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LIPSCHITZ FUNCTIONS AND APPROXIMATE RESOLUTIONS

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Abstract. A Lipschitz function between metric spaces is an important notion in fractal geometry as it is well-known to have a close connection to fractal dimension. On the other hand, the theory of approximate resolutions has been developed by Mardešić and Watanabe. In this theory maps $f : X \to Y$ between general spaces are represented by approximate maps $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ between approximate systems for any approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$, and the approximate maps $\boldsymbol{f}$ give useful information about the properties of the maps f. In this paper, we describe a new method of using the theory of approximate resolutions to study Lipschitz functions. More precisely, first of all, given a Hausdorff space X and a normal sequence $\mathbb{U}$ with a reasonable condition, a new metric $\mathrm{d}_{\mathbb{U}}$ which induces the given topology is defined, and Lipschitz functions with respect to the metrics induced by normal sequences are characterized by a property of the normal sequences. Secondly, using this metric, for each compact metric space X and for each approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ of X with a reasonable condition, a new metric $\mathrm{d}_{\boldsymbol{p}}$ which is topologically equivalent to the given metric is defined, and the properties of those metrics are investigated. Lipschitz functions between continua with the metrics induced by approximate resolutions are characterized by approximate resolutions. As an application, contraction maps are characterized, and a sufficient condition in terms of approximate resolutions for the existence of a unique fixed point is obtained.

1. Introduction

Recall that a function $f : X \to Y$ between metric spaces X and Y is a *Lipschitz function* provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}(f(x), f(x')) \le \alpha\, \mathrm{d}(x, x') \text{ for } x, x' \in X.$$

Being a Lipschitz function is an important property in fractal geometry, especially, in fractal dimensions. For example, one of the required conditions for a fractal dimension is the Lipschitz invariance (see [2, p. 37], and also [3, §1]), i.e., if a map $f : X \to Y$ is a bi-Lipschitz function, then X and its image $f(X)$ have the same fractal dimension. In this paper, we describe a new method of using the theory of approximate resolutions to study Lipschitz functions.

Mardešić and Watanabe [10] introduced the notion of approximate resolutions, which generalizes all compact limits, approximate limits of Mardešić and Rubin [6] and resolutions of Mardešić [5]. This notion has proved to be userful in many problems in

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topology especially for nonmetric or noncompact spaces [15, 16, 17, 14, 12]. However, even for compact metric spaces, approximate resolutions are essential [6, 11, 15, 16]. In fact, when we are given a map $f : X \to Y$ between compact metric spaces and limits $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, p_{ii+1}\}$ and $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, q_{jj+1}\}$, there may not exist a map of systems $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$, i.e., a function $f : \mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers, and maps $f_j : X_{f(j)} \to Y_j,\ j \in \mathbb{N}$, with the property that for any $j < j'$, there is $i > f(j), f(j')$ such that

(M): $f_j p_{f(j)i} = q_{jj'} f_{j'} p_{f(j')i}$; and
(LM): $f_j p_{f(j)} = q_j f,\ j \in \mathbb{N}$.

In the theory of approximate resolutions, we replace those commutativity conditions by approximate commutativity conditions so that a map of systems $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ exists. This is the fact we use for our purpose.

To each normal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ on a Hausdorff space X so that for each $x \in X$, the stars of x with respect to $\mathcal{U}_i$, $i = 1, 2, ...$, form a base at x, is associated a metric $\mathrm{d}_{\mathbb{U}}$ which induces the given topology by an approach similar to Alexandroff and Urysohn [1] (see also [13]). Let $f : X \to Y$ be a continuous map between Hausdorff spaces X and Y with normal sequences $\mathbb{U}$ and $\mathbb{V}$ as in the above, respectively. We give a necessary and sufficient condition on the normal sequences $\mathbb{U}$ and $\mathbb{V}$ for the map f to be a Lipschitz function with respect to the metrics $\mathrm{d}_{\mathbb{U}}$ and $\mathrm{d}_{\mathbb{V}}$.

Those metrics induced by normal sequences have a close connection to approximate resolutions. For each compact metric space X and for each approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ of X into an approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ so that $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i : i = 1, 2, ...\}$ forms a normal sequence on X and for each $x \in X$ the stars of x with respect to $p_i^{-1}\mathcal{U}_i$, $i = 1, 2, ...$, form a base at x, we define a new metric $\mathrm{d}_{\boldsymbol{p}}$ which is topologically equivalent to the given metric. For each continuous map $f : X \to Y$ between continua X and Y with approximate resolutions $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ as in the above, we give a necessary and sufficient condition on the approximate resolutions for the map f to be a Lipschitz function with respect to the metrics $\mathrm{d}_{\boldsymbol{p}}$ and $\mathrm{d}_{\boldsymbol{q}}$. Moreover, contraction maps $f : X \to Y$ with respect to the metrics $\mathrm{d}_{\boldsymbol{p}}$ and $\mathrm{d}_{\boldsymbol{q}}$ are characterized in terms of approximate resolutions. Using this result, we obtain a sufficient condition on the approximate resolutions that imply that the map f has a unique fixed point.

The paper is organized as follows: After the definitions and basic properties of approximate resolutions are recalled in the following section, in Section 3, the definition of the metric induced by normal sequence is given, and its fundamental properties and some examples are presented. In Section 4, the metric induced by approximate resolution is defined, and its fundamental properties are obtained. In Section 5, Lipschitz functions with respect to the metrics induced by normal sequences are characterized by a property of the normal sequences, and in the following section, Lipschitz functions with respect to the metrics induced by approximate resolutions are charecterized by the approximate resolutions. In the final section, contraction maps are characterized, and a unique fixed point property is discussed.

Throughout the paper, a map means a continuous map unless otherwise stated.

For any topological space X, let $\mathrm{Cov}(X)$ denote the set of all normal open coverings of X. For any subset A of X and $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ and $\mathcal{U}|A = \{U \cap A : U \in \mathcal{U}\}$. If $A = \{x\}$, we write $\mathrm{st}(x,\mathcal{U})$ for $\mathrm{st}(\{x\},\mathcal{U})$. For each $\mathcal{U} \in \mathrm{Cov}(X)$, let $\mathrm{st}\,\mathcal{U} = \{\mathrm{st}(U,\mathcal{U}) : U \in \mathcal{U}\}$ and $\mathcal{U}^{\Delta} = \{\mathrm{st}(x,\mathcal{U}) : x \in X\}$. Let $\mathrm{st}^{n+1}\mathcal{U} = \mathrm{st}(\mathrm{st}^{n}\mathcal{U})$ for each $n = 1, 2, ...$ and $\mathrm{st}^{1}\mathcal{U} = \mathrm{st}\,\mathcal{U}$, and for each subset A of X let $\mathrm{st}^{n+1}(A,\mathcal{U}) = \mathrm{st}(\mathrm{st}^{n}(A,\mathcal{U}), \mathrm{st}^{n}\mathcal{U})$ for each $n = 1, 2, ...$ and $\mathrm{st}^{1}(A,\mathcal{U}) = \mathrm{st}(A,\mathcal{U})$. Also, for any $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$, let $\mathrm{st}(\mathcal{U}, \mathcal{V})$ denote the normal open covering $\{\mathrm{st}(U, \mathcal{V}) : U \in \mathcal{U}\}$. For any metric space (X, d) and $r > 0$, let $\mathrm{U}_{\mathrm{d}}(x, r) = \{y \in X : \mathrm{d}(x, y) < r\}$, and write $\mathrm{U}(x, r)$ if there is no confusion on the choice of the metric d. For any $\mathcal{U} \in \mathrm{Cov}(X)$, two points $x, x' \in X$ are *$\mathcal{U}$-near*, denoted $(x, x') < \mathcal{U}$, provided $x, x' \in U$ for some $U \in \mathcal{U}$. For any $\mathcal{V} \in \mathrm{Cov}(Y)$, two maps $f, g : X \to Y$ between topological spaces are *$\mathcal{V}$-near*, denoted $(f, g) < \mathcal{V}$, provided $(f(x), g(x)) < \mathcal{V}$ for each $x \in X$. For each $\mathcal{U} \in \mathrm{Cov}(X)$ and $\mathcal{V} \in \mathrm{Cov}(Y)$, let $f\mathcal{U} = \{f(U) : U \in \mathcal{U}\}$ and $f^{-1}\mathcal{V} = \{f^{-1}(V) : V \in \mathcal{V}\}$.

