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APPROXIMATE RESOLUTIONS AND BOX-COUNTING DIMENSION

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Abstract. In an earlier paper the authors introduced a new approach using normal sequences and approximate resolutions to study Lipschitz maps between compact metric spaces. In this paper we intoduce two kinds of box-counting dimension, which are defined for every compact metric space with a normal sequence and for every approximate resolution of any compact metric space, and investigate their properties. In a special case those notions coincide with the usual box-counting dimension for compact subsets of $\mathbb{R}^n$. Our box-counting dimensions are Lipschitz subinvariant, where Lipschitz maps are in the sense of the earlier paper. Moreover, we obtain fundamental theorems such as the subset theorem, the product theorem and the sum theorem. As an example, for each r with $0 \leq r \leq \infty$, we present a systematic way to construct a compact metric space with an approximate resolution whose box-counting dimension equals r.

1. Introduction

For each non-empty subset X of $\mathbb{R}^m$ the lower and upper box-counting dimensions of X are respectively defined as

$$\underline{\dim}_B X = \varliminf_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta} \text{ and } \overline{\dim}_B X = \varlimsup_{\delta \to 0} \frac{\log N_\delta(X)}{-\log \delta},$$

where $N_\delta(X)$ is the smallest number of open balls with radius at most δ which can cover X (see [2]). If these values coincide, the common value is called the box-counting dimension of X and denoted by $\dim_B X$. In this paper we introduce a new method of using the theory of approximate resolutions to study box-counting dimension.

The notion of approximate resolution, which was introduced by Mardešić and Watanabe [6], is useful in many problems in topology [4, 5, 15, 16, 17, 14, 9] and is essential even for compact metric spaces [3, 7, 15, 16]. In the theory of approximate resolutions, given a map $f : X \to Y$ and polyhedral approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ of X and Y, respectively, there exists an approximate map of systems $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ representing f, which is called an approximate resolution of f and is used to study the properties of the map $f : X \to Y$. This idea made it possible to characterize Lipschitz maps in terms of approximate resolutions [10].

In [10] it is shown that every normal sequence $\mathbb{U}$ on Hausdorff space X and every approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ of a compact metric space X induce metrics $\mathrm{d}_{\mathbb{U}}$ and $\mathrm{d}_{\boldsymbol{p}}$ under some reasonable conditions on $\mathbb{U}$ and $\boldsymbol{p}$, respectively. For such normal sequences $\mathbb{U}$ and $\mathbb{V}$ on Hausdorff spaces X and Y, respectively, one can speak of Lipschitz maps with respect to the metrics $\mathrm{d}_{\mathbb{U}}$ and $\mathrm{d}_{\mathbb{V}}$, which are called $(\mathbb{U}, \mathbb{V})$-Lipschitz maps, and they are characerized by a property on the normal sequences $\mathbb{U}$ and $\mathbb{V}$. Moreover, for such approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ of compact metric spaces X and Y, respectively, one

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can speak of Lipschitz maps $f : X \to Y$ with respect to the metrics $\mathrm{d}_{\boldsymbol{p}}$ and $\mathrm{d}_{\boldsymbol{q}}$, which are called $(\boldsymbol{p}, \boldsymbol{q})$-Lipschitz maps, and they are characterized by a property on the approximate resolution $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ of f.

In the present paper, we intoduce two kinds of box-counting dimension. First, we define a box-counting dimension for every compact metric space X with a normal sequence $\mathbb{U}$ (Section 4). If X is a compact subset of an Euclidean space $\mathbb{R}^n$ the usual box-counting dimension of X coincides with the box-counting dimension of X with some normal sequence in our sense. We obtain its properties which include Lipschitz subinvariance, i.e., if $f : X \to Y$ is a $(\mathbb{U}, \mathbb{V})$-Lipschitz map for some normal sequences $\mathbb{U}$ and $\mathbb{V}$ on compact metric spaces X and Y, respectively, then the box-counting dimension of $(Y, \mathbb{V})$ is not greater than that of $(X, \mathbb{U})$. Secondly, we define a box-counting dimension for every approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ of any compact metric space X (Section 5). We show that for each approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ with some reasonable condition this box-counting dimension equals that of X with the normal sequence which is induced by $\boldsymbol{p} : X \to \boldsymbol{X}$. We demonstrate how some particular operations on the approximate resolutions change the box-counting dimension. We then prove Lipschitz subinvariance (Section 6), and obtain fundamental theorems such as the subset theorem, the product theorem and the sum theorem (Section 7). As an example, for each r with $0 \le r \le \infty$, we present a systematic way to construct a compact metric space with an approximate resolution whose box-counting dimension equals r (Section 8).

Throughout the paper, *a map means a continuous map.*

For any space X, let $\mathrm{Cov}(X)$ denote the set of all open coverings of X. For $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$, $\mathcal{U}$ is said to *refine* $\mathcal{V}$, or $\mathcal{U}$ *refines* $\mathcal{V}$, in notation, $\mathcal{U} < \mathcal{V}$, provided for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subseteq V$. For any subset A of X and $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $\mathcal{U}|A = \{U \cap A : U \in \mathcal{U}\}$. If $A = \{x\}$, we write $\mathrm{st}(x, \mathcal{U})$ for $\mathrm{st}(\{x\}, \mathcal{U})$. For each $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}\,\mathcal{U} = \{\mathrm{st}(U, \mathcal{U}) : U \in \mathcal{U}\}$. Let $\mathrm{st}^{n+1}\mathcal{U} = \mathrm{st}(\mathrm{st}^n\mathcal{U})$ for each $n = 1, 2, ...$ and $\mathrm{st}^1\mathcal{U} = \mathrm{st}\,\mathcal{U}$. For any metric space (X, d) and $r > 0$, let $\mathrm{U}_{\mathrm{d}}(x, r) = \{y \in X : \mathrm{d}(x, y) < r\}$. For any $\mathcal{U} \in \mathrm{Cov}(X)$, two points $x, x' \in X$ are $\mathcal{U}$*-near*, denoted $(x, x') < \mathcal{U}$, provided $x, x' \in U$ for some $U \in \mathcal{U}$. For any $\mathcal{V} \in \mathrm{Cov}(Y)$, two maps $f, g : X \to Y$ between spaces are $\mathcal{V}$*-near*, denoted $(f, g) < \mathcal{V}$, provided $(f(x), g(x)) < \mathcal{V}$ for each $x \in X$. For each $\mathcal{U} \in \mathrm{Cov}(X)$ and $\mathcal{V} \in \mathrm{Cov}(Y)$, let $f\mathcal{U} = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}\mathcal{V} = \{f^{-1}(V) : V \in \mathcal{V}\}$.

Let $\mathbb{N}$ denote the ordered set of all natural numbers.

2. Approximate resolutions

In this section we recall the definitions and properties of approximate resolutions which will be needed in later sections. Although approximate resolutions are defined and useful for arbitrary topological spaces, in this section *all spaces are assumed to be compact metric spaces* for our purpose. For more details, the reader is referred to [6].

An *approximate inverse sequence* (*approximate sequence*, in short) $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ consists of

i) a sequence of spaces X_i (called *coordinate spaces*), $i \in \mathbb{N}$;

ii) a sequence of $\mathcal{U}_i \in \mathrm{Cov}(X_i)$, $i \in \mathbb{N}$; and

iii) maps $p_{ii'} : X_{i'} \to X_i$ for $i < i'$ where $p_{ii} = 1_{X_i}$ the identity map on X_i.

It must satisfy the following three conditions:

(A1): $(p_{ii'}p_{i'i''}, p_{ii''}) < \mathcal{U}_i$ for $i < i' < i''$;

(A2): For each $i \in \mathbb{N}$ and $\mathcal{U} \in \operatorname{Cov}(X_i)$, there exists $i' > i$ such that $(p_{ii_1}p_{i_1i_2}, p_{ii_2}) < \mathcal{U}$ for $i' < i_1 < i_2$; and

(A3): For each $i \in \mathbb{N}$ and $\mathcal{U} \in \operatorname{Cov}(X_i)$, there exists $i' > i$ such that $\mathcal{U}_{i''} < p_{ii''}^{-1}\mathcal{U}$ for $i' < i''$.

An *approximate map* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ of a space X into an approximate sequence $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ consists of maps $p_i : X \to X_i$ for $i \in \mathbb{N}$ with the following property:

(AS): For each $i \in \mathbb{N}$ and $\mathcal{U} \in \operatorname{Cov}(X_i)$, there exists $i' > i$ such that $(p_{ii''}p_{i''}, p_i) < \mathcal{U}$ for $i'' > i'$.

An *approximate resolution* of a space X is an approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ of X into an approximate sequence $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ which satisfies the following two conditions:

(R1): For each ANR P, $\mathcal{V} \in \operatorname{Cov}(P)$ and map $f : X \to P$, there exist $i \in \mathbb{N}$ and a map $g : X_i \to P$ such that $(gp_i, f) < \mathcal{V}$; and

(R2): For each ANR P and $\mathcal{V} \in \operatorname{Cov}(P)$, there exists $\mathcal{V}' \in \operatorname{Cov}(P)$ such that whenever $i \in \mathbb{N}$ and $g, g' : X_i \to P$ are maps with $(gp_i, g'p_i) < \mathcal{V}'$, then $(gp_{ii'}, g'p_{ii'}) < \mathcal{V}$ for some $i' > i$.

The approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ is called a *polyhedral approximate resolution* if all the coordinate spaces are polyhedra.

