppresses the values of k, which means that the lower flavor number is enough for the Higgs mass to satisfy the experimental lower bound.

Let us present the flavor number set for $(k, l) = (-4, -1)$. There is another free integer m, which is defined by Eq. (26). We choose $m = 0$ as a demonstration. Then, from Eqs. (26)–(28) and (49) we have

$$(N_{\rm adj}^{(-)}, dN_{\rm adj}^{(-)s}) = (1, 3), (2, 7), \ldots, \tag{55}$$

$$(N_{fd}^{(-)}, N_{fd}^{(-)s}) = (7, 1), (8, 3), \ldots, \tag{56}$$

$$(N_{fd}^{(+)}, N_{fd}^{(+)s}) = (6, 0), (7, 2), \ldots, \tag{57}$$

$$(N_{\rm adj}^{(+)}, dN_{\rm adj}^{(+)s}) = (1, 1), (2, 5), \ldots. \tag{58}$$

The Higgs mass in GeV units is calculated as

$$m_H^2 = \frac{g_4^2}{16\pi^2} M_W^2 (|k|(6Y + 8(1 - \ln 2)) - |l| \times 12 \ln 2)|_{k=-4,l=-1} \to m_H \simeq \begin{cases} 119.5 & \text{for } p = 17, \\ 115.5 & \text{for } p = 16, \\ 102.4 & \text{for } p = 13, \\ 97.6 & \text{for } p = 12, \\ 92.6 & \text{for } p = 11, \end{cases} \tag{59}$$

where we have used $g_4^2 \simeq 0.42$.[8] We observe that, for the fixed integers (k, l), the large gauge hierarchy, that is, large $\ln(\pi a_0) = -Y$, enhances the size of the Higgs mass.

Let us next consider non-negative l with $k < 0$. In this case it is obvious from Eq. (50) that the Higgs mass is positive definite. We first consider the $l = 0$ case. The requirement of $m_H \gtrsim 114$ GeV gives us the allowed values of $k = -|k|$,

$$|k| \gtrsim \frac{16\pi^2}{g_4^2 M_W^2} \frac{(114 \text{ GeV})^2}{6Y + 8(1 - \ln 2)} = \begin{cases} 3.61 & \text{for } p = 17, \\ 3.87 & \text{for } p = 16, \\ 4.91 & \text{for } p = 13, \\ 5.39 & \text{for } p = 12, \\ 5.98 & \text{for } p = 11. \end{cases} \tag{60}$$

The flavor number set for $(k, l) = (-4, 0)$ with $m = 0$ is given by

$$(N_{\rm adj}^{(-)}, dN_{\rm adj}^{(-)s}) = (1, 4), (2, 8), \ldots, \tag{61}$$

$$(N_{fd}^{(-)}, N_{fd}^{(-)s}) = (8, 0), (9, 2), \ldots, \tag{62}$$

$$(N_{fd}^{(+)}, N_{fd}^{(+)s}) = (6, 0), (7, 2), \ldots, \tag{63}$$

$$(N_{\rm adj}^{(+)}, dN_{\rm adj}^{(+)s}) = (1, 1), (2, 5), \ldots. \tag{64}$$

The Higgs mass in GeV units is obtained as

$$m_H^2 = \frac{g_4^2}{16\pi^2} M_W^2 (|k|(6Y + 8(1 - \ln 2)) + l \times 12 \ln 2)|_{k=-4,l=0} \to m_H \simeq \begin{cases} 119.9 & \text{for } p = 17, \\ 115.9 & \text{for } p = 16, \\ 102.9 & \text{for } p = 13, \\ 98.2 & \text{for } p = 12, \\ 93.2 & \text{for } p = 11. \end{cases} \tag{65}$$

We again observe that for the fixed integers (k, l) the large gauge hierarchy enhances the size of the Higgs mass thanks to the large $\ln(\pi a)$. We also confirm that the dominant contribution to the Higgs mass is given by $\ln(\pi a)$ if we compare Eq. (59) with Eq. (65).

If $k = 0$, there is no logarithmic term, and the Higgs mass is inevitably light. Instead of showing the Higgs mass in the $k = 0$ case, we show that many flavor numbers are necessary to enhance the size of the Higgs mass, though such a large flavor number violates the validity of perturbation theory. For $k = 0$, the Higgs mass is reduced to

$$m_H^2|_{k=0} = \frac{g_4^2}{16\pi^2} M_W^2 \times l \times 12 \ln 2. \tag{66}$$

The allowed value of l consistent with $m_H \gtrsim 114$ GeV is $l \gtrsim 92$. One needs many flavor numbers, which should be avoided.