2. Approximate resolutions

In this section we recall the definitions and properties of approximate resolutions which will be needed in later sections. For more details, the reader is referred to [10].

An *approximate inverse system* (*approximate system*, in short) $\boldsymbol{X} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}$ consists of

i) a directed preordered set $A = (A, <)$ with no maximal element;

ii) topological spaces X_a for $a \in A$;

iii) $\mathcal{U}_a \in \mathrm{Cov}(X_a)$ for $a \in A$; and

iv) maps $p_{aa'} : X_{a'} \to X_a$ for $a < a'$ and $p_{aa} = 1_{X_a}$ the identity map on X_a.

It must satisfy the following three conditions:

(A1): $(p_{aa'}p_{a'a''}, p_{aa''}) < \mathcal{U}_a$ for $a < a' < a''$;

(A2): For each $a \in A$ and $\mathcal{U} \in \mathrm{Cov}(X_a)$, there exists $a' > a$ such that $(p_{aa_1}p_{a_1a_2}, p_{aa_2}) < \mathcal{U}$ for $a' < a_1 < a_2$; and

(A3): For each $a \in A$ and $\mathcal{U} \in \mathrm{Cov}(X_a)$, there exists $a' > a$ such that $\mathcal{U}_{a''} < p^{-1}_{aa''}\mathcal{U}$ for $a' < a''$.

An *approximate map* $\boldsymbol{p} = \{p_a : a \in A\} : X \to \boldsymbol{X}$ of a topological space X into an approximate system $\boldsymbol{X} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}$ consists of maps $p_a : X \to X_a$ for $a \in A$ with the following property:

(AS): For each $a \in A$ and $\mathcal{U} \in \mathrm{Cov}(X_a)$, there exists $a' > a$ such that $(p_{aa''}p_{a''}, p_a) < \mathcal{U}$ for $a'' > a'$.

An *approximate resolution* of a topological space X is an approximate map $\boldsymbol{p} = \{p_a : a \in A\} : X \to \boldsymbol{X}$ of X into an approximate system $\boldsymbol{X} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}$ which satisfies the following two conditions:

(R1): For each ANR P, $\mathcal{V} \in \mathrm{Cov}(P)$ and map $f : X \to P$, there exist $a \in A$ and a map $g : X_a \to P$ such that $(gp_a, f) < \mathcal{V}$; and

(R2): For each ANR P and $\mathcal{V} \in \mathrm{Cov}(P)$, there exists $\mathcal{V}' \in \mathrm{Cov}(P)$ such that whenever $a \in A$ and $g, g' : X_a \to P$ are maps with $(gp_a, g'p_a) < \mathcal{V}'$, then $(gp_{aa'}, g'p_{aa'}) < \mathcal{V}$ for some $a' > a$.

If $\mathcal{C}$ is a collection of topological spaces, and if all X_a belong to $\mathcal{C}$, then an approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ is called an *approximate $\mathcal{C}$-resolution.*

Theorem 2.1. *An approximate map $\boldsymbol{p} = \{p_a : a \in A\} : X \to \boldsymbol{X} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}$ is an approximate resolution of a topological space X if and only if it satisfies the following two conditions:*

(B1): *For each $\mathcal{U} \in \mathrm{Cov}(X)$, there exists $a \in A$ such that $p_a^{-1}\mathcal{V} < \mathcal{U}$ for some $\mathcal{V} \in \mathrm{Cov}(X_a)$; and*

(B2): *For each $a \in A$ and $\mathcal{U} \in \mathrm{Cov}(X_a)$, there exists $a' > a$ such that $p_{aa'}(X_{a'}) \subseteq \mathrm{st}(p_a(X), \mathcal{U})$.*

Let **Pol** and **APol** respectively denote the collections of polyhedra and approximate polyhedra (see [5]).

Theorem 2.2. ([10, 2.19]) *Every topological space admits an approximate* **Pol***-resolution with a cofinite index set.*

If $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}, \mathbb{N}\}$ has the index set $\mathbb{N}$ of positive integers with the usual order, we simply write $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$.

Theorem 2.3. i) *Every compact metric space X admits an approximate* **Pol***-resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ such that all X_i are finite polyhedra.*
ii) *Every continuum, i.e., connected compact metric space, X admits an approximate* **Pol***-resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ such that all X_i are connected finite polyhedra, and all p_i and $p_{ii'}$ are surjective.*

Proof. If X is a compact metric space, then there exists an inverse limit $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ into an inverse system $\boldsymbol{X} = \{X_i, p_{ii'}\}$ such that all X_i are finite polyhedra. If X is a continuum, then since X is $\mathcal{P}$-like where $\mathcal{P}$ is the class of connected finite polyhedra, [7, Theorem 1] implies that there exists an inverse limit $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ into an inverse system $\boldsymbol{X} = \{X_i, p_{ii'}\}$ such that all $X_i \in \mathcal{P}$ and all $p_{ii'}$ are surjective. In either case, by [15, 3.8], there exist $\mathcal{U}_i \in \mathrm{Cov}(X_i)$ so that $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ forms an approximate system and that $\boldsymbol{p} : X \to \boldsymbol{X}$ forms an approximate **Pol**-resolution. In the case of continuum, [9, Corollary 1] implies that all $p_i : X \to X_i$ are also surjective. This concludes the theorem. $\square$

Let $\boldsymbol{X} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}$ and $\boldsymbol{Y} = \{Y_b, \mathcal{V}_b, q_{bb'}, B\}$ be approximate systems of topological spaces. An *approximate map* $\boldsymbol{f} = \{f_a, f : a \in A\} : \boldsymbol{X} \to \boldsymbol{Y}$ consists of a function $f : B \to A$ and maps $f_b : X_{f(b)} \to Y_b, b \in B$, with the following condition:

(AM): For any $b, b' \in B$ with $b < b'$, there exists $a \in A$ with $a > f(b), f(b')$ such that

$$(q_{bb'} f_{b'} p_{f(b')a'}, f_b p_{f(b)a'}) < \mathrm{st}\, \mathcal{V}_b \text{ for all } a' > a.$$

A map $f : X \to Y$ is a *limit* of $\boldsymbol{f}$ provided the following condition is satisfied:

(LAM): For each $b \in B$ and $\mathcal{V} \in \mathrm{Cov}(Y_b)$, there exists $b' > b$ such that

$$(q_{bb''} f_{b''} p_{f(b'')}, q_b f) < \mathcal{V} \text{ for all } b'' > b'.$$

For each map $f : X \to Y$, an *approximate resolution* of f is a triple $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ consisting of approximate resolutions $\boldsymbol{p} = \{p_a : a \in A\} : X \to \boldsymbol{X} = \{X_a, \mathcal{U}_a, p_{aa'}, A\}$ of X and $\boldsymbol{q} = \{q_b : b \in B\} : Y \to \boldsymbol{Y} = \{Y_b, \mathcal{V}_b, q_{bb'}, B\}$ of Y and of an approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ with property (LAM).

Theorem 2.4. *Let X and Y be topological spaces. For any approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ and approximate* **APol***-resolution $\boldsymbol{q} : Y \to \boldsymbol{Y}$ over a cofinite index set, every map $f : X \to Y$ admits an approximate map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ such that $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ is an approximate resolution of f.*

Theorem 2.4 fails even for compact metric spaces X and Y if we require the following commutativity relations instead of (AM) and (LAM) ([4], [11], [15], [16]):

(M): $f_b p_{f(b)a} = q_{bb'} f_{b'} p_{f(b')a}$; and
(LM): $f_b p_{f(b)} = q_b f$ for $b \in B$.

3. Metrics induced by normal open coverings

Throughout this section, *a space means a Hausdorff space* unless otherwise stated. In this section, following the approach of Alexandroff and Urysohn [1], [13, 2-16], given a space X and a normal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ on X, we define a pseudometric $\mathrm{d}_{\mathbb{U}}$ on X and obtain its properties.