Theorem 2.1 ([6])**.** *An approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is an approximate resolution of a space X if and only if it satisfies the following two conditions:*

(B1): *For each $\mathcal{U} \in \operatorname{Cov}(X)$, there exists $i_0 \in \mathbb{N}$ such that $p_i^{-1}\mathcal{U}_i < \mathcal{U}$ for $i > i_0$; and*

(B2): *For each $i \in \mathbb{N}$ and $\mathcal{U} \in \operatorname{Cov}(X_i)$, there exists $i_0 > i$ such that $p_{ii'}(X_{i'}) \subseteq \operatorname{st}(p_i(X), \mathcal{U})$ for $i' > i_0$.*

Theorem 2.2 ([16])**.** *Every space X admits a polyhedral approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ such that all X_i are finite polyhedra.*

Theorem 2.3 ([5])**.** *Every connected space X admits a polyhedral approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ such that all X_i are connected finite polyhedra, and all p_i and $p_{ii'}$ are surjective.*

Let $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ be approximate sequences of spaces. An *approximate map* $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ consists of an increasing function $f : \mathbb{N} \to \mathbb{N}$ and maps $f_j : X_{f(j)} \to Y_j$, $j \in \mathbb{N}$, with the following condition:

(AM): For any $j, j' \in \mathbb{N}$ with $j < j'$, there exists $i \in \mathbb{N}$ with $i > f(j')$ such that

$$(q_{jj'}f_{j'}p_{f(j')i'}, f_jp_{f(j)i'}) < \operatorname{st}\mathcal{V}_j \text{ for } i' > i.$$

A map $f : X \to Y$ is a *limit* of $\boldsymbol{f}$ provided the following condition is satisfied:

(LAM): For each $j \in \mathbb{N}$ and $\mathcal{V} \in \operatorname{Cov}(Y_j)$, there exists $j' > j$ such that

$$(q_{jj''}f_{j''}p_{f(j'')}, q_jf) < \mathcal{V} \text{ for } j'' > j'.$$

For each map $f : X \to Y$, an *approximate resolution* of f is a triple $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ consisting of approximate resolutions $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ of X and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ of Y and of an approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ with property (LAM).

Theorem 2.4 ([6])**.** *Let X and Y be spaces. For any approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ and polyhedral approximate resolution $\boldsymbol{q} : Y \to \boldsymbol{Y}$, every map $f : X \to Y$ admits an approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ such that $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ is an approximate resolution of f.*

For each approximate sequence $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, let $\operatorname{st} \boldsymbol{X}$ denote the approximate system $\{X_i, \operatorname{st} \mathcal{U}_i, p_{ii'}\}$. Then there is a natural approximate map $\boldsymbol{i_X} = \{1_{X_i}\} : \boldsymbol{X} \to \operatorname{st} \boldsymbol{X}$, where $1_{X_i} : X_i \to X_i$ is the identity map. For each approximate map $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, the map $\operatorname{st} \boldsymbol{p} = \{p_i\} : X \to \operatorname{st} \boldsymbol{X} = \{X_i, \operatorname{st} \mathcal{U}_i, p_{ii'}\}$ also satisfies (AS) and hence is an approximate map. Moreover, if $\boldsymbol{p} : X \to \boldsymbol{X}$ is an approximate resolution, so is $\operatorname{st} \boldsymbol{p} : X \to \operatorname{st} \boldsymbol{X}$.

For any approximate sequences $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ and for each approximate map $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$, the map $\operatorname{st} \boldsymbol{f} = \{f_j, f\} : \operatorname{st} \boldsymbol{X} \to \operatorname{st} \boldsymbol{Y}$ also satisfies (AM) and hence is an approximate map. Moreover, if $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is an approximate resolution of a map $f : X \to Y$, then $\operatorname{st} \boldsymbol{f} : \operatorname{st} \boldsymbol{X} \to \operatorname{st} \boldsymbol{Y}$ also satisfies (LAM) and hence $(\operatorname{st} \boldsymbol{f}, \operatorname{st} \boldsymbol{p}, \operatorname{st} \boldsymbol{q})$ is an approximate resolution of f.

For each approximate map $\boldsymbol{f} = \{f_j\} : \boldsymbol{X} \to \boldsymbol{Y}$ between approximate sequences $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$, consider the following property:

(APS): $(\forall j \in \mathbb{N})(\forall \mathcal{V} \in \operatorname{Cov}(Y_j))(\exists j_0 > j)(\forall j' > j_0)(\exists j_0' > j')(\forall j'' > j_0')(\exists i_0 > f(j'))(\forall i > i_0)$:

$$q_{jj''}(Y_{j''}) \subseteq \operatorname{st}(q_{jj'} f_{j'} p_{f(j')i}(X_i), \mathcal{V}).$$

Theorem 2.5 ([12]). *Let $f : X \to Y$ be a map, and $\boldsymbol{f} = \{f_j\} : \boldsymbol{X} \to \boldsymbol{Y}$ be an approximate map such that $(\boldsymbol{f}, \boldsymbol{p}, \boldsymbol{q})$ is an approximate resolution of f where $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ are approximate resolutions of X and Y, respectively. Then f is surjective if and only if $\boldsymbol{f}$ satisfies (APS).*

Throughout the rest of the paper, *an approximate resolution means a polyhedral approximate resolution* unless otherwise stated.

3. Lipschitz maps

A new approach to Lipschitz maps using normal sequences and approximate resolutions was first introduced in [10]. In this seciton we recall and improve some of the important results for later sections.

A normal sequence approach. Let X be a Hausdorff space. A family $\mathbb{U} = \{\mathcal{U}_i : i \in \mathbb{N}\}$ of open coverings on X is said to be a *normal sequence* provided $\operatorname{st} \mathcal{U}_{i+1} < \mathcal{U}_i$ for each i. Let $\Sigma\mathbb{U}$ denote the normal sequence $\{\mathcal{V}_i : \mathcal{V}_i = \mathcal{U}_{i+1}, i \in \mathbb{N}\}$ and $\operatorname{st} \mathbb{U}$ the normal sequence $\{\operatorname{st} \mathcal{U}_i : i \in \mathbb{N}\}$. For any normal sequences $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{\mathcal{V}_i\}$, we write $\mathbb{U} < \mathbb{V}$ provided $\mathcal{U}_i < \mathcal{V}_i$ for each i. Let $\Sigma^0\mathbb{U} = \mathbb{U}$ and $\operatorname{st}^0 \mathbb{U} = \mathbb{U}$, and for each $n \in \mathbb{N}$, let $\Sigma^n\mathbb{U} = \Sigma(\Sigma^{n-1}\mathbb{U})$ and $\operatorname{st}^n \mathbb{U} = \operatorname{st}(\operatorname{st}^{n-1} \mathbb{U})$. For each map $f : X \to Y$ and for each normal sequence $\mathbb{V} = \{V_i\}$, let $f^{-1}\mathbb{V} = \{f^{-1}\mathcal{V}_i\}$. For each closed subset A of X and for each normal sequence $\mathbb{U} = \{\mathcal{U}_i\}$ on X, let $\mathbb{U}|A = \{\mathcal{U}_i|A\}$.

Given a normal sequence $\mathbb{U} = \{\mathcal{U}_i\}$ on X, we define the function $\mathcal{D}_\mathbb{U} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_\mathbb{U}(x, x') = \begin{cases} 9 & \text{if } (x, x') \not< \mathcal{U}_1; \\ \frac{1}{3^{i-2}} & \text{if } (x, x') < \mathcal{U}_i \text{ but } (x, x') \not< \mathcal{U}_{i+1}\ ; \\ 0 & \text{if } (x, x') < \mathcal{U}_i \text{ for all } i \in \mathbb{N}\ , \end{cases}$$

and the function $\mathrm{d}_\mathbb{U} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_\mathbb{U}(x, x') = \inf\{\mathcal{D}_\mathbb{U}(x, x_1) + \mathcal{D}_\mathbb{U}(x_1, x_2) + \cdots + \mathcal{D}_\mathbb{U}(x_n, x')\},$$

where the infimum is taken over all points $x_1, x_2, ..., x_n$ in X and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. Then the function $\mathrm{d}_\mathbb{U} : X \times X \to \mathbb{R}_{\geq 0}$ defines a pseudometric on

X with the property that

$$\mathrm{st}(x,\mathcal{U}_{i+3}) \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{U}}}(x,\frac{1}{3^i}) \subseteq \mathrm{st}(x,\mathcal{U}_i) \text{ for each } x \in X \text{ and } i. \tag{3.1}$$

Moreover, if $\mathbb{U}$ has the following property:

(B): $\{\mathrm{st}(x,\mathcal{U}_i) : i \in \mathbb{N}\}$ is a base at x for each $x \in X$.

then $\mathrm{d}_{\mathbb{U}}$ defines a metric on X, which we call the *metric induced by the normal sequence* $\mathbb{U}$. In particular, if (X, d) is a metric space and if $\mathbb{U} = \{\mathcal{U}_i\}$ is the normal sequence such that $\mathcal{U}_i = \{\mathrm{U}_{\mathrm{d}}(x, \frac{1}{3^i}) : x \in X\}$, then the metric $\mathrm{d}_{\mathbb{U}}$ induced by the nomal sequence $\mathbb{U}$ induces the uniformity which is isomorphic to that induced by the metric d.

Proposition 3.1. *Let X be a Hausdorff space, and let $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{\mathcal{V}_i\}$ be normal sequences on X. Then we have the following properties:*

1) *If A is a closed subset of X, then $\mathrm{d}_{\mathbb{U}|A}(x,x') \geq \mathrm{d}_{\mathbb{U}}(x,x')$ for all $x, x' \in A$.*
2) *If $\mathbb{U} < \mathbb{V}$, then $\mathrm{d}_{\mathbb{U}}(x,x') \geq \mathrm{d}_{\mathbb{V}}(x,x')$ for all $x, x' \in X$.*
3) *$\mathrm{d}_{\Sigma\mathbb{U}}(x,x') = 3\,\mathrm{d}_{\mathbb{U}}(x,x')$ for all $x, x' \in X$.*
4) *$\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x,x') \leq \mathrm{d}_{\mathbb{U}}(x,x') \leq 3\,\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x,x')$ for all $x, x' \in X$.*

Let X and Y be Hausdorff spaces, and let $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{\mathcal{V}_i\}$ be normal sequences on X and Y, respectively, which have property (B). A map $f : X \to Y$ is said to be a $(\mathbb{U}, \mathbb{V})$-*Lipschitz map* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \alpha\,\mathrm{d}_{\mathbb{U}}(x,x') \text{ for } x, x' \in X.$$

In particular, if we can choose α such that $0 < \alpha < 1$, the map $f : X \to Y$ is said to be a $(\mathbb{U}, \mathbb{V})$-*contraction map.*

Lipschitz maps and contraction maps between spaces are characterized in terms of normal sequences as follows:

For $m \in \mathbb{Z}$ consider the following statement:

(L)$_m$: $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq 3^m\,\mathrm{d}_{\mathbb{U}}(x,x')$ for $x, x' \in X$,

and for $m, n \geq 0$ consider the following statements:

(M)$_{m,n}$: $\Sigma^m\mathbb{U} < f^{-1}\,\mathrm{st}^n\,\mathbb{V}$; and
(N)$_{m,n}$: $\Sigma^m\mathbb{U} < f^{-1}\Sigma^n\mathbb{V}$.