We have assumed $m = 0$ in the above two examples, $(k, l) = (-4, -1), (-4, 0)$. If we take another value for m, we can further reduce the number of flavors, which is desirable from the point of view of perturbation theory. Let us show the result when we take $m = -1$. Equations (55)–(58) change to

$$(N_{\rm adj}^{(-)}, dN_{\rm adj}^{(-)s}) = (1, 3), (2, 7), \ldots, \tag{67}$$

$$(N_{fd}^{(-)}, N_{fd}^{(-)s}) = (3, 1), (4, 3), \ldots, \tag{68}$$

$$(N_{fd}^{(+)}, N_{fd}^{(+)s}) = (2, 1), (3, 3), \ldots, \tag{69}$$

$$(N_{\rm adj}^{(+)}, dN_{\rm adj}^{(+)s}) = (1, 0), (2, 4), \ldots. \tag{70}$$

We observe that the flavor numbers $N_{fd}^{(\pm)}$, $N_{fd}^{(\pm)s}$ are reduced. Likewise, Eqs. (61)–(64) change to

[8] In the usual scenario of the gauge-Higgs unification, one requires $g_4 \sim O(1)$ in order to have a heavy enough Higgs mass [14,17]. Our scenario can give heavy Higgs masses compatible with experiments, even for the weak coupling.

$$(N_{\rm adj}^{(-)}, dN_{\rm adj}^{(-)s}) = (1,4), (2,8), \ldots, \tag{71}$$

$$(N_{fd}^{(-)}, N_{fd}^{(-)s}) = (4,0), (5,2), \ldots, \tag{72}$$

$$(N_{fd}^{(+)}, N_{fd}^{(+)s}) = (2,1), (3,3), \ldots, \tag{73}$$

$$(N_{\rm adj}^{(+)}, dN_{\rm adj}^{(+)s}) = (1,0), (2,4), \ldots. \tag{74}$$

Again, the flavor numbers $N_{fd}^{(\pm)}$, $N_{fd}^{(\pm)s}$ are reduced. The Higgs mass does not depend on the integer m, so that the results, Eqs. (59) and (65), do not change.

In model II, the large gauge hierarchy and the heavy Higgs mass are compatible. The massive bulk fermion plays the role of generating the large gauge hierarchy. Once the large gauge hierarchy is achieved, the consistent Higgs mass can be obtained for the fixed set of a reasonable flavor number. The larger the gauge hierarchy is, the heavier the Higgs mass tends to be.

IV. CONCLUSIONS AND DISCUSSIONS

We have considered the five dimensional nonsupersymmetric $SU(3)$ model compactified on $M^4 \times S^1/Z_2$, which is the simplest model to realize the scenario of the gauge-Higgs unification. We have discussed whether the large gauge hierarchy is achieved in the scenario or not. The Higgs potential is generated at the one-loop level and is obtained in a finite form, reflecting the nonlocal nature that the Higgs field is the Wilson line phase in the gauge-Higgs unification. The Higgs potential is calculable and, accordingly, so is the Higgs mass.

The Higgs potential is expanded in terms of the logarithm and the polynomials for the small VEV of the Higgs field. Then, the coefficient is written in terms of the flavor number of the massless bulk matter introduced into the theory. This means that the coefficient is the discrete value, unlike the usual quantum field theory, in which the coefficient is a scale-dependent parameter. It is possible to have the vanishing mass term in the Higgs potential by choosing the appropriate flavor number set. This is not the fine-tuning of the parameters. We have found two models (models I, II), in which the large gauge hierarchy is realized.

In model I, the Higgs potential consists of the logarithmic and quartic terms, both of which are generated at the one-loop level. Perturbation theory is not spoiled even in the nontrivial vacuum configuration, for which the size of both terms is the same order. This is a different point than the Coleman-Weinberg paper [9], in which the nontrivial vacuum is outside of the validity of perturbation theory. We have found that in order to realize the large gauge hierarchy we need large (small) $C^{(4)}(C^{(3)})$. This, in turn, requires the $O(10)$ numbers of the flavor. From the point of view of perturbation theory, such a large flavor number is not favored.

The Higgs mass in model I is inevitably light, lighter than M_W, which is the same result as in the Coleman-Weinberg paper. The small $C^{(3)}$ tends to realize the large gauge hierarchy, while the heavy Higgs mass needs the large $C^{(3)}$. Therefore, in model I, the large gauge hierarchy is realized, but the Higgs mass is too light.

In model II, we have considered the massive bulk fermion in addition to the massless bulk matter. If we assume that the contribution to the mass term in the Higgs potential from the massless bulk matter vanishes, the dominant contribution to the mass term comes from the massive bulk fermion alone, whose coefficient is given by the Boltzmann-like suppression factor. As a result, the VEV of the Higgs field is exponentially small and the large gauge hierarchy is realized for the appropriate values of $z = M/M_c$. The flavor number of the massless bulk matter does not concern the gauge hierarchy.

It is interesting to note that the Higgs mass in model II becomes heavier as the compactified scale R becomes smaller. We have shown that the Higgs mass can be consistent with the experimental lower bound without introducing many flavor numbers.