We call the family $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ of open coverings on a space X a *normal sequence* provided $\mathrm{st}\,\mathcal{U}_{i+1} < \mathcal{U}_i$ for each i. Given a normal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ on X, we define the function $\mathcal{D}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_{\mathbb{U}}(x, x') = \begin{cases} 9 & \text{if } (x, x') \not< \mathcal{U}_1; \\ \frac{1}{3^{i-2}} & \text{if } (x, x') < \mathcal{U}_i \text{ but } (x, x') \not< \mathcal{U}_{i+1}\ ; \\ 0 & \text{if } (x, x') < \mathcal{U}_i \text{ for all } i = 1, 2, ...\ , \end{cases}$$

and the function $\mathrm{d}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_{\mathbb{U}}(x, x') = \inf\{\mathcal{D}_{\mathbb{U}}(x, x_1) + \mathcal{D}_{\mathbb{U}}(x_1, x_2) + \cdots + \mathcal{D}_{\mathbb{U}}(x_n, x')\}$$

where the infimum is taken over all points $x_1, x_2, ..., x_n$ in X and $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers.

Proposition 3.1. *Let X be a space, and let $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ be a normal sequence. Then the function $\mathrm{d}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ is a pseudometric on X with the property that*

$$\mathrm{st}(x, \mathcal{U}_{i+3}) \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{U}}}(x, \frac{1}{3^i}) \subseteq \mathrm{st}(x, \mathcal{U}_i) \text{ for each } x \in X \text{ and } i.$$

Proof. That $\mathrm{d}_{\mathbb{U}}$ is a pseudometric on X is obvious. It is also obvious from the definition that

$$\mathrm{d}_{\mathbb{U}}(x,x') \leq \mathcal{D}_{\mathbb{U}}(x,x') \text{ for } x,x' \in X. \tag{3.1}$$

Using the fact that a star refinement is a 2-refinement in the sense of [13, p.13] and following the proof of [13, 2-16], we can show

$$\mathcal{D}_{\mathbb{U}}(x,x') \leq 4\,\mathrm{d}_{\mathbb{U}}(x,x') \text{ for } x,x' \in X. \tag{3.2}$$

(3.1) and (3.2) imply that $\mathrm{d}_{\mathbb{U}}$ has the desired property. □

Proposition 3.2. *Let X be a space, and suppose a normal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ has the following property:*

(B): *$\{\mathrm{st}(x,\mathcal{U}_i) : i = 1,2,...\}$ is a base at x for each $x \in X$.*

Then $\mathrm{d}_{\mathbb{U}}$ is a metric on X. In this case we call $\mathrm{d}_{\mathbb{U}}$ the metric induced by the normal sequence $\mathbb{U}$.

Proof. $\mathrm{d}_{\mathbb{U}}(x,x') = 0$ implies $x = x'$ by Proposition 3.1 and the fact that X is a Hausdorff. □

Proposition 3.3. *Let (X,d) be a metric space, and let $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ be the normal sequence such that $\mathcal{U}_i = \{\mathrm{U}_{\mathrm{d}}(x,\frac{1}{3^i}) : x \in X\}$ for each $i = 1,2,....$ Then the metric $\mathrm{d}_{\mathbb{U}}$ induced by the nomal sequence $\mathbb{U}$ induces the uniformity which is isomorphic to that induced by the metric d.*

Proof. Proposition 3.2 implies that $\mathrm{d}_{\mathbb{U}}$ defines a metric on X, and it follows from Proposition 3.1 that the uniformity induced by $\mathrm{d}_{\mathbb{U}}$ is isomorphic to that induced by d. □

Proposition 3.4. *Let X be a space.*

i) *If A is a subset of X, and if $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ is a normal sequence, then*

$$\mathrm{d}_{\mathbb{U}|A}(x,x') \geq \mathrm{d}_{\mathbb{U}}(x,x') \textit{ for all } x,x' \in A.$$

Here $\mathbb{U}|A$ denotes the normal sequence $\{\mathcal{U}_i|A : i = 1,2,...\}$ restricted to A.

ii) *If $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1,2,...\}$ are normal sequences on X such that $\mathcal{U}_i < \mathcal{V}_i$ for each i, denoted $\mathbb{U} < \mathbb{V}$, then*

$$\mathrm{d}_{\mathbb{U}}(x,x') \geq \mathrm{d}_{\mathbb{V}}(x,x') \textit{ for all } x,x' \in X.$$

iii) *For each normal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ on X, let $\Sigma\mathbb{U}$ be the normal sequence $\{\mathcal{V}_i : \mathcal{V}_i = \mathcal{U}_{i+1}, i = 1,2,...\}$ on X. Then*

$$\mathrm{d}_{\Sigma\mathbb{U}}(x,x') = 3\,\mathrm{d}_{\mathbb{U}}(x,x') \textit{ for all } x,x' \in X.$$

iv) *For each normal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ on X, let $\mathrm{st}\,\mathbb{U}$ be the normal sequence $\{\mathrm{st}\,\mathcal{U}_i : i = 1,2,...\}$ on X. Then*

$$\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x,x') \leq \mathrm{d}_{\mathbb{U}}(x,x') \leq 3\,\mathrm{d}_{\mathrm{st}\,\mathbb{U}}(x,x') \textit{ for all } x,x' \in X.$$

v) *For each normal sequence* $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ *on* X*, let* $\mathbb{U}^\Delta$ *be the normal sequence* $\{\mathcal{U}_i^\Delta : i = 1, 2, ...\}$ *on* X*. Then*

$$\mathrm{d}_{\mathbb{U}^\Delta}(x, x') \leq \mathrm{d}_{\mathbb{U}}(x, x') \leq 2\,\mathrm{d}_{\mathbb{U}^\Delta}(x, x') \text{ for all } x, x' \in X.$$

Proof. i), ii) and iii) are obvious from the definitions of $\mathrm{d}_{\mathbb{U}}$ and $\mathcal{D}_{\mathbb{U}}$. iv) follows from the fact that $\mathrm{st}\,\mathcal{U}_{i+1} < \mathcal{U}_i < \mathrm{st}\,\mathcal{U}_i$ for each i, the definition of $\mathcal{D}_{\mathbb{U}}$ and iii). For v), for any $x, x' \in X$, if $(x, x') < \mathcal{U}_i^\Delta$ and $(x, x') \not< \mathcal{U}_{i+1}^\Delta$, then there is $x'' \in X$ such that $(x, x'') < \mathcal{U}_i$ and $(x'', x') < \mathcal{U}_i$. So, $\mathcal{D}_{\mathbb{U}}(x, x'') + \mathcal{D}_{\mathbb{U}}(x'', x') \leq \frac{2}{3^{i-2}} = 2\mathcal{D}_{\mathbb{U}^\Delta}(x, x')$. This together with the definition of $\mathrm{d}_{\mathbb{U}}$ implies that $\mathrm{d}_{\mathbb{U}}(x, x') \leq 2\,\mathrm{d}_{\mathbb{U}^\Delta}(x, x')$ for all $x, x' \in X$. That $\mathrm{d}_{\mathbb{U}^\Delta}(x, x') \leq \mathrm{d}_{\mathbb{U}}(x, x')$ follows from ii). □

Let $\Sigma^1\mathbb{U} = \Sigma\mathbb{U}$, and for each $n = 1, 2, ...$, let $\Sigma^{n+1}\mathbb{U} = \Sigma(\Sigma^n\mathbb{U})$, and also let $\mathrm{st}^1\,\mathbb{U} = \mathrm{st}\,\mathbb{U}$ and $\mathrm{st}^{n+1}\,\mathbb{U} = \mathrm{st}(\mathrm{st}^n\,\mathbb{U})$.

Next, we wish to give a simpler description of the metric $\mathrm{d}_{\mathbb{U}}$ for special cases. For any space X and for any nomal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ on X, we define the function $\overline{\mathrm{d}}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\overline{\mathrm{d}}_{\mathbb{U}}(x, x') = \inf\ \frac{n}{3^{i-2}}$$

where the infimum is taken over all the choices of points $x = x_0, x_1, ..., x_n = x'$ in X such that $(x_j, x_{j+1}) < \mathcal{U}_i$ but $(x_j, x_{j+1}) \not< \mathcal{U}_i$ for each $j = 0, 1, ..., n-1$.