Theorem 3.2. *The following implications hold for any $m, n \geq 0$:*

1) *(M)$_{m,n}$ $\Rightarrow$ (L)$_{m+n}$;*
2) *(N)$_{m,n}$ $\Rightarrow$ (L)$_{n-m}$;*
3) *(L)$_m$ $\Rightarrow$ (M)$_{m+4,0}$ $=$ (N)$_{m+4,0}$; and*
4) *(L)$_{-m}$ $\Rightarrow$ (N)$_{4,m}$.*

Proof. See [10, §5, 7]. □

An approximate sequence approach. Let X be a compact metric space. For each approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, consider the following three conditions:

(U): $\mathrm{st}^2\,\mathcal{U}_j < p_{ij}^{-1}\mathcal{U}_i$ for $i < j$;
(A): $(p_{ij}p_j, p_i) < \mathcal{U}_i$ for $i < j$; and
(NR): $p_j^{-1}\,\mathrm{st}\,\mathcal{U}_j < p_i^{-1}\mathcal{U}_i$ for $i < j$.

An approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is *normal* provided the family $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$ is a normal sequence and has property (B), and it is *admissible* provided it pocesses properties (U), (A), (NR).

Lemma 3.3. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution of X. Then the following properties hold:*

1) $p_j^{-1} \operatorname{st}^2 \mathcal{U}_j < p_i^{-1} \operatorname{st} \mathcal{U}_i$ *for $i < j$; and*
2) $\operatorname{st} p_j^{-1} \operatorname{st} \mathcal{U}_j < p_i^{-1} \operatorname{st} \mathcal{U}_i$ *for $i < j$.*

Proof. Let $U \in \mathcal{U}_j$, and let $x \in p_j^{-1}(\operatorname{st}(\operatorname{st}(U,\mathcal{U}_j),\mathcal{U}_j))$. By (U), there is $U' \in \mathcal{U}_i$ such that $p_j(x) \in \operatorname{st}(\operatorname{st}(U,\mathcal{U}_j),\mathcal{U}_j) \subseteq p_{ij}^{-1}(U')$. So $p_{ij}p_j(x) \in U'$. But this implies by (A) that $p_i(x) \in \operatorname{st}(U',\mathcal{U}_i)$ and hence $x \in p_i^{-1}(\operatorname{st}(U',\mathcal{U}_i))$. This verifies 1). 1) then immediately implies 2) since $\operatorname{st} p_j^{-1} \operatorname{st} \mathcal{U}_j < p_j^{-1} \operatorname{st}^2 \mathcal{U}_j$. □

By Lemma 3.3 2), every admissible approximate resolution is normal.

Proposition 3.4. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution of X. Then the following properties hold:*

1) *The family $\mathbb{U}_k = \{p_i^{-1} \operatorname{st}^k \mathcal{U}_i : i \in \mathbb{N}\}$ forms a normal sequence on X for $k \geq 0$;*
2) *The approximate resolution $\operatorname{st}^k \boldsymbol{p} = \{p_i\} : X \to \operatorname{st}^k \boldsymbol{X} = \{X_i, \operatorname{st}^k \mathcal{U}_i, p_{ii'}\}$ is admissible for $k \geq 1$.*

Proof. (NR) immediately implies 1) for $k = 0$ since $\operatorname{st} p_j^{-1}\mathcal{U}_j < p_j^{-1} \operatorname{st} \mathcal{U}_j$ for each j. Lemma 3.3 1) means (NR) for $\operatorname{st} \boldsymbol{p} : X \to \operatorname{st} \boldsymbol{X}$, which also implies 1) for $k = 1$. It is easy to see that $\operatorname{st} \boldsymbol{p} : X \to \operatorname{st} \boldsymbol{X}$ has properties (U) and (A), so 2) holds for $k = 1$. We then inductively obtain 1) and 2) for $k \geq 2$, as required. □

For any approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we can always find an admissible approximate resolution $\boldsymbol{p}' = \{p_{k_i}\} : X \to \boldsymbol{X}' = \{X_{k_i}, \mathcal{U}_{k_i}, p_{k_ik_j}\}$ by taking a subsystem.

Let $\boldsymbol{p} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be any normal approximate resolution of X. Then for any $x, x' \in X$, we define the function $\mathcal{D}_{\boldsymbol{p}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_{\boldsymbol{p}}(x,x') = \begin{cases} 9 & \text{if } (p_i(x), p_i(x')) \not< \mathcal{U}_i \text{ for any } i; \\ \frac{1}{3^{i-2}} & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ but } (p_i(x), p_i(x')) \not< \mathcal{U}_{i+1}\ ; \\ 0 & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ for all } i\ , \end{cases}$$

and the function $\mathrm{d}_{\boldsymbol{p}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_{\boldsymbol{p}}(x,x') = \inf\{\mathcal{D}_{\boldsymbol{p}}(x,x_1) + \mathcal{D}_{\boldsymbol{p}}(x_1,x_2) + \cdots + \mathcal{D}_{\boldsymbol{p}}(x_n,x')\}$$

where the infimum is taken over all finitely many points $x_1, x_2, ..., x_n$ of X. Note that property (B1) implies that the family $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$ has property (B) and that $\mathrm{d}_{\boldsymbol{p}}(x,x') = \mathrm{d}_{\mathbb{U}}(x,x')$ for any $x, x' \in X$.

For each approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we define the approximate sequence $\Sigma\boldsymbol{X}$ as $\{Z_i, \mathcal{W}_i, r_{ii'}\}$ where $Z_i = X_{i+1}$, $\mathcal{W}_i = \mathcal{U}_{i+1}$, $r_{ii'} = p_{i+1i'+1} : Z_{i'} \to Z_i$ and the approximate resolution $\Sigma\boldsymbol{p}$ as $\{r_i : i \in \mathbb{N}\} : X \to \Sigma\boldsymbol{X}$ where $r_i = p_{i+1} : X \to X_{i+1}$. Let $\Sigma^0\boldsymbol{X} = \boldsymbol{X}$ and $\Sigma^0\boldsymbol{p} = \boldsymbol{p}$, and for each $n \in \mathbb{N}$, let $\Sigma^n\boldsymbol{X} = \Sigma(\Sigma^{n-1}\boldsymbol{X})$ and $\Sigma^n\boldsymbol{p} = \Sigma(\Sigma^{n-1}\boldsymbol{p})$.

Proposition 3.5. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be a normal approximate resolution of X. Then*

1) $\mathrm{d}_{\Sigma^n\boldsymbol{p}}(x,x') = 3^n \mathrm{d}_{\boldsymbol{p}}(x,x')$ *for $x, x' \in X$ and for each $n \in \mathbb{N}$; and*

2) $\mathrm{d}_{\mathrm{st}\,\boldsymbol{p}}(x,x') \le \mathrm{d}_{\boldsymbol{p}}(x,x') \le 3\,\mathrm{d}_{\mathrm{st}\,\boldsymbol{p}}(x,x')$ *for* $x,x' \in X$ *if* $\boldsymbol{p}$ *has property (NR).*

Proof. 1) is obvious, and 2) follows from the fact that $p_{i+1}^{-1}\,\mathrm{st}\,\mathcal{U}_{i+1} < p_i^{-1}\mathcal{U}_i < p_i^{-1}\,\mathrm{st}\,\mathcal{U}_i$ for each i. $\square$

Let X and Y be compact metric spaces, and let $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ be normal approximate resolutions of X and Y, respectively. A map $f : X \to Y$ is said to be a $(\boldsymbol{p},\boldsymbol{q})$-*Lipschitz map* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \le \alpha\,\mathrm{d}_{\boldsymbol{p}}(x,x') \text{ for } x,x' \in X.$$

In particular, if we can choose α such that $0 < \alpha < 1$, a map $f : X \to Y$ is said to be a $(\boldsymbol{p},\boldsymbol{q})$-*contraction map.*

For each $m \in \mathbb{Z}$, consider the following condition:

(Lip)$_m$: $\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \le 3^m\,\mathrm{d}_{\boldsymbol{p}}(x,x')$ for $x,x' \in X$,

and for each $m \ge 0$ and for each approximate map $\boldsymbol{f} = \{f_i, f\} : \boldsymbol{X} \to \boldsymbol{Y}$, consider the following condition:

(ALip)$_m$: For each i, there exists $j_0 > i$ with the property that each $j > j_0$ admits $i_0 > f(j), i+m$ such that for each $i' > i_0$,

$$p_{i+m,i'}^{-1}\mathcal{U}_{i+m} < p_{f(j)i'}^{-1} f_j^{-1} q_{ij}^{-1}\mathcal{V}_i.$$

$(\boldsymbol{p},\boldsymbol{q})$-Lipschitz maps are characterized in terms of condition (ALip)$_m$ for approximate resolutions as follows:

Theorem 3.6. *Let* $f : X \to Y$ *be a map, and let* $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ *be an approximate map such that* $(\boldsymbol{f},\boldsymbol{p},\boldsymbol{q})$ *is an approximate resolution of* f *where* $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ *and* $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ *are admissible approximate resolutions of* X *and* Y*, respectively. Then the following implications hold for* $m \ge 0$*:*

1) *(ALip)$_m$ for* $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y}$ $\Rightarrow$ *(Lip)$_m$ for* $\boldsymbol{p}$ *and* $\mathrm{st}^2\,\boldsymbol{q}$ $\Rightarrow$ *(Lip)$_{m+2}$ for* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*.*

Moreover, if each p_i *is surjective, the following implication also holds:*

2) *(Lip)$_m$ for* $\boldsymbol{p}$ *and* $\boldsymbol{q}$ $\Rightarrow$ *(ALip)$_{m+4}$ for* $\boldsymbol{i}_{\mathrm{st}\,\boldsymbol{Y}}\boldsymbol{i}_{\boldsymbol{Y}}\boldsymbol{f} : \boldsymbol{X} \to \mathrm{st}^2\,\boldsymbol{Y}$*.*

Proof. To show 1), it suffices to show that for each i,

$$p_{i+m}^{-1}\mathcal{U}_{i+m} < f^{-1}q_i^{-1}\,\mathrm{st}^2\,\mathcal{V}_i. \tag{3.2}$$

Indeed, this means (M)$_{m,0}$ for $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$ and $\mathbb{V} = \{q_i^{-1}\,\mathrm{st}^2\,\mathcal{V}_i\}$, which implies by Theorem 3.2 1) that (L)$_m$ for $\mathbb{U}$ and $\mathbb{V}$. But this means (Lip)$_m$ for $\boldsymbol{p}$ and $\mathrm{st}^2\,\boldsymbol{q}$. Also since $3^{-2}\,\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \le \mathrm{d}_{\mathrm{st}^2\,\boldsymbol{q}}(f(x), f(x'))$ for $x,x' \in X$ (Proposition 3.5 2)), then this implies (Lip)$_{m+2}$ for $\boldsymbol{p}$ and $\boldsymbol{q}$.