We have introduced the scalar fields in the bulk. In general, scalar fields receive large radiative corrections. One might ask whether the large gauge hierarchy can be realized without scalar fields in the bulk. Unfortunately, one can neither realize the large gauge hierarchy nor have a stable nontrivial vacuum in model I. The relevant quantity for the large gauge hierarchy is reduced to

$$\frac{C^{(4)}}{|C^{(3)}|} \to \frac{2}{k}(-4N_{\rm adj}^{(-)}) + \frac{4}{3} \tag{75}$$

if there is no scalar field in the bulk. One needs $C^{(4)}/|C^{(3)}| \gg 1$ for the large gauge hierarchy, but this requires negative k (or, equivalently, positive $C^{(3)}$), for which the Higgs mass squared becomes negative, as seen from Eq. (37). Therefore, one cannot realize the large gauge hierarchy with only the fermions in the bulk. One can say that, in model I, the scalar fields in the bulk are essential to realize the large gauge hierarchy. If we require the stability of the nontrivial vacuum, that is, the negative $C^{(3)}$, we have an inverse hierarchy $M_W \gg M_c$. As for model II, the flavor number of the massless bulk matter has no effect on the gauge hierarchy. As long as the bulk mass takes the appropriate values, the large gauge hierarchy is achieved. We care about the stability of the vacuum configuration, $F \geq 0$. Since $l = -4N_{\rm adj}^{(-)}$ for the present case, l is negative, so that k (or equivalently $C^{(3)}$) must be negative (positive). One easily finds a possible flavor set for $(k, l, m) = (-5, -4, -1)$, as an example,

$$(N_{\rm adj}^{(+)}, N_{fd}^{(+)}, N_{\rm adj}^{(-)}, N_{fd}^{(-)}) = (1,3,1,0), \tag{76}$$

and the Higgs mass is obtained as

$$m_H \simeq \begin{cases} 132.1 \text{ GeV} & \text{for } p = 17, \\ 127.6 \text{ GeV} & \text{for } p = 16, \\ 112.7 \text{ GeV} & \text{for } p = 13, \\ 107.3 \text{ GeV} & \text{for } p = 12, \\ 101.6 \text{ GeV} & \text{for } p = 11. \end{cases} \quad (77)$$

Hence, the large gauge hierarchy and the Higgs mass are compatible in model II even though the massless bulk matter is given by the fermions alone.

Let us discuss an important point for the scenario we have considered—whether or not the condition $C^{(2)} = 0$ is stable against higher-loop corrections. The condition $C^{(2)} = 0$ is realized by choosing the appropriate flavor number set at the one-loop level. If we have nonzero finite corrections at the two (higher)-loop level, our scenario considered in this paper no longer holds. In general, it may be natural to expect that we have nonzero finite corrections to the mass term at the higher-loop level, but, at the moment, it may be too hasty to exclude the possibility of $C^{(2)} = 0$ at the two (higher)-loop level.

Recently, a two-loop calculation has been carried out in the five dimensional QED compactified on $M^4 \times S^1$ [21]. In this paper, it has been reported that there is no finite correction to the Higgs mass from the two-loop level, even though there is no concrete discussion to understand why this is so. As long as we have an example of the vanishing finite correction at the two-loop level, it does not seem unreasonable to expect the stability of the condition against the higher-loop corrections.

In connection with the above discussion, it may be worth mentioning that there are examples in which the loop correction is exhausted at the one-loop level (without supersymmetry). They are the coefficient of the axial anomaly [22] and the Chern-Simons coupling [23]. As for the latter case, a simple reason for the two (higher)-loop correction not to be generated comes from the invariance of the action under the large gauge transformation. Since the shift symmetry of the Higgs potential can be regarded as the invariance under the large gauge transformation, one may be able to prove that there is no two (higher)-loop correction to the mass term of the Higgs potential. In order to confirm this, we need more studies of the higher-loop corrections to the Higgs potential (mass) in the gauge-Higgs unification [24].

We have considered the massless bulk matter belonging to the adjoint and fundamental representation under the gauge group $SU(3)$. One can consider the higher dimensional representations such as **10**, **15** of $SU(3)$ [25]. In fact, these higher dimensional fields have been known to play an important role in enhancing the Higgs mass and obtaining the large top Yukawa coupling for constructing realistic models. Therefore, it is interesting to study the effect of the higher dimensional field on the large gauge hierarchy in the scenario of the gauge-Higgs unification.

Finally, let us comment on the possibility of realizing the large gauge hierarchy if we have a term like

$$B(\pi a)^2 \ln(\pi a)^2, \quad (78)$$

where B is a constant. Since we are interested in very small values of a, the Higgs potential is approximately given by

$$V = A(\pi a)^2 + B(\pi a)^2 \ln(\pi a)^2 + O((\pi a)^4, (\pi a)^4 \ln(\pi a)^2). \quad (79)$$

Then, the VEV is obtained in the desirable form as

$$\pi a_0 = \exp\left(-\frac{A}{2B} - \frac{1}{2}\right), \quad (80)$$

and if $A/B \gg 1$, the large gauge hierarchy is realized. This is also an interesting possibility, but, unfortunately, we do not yet understand the origin of the second term in Eq. (79).

Finally, let us make a brief comment on the fermion masses in the models we have studied. The mass scale for the matter in the models is of the order of the weak scale or the compactification scale M_c. Hence, it is difficult to reproduce the realistic fermion mass spectrum in the models. This is the common problem of the gauge-Higgs unification. Some attempts have been made to overcome this problem in [12,25,26].

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