Proposition 3.5. *Let* X *be a space, and let* $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ *be a normal sequence. Then the function* $\overline{\mathrm{d}}_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ *is a pseudometric on* X *with the following properties:*

$$\overline{\mathrm{d}}_{\mathbb{U}}(x, x') \geq \mathrm{d}_{\mathbb{U}}(x, x') \quad \text{for all } x, x' \in X \tag{3.3}$$

and

$$\mathrm{st}(x, \mathcal{U}_{i+3}) \subseteq \mathrm{U}_{\overline{\mathrm{d}}_{\mathbb{U}}}(x, \frac{1}{3^i}) \subseteq \mathrm{st}(x, \mathcal{U}_i) \text{ for each } x \in X \text{ and } i. \tag{3.4}$$

Moreover, if $\mathbb{U}$ *has property (B) (see Proposition 3.2), then* $\overline{\mathrm{d}}_{\mathbb{U}}$ *is a metric on* X*.*

Proof. Obviously, $\overline{\mathrm{d}}_{\mathbb{U}}$ is a pseudometric on X. It is also obvious from the definitions of $\overline{\mathrm{d}}_{\mathbb{U}}$ and $\mathrm{d}_{\mathbb{U}}$ that $\overline{\mathrm{d}}_{\mathbb{U}}$ has properties (3.3) and

$$\overline{\mathrm{d}}_{\mathbb{U}}(x, x') \leq \mathcal{D}_{\mathbb{U}}(x, x') \text{ for each } x \in X. \tag{3.5}$$

(3.3) and (3.2) (in the proof of Proposition 3.1) imply that

$$\mathcal{D}_{\mathbb{U}}(x, x') \leq 4\overline{\mathrm{d}}_{\mathbb{U}}(x, x') \text{ for each } x \in X. \tag{3.6}$$

(3.5) and (3.6) imply property (3.4). Finally, (3.4) immediately implies the last assertion. □

Proposition 3.6. *Let* X *be a convex subset of a linear topological space* L*. Then*

$$\overline{\mathrm{d}}_{\mathbb{U}}(x, x') = \mathrm{d}_{\mathbb{U}}(x, x') \quad \text{for all } x, x' \in X$$

where $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ *and* $\mathcal{U}_i = \{\mathrm{U}(x, \frac{1}{3^i}) : x \in X\}$ *for each* i*.*

Proof. By (3.3) of Proposition 3.5, it suffices to show that $\overline{\mathrm{d}}_{\mathbb{U}}(x,x') \leq \mathrm{d}_{\mathbb{U}}(x,x')$. Let $x = x_0, x_1, ..., x_n = x'$ be points in X such that $(x_j, x_{j+1}) < \mathcal{U}_{k_j}$ but $(x_j, x_{j+1}) \not< \mathcal{U}_{k_j+1}$ for each $j = 0, 1, ..., n-1$. Let $k = \max\{k_j : j = 0, 1, ..., n-1\}$, and take $i > k$ so that

$$\frac{1}{3^i} \leq \min \left\{ \frac{1}{2} \left(\frac{2}{3^{k_j}} - \|x_{j+1} - x_j\| \right) : j = 0, 1, ..., n-1 \right\}. \tag{3.7}$$

Claim: For each j, there exist points $x_j = x_{j,0}, x_{j,1}, ..., x_{j,l_j} = x_{j+1} \in X$ such that

$$(x_{j,p}, x_{j,p+1}) < \mathcal{U}_i \text{ but } (x_{j,p}, x_{j,p+1}) \not< \mathcal{U}_{i+1} \text{ for } p = 0, 1, ..., l_j - 1 \tag{3.8}$$

and

$$\mathcal{D}_{\mathbb{U}}(x_j, x_{j+1}) = \frac{1}{3^{k_j-2}} \geq \frac{l_j}{3^{i-2}}. \tag{3.9}$$

Indeed, since X is a convex subset, there is a line segment from x_j to x_{j+1} in X. Let

$$x_{j,p} = \frac{p}{\lfloor \frac{R_j}{2/3^i} \rfloor + 1}(x_{j+1} - x_j) + x_j \ (p = 0, 1, ..., l_j = \left\lfloor \frac{R_j}{2/3^i} \right\rfloor + 1)$$

where $R_j = \|x_{j+1} - x_j\|$. Then we have

$$\frac{2}{3^{i+1}} \leq \|x_{j,p+1} - x_{j,p}\| < \frac{2}{3^i}.$$

For,

$$\|x_{j,p+1} - x_{j,p}\| = \frac{R_j}{\left\lfloor \frac{R_j}{2/3^i} \right\rfloor + 1}$$

and

$$\frac{2}{3^{i+1}} \leq \frac{R_j}{\frac{R_j}{2/3^i} + 1} \leq \frac{R_j}{\left\lfloor \frac{R_j}{2/3^i} \right\rfloor + 1} < \frac{R_j}{(\frac{R_j}{2/3^i} - 1) + 1} = \frac{2}{3^i}$$

where the first inequality follows from the fact that $R_j \geq \frac{2}{3^i}$. This relation implies (3.8). (3.9) follows since by (3.7),

$$\frac{1}{3^{k_j-2}} - \frac{l_j}{3^{i-2}} \geq \frac{1}{3^{k_j-2}} - \frac{1}{3^{i-2}} \left(\frac{R_j}{2/3^i} + 1 \right) = \frac{9}{2} \left(\frac{2}{3^{k_j}} - R_j - \frac{2}{3^i} \right) \geq 0.$$

Now by Claim,

$$\sum_{j=0}^{n-1} \mathcal{D}_{\mathbb{U}}(x_j, x_{j+1}) \geq \frac{1}{3^{i-2}} \sum_{j=0}^{n-1} l_j \geq \overline{\mathrm{d}}_{\mathbb{U}}(x,x').$$

Since this is true for any choice of the points $x = x_0, x_1, ..., x_n = x' \in X$, we have $\mathrm{d}_{\mathbb{U}}(x,x') \geq \overline{\mathrm{d}}_{\mathbb{U}}(x,x')$. □

Proposition 3.7. *Let X be a convex subset of a linear topological space L, and let $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ where $\mathcal{U}_i = \{\mathrm{U}(x, \frac{1}{3^i}) : x \in X\}$. Then*

$$\mathrm{d}_{\mathbb{U}}(x,x') = \frac{9}{2}\|x - x'\| \text{ for all } x, x' \in X.$$

Proof. By Proposition 3.6,

$$\mathrm{d}_{\mathbb{U}}(x,y) = \inf\Big\{ \frac{\lfloor \frac{\|x-y\|}{2/3^i} \rfloor + 1}{3^{i-2}} : i = 1,2,...\Big\} = \frac{9}{2}\|x-y\|.$$

□

Finally in this section, we wish to compare our definition of the metric $\mathrm{d}_{\mathbb{U}}$ with that in [13, 2-16].

We call the family $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ a 2-*refinement sequence* provided for each i, if U_1, $U_2 \in \mathcal{U}_{i+1}$ have a nonempty intersection, then $U_1 \cup U_2$ refines $\mathcal{U}_i$. Given a 2-refinement sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ on X, we define the function $\mathcal{D}'_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}'_{\mathbb{U}}(x,y) = \begin{cases} 4 & \text{if } (x,y) \not< \mathcal{U}_1; \\ \frac{1}{2^{i-2}} & \text{if } (x,y) < \mathcal{U}_i \text{ but } (x,y) \not< \mathcal{U}_{i+1} \text{ ;} \\ 0 & \text{if } (x,y) < \mathcal{U}_i \text{ for all } i = 1,2,... \text{ ,} \end{cases}$$

and the function $\mathrm{d}'_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}'_{\mathbb{U}}(x,x') = \inf\{\mathcal{D}'_{\mathbb{U}}(x,x_1) + \mathcal{D}'_{\mathbb{U}}(x_1,x_2) + \cdots + \mathcal{D}'_{\mathbb{U}}(x_n,x')\}$$

where the infimum is taken over all finitely many points $x_1, x_2, ..., x_n$ of X. Then by [13, 2-16,2-18], the results analogous to Propositions 3.1 and 3.2 hold. Similarly to $\overline{\mathrm{d}}_{\mathbb{U}}$ for any normal sequence $\mathbb{U}$, we can define the function $\overline{\mathrm{d}}'_{\mathbb{U}} : X \times X \to \mathbb{R}_{\geq 0}$ for each 2-refinement sequence $\mathbb{U}$ on X, and the result analogous to Proposition 3.5 holds. Also, for the 2-refinement sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ where $\mathcal{U}_i = \{\mathrm{U}(x, \frac{1}{2^i}) : x \in X\}$, the results analogous to Propositions 3.6 and 3.7 hold, where in Proposition 3.7 the constant multiple is 2 instead of $\frac{9}{2}$.