Now let $i \in \mathbb{N}$, and take $\mathcal{V} \in \mathrm{Cov}(Y_i)$ such that

$$\mathrm{st}\,\mathcal{V} < \mathcal{V}_i. \tag{3.3}$$

By (ALip)$_m$ for $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y}$, take $j_0 > i$ as in (ALip)$_m$. Then (LAM) implies that there exists $j_1 > j_0$ such that for each $j > j_1$

$$(q_i f, q_{ij} f_j p_{f(j)}) < \mathcal{V}. \tag{3.4}$$

Fix $j > j_1$, and by the choice of j_0, there exists $i_0 > f(j), i+m$ such that for each $i' > i_0$

$$p_{i+m,i'}^{-1}\,\mathrm{st}\,\mathcal{U}_{i+m} < p_{f(j)i'}^{-1} f_j^{-1} q_{ij}^{-1}\,\mathrm{st}\,\mathcal{V}_i. \tag{3.5}$$

By (AS), there exists $i' > i_0$ such that

$$(p_{f(j)}, p_{f(j)i'}p_{i'}) < f_j^{-1}q_{ij}^{-1}\mathcal{V}, \tag{3.6}$$

and

$$(p_{i+m}, p_{i+m,i'}p_{i'}) < \mathcal{U}_{i+m}. \tag{3.7}$$

To verify (3.2), let $U \in \mathcal{U}_{i+m}$. Then by (3.5) there exists $V \in \mathcal{V}_i$ such that

$$q_{ij}f_jp_{f(j)i'}(p_{i+m,i'}^{-1}(\mathrm{st}(U, \mathcal{U}_{i+m}))) \subseteq \mathrm{st}(V, \mathcal{V}_i). \tag{3.8}$$

By (3.7),

$$p_{i+m}^{-1}(U) \subseteq p_{i'}^{-1}p_{i+m,i'}^{-1}(\mathrm{st}(U, \mathcal{U}_{i+m})).$$

This together with (3.8) implies

$$q_{ij}f_jp_{f(j)i'}p_{i'}(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}(V, \mathcal{V}_i).$$

Applying (3.6) to this, we get

$$q_{ij}f_jp_{f(j)}(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}(\mathrm{st}(V, \mathcal{V}_i), \mathcal{V}).$$

Applying (3.4) to this and using (3.3), we have

$$q_if(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}(\mathrm{st}(\mathrm{st}(V, \mathcal{V}_i), \mathcal{V}), \mathcal{V}) \subseteq \mathrm{st}(\mathrm{st}(V, \mathcal{V}_i), \mathcal{V}_i),$$

verifying (3.2).

2) easily follows from [10, Theorems 6.1, 6.2]. □

In a similar way $(\boldsymbol{p}, \boldsymbol{q})$-contraction maps are characterized in terms of the following condition for $m \geq 0$:

(ACon)$_m$: For each i there exists $j_0 > i$ with the property that each $j > j_0$ admits $i_0 > f(j), i$ such that for each $i' > i_0$

$$p_{ii'}^{-1}\mathcal{U}_i < p_{f(j)i'}^{-1}f_j^{-1}q_{i+m,j}^{-1}\mathcal{V}_{i+m}.$$

Theorem 3.7. *Let $f : X \to Y$ be a map, and let $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ be an approximate resolution of f where $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ are admissible approximate resolutions of X and Y, respectively. Then the following implications hold for $m \geq 0$:*

1) *(ACon)$_m$ for $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y}$ $\Rightarrow$ (Lip)$_{-m}$ for $\boldsymbol{p}$ and $\mathrm{st}^2\,\boldsymbol{q}$ $\Rightarrow$ (Lip)$_{-m+2}$ for $\boldsymbol{p}$ and $\boldsymbol{q}$.*

Moreover, if each p_i is surjective, the following implication also holds:

2) *(Lip)$_{-m}$ for $\boldsymbol{p}$ and $\boldsymbol{q}$ $\Rightarrow$ (ACon)$_{m-4}$ for $\boldsymbol{i}_{\mathrm{st}\,\boldsymbol{Y}}\boldsymbol{i}_{\boldsymbol{Y}}\boldsymbol{f} : \boldsymbol{X} \to \mathrm{st}^2\,\boldsymbol{Y}$.*

Proof. 1) is similar to Theorem 3.6 1), and 2) follows from [10, Theorems 7.2, 7.3]. □

Remark. In [10], Theorems 3.6 1) and 3.7 1) are shown under the assumption that each p_i is surjective. Here note that if X is connected, there exists an approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ with each p_i being surjective (see Theorem 2.3).

4. Box-counting dimension of spaces with normal open coverings

Throughout the rest of the paper, *spaces mean compact metric spaces.*

Let X be a space. For each $\mathcal{U} \in \mathrm{Cov}(X)$ and for each closed subset A of X, let

$$N_{\mathcal{U}}(A) = \min\{n : A \subseteq U_1 \cup \cdots \cup U_n, U_i \in \mathcal{U}\}.$$

For each normal sequence $\mathbb{U} = \{\mathcal{U}_i\}$ on a space X, we respectively define the *lower and upper box-counting dimensions* of $(X, \mathbb{U})$ by

$$\underline{\dim}_B(X, \mathbb{U}) = \varliminf_{i\to\infty} \frac{\log_3 N_{\mathcal{U}_i}(X)}{i}$$

and

$$\overline{\dim}_B(X, \mathbb{U}) = \varlimsup_{i\to\infty} \frac{\log_3 N_{\mathcal{U}_i}(X)}{i}.$$

If the two values coincide, the common value is called the *box-counting dimension* of $(X, \mathbb{U})$ and denoted by $\dim_B(X, \mathbb{U})$. In a similar way, we can define the box-counting dimension for each closed subspace A of X with a normal sequence $\mathcal{U}$ on X.

Proposition 4.1. *Let $\mathbb{U} = \{\mathcal{U}_i\}$ and $\mathbb{V} = \{V_i\}$ be normal sequences on X. If $\mathbb{U} < \mathbb{V}$, then $\underline{\dim}_B(X, \mathbb{V}) \le \underline{\dim}_B(X, \mathbb{U})$ and $\overline{\dim}_B(X, \mathbb{V}) \le \overline{\dim}_B(X, \mathbb{U})$.*

Proof. By assumption, $\mathcal{U}_{i+m} < \mathcal{V}_i$ for each i, and hence $N_{\mathcal{V}_i}(X) \le N_{U_{i+m}}(X)$ for each i, which implies the assertions. □

Proposition 4.2. *Let $\mathbb{U} = \{\mathcal{U}_i\}$ be a normal sequence on X. Then for any $m \ge 0$,*

$$\underline{\dim}_B(X, \mathbb{U}) = \underline{\dim}_B(X, \Sigma^m \mathbb{U})$$

and

$$\overline{\dim}_B(X, \mathbb{U}) = \overline{\dim}_B(X, \Sigma^m \mathbb{U}).$$

Proof. The first equality follows from the fact

$$\varliminf_{i\to\infty} \frac{\log_3 N_{\mathcal{U}_{i+m}}(X)}{i} = \varliminf_{i\to\infty} \frac{\log_3 N_{\mathcal{U}_i}(X)}{i}$$

and the second equality is similar. □

Corollary 4.3. *Let $\mathbb{U} = \{\mathcal{U}_i\}$ be a normal sequence on X. Then*

$$\underline{\dim}_B(X, \mathbb{U}) = \underline{\dim}_B(X, \mathrm{st}\, \mathbb{U})$$

and

$$\overline{\dim}_B(X, \mathbb{U}) = \overline{\dim}_B(X, \mathrm{st}\, \mathbb{U}).$$

Proof. Since $\mathbb{U} < \mathrm{st}\, \mathbb{U}$, by Proposition 4.1, $\underline{\dim}_B(X, \mathrm{st}\, \mathbb{U}) \le \underline{\dim}_B(X, \mathbb{U})$. On the other hand, since $\mathbb{U}$ is a normal sequence, $\Sigma\, \mathrm{st}\, \mathbb{U} < \mathbb{U}$, which implies by Propositions 4.1 and 4.2 that $\underline{\dim}_B(X, \mathbb{U}) \le \underline{\dim}_B(X, \mathrm{st}\, \mathbb{U})$. Similarly for the other equality. □

Proposition 4.4. *Let X be a compact subset of $\mathbb{R}^m$ with the usual metric, and let $\mathbb{U} = \{\mathcal{U}_i\}$ be the normal sequence of open coverings $\mathcal{U}_i$ by open balls with radius $\frac{1}{3^i}$. Then $\underline{\dim}_B(X, \mathbb{U}) = \underline{\dim}_B X$ and $\overline{\dim}_B(X, \mathbb{U}) = \overline{\dim}_B X$.*

Proof. The first equality follows from the following:

$$\underline{\dim}_B X = \varliminf_{\delta\to 0}\frac{\log N_\delta(X)}{-\log\delta} = \varliminf_{i\to\infty}\frac{\log N_{\frac{1}{3^i}}(X)}{-\log\frac{1}{3^i}} = \varliminf_{i\to\infty}\frac{\log_3 N_{\mathcal{U}_i}(X)}{i} = \underline{\dim}_B(X,\mathbb{U}).$$

Similarly for the other equality. □

Corollary 4.5. *Let X be a compact smooth m-dimensional submanifold of $\mathbb{R}^n$, and let $\mathbb{U}$ be the normal sequence as in Proposition 4.4. Then $\dim_B(X,\mathbb{U}) = m$.*