Example. Consider the real number $\mathbb{R}$ as a linear toplogical space and the normal sequence $\mathbb{U} = \{\mathcal{U}_i : i = 1,2,...\}$ where $\mathcal{U}_i = \{\mathrm{U}(x, \frac{1}{3^i}) : x \in X\}$ and the 2-refinement sequence $\mathbb{U}' = \{\mathcal{U}'_i : i = 1,2,...\}$ where $\mathcal{U}'_i = \{\mathrm{U}(x, \frac{1}{2^i}) : x \in X\}$ on $\mathbb{R}$. Then, by Proposition 3.7,

$$\mathrm{d}_{\mathbb{U}}(0,1) = \frac{9}{2}$$

and

$$\mathrm{d}'_{\mathbb{U}'}(0,1) = \inf\Big\{ \frac{\lfloor \frac{1}{2/2^i} \rfloor + 1}{2^{i-2}} : i = 1,2,... \Big\}$$

which equals 2. However, if we take the normal sequence $\mathbb{U}$ as a 2-refinement sequence, then

$$\mathrm{d}'_{\mathbb{U}}(0,1) = \inf\Big\{ \frac{\lfloor \frac{1}{2/3^i} \rfloor + 1}{2^{i-2}} : i = 1,2,... \Big\}$$

which equals 4 since the sequence monotonically increases.

4. Metrics induced by approximate resolutions

In Section 3, we studied how normal sequences induce metrics. In this section, we extend this idea to the theory of approximate resolutions. Throughout this section, *a space means a compact metric space* unless otherwise stated.

Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be an approximate **Pol**-resolution of a space X. Assume the approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ satisfies the following condition:

(U): $\mathrm{st}\,\mathcal{U}_{i'} < p_{ii'}^{-1}\mathcal{U}_i$ for $i < i'$.

Note that with any approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is associated an approximate system $\boldsymbol{X}^* = \{X_{k_i}, \mathcal{U}_{k_i}, p_{k_ik_{i'}}\}$ with property (U) by taking a cofinal subsystem of $\boldsymbol{X}$ as in [10, 1.6]. For each $n \geq 2$, we define the function $\mathcal{D}_{\boldsymbol{X},n} : X_n \times X_n \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_{\boldsymbol{X},n}(z,z') = \begin{cases} 9 & \text{if } (p_{in}(z), p_{in}(z')) \not< \mathcal{U}_i \text{ for any } i \leq n; \\ \frac{1}{3^{i-2}} & \text{if } (p_{in}(z), p_{in}(z')) < \mathcal{U}_i \text{ but} \\ & (p_{i+1n}(z), p_{i+1n}(z')) \not< \mathcal{U}_{i+1} \text{ for some } i < n; \\ 0 & \text{if } (p_{in}(z), p_{in}(z')) < \mathcal{U}_i \text{ for all } i \leq n\ , \end{cases}$$

and the function $\mathrm{d}_{\boldsymbol{X}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_{\boldsymbol{X}}(x,x') = \inf\{\mathcal{D}_{\boldsymbol{X},n}(p_n(x), z_1) + \mathcal{D}_{\boldsymbol{X},n}(z_1, z_2) + \cdots + \mathcal{D}_{\boldsymbol{X},n}(z_k, p_n(x'))\}$$

where the infimum is taken over all $n \geq 2$ and all finitely many points $z_1, z_2, ..., z_k$ of X_n. Then it is easy to see that $\mathrm{d}_{\boldsymbol{X}}$ is a pseudometric on X.

An approximate resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ into an approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ is said to be *normal* provided the family of open coverings $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i : i = 1, 2, ...\}$ is a normal sequence and has property (B) (see Proposition 3.2). Note that with any approximate system $\boldsymbol{X}$ is associated a normal approximate system $\boldsymbol{X}' = \{X_{l_i}, \mathcal{U}_{l_i}, p_{l_il_{i'}}\}$ by taking a cofinal subsystem of $\boldsymbol{X}$ (use (B1)).

Let $\boldsymbol{p} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be any normal approximate **Pol**-resolution of a space. Then for any $x, x' \in X$, we define the function $\mathcal{D}_{\boldsymbol{p}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathcal{D}_{\boldsymbol{p}}(x,x') = \begin{cases} 9 & \text{if } (p_i(x), p_i(x')) \not< \mathcal{U}_i \text{ for any } i; \\ \frac{1}{3^{i-2}} & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ but } (p_i(x), p_i(x')) \not< \mathcal{U}_{i+1}\ ; \\ 0 & \text{if } (p_i(x), p_i(x')) < \mathcal{U}_i \text{ for all } i\ , \end{cases}$$

and the function $\mathrm{d}_{\boldsymbol{p}} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$\mathrm{d}_{\boldsymbol{p}}(x,x') = \inf\{\mathcal{D}_{\boldsymbol{p}}(x, x_1) + \mathcal{D}_{\boldsymbol{p}}(x_1, x_2) + \cdots + \mathcal{D}_{\boldsymbol{p}}(x_n, x')\}$$

where the infimum is taken over all finitely many points $x_1, x_2, ..., x_n$ of X. Note that $\mathrm{d}_{\boldsymbol{p}}(x,x') = \mathrm{d}_{\mathbb{U}}(x,x')$ for any $x, x' \in X$, where $\mathbb{U} = \{p_i^{-1}\mathcal{U}_i : i = 1, 2, ...\}$.

Proposition 4.1. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be a normal approximate* **Pol**-*resolution with property (U) of a space X. Then, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

$$\mathcal{D}_{\boldsymbol{X},n}(p_n(x), p_n(x')) \leq \mathcal{D}_{\boldsymbol{p}}(x,x') \text{ for all } x, x' \in X.$$

Proof. Suppose $(p_i(x), p_i(x')) < \mathcal{U}_i$ for some i. Then $p_i(x), p_i(x') \in U$ for some $U \in \mathcal{U}_i$. Let V be open such that $p_i(x), p_i(x') \in V \subseteq \overline{V} \subseteq U$, and let $\mathcal{V} \in \mathrm{Cov}(X_i)$ such that $\mathcal{V} < \{U, X_i \setminus \overline{V}\}$. By (AS), there is $n_0 \geq i$ such that $(p_i, p_{in}p_n) < \mathcal{V}$ for $n \geq n_0$. So, $p_{in}(p_n(x)), p_{in}(p_n(x')) \in U$ for all $n \geq n_0$. So $(p_{in}(p_n(x)), p_{in}(p_n(x'))) < \mathcal{U}_i$. This shows our assertion. □

Proposition 4.2. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be a normal approximate* **Pol***-resolution with property (U) of a space X. Then, for each $n \geq 2$,*

$$\mathcal{D}_{\boldsymbol{p}}(x, x') \leq 3\mathcal{D}_{\boldsymbol{X},n}(p_n(x), p_n(x')) \text{ for all } x, x' \in X.$$

Proof. Suppose $(p_{in}(p_n(x)), p_{in}(p_n(x'))) < \mathcal{U}_i$ for some i with $2 \leq i < n$. Then there is $U \in \mathcal{U}_i$ such that $p_{in}(p_n(x)), p_{in}(p_n(x')) \in U$. By (A1), $x, x' \in p_i^{-1}(\mathrm{st}(U, \mathcal{U}_i))$. But since $\boldsymbol{p}$ is normal, $p_i^{-1}(\mathrm{st}(U, \mathcal{U}_i)) \subseteq p_{i-1}^{-1}(U')$ for some $U' \in \mathcal{U}_{i-1}$. So, $p_{i-1}(x), p_{i-1}(x') \in U'$ and hence $(p_{i-1}(x), p_{i-1}(x')) < \mathcal{U}_{i-1}$. This fact implies that $3\mathcal{D}_{\boldsymbol{X},n}(p_n(x), p_n(x')) \geq \mathcal{D}_{\boldsymbol{p}}(x, x')$ as required. □