Proposition 4.6. *Let $\mathbb{U}=\{\mathcal{U}_i\}$ be a normal sequence on X with property (B). Consider X as the compact metric space with the metric $\mathrm{d}_\mathbb{U}$ induced by $\mathbb{U}$. Then $\underline{\dim}_B(X,\mathbb{U}) = \underline{\dim}_B X$ and $\overline{\dim}_B(X,\mathbb{U}) = \overline{\dim}_B X$.*

Proof. For each $r>0$, let $\mathcal{B}(r)$ denote the open covering of X consisting of open balls with radius r. Then since by (3.1)

$$\mathrm{st}(x,\mathcal{U}_{i+3}) \subseteq U_{\mathrm{d}_\mathbb{U}}(x,\frac{1}{3^i}) \subseteq \mathrm{st}(x,\mathcal{U}_i) \text{ for each } i,$$

then $\mathcal{U}_{i+3} < \mathcal{B}(\frac{1}{3^i})$ and $\mathcal{B}(\frac{1}{3^{i+1}}) < \mathrm{st}\,\mathcal{U}_{i+1} < \mathcal{U}_i$ for each i. Let $\mathbb{B}(\frac{1}{3^i})$ denote the normal sequence $\{\mathcal{B}(\frac{1}{3^i})\}$. Then $\Sigma^3\mathbb{U} < \mathbb{B}(\frac{1}{3^i})$ and $\Sigma\mathbb{B}(\frac{1}{3^i}) < \mathbb{U}$, and hence, by Propositions 4.1 and 4.2,

$$\underline{\dim}_B(X,\mathbb{U}) = \underline{\dim}_B(X,\mathbb{B}(\frac{1}{3^i})).$$

But,

$$\underline{\dim}_B X = \varliminf_{\delta\to 0}\frac{\log N_\delta(X)}{-\log\delta} = \varliminf_{i\to\infty}\frac{\log_3 N_{\frac{1}{3^i}}(X)}{-\log_3\frac{1}{3^i}} = \varliminf_{i\to\infty}\frac{\log_3 N_{\mathcal{B}(\frac{1}{3^i})}(X)}{i}.$$

Hence we have the first equality, and the second equality is proved similarly. □

Proposition 4.7. *Let $\mathbb{U}=\{\mathcal{U}_i\}$ and $\mathbb{V}=\{\mathcal{V}_i\}$ be normal sequences on X and Y, respectively. If $f: X\to Y$ is a $(\mathbb{U},\mathbb{V})$-Lipschitz map, then*

$$\underline{\dim}_B(f(X),\mathbb{V}) \le \underline{\dim}_B(X,\mathbb{U})$$

and

$$\overline{\dim}_B(f(X),\mathbb{V}) \le \overline{\dim}_B(X,\mathbb{U}).$$

Proof. By hypothesis and Theorem 3.2, $\Sigma^m\mathbb{U} < f^{-1}\mathbb{V}$ for some $m\ge 0$, and hence $N_{\mathcal{V}_i}(f(X)) \le N_{\mathcal{U}_{i+m}}(X)$. This implies $\underline{\dim}_B(f(X),\mathbb{V}) \le \underline{\dim}_B(X,\Sigma^m\mathbb{U}) = \underline{\dim}_B(X,\mathbb{U})$ (see Proposition 4.2). Similarly for the other. □

Proposition 4.8. *Let $X = X_1\cup X_2$ where X_1 and X_2 are closed subsets of X, and let $\mathbb{U}=\{\mathcal{U}_i\}$ be a normal sequence on X. Then*

$$\overline{\dim}_B(X,\mathbb{U}) = \max\{\overline{\dim}_B(X_1,\mathbb{U}),\overline{\dim}_B(X_2,\mathbb{U})\}.$$

Proof. For each i, $N_{\mathcal{U}_i}(X) \ge N_{\mathcal{U}_i}(X_1), N_{\mathcal{U}_i}(X_2)$, and hence "$\ge$" holds. For the other inequality,

$$N_{\mathcal{U}_i}(X) \le N_{\mathcal{U}_i}(X_1) + N_{\mathcal{U}_i}(X_2) \le 2\max\{N_{\mathcal{U}_i|X_1}(X_1), N_{\mathcal{U}_i|X_2}(X_2)\}.$$

So, for each i,

$$\sup_{k\ge i}\frac{\log_3 N_{\mathcal{U}_k}(X)}{k} \le \sup_{k\ge i}\frac{\log_3 2}{k} + \max\left\{\sup_{k\ge i}\frac{\log_3 N_{\mathcal{U}_i}(X_1)}{k}, \sup_{k\ge i}\frac{\log_3 N_{\mathcal{U}_i}(X_2)}{k}\right\}.$$

Taking limits, we have "$\le$". □

5. Box-counting dimension of approximate resolutions

For each approximate sequence $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we define the *upper* and *lower box-counting dimensions* of $\boldsymbol{X}$ by

$$\overline{\dim}_B \boldsymbol{X} = \varlimsup_{i\to\infty} \frac{\log_3 \beta_i(\boldsymbol{X})}{i}$$

and

$$\underline{\dim}_B \boldsymbol{X} = \varliminf_{i\to\infty} \frac{\log_3 \beta_i(\boldsymbol{X})}{i},$$

where

$$\beta_i(\boldsymbol{X}) = \varlimsup_{j\to\infty} N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \text{ for each } i \in \mathbb{N}.$$

If the two values coincide, then we write $\dim_B \boldsymbol{X}$ for the common value and call it the *box-counting dimension* of $\boldsymbol{X}$.

Proposition 5.1. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an approximate resolution of a space X. Then we have the followind properties:*

1) *For each i, there is $i_0 \geq i$ such that*
$$N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \leq N_{p_i^{-1}\mathcal{U}_i}(X) \textit{ for } j \geq i_0;$$
2) *For each i and for each $\mathcal{U} \in \mathrm{Cov}(X_i)$, there is $i_0 \geq i$ such that*
$$N_{p_i^{-1}\mathrm{st}(\mathcal{U}_i,\mathcal{U})}(X) \leq N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \textit{ for } j \geq i_0,$$
where
$$\mathrm{st}(\mathcal{U}_i,\mathcal{U}) = \{\mathrm{st}(U,\mathcal{U}) : U \in \mathcal{U}_i\}; \textit{ and}$$
3) *For each i and for each $\mathcal{U} \in \mathrm{Cov}(X_i)$, there is $i_0 \geq i$ such that*
$$N_{p_{ik}^{-1}\mathrm{st}(\mathcal{U}_i,\mathcal{U})}(X_k) \leq N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \textit{ for } i_0 \leq j \leq k.$$

Proof. For 1), let $n = N_{p_i^{-1}\mathcal{U}_i}(X)$ and choose $U_1, ..., U_n \in \mathcal{U}_i$ such that $X = p_i^{-1}(U_1) \cup \cdots \cup p_i^{-1}(U_n)$. Then $p_i(X) \subseteq U_1 \cup \cdots \cup U_n$, and there is $\mathcal{U} \in \mathrm{Cov}(X_i)$ such that $\mathcal{U}$ is of the form $\{U_1, ..., U_n\} \cup \mathcal{U}'$ where $\mathcal{U}'$ consists of some open subsets U' of X_i such that $U' \cap p_i(X) = \emptyset$. So we have

$$N_{p_i^{-1}\mathcal{U}_i}(X) = N_{\mathcal{U}_i}(p_i(X)) = N_{\mathcal{U}_i}(\mathrm{st}(p_i(X),\mathcal{U})). \tag{5.1}$$

But (B2) implies that there is $i_0 \geq i$ such that for $j \geq i_0$,

$$p_{ij}(X_j) \subseteq \mathrm{st}(p_i(X),\mathcal{U}),$$

and hence

$$N_{\mathcal{U}_i}(\mathrm{st}(p_i(X),\mathcal{U})) \geq N_{\mathcal{U}_i}(p_{ij}(X_j)) = N_{p_{ij}^{-1}\mathcal{U}_i}(X_j). \tag{5.2}$$

(5.1) and (5.2) imply 1).

For 2), let $i \in \mathbb{N}$, and let $\mathcal{U} \in \mathrm{Cov}(X_i)$. Then by (AS) there is $i_0 \geq i$ such that

$$(p_{ij}p_j, p_i) < \mathcal{U} \text{ for } j \geq i_0. \tag{5.3}$$

Let $j \geq i_0$, and suppose $n = N_{p_{ij}^{-1}\mathcal{U}_i}(X_j)$. Choose $U_1, ..., U_n \in \mathcal{U}_i$ such that $X_j = p_{ij}^{-1}(U_1) \cup \cdots \cup p_{ij}^{-1}(U_n)$. Then $p_j^{-1}p_{ij}^{-1}(U_1) \cup \cdots \cup p_j^{-1}p_{ij}^{-1}(U_n) = X$. This together with (5.3) implies $p_i^{-1}(\mathrm{st}(U_1,\mathcal{U})) \cup \cdots \cup p_i^{-1}(\mathrm{st}(U_n,\mathcal{U})) = X$. Thus $N_{p_i^{-1}\mathrm{st}(\mathcal{U}_i,\mathcal{U})}(X) \leq n$, which verifies 2).