Proposition 4.3. *Let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be a normal approximate* **Pol***-resolution with property (U) of a space X. Then*

i) $\mathrm{d}_{\boldsymbol{X}}(x, x') \leq \mathrm{d}_{\boldsymbol{p}}(x, x')$ *for all $x, x' \in X$; and*

ii) *If each p_i is surjective, then*

$$\mathrm{d}_{\boldsymbol{p}}(x, x') \leq 3\,\mathrm{d}_{\boldsymbol{X}}(x, x') \text{ for all } x, x' \in X.$$

Proof. i) immediately follows from Proposition 4.1, and ii) immediately follows from Proposition 4.2 and the assumption that each p_i is surjective. □

For each approximate resolution $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$, we define the approximate system $\Sigma^n \boldsymbol{X}$ as $\{Z_i, \mathcal{W}_i, r_{ii'}, \mathbb{N}\}$ where $Z_i = X_{i+n}$, $\mathcal{W}_i = \mathcal{U}_{i+n}$, $r_{ii'} = p_{i+ni'+n} : Z_{i'} \to Z_i$ and the approximate resolution $\Sigma^n \boldsymbol{p}$ as $\{r_i : i \in \mathbb{N}\} : X \to \Sigma^n \boldsymbol{X}$ where $r_i = p_{i+n} : X \to X_{i+n}$.

Proposition 4.4. *Let X be a space, and let $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ be a normal approximate* **Pol***-resolution of X. Then, for each $n \in \mathbb{N}$,*

$$\mathrm{d}_{\Sigma^n \boldsymbol{p}}(x, x') = 3^n\,\mathrm{d}_{\boldsymbol{p}}(x, x') \text{ for } x, x' \in X.$$

Proof. This follows from Proposition 3.4 iii). □

5. Lipschitz functions and normal sequences

Throughout this section, *a space means a Hausforff space* and *all normal sequences are assumed to have property (B)* unless otherwise stated.

Let X and Y be any spaces, and let $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$ be normal sequences on X and Y, respectively. Then a map $f : X \to Y$ is called a $(\mathbb{U}, \mathbb{V})$-*Lipschitz function* (alternatively, *Lipschitz function with respect to* $\mathbb{U}$ *and* $\mathbb{V}$) provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \alpha\,\mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

In this section we give a characterization for $(\mathbb{U},\mathbb{V})$-Lipschitz functions in terms of the normal sequences $\mathbb{U}$ and $\mathbb{V}$.

For each map $f : X \to Y$ and for any normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$, we write $\mathbb{U} < f^{-1}\mathbb{V}$ if $\mathcal{U}_i < f^{-1}\mathcal{V}_i$ for each $i = 1, 2, ...$

Theorem 5.1. *Let X and Y be spaces with normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$, respectively, and let $f : X \to Y$ be a map. Consider the following statements:*

i) $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \mathrm{d}_{\mathbb{U}}(x, x')$ *for all* $x, x' \in X$*;*

ii) $\mathbb{U} < f^{-1}\mathbb{V}$*; and*

iii) $\Sigma^2\mathbb{U} < f^{-1}\mathbb{V}$.

Then the implications ii) $\Rightarrow$ i) $\Rightarrow$ iii) hold.

Proof. ii) $\Rightarrow$ i): Let $x, x' \in X$, and let $x = x_0, x_1, ..., x_n = x'$ be points in X such that $\mathcal{D}_{\mathbb{U}}(x_i, x_{i+1}) = \frac{1}{3^{k_i-2}}$ for some $k_i \geq 0$, $i = 0, 1, ..., n-1$. Since $\mathcal{U}_j < f^{-1}\mathcal{V}_j$ for each j, then $\mathcal{D}_{\mathbb{V}}(f(x_i), f(x_{i+1})) \leq \frac{1}{3^{k_i-2}}$, $i = 0, 1, ..., n-1$, which implies that

$$\mathrm{d}_{\mathbb{V}|f(X)}(f(x), f(x')) \leq \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

By Proposition 3.4 i),

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \mathrm{d}_{\mathbb{V}|f(X)}(f(x), f(x')) \text{ for } x, x' \in X.$$

Hence i) holds.

i) $\Rightarrow$ iii): Let $x \in X$, and take $U \in \mathcal{U}_{i+2}$ such that $x \in U$. Then Proposition 3.1 implies that

$$U \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{U}}}(x, \frac{1}{3^{i-1}}). \tag{5.1}$$

By i) and (5.1),

$$f(U) \subseteq \mathrm{U}_{\mathrm{d}_{\mathbb{V}}}(f(x), \frac{1}{3^{i-1}}). \tag{5.2}$$

Again by Proposition 3.1,

$$\mathrm{U}_{\mathrm{d}_{\mathbb{V}}}(f(x), \frac{1}{3^{i-1}}) \subseteq \mathrm{st}(f(x), \mathcal{V}_{i-1}). \tag{5.3}$$

By (5.2) and (5.3), there exists $V \in \mathcal{V}_i$ such that $f(U) \subseteq V$. Hence $\mathcal{U}_{i+2} < f^{-1}\mathcal{V}_i$ for each i, which means iii). $\square$

Theorem 5.2. *Let X and Y be spaces with normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$, respectively, and let $f : X \to Y$ be a map. Consider the following statements for $m \geq 0$:*

(L)$_m$: $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq 3^m\, \mathrm{d}_{\mathbb{U}}(x, x')$ *for all* $x, x' \in X$*; and*
(C)$_m$: $\Sigma^m\mathbb{U} < f^{-1}\mathbb{V}$.

Then the following implications hold:

i) *(C)$_m$ $\Rightarrow$ (L)$_m$; and*

ii) *(L)$_m$ $\Rightarrow$ (C)$_{m+2}$.*

Proof. Suppose that $\Sigma^m \mathbb{U} < f^{-1}\mathbb{V}$ for some $m \geq 0$. Then by Theorem 5.1,

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq \mathrm{d}_{\Sigma^m \mathbb{U}}(x, x') \text{ for all } x, x' \in X. \tag{5.4}$$

But by Proposition 3.4 iii),

$$\mathrm{d}_{\Sigma^m \mathbb{U}}(x, x') = 3^m \,\mathrm{d}_{\mathbb{U}}(x, x') \text{ for all } x, x' \in X. \tag{5.5}$$

(5.4) and (5.5) imply (L)$_m$. Conversely, suppose $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq 3^m \,\mathrm{d}_{\mathbb{U}}(x, x')$ for all $x, x' \in X$. Then by (5.5), we have (5.4). Now this together with Theorem 5.1 implies (C)$_{m+2}$. □

Theorem 5.2 immediately implies

Corollary 5.3. *Let X and Y be spaces with normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$, respectively. Then a map $f : X \to Y$ is a $(\mathbb{U}, \mathbb{V})$-Lipschitz function if and only if $\Sigma^m \mathbb{U} < f^{-1}\mathbb{V}$ for some $m \geq 0$.*

and

Corollary 5.4. *Let X be a space with normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{U}' = \{\mathcal{U}'_i : i = 1, 2, ...\}$ on X. Then there exists $\alpha > 0$ such that $\mathrm{d}_{\mathbb{U}'}(x, x') \leq \alpha \,\mathrm{d}_{\mathbb{U}}(x, x')$ for $x, x' \in X$ if and only if there exists $m \geq 0$ such that $\Sigma^m \mathbb{U} < \mathbb{U}'$.*

Corollary 5.5. *Let X and Y be spaces with normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$, respectively, and let $f : X \to Y$ be a map. Consider statement (L)$_m$ and the following statement for $m \geq 0$:*

(C')$_m$: $\mathbb{U} < f^{-1} \,\mathrm{st}^m \,\mathbb{V}$.

Then the implication (C')$_m$ $\Rightarrow$ (L)$_m$ holds.

Proof. By Theorem 5.1, (C')$_m$ implies that

$$\mathrm{d}_{\mathrm{st}^m \mathbb{V}}(f(x), f(x')) \leq \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

But by Proposition 3.4 iv),

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq 3^m \,\mathrm{d}_{\mathrm{st}^m \mathbb{V}}(f(x), f(x')) \text{ for } x, x' \in X.$$

Those two inequalities imply (L)$_m$. □

Corollary 5.6. *Let X and Y be spaces with normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$, respectively, and let $f : X \to Y$ be a map. Consider statement (L)$_{m+n}$ and the following statement for $m, n \geq 0$:*

(C)$_{m,n}$: $\Sigma^m \mathbb{U} < f^{-1} \,\mathrm{st}^n \,\mathbb{V}$.

Then the implication (C)$_{m,n}$ $\Rightarrow$ (L)$_{m+n}$ holds.

Proof. By Corollary 5.5, $(\mathrm{C})_{m,n}$ implies

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \leq 3^n \,\mathrm{d}_{\Sigma^m \mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

By Proposition 3.4 iii),

$$\mathrm{d}_{\Sigma^m \mathbb{U}}(x, x') = 3^m \,\mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

Hence those two statements imply $(\mathrm{L})_{m+n}$. □

6. Characterizations of Lipschitz functions

In this section we give characterizations in terms of approximate resolutions for the Lipschitz functions which were discussed in the previous section. Throughout this section, *a space means a continuum*, i.e., *connected compact metric space*, unless otherwise stated.

For any space X, let $\mathcal{APRES}(X)$ denote the collection of all normal approximate **Pol**-resolutions $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$ of X into an approximate system $\boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ such that all p_i and $p_{ii'}$ are surjective. By Theorem 2.3, $\mathcal{APRES}(X) \neq \emptyset$. For each map $f : X \to Y$, let $\mathcal{APRES}(f)$ denote the collection of all approximate resolutions $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f})$ of f.

For any spaces X and Y and for any $\boldsymbol{p} \in \mathcal{APRES}(X)$ and $\boldsymbol{q} \in \mathcal{APRES}(Y)$, a map $f : X \to Y$ is said to be a $(\boldsymbol{p}, \boldsymbol{q})$*-Lipschitz function* (alternatively, *Lipschitz function with respect to* $\boldsymbol{p}$ *and* $\boldsymbol{q}$) provided there exists a constant $\alpha > 0$ such that

$$\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \leq \alpha \,\mathrm{d}_{\boldsymbol{p}}(x, x') \text{ for } x, x' \in X.$$

Theorem 6.1. *Let X and Y be spaces, and let $f : X \to Y$ be a map. Let $\boldsymbol{p} \in \mathcal{APRES}(X)$, $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} \in \mathcal{APRES}(Y)$, $\boldsymbol{q} = \{q_i\} : Y \to \boldsymbol{Y} = \{Y_i, \mathcal{V}_i, q_{ii'}\}$, and let $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}) \in \mathcal{APRES}(f)$ such that $\boldsymbol{f} = \{f_i\} : \boldsymbol{X} \to \boldsymbol{Y}$ is a level morphism. For $m, n \geq 0$, consider the following statements:*

$(\mathrm{P}_1)_m$: $\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \leq 3^m \,\mathrm{d}_{\boldsymbol{p}}(x, x')$ *for all* $x, x' \in X$*;*

$(\mathrm{P}_2)_{m,n}$: $p_{i+m}^{-1}\mathcal{U}_{i+m} < f^{-1}q_i^{-1}(\mathrm{st}^n \,\mathcal{V}_i)$ *for each* $i \in \mathbb{N}$*; and*

$(\mathrm{P}_3)_{m,n}$: *For each $i \in \mathbb{N}$, there exists $j_0 > i + m$ such that*

$$p_{i+mj}^{-1}\mathcal{U}_{i+m} < f_j^{-1}q_{ij}^{-1}(\mathrm{st}^n \,\mathcal{V}_i) \textit{ for } j > j_0.$$

Then the following implications hold:

i) *$(P_1)_m \Rightarrow (P_2)_{m+2,0}$;*

ii) *$(P_2)_{m,n} \Rightarrow (P_3)_{m,n+2}$;*

iii) *$(P_3)_{m,n} \Rightarrow (P_2)_{m,n+2}$; and*

iv) *$(P_2)_{m,n} \Rightarrow (P_1)_{m+n}$.*

Proof. i) and iv) follow from $(\mathrm{L})_m \Rightarrow (\mathrm{C})_{m+2}$ of Theorem 5.2 and Corollary 5.6, respectively. For ii), suppose $(\mathrm{P}_2)_{m,n}$ holds, and let $i \in \mathbb{N}$. By (LAM) and(AS), there exists $j_0 > i + m$ such that

$$(q_i f, q_{ij} f_j p_j) < \mathcal{V}_i \text{ for } j > j_0, \tag{6.1}$$

and

$$(p_{i+m}, p_{i+mj}p_j) < \mathcal{U}_{i+m} \text{ for } j > j_0. \tag{6.2}$$

By $(\mathrm{P}_2)_{m,n}$,

$$p_{i+m}^{-1}\mathcal{U}_{i+m} < f^{-1}q_i^{-1}(\mathrm{st}^n \mathcal{V}_i). \tag{6.3}$$

Let $U \in \mathcal{U}_{i+m}$. Then (6.3) implies that there exists $V_U \in \mathcal{V}_i$ such that

$$q_i f(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}^n(V_U, \mathcal{V}_i). \tag{6.4}$$

By (6.1) and (6.4),

$$q_{ij} f_j p_j(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}^{n+1}(V_U, \mathcal{V}_i). \tag{6.5}$$

On the other hand, by (6.2),

$$p_j^{-1}p_{i+mj}^{-1}(U) \subseteq p_{i+m}^{-1}(\mathrm{st}(U,\mathcal{U}_{i+m})).$$

So, since p_j is surjective, we have

$$p_{i+mj}^{-1}(U) \subseteq p_j(p_{i+m}^{-1}(\mathrm{st}(U,\mathcal{U}_{i+m}))).$$

This and (6.5) together with the surjectivity of p_{i+m} imply

$$\begin{array}{rcl} q_{ij}f_j(p_{i+mj}^{-1}(U)) & \subseteq & q_{ij}f_jp_j(p_{i+m}^{-1}(\mathrm{st}(U,\mathcal{U}_{i+m}))) \\ & \subseteq & \mathrm{st}(q_{ij}f_jp_j(p_{i+m}^{-1}(U)), q_{ij}f_jp_jp_{i+m}^{-1}\mathcal{U}_{i+m}) \\ & \subseteq & \mathrm{st}(\mathrm{st}^{n+1}(V_U,\mathcal{V}_i), q_{ij}f_jp_jp_{i+m}^{-1}\mathcal{U}_{i+m}). \end{array} \tag{6.6}$$

But by (6.1) and (6.3),

$$q_{ij}f_jp_jp_{i+m}^{-1}\mathcal{U}_{i+m} < \mathrm{st}^{n+1}\mathcal{V}_i. \tag{6.7}$$

By (6.6) and (6.7),

$$q_{ij}f_j(p_{i+mj}^{-1}(U)) \subseteq \mathrm{st}(\mathrm{st}^{n+1}(V_U,\mathcal{V}_i), \mathrm{st}^{n+1}\mathcal{V}_i) = \mathrm{st}^{n+2}(V_U,\mathcal{V}_i).$$

This shows that

$$p_{i+mj}^{-1}\mathcal{U}_{i+m} < f_j^{-1}q_{ij}^{-1}(\mathrm{st}^{n+2}\mathcal{V}_i)$$

as required.