For 3), let $i \in \mathbb{N}$, and let $\mathcal{U} \in \mathrm{Cov}(X_i)$. Then by (A2) there is $i_0 \geq i$ such that

$$(p_{ik}, p_{ij}p_{jk}) < \mathcal{U} \text{ for } i_0 \leq j \leq k. \tag{5.4}$$

Let $j \geq i_0$, and suppose $n = N_{p_{ij}^{-1}\mathcal{U}_i}(X_j)$. Choose $U_1, ..., U_n \in \mathcal{U}_i$ such that $p_{ij}^{-1}(U_1) \cup \cdots \cup p_{ij}^{-1}(U_n) = X_j$. Then $p_{jk}^{-1}p_{ij}^{-1}(U_1) \cup \cdots \cup p_{jk}^{-1}p_{ij}^{-1}(U_n) = X_k$ for $j \leq k$. This together with (5.4) implies $p_{ik}^{-1}(\mathrm{st}(U_1, \mathcal{U})) \cup \cdots \cup p_{ik}^{-1}(\mathrm{st}(U_n, \mathcal{U})) = X_k$ and hence $N_{p_{ik}^{-1}\,\mathrm{st}(\mathcal{U}_i,\mathcal{U})}(X_k) \leq n$, which verifies 3). $\square$

Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an approximate resolution. Then, for each i, $\beta_i(\boldsymbol{X}) < \infty$ since each i admits $i_0 \geq i$ such that $N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \leq N_{p_i^{-1}\mathcal{U}_i}(X)$ for $j \geq i_0$ by Proposition 5.1 1). We define the *upper* and *lower box-counting dimensions* of $\boldsymbol{p} : X \to \boldsymbol{X}$ by

$$\overline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \overline{\dim}_B \boldsymbol{X}$$

and

$$\underline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) = \underline{\dim}_B \boldsymbol{X}.$$

If the two values coincide, then we write $\dim_B(\boldsymbol{p} : X \to \boldsymbol{X})$ for the common value and call it the *box-counting dimension* of $\boldsymbol{p} : X \to \boldsymbol{X}$. If there is no confusion, we write $\overline{\dim}_B(\boldsymbol{p})$, $\underline{\dim}_B(\boldsymbol{p})$, $\dim_B(\boldsymbol{p})$ respectively for $\overline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X})$, $\underline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X})$, $\dim_B(\boldsymbol{p} : X \to \boldsymbol{X})$.

Proposition 5.2. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an approximate resolution of X. Then if each p_i is onto, then*

$$\overline{\dim}_B(\boldsymbol{p}) = \overline{\lim_{i \to \infty}} \frac{\log_3 N_{\mathcal{U}_i}(X_i)}{i}$$

and

$$\underline{\dim}_B(\boldsymbol{p}) = \underline{\lim_{i \to \infty}} \frac{\log_3 N_{\mathcal{U}_i}(X_i)}{i}.$$

Proof. For each i, $N_{p_i^{-1}\mathcal{U}_i}(X) \leq N_{\mathcal{U}_i}(X_i)$, and since p_i is onto, $N_{\mathcal{U}_i}(X_i) \leq N_{p_i^{-1}\mathcal{U}_i}(X)$. Those facts easily imply the equalities. $\square$

Theorem 5.3. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution, and let $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i\}$. Then*

$$\underline{\dim}_B(\mathrm{st}\,\boldsymbol{p}) = \underline{\dim}_B(X, \mathrm{st}\,\mathbb{U})$$

and

$$\overline{\dim}_B(\mathrm{st}\,\boldsymbol{p}) = \overline{\dim}_B(X, \mathrm{st}\,\mathbb{U}).$$

Proof. Let $i \in \mathbb{N}$. By Proposition 5.1 1) and 2), there is $i_0 \geq i$ such that

$$N_{p_i^{-1}\,\mathrm{st}^2\mathcal{U}_i}(X) \leq N_{p_{ij}^{-1}\,\mathrm{st}\,\mathcal{U}_i}(X_j) \leq N_{p_i^{-1}\,\mathrm{st}\,\mathcal{U}_i}(X) \text{ for } j \geq i_0. \tag{5.5}$$

But by Lemma 3.3 1) and (NR), $p_i^{-1}\,\mathrm{st}^2\,\mathcal{U}_i < p_{i-1}^{-1}\,\mathrm{st}\,\mathcal{U}_{i-1} < p_{i-2}^{-1}\mathcal{U}_{i-2}$, and so

$$N_{p_{i-2}^{-1}\mathcal{U}_{i-2}}(X) \leq N_{p_i^{-1}\,\mathrm{st}^2\,\mathcal{U}_i}(X). \tag{5.6}$$

By (5.5) and (5.6),

$$N_{\mathrm{st}\,p_{i-2}^{-1}\mathcal{U}_{i-2}}(X) \leq N_{p_{i-2}^{-1}\mathcal{U}_{i-2}}(X) \leq \beta_i(\mathrm{st}\,\boldsymbol{X}) \leq N_{p_i^{-1}\,\mathrm{st}\,\mathcal{U}_i}(X) \leq N_{\mathrm{st}\,p_i^{-1}\mathcal{U}_i}(X),$$

and hence

$$\underline{\lim}_{i\to\infty}\frac{N_{\mathrm{st}\,p_{i-2}^{-1}\mathcal{U}_{i-2}}(X)}{i} \le \underline{\lim}_{i\to\infty}\frac{\beta_i(\mathrm{st}\,\boldsymbol{X})}{i} \le \underline{\lim}_{i\to\infty}\frac{N_{\mathrm{st}\,p_i^{-1}\mathcal{U}_i}(X)}{i}.$$

This implies the first assertion, and the second assertion is similarly proved. □

For the rest of this section, we investigate the fundamental properties of box-counting dimension, especially, the relationship to some operations on the approximate resolutions.

Proposition 5.4. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an approximate resolution of X. Then, for each $k \ge 1$,*

$$\underline{\dim}_B(\boldsymbol{p}) = \underline{\dim}_B(\Sigma^k \boldsymbol{p})$$

and

$$\overline{\dim}_B(\boldsymbol{p}) = \overline{\dim}_B(\Sigma^k \boldsymbol{p}).$$

Proof. For each i, $\beta_i(\Sigma^k \boldsymbol{X}) = \beta_{i+k}(\boldsymbol{X})$, and hence

$$\begin{aligned}\underline{\dim}_B(\Sigma^k \boldsymbol{p} : X \to \Sigma^k \boldsymbol{X}) &= \underline{\lim}_{i\to\infty}\frac{\log_3 \beta_i(\Sigma^k \boldsymbol{X})}{i} = \underline{\lim}_{i\to\infty}\frac{\log_3 \beta_{i+k}(\boldsymbol{X})}{i}\\ &= \underline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}).\end{aligned}$$

Similarly for the other equality. □

Proposition 5.5. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution of X. Then*

$$\underline{\dim}_B(\boldsymbol{p}) \ge \underline{\dim}_B(\mathrm{st}\,\boldsymbol{p}) = \underline{\dim}_B(\mathrm{st}^2\,\boldsymbol{p})$$

and

$$\overline{\dim}_B(\boldsymbol{p}) \ge \overline{\dim}_B(\mathrm{st}\,\boldsymbol{p}) = \overline{\dim}_B(\mathrm{st}^2\,\boldsymbol{p}).$$

Proof. The two "$\ge$" follow from the fact that

$$N_{p_{ij}^{-1}\mathrm{st}\,\mathcal{U}_i}(X_j) \le N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \text{ for } i \le j.$$

To verify the two equalities, let $i \in \mathbb{N}$. By Proposition 5.1 2), there is $i_0 \ge i$ such that

$$N_{p_i^{-1}\mathrm{st}^3\mathcal{U}_i}(X) \le N_{p_{ij}^{-1}\mathrm{st}^2\mathcal{U}_i}(X_j) \text{ for } j \ge i_0. \tag{5.7}$$

Since $\boldsymbol{p} : X \to \boldsymbol{X}$ has properties (U) and (A), by Lemma 3.3 1),

$$p_{i-1}^{-1}\,\mathrm{st}^2\,\mathcal{U}_{i-1} < p_{i-2}^{-1}\,\mathrm{st}\,\mathcal{U}_{i-2}.$$

Since $\mathrm{st}\,\boldsymbol{p} : X \to \mathrm{st}\,\boldsymbol{X}$ also has properties (U) and (A) (Proposition 3.4 2)), by Lemma 3.3 1),

$$p_i^{-1}\,\mathrm{st}^3\,\mathcal{U}_i < p_{i-1}^{-1}\,\mathrm{st}^2\,\mathcal{U}_{i-1}.$$

So $p_i^{-1}\,\mathrm{st}^3\,\mathcal{U}_i < p_{i-2}^{-1}\,\mathrm{st}\,\mathcal{U}_{i-2}$ and hence

$$N_{p_{i-2}^{-1}\mathrm{st}\,\mathcal{U}_{i-2}}(X) \le N_{p_i^{-1}\mathrm{st}^3\mathcal{U}_i}(X). \tag{5.8}$$

By Proposition 5.1 1), there is $i_0' \ge i_0$ such that for $j \ge i_0'$

$$N_{p_{i-2,j}^{-1}\mathrm{st}\,\mathcal{U}_{i-2}}(X_j) \le N_{p_{i-2}^{-1}\mathrm{st}\,\mathcal{U}_{i-2}}(X). \tag{5.9}$$

By (5.7), (5.8) and (5.9),

$$N_{p_{i-2,j}^{-1}\mathrm{st}\,\mathcal{U}_{i-2}}(X_j) \le N_{p_{ij}^{-1}\mathrm{st}^2\mathcal{U}_i}(X_j) \text{ for } j \ge i_0'.$$

On the other hand, we also have $N_{p_{ij}^{-1}\operatorname{st}^2\mathcal{U}_i}(X_j) \le N_{p_{ij}^{-1}\operatorname{st}\mathcal{U}_i}(X_j)$. So,

$$\beta_{i-2}(\operatorname{st}\boldsymbol{X}) \le \beta_i(\operatorname{st}^2\boldsymbol{X}) \le \beta_i(\operatorname{st}\boldsymbol{X})$$

and hence

$$\varliminf_{i\to\infty}\frac{\beta_{i-2}(\operatorname{st}\boldsymbol{X})}{i} \le \varliminf_{i\to\infty}\frac{\beta_i(\operatorname{st}^2\boldsymbol{X})}{i} \le \varliminf_{i\to\infty}\frac{\beta_i(\operatorname{st}\boldsymbol{X})}{i}.$$

This means the equality in the first assertion, and similarly for the equality in the second assertion. □

Proposition 5.6. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution. For each $k \ge 1$, define the approximate resolution $M^k\boldsymbol{p} = \{p_{k_i}\} : X \to M^k\boldsymbol{X} = \{X_{ki}, \mathcal{U}_{ki}, p_{ki,kj}\}$. Then, if $\underline{\dim}_B(\operatorname{st}\boldsymbol{p}) = \overline{\dim}_B(\operatorname{st}\boldsymbol{p})$, then*

$$\dim_B(M^k\operatorname{st}^2\boldsymbol{p}) = k\dim_B(\operatorname{st}^2\boldsymbol{p}) = k\dim_B(\operatorname{st}\boldsymbol{p}).$$

Proof. Let $i \in \mathbb{N}$. Then by Proposition 5.1 2) and 1), there is $i_0 \ge i$ such that for $j > i_0$,

$$N_{p_{ki}^{-1}\operatorname{st}^3\mathcal{U}_{ki}}(X) \le \left\{\begin{array}{c} N_{p_{ki,j}^{-1}\operatorname{st}^2\mathcal{U}_{ki}}(X_j) \\ N_{p_{ki,kj}^{-1}\operatorname{st}^2\mathcal{U}_{ki}}(X_{kj}) \end{array}\right\} \le N_{p_{ki}^{-1}\operatorname{st}^2\mathcal{U}_{ki}}(X) \le N_{p_{ki,j}^{-1}\operatorname{st}\mathcal{U}_{ki}}(X_j). \tag{5.10}$$

Since $\boldsymbol{p} : X \to \boldsymbol{X}$ has properties (U) and (A), by Proposition 3.4 2) and Lemma 3.3 1), we have

$$p_{ki}^{-1}\operatorname{st}^3\mathcal{U}_{ki} < p_{k(i-1)}^{-1}\operatorname{st}^2\mathcal{U}_{k(i-1)} < p_{k(i-2)}^{-1}\operatorname{st}\mathcal{U}_{k(i-2)},$$

and hence

$$N_{p_{k(i-2)}^{-1}\operatorname{st}\mathcal{U}_{k(i-2)}}(X) \le N_{p_{ki}^{-1}\operatorname{st}^3\mathcal{U}_{ki}}(X). \tag{5.11}$$

By Lemma 5.1 1), there is $i_0' \ge i_0$ such that for $j \ge i_0'$,

$$N_{p_{k(i-2),j}^{-1}\operatorname{st}\mathcal{U}_{k(i-2)}}(X_j) \le N_{p_{k(i-2)}^{-1}\operatorname{st}\mathcal{U}_{k(i-2)}}(X). \tag{5.12}$$

(5.12), (5.11) and (5.10) imply that

$$N_{p_{k(i-2),j}^{-1}\operatorname{st}\mathcal{U}_{k(i-2)}}(X_j) \le \left\{\begin{array}{c} N_{p_{ki,j}^{-1}\operatorname{st}^2\mathcal{U}_{ki}}(X_j) \\ N_{p_{ki,kj}^{-1}\operatorname{st}^2\mathcal{U}_{ki}}(X_{kj}) \end{array}\right\} \le N_{p_{ki,j}^{-1}\operatorname{st}\mathcal{U}_{ki}}(X_j) \text{ for } j > i_0'.$$

This implies that for each i,

$$\beta_{k(i-2)}(\operatorname{st}\boldsymbol{X}) \le \left\{\begin{array}{c} \beta_{ki}(\operatorname{st}^2\boldsymbol{X}) \\ \beta_i(M^k\operatorname{st}^2\boldsymbol{X}) \end{array}\right\} \le \beta_{ki}(\operatorname{st}\boldsymbol{X}),$$

and so

$$\frac{\log_3\beta_{k(i-2)}(\operatorname{st}\boldsymbol{X})}{k(i-2)}\cdot\frac{k(i-2)}{i} \le \left\{\begin{array}{c} \frac{\log_3\beta_{ki}(\operatorname{st}^2\boldsymbol{X})}{ki}\cdot k \\ \frac{\log_3\beta_i(M^k\operatorname{st}^2\boldsymbol{X})}{i} \end{array}\right\} \le \frac{\log_3\beta_{ki}(\operatorname{st}\boldsymbol{X})}{ki}\cdot k.$$

Since $\underline{\dim}_B(\boldsymbol{p} : X \to \operatorname{st}\boldsymbol{X}) = \overline{\dim}_B(\boldsymbol{p} : X \to \operatorname{st}\boldsymbol{X})$, the limits as $i \to \infty$ of the left and right hand sides of the above exist and coincide. Hence the limits in the middle exist and the two equalities in the assertion hold. □

Theorem 6.1. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = (Y_j, \mathcal{V}_j, q_{jj'})$ be admissible approximate resolutions, and let $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ be an approximate map with property (APS) (see Theorem 2.5). If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ satisfies (ALip)$_m$ for some $m \geq 0$, then we have the following inequalities:*

$$\underline{\dim}_B(\boldsymbol{p}) \geq \underline{\dim}_B(\mathrm{st}\,\boldsymbol{q})$$

and

$$\overline{\dim}_B(\boldsymbol{p}) \geq \overline{\dim}_B(\mathrm{st}\,\boldsymbol{q}).$$

Proof. Let $i \in \mathbb{N}$. By (A1), (ALip)$_m$ and (APS), there exist j, j_0 with $i < j < j_0$ and $f(j) > i + m$ with the property that for each $j' > j_0$, there exists i_0 with $i_0 > f(j), i + m$ such that for each $k > i_0$,

$$p_{i+m,k}^{-1}\mathcal{U}_{i+m} < p_{f(j)k}^{-1} f_j^{-1} q_{ij}^{-1} \mathcal{V}_i \tag{6.1}$$

and

$$q_{ij'}(Y_{j'}) \subseteq \mathrm{st}(q_{ij} f_j p_{f(j)k}(X_k), \mathcal{V}_i). \tag{6.2}$$

Then, by (6.1), for $k > i_0$,

$$N_{p_{i+m,k}^{-1}\mathcal{U}_{i+m}}(X_k) \geq N_{p_{f(j)k}^{-1} f_j^{-1} q_{ij}^{-1} \mathcal{V}_i}(X_k) = N_{\mathcal{V}_i}(q_{ij} f_j p_{f(j)k}(X_k)). \tag{6.3}$$

If $m = N_{\mathcal{V}_i}(q_{ij} f_j p_{f(j)k}(X_k))$, then there exist $V_1, ..., V_m \in \mathcal{V}_i$ such that

$$q_{ij} f_j p_{f(j)k}(X_k) \subseteq V_1 \cup \cdots \cup V_m,$$

which implies that

$$\mathrm{st}(q_{ij} f_j p_{f(j)k}(X_k), \mathcal{V}_i) \subseteq \mathrm{st}(V_1, \mathcal{V}_i) \cup \cdots \cup \mathrm{st}(V_m, \mathcal{V}_i).$$

This together with (6.2) implies that for each $j' > j_0$,

$$m \geq N_{\mathrm{st}\,\mathcal{V}_i}(\mathrm{st}(q_{ij} f_j p_{f(j)k}(X_k), \mathcal{V}_i)) \geq N_{\mathrm{st}\,\mathcal{V}_i}(q_{ij'}(Y_{j'})) = N_{q_{ij'}^{-1}\,\mathrm{st}\,\mathcal{V}_i}(Y_{j'}) \text{ for } k > i_0. \tag{6.4}$$

Thus, by (6.3) and (6.4), for each $j' > j_0$,

$$N_{p_{i+m,k}^{-1}\mathcal{U}_{i+m}}(X_k) \geq N_{q_{ij'}^{-1}\,\mathrm{st}\,\mathcal{V}_i}(Y_{j'}) \text{ for } k > i_0,$$

which implies that for each $j' > j_0$,

$$\lim_{n\to\infty} \sup_{k\geq n} N_{p_{i+m,k}^{-1}\mathcal{U}_{i+m}}(X_k) \geq N_{q_{ij'}^{-1}\,\mathrm{st}\,\mathcal{V}_i}(Y_{j'}).$$

and hence we have $\beta_{i+m}(\boldsymbol{X}) \geq \beta_i(\mathrm{st}\,\boldsymbol{Y})$ for each i. Thus we have the assertion. □

Corollary 6.2. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ be admissible approximate resolutions, and let $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$ be an approximate map with property (APS). If $\mathrm{st}\,\boldsymbol{f} : \mathrm{st}\,\boldsymbol{X} \to \mathrm{st}\,\boldsymbol{Y}$ satisfies (ALip)$_m$ for some $m \geq 0$, then*

$$\underline{\dim}_B(\mathrm{st}\,\boldsymbol{p}) \geq \underline{\dim}_B(\mathrm{st}\,\boldsymbol{q})$$

and

$$\overline{\dim}_B(\mathrm{st}\,\boldsymbol{p}) \geq \overline{\dim}_B(\mathrm{st}\,\boldsymbol{q}).$$

Proof. This easily follows from Proposition 5.5 and Theorem 6.1. □

An approximate resolution $\boldsymbol{p}' = \{p'_i\} : X' \to \boldsymbol{X}' = \{X'_i, \mathcal{U}'_i, p'_{ii'}\}$ is said to be *contained in* an approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ provided $X' \subseteq X$, $X'_i \subseteq X_i$, $\mathcal{U}'_i = \mathcal{U}_i | X'_i$, $p'_{ii'} = p_{ii'} | X'_{i'}$. Note if $\boldsymbol{p}$ is admissible, so is $\boldsymbol{p}'$

Theorem 7.1 (Subset theorem). *If an approximate resolution $\boldsymbol{p}' = \{p'_i\} : X' \to \boldsymbol{X}' = \{X'_i, \mathcal{U}'_i, p_{ii'}\}$ is contained in an admissible approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, then*

$$\underline{\dim}_B(\boldsymbol{p}') \leq \underline{\dim}_B(\boldsymbol{p})$$

and

$$\overline{\dim}_B(\boldsymbol{p}') \leq \overline{\dim}_B(\boldsymbol{p}).$$

Proof. This easily follows from the fact that $N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \geq N_{p'^{-1}_{ij}\mathcal{U}'_i}(X'_j)$ for $i < j$. □

Theorem 7.2 (Product theorem). *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$ be admissible approximate resolutions. Then the map $\boldsymbol{p} \times \boldsymbol{q} = \{p_i \times q_i\} : X \times Y \to \boldsymbol{X} \times \boldsymbol{Y} = \{X_i \times Y_i, \mathcal{U}_i \times \mathcal{V}_i, p_{ii'} \times q_{ii'}\}$ is an admissible approximate resolution, and*

$$\overline{\dim}_B(\boldsymbol{p} \times \boldsymbol{q} : X \times Y \to \boldsymbol{X} \times \boldsymbol{Y}) \leq \overline{\dim}_B(\boldsymbol{p} : X \to \boldsymbol{X}) + \overline{\dim}_B(\boldsymbol{q} : Y \to \boldsymbol{Y}).$$

Proof. It is easy to check (B1) and (B2) for $\boldsymbol{p}$ and $\boldsymbol{q}$ imply (B1) and (B2) for $\boldsymbol{p} \times \boldsymbol{q}$, using the fact that any open covering of the product $X \times Y$ is refined by the product of finite open coverings of X and Y. It is easy to see that the approximate resolution $\boldsymbol{p} \times \boldsymbol{q}$ is admissible. For each i, we have

$$N_{(p_{ii'} \times p_{ii'})^{-1}\mathcal{U}_i \times \mathcal{V}_i}(X_{i'} \times Y_{i'}) \leq N_{p_{ii'}^{-1}\mathcal{U}_i}(X_{i'}) \cdot N_{q_{ii'}^{-1}\mathcal{V}_i}(Y_{i'}) \text{ for } i < i',$$

and hence for each k,

$$\sup_{i' \geq k} N_{(p_{ii'} \times p_{ii'})^{-1}\mathcal{U}_i \times \mathcal{V}_i}(X_{i'} \times Y_{i'}) \leq \sup_{i' \geq k} N_{p_{ii'}^{-1}\mathcal{U}_i}(X_{i'}) \cdot \sup_{i' \geq k} N_{q_{ii'}^{-1}\mathcal{V}_i}(Y_{i'}).$$

This implies that, for each i, we have $\beta_i(\boldsymbol{X} \times \boldsymbol{Y}) \leq \beta_i(\boldsymbol{X}) \cdot \beta_i(\boldsymbol{Y})$, and hence

$$\overline{\lim_{i \to \infty}} \frac{\log_3 \beta_i(\boldsymbol{X} \times \boldsymbol{Y})}{i} \leq \overline{\lim_{i \to \infty}} \frac{\log_3 \beta_i(\boldsymbol{X})}{i} + \overline{\lim_{i \to \infty}} \frac{\log_3 \beta_i(\boldsymbol{Y})}{i},$$

which means the assertion. □

Theorem 7.3 (Sum theorem). *Let X_0 and X_1 be closed subsets of X such that $X = X_0 \cup X_1$, and let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an admissible approximate resolution of X such that the restrictions $\boldsymbol{p}|X_0 = \{p_i|X_0\} : X_0 \to \boldsymbol{X}_0 = \{X_{0i}, \mathcal{U}_i|X_{0i}, p_{ii'}|X_{0i'}\}$ and $\boldsymbol{p}|X_1 = \{p_i|X_1\} : X_1 \to \boldsymbol{X}_1 = \{X_{1i}, \mathcal{U}_i|X_{1i}, p_{ii'}|X_{1i'}\}$ are admissible approximate resolutions of X_0 and X_1, respectively, where X_{0i}, X_{1i} are subpolyhedra of X_i such that $X_{0i} \cup X_{1i} = X_i$. Then*

$$\overline{\dim}_B(\boldsymbol{p}) = \max\left\{\overline{\dim}_B(\boldsymbol{p}|X_0), \overline{\dim}_B(\boldsymbol{p}|X_1)\right\}.$$

Proof. For each i, we have

$$N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \leq N_{p_{ij}^{-1}\mathcal{U}_i}(X_{0j}) + N_{p_{ij}^{-1}\mathcal{U}_i}(X_{1j}) \leq 2\max\{N_{p_{ij}^{-1}\mathcal{U}_i}(X_{0j}), N_{p_{ij}^{-1}\mathcal{U}_i}(X_{1j})\} \text{ for } i < j.$$

Then, for $n > i$,

$$\sup_{j \geq n} N_{p_{ij}^{-1}\mathcal{U}_i}(X_j) \leq 2\max\{\sup_{j \geq n} N_{p_{ij}^{-1}\mathcal{U}_i}(X_{0j}), \sup_{j \geq n} N_{p_{ij}^{-1}\mathcal{U}_i}(X_{1j})\},$$

which implies

$$\beta_i(\boldsymbol{X}) \leq 2\max\{\beta_i(\boldsymbol{X}_0), \beta_i(\boldsymbol{X}_1)\},$$

and hence

$$\frac{\log_3 \beta_i(\boldsymbol{X})}{i} \leq \frac{\log_3 2}{i} + \max\left\{\frac{\log_3 \beta_i(\boldsymbol{X}_0)}{i}, \frac{\log_3 \beta_i(\boldsymbol{X}_1)}{i}\right\}.$$

Taking $\overline{\lim\limits_{i\to\infty}}$, we have

$$\overline{\dim}_B(\boldsymbol{p}) \leq \max\{\overline{\dim}_B(\boldsymbol{p}|X_0), \overline{\dim}_B(\boldsymbol{p}|X_1)\}.$$

The other direction follows from the fact that $N_{p_{ij}^{-1}\mathcal{U}_i}(X_{0j}), N_{p_{ij}^{-1}\mathcal{U}_i}(X_{1j}) \leq N_{p_{ij}^{-1}\mathcal{U}_i}(X_j)$ for $i < j$. $\square$

Remark. There exists such an approximate resolution of X as in Theorem 7.3. Indeed, there exists a resolution of the triad $(X; X_0, X_1)$ $\boldsymbol{p} = \{p_i\} : (X; X_0, X_1) \to (\boldsymbol{X}; \boldsymbol{X}_0, \boldsymbol{X}_1) = \{(X_i; X_{0i}, X_{1i}), p_{ii'}\}$ such that the restrictions $\boldsymbol{p}|X = \{p_i|X\} : X \to \boldsymbol{X}$, $\boldsymbol{p}|X_0 = \{p_i|X_0\} : X_0 \to \boldsymbol{X}_0 = \{X_{0i}, p_{ii'}|X_{0i'}\}$ and $\boldsymbol{p}|X_1 = \{p_i|X_1\} : X_1 \to \boldsymbol{X}_1 = \{X_{1i}, p_{ii'}|X_{1i'}\}$ are resolutions of X, X_0, X_1, respectively (see [8]). For each i, there exists a finite open covering $\mathcal{U}_i$ so that $\boldsymbol{p}|X = \{p_i|X\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is an admissible approximate resolution (see [16]) by taking a subsystem if necessary. Then it is easy to check $\boldsymbol{p}|X_0 = \{p_i|X_0\} : X_0 \to \{X_{0i}, \mathcal{U}_i|X_{0i}, p_{ii'}|X_{0i'}\}$ and $\boldsymbol{p}|X_1 = \{p_i|X_1\} : X_1 \to \{X_{1i}, \mathcal{U}_i|X_{1i}, p_{ii'}|X_{1i'}\}$ are admissible approximate resolutions as required.

8. Example

In this section we prove

Theorem 8.1. *For each $r \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ there exist a Cantor set X and an admissible approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ such that $\dim_B(\boldsymbol{p} : X \to \boldsymbol{X}) = r$.*

Proof. First assume $0 < r < \infty$. For each $n \geq 1$, let a_n be the smallest integer that is not less than nr. So, $a_n - 1 < nr \leq a_n$ and $a_n \leq a_{n+1}$ for each $n \geq 1$, and hence $\lim\limits_{i\to\infty} \frac{a_n}{n} = r$. Define the sequence $\{b_n : n \geq 1\}$ by $b_1 = a_1$ and $b_{n+1} = a_{n+1} - a_n$. Now we write

$$\frac{a_n}{n} = \frac{\log_3 3^{a_n}}{n} = \frac{\log_3 3^{b_1} \cdot 3^{b_2} \cdot \dots \cdot 3^{b_n}}{n}.$$

and define the inverse sequence $\{X_n, p_{nn'}\}$ as follows (see Figure 1): Let $X_0 = \{*\}$, and let X_n be the discrete space consisting of 3^{a_n} points $\{x_{i_1 i_2 \cdots i_n} : 1 \leq i_k \leq 3^{b_k}, k = 1, 2, \cdots, n\}$. Define the map $p_{nn+1} : X_{n+1} \to X_n$ by $p_{nn+1}(x_{i_1 i_2 \cdots i_n i_{n+1}}) = x_{i_1 i_2 \cdots i_n}$. Let X be the limit of $\{X_n, p_{nn'}\}$, and let $p_n : X \to X_n$ be the projection map. For each n, let $\mathcal{U}_n$ be the finite open covering of X_n consisting of the discrete 3^{a_n} points as elements. Then $\boldsymbol{p} = \{p_n\} : X \to \boldsymbol{X} = \{X_n, \mathcal{U}_n, p_{nn'}\}$ is an admissible approximate resolution.

Claim. $\dim_B(\boldsymbol{p}) = r$.

Indeed, for each n, $\beta_n(\boldsymbol{X}) = 3^{b_1} \cdot 3^{b_2} \cdots 3^{b_n} = 3^{a_n}$, so $\frac{\log_3 \beta_i(\boldsymbol{X})}{n} = \frac{a_n}{n}$ which converges to r.

Thus obtained X is totally disconnected and perfect and hence is a Cantor set. For the case $r = \infty$ (resp., 0), for each i, let $a_n = n^2$ (resp., the smallest integer that is not greater than $\log_3 n$), and construct an approximate resolution $\boldsymbol{p} = \{p_n\} : X \to \boldsymbol{X} = \{X_n, \mathcal{U}_n, p_{nn'}\}$ by the same procedure as above. Then, since $\lim\limits_{n\to\infty} \frac{a_n}{n} = \infty$ (resp., 0), $\dim_B(\boldsymbol{p}) = \infty$ (resp., 0). $\square$

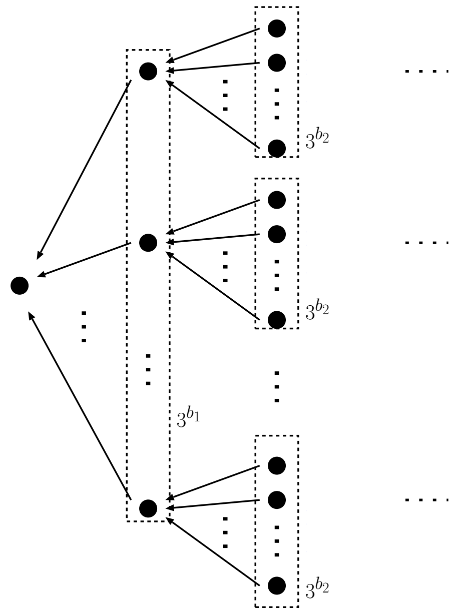

FIGURE 1. A system with box-counting dimension r

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