For iii), suppose $(\mathrm{P}_3)_{m,n}$ holds, and let $i \in \mathbb{N}$. By (LAM) and (AS), there exists $j_0 > i+m$ so that (6.1) and (6.2) hold. By $(\mathrm{P}_3)_{m,n}$, there exists $j_1 > j_0$ so that for each $j > j_1$,

$$p_{i+mj}^{-1}\mathcal{U}_{i+m} < f_j^{-1}q_{ij}^{-1}(\mathrm{st}^n \mathcal{V}_i). \tag{6.8}$$

Let $j > j_1$, and let $U \in \mathcal{U}_{i+m}$. Then (6.8) implies that there exists $V_U \in \mathcal{V}_i$ such that

$$q_{ij}f_j(p_{i+mj}^{-1}(U)) \subseteq \mathrm{st}^n(V_U,\mathcal{V}_i). \tag{6.9}$$

By (6.2),

$$p_{i+m}^{-1}(U) \subseteq p_j^{-1}p_{i+mj}^{-1}(\mathrm{st}(U,\mathcal{U}_{i+m})).$$

Since p_{i+mj} is surjecive, this implies

$$p_j(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}(p_{i+mj}^{-1}(U), p_{i+mj}^{-1}\mathcal{U}_{i+m}).$$

So,

$$q_{ij}f_jp_j(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}(q_{ij}f_j(p_{i+mj}^{-1}(U)), q_{ij}f_jp_{i+mj}^{-1}\mathcal{U}_{i+m}). \tag{6.10}$$

But, by (6.8),

$$q_{ij}f_jp_{i+mj}^{-1}\mathcal{U}_{i+m} < \mathrm{st}^n \mathcal{V}_i. \tag{6.11}$$

So, by (6.10), (6.9) and (6.11),

$$q_{ij}f_jp_j(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}(\mathrm{st}^n(V_U, \mathcal{V}_i), \mathrm{st}^n \mathcal{V}_i) = \mathrm{st}^{n+1}(V_U, \mathcal{V}_i). \tag{6.12}$$

By (6.12) and (6.1),

$$q_if(p_{i+m}^{-1}(U)) \subseteq \mathrm{st}^{n+2}(V_U, \mathcal{V}_i).$$

This shows that

$$p_{i+m}^{-1}\mathcal{U}_{i+m} < f^{-1}q_i^{-1}(\mathrm{st}^{n+2}\mathcal{V}_i)$$

as required. □

Theorem 6.2. *Let X and Y be spaces, and let $f : X \to Y$ be a map. Let $\boldsymbol{p} \in \mathcal{APRES}(X)$, $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} \in \mathcal{APRES}(Y)$, $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$, and let $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}) \in \mathcal{APRES}(f)$, $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$. For $m, n \geq 0$, consider statement $(P_2)_{m,n}$ and the following statement:*

- **$(\mathbf{P}_4)_{m,n}$:** *For each $j \in \mathbb{N}$, there exists $j_0 > j$ with the property that each $j' > j_0$ admits $i_0 > f(j'), j+m$ such that for each $i > i_0$,*

$$p_{j+mi}^{-1}\mathcal{U}_{j+m} < p_{f(j')i}^{-1}f_{j'}^{-1}q_{jj'}^{-1}(\mathrm{st}^n \mathcal{V}_j).$$

Then the following implications hold:

i) *$(P_2)_{m,n} \Rightarrow (P_4)_{m,n+2}$; and*

ii) *$(P_4)_{m,n} \Rightarrow (P_2)_{m,n+2}$.*

Proof. For i), suppose $(\mathrm{P}_2)_{m,n}$, and let $j \in \mathbb{N}$. For each $U \in \mathcal{U}_{j+m}$, there exists $V_U \in \mathcal{V}_j$ such that

$$q_jf(p_{j+m}^{-1}(U)) \subseteq \mathrm{st}^n(V_U, \mathcal{V}_j). \tag{6.13}$$

Take $\mathcal{V} \in \mathrm{Cov}(Y_j)$ such that

$$\mathrm{st}^2 \mathcal{V} < \mathcal{V}_j. \tag{6.14}$$

By (LAM), there exists $j_0 > j$ such that for each $j' > j_0$,

$$(q_jf, q_{jj'}f_{j'}p_{f(j')}) < \mathcal{V}. \tag{6.15}$$

Let $j' > j_0$, and take $\mathcal{U} \in \mathrm{Cov}(X_{f(j')})$ such that

$$\mathcal{U} < f_{j'}^{-1}q_{jj'}^{-1}\mathcal{V}. \tag{6.16}$$

By (AS), there exists $i_0 > j+m, f(j')$ such that for each $i > i_0$

$$(p_{j+m}, p_{j+mi}p_i) < \mathcal{U}_{j+m}, \tag{6.17}$$

and

$$(p_{f(j')}, p_{f(j')i}p_i) < \mathcal{U}. \tag{6.18}$$

Now let $U \in \mathcal{U}_{j+m}$. By (6.13) and (6.15),

$$q_{jj'}f_{j'}p_{f(j')}(p_{j+m}^{-1}(U)) \subseteq \mathrm{st}(\mathrm{st}^n(V_U, \mathcal{V}_j), \mathcal{V}) \tag{6.19}$$

and

$$q_{jj'}f_{j'}p_{f(j')}p_{j+m}^{-1}\mathcal{U}_{j+m} < \mathrm{st}(\mathrm{st}^n \mathcal{V}_j, \mathcal{V}). \tag{6.20}$$

On the other hand, since p_{j+m} is surjective, by (6.17),

$$p_i^{-1}p_{j+mi}^{-1}(U) \subseteq p_{j+m}^{-1}(\mathrm{st}(U, \mathcal{U}_{j+m})) = \mathrm{st}(p_{j+m}^{-1}(U), p_{j+m}^{-1}\mathcal{U}_{j+m}).$$

So, since p_i is surjective, we have

$$p_{j+mi}^{-1}(U) \subseteq p_i(\mathrm{st}(p_{j+m}^{-1}(U), p_{j+m}^{-1}\mathcal{U}_{j+m})) \subseteq \mathrm{st}(p_i(p_{j+m}^{-1}(U)), p_ip_{j+m}^{-1}\mathcal{U}_{j+m}).$$

This together with (6.18) implies

$$\begin{aligned} p_{f(j')i}(p_{j+mi}^{-1}(U)) &\subseteq \mathrm{st}(p_{f(j')i}p_i(p_{j+m}^{-1}(U)), p_{f(j')i}p_ip_{j+m}^{-1}\mathcal{U}_{j+m}) \\ &\subseteq \mathrm{st}(\mathrm{st}(p_{f(j')}(p_{j+m}^{-1}(U)), \mathcal{U}), \mathrm{st}(p_{f(j')}p_{j+m}^{-1}\mathcal{U}_{j+m}, \mathcal{U})). \end{aligned}$$

So this together with (6.19), (6.20), (6.16) and (6.14) implies

$$\begin{aligned} & q_{jj'}f_{j'}p_{f(j')i}(p_{j+mi}^{-1}(U)) \\ \subseteq\ & \mathrm{st}(\mathrm{st}(q_{jj'}f_{j'}p_{f(j')}(p_{j+m}^{-1}(U)), q_{jj'}f_{j'}\mathcal{U}), \mathrm{st}(q_{jj'}f_{j'}p_{f(j')}p_{j+m}^{-1}\mathcal{U}_{j+m}, q_{jj'}f_{j'}\mathcal{U})) \\ \subseteq\ & \mathrm{st}(\mathrm{st}(\mathrm{st}(\mathrm{st}^n(V_U, \mathcal{V}_j), \mathcal{V}), \mathcal{V}), \mathrm{st}(\mathrm{st}(\mathrm{st}^n \mathcal{V}_j, \mathcal{V}), \mathcal{V})) \\ \subseteq\ & \mathrm{st}(\mathrm{st}^{n+1}(V_U, \mathcal{V}_j), \mathrm{st}^{n+1} \mathcal{V}_j) \\ =\ & \mathrm{st}^{n+2}(V_U, \mathcal{V}_j). \end{aligned}$$

Hence $(\mathrm{P}_4)_{m,n+2}$ holds.

For ii), suppose $(\mathrm{P}_4)_{m,n}$, and let $j \in \mathbb{N}$. Take $\mathcal{V} \in \mathrm{Cov}(Y_j)$ such that

$$\mathrm{st}^2 \mathcal{V} < \mathcal{V}_j, \tag{6.21}$$

and take $j_0 > j$ as in $(\mathrm{P}_4)_{m,n}$. By (LAM), there exists $j_1 > j_0$ such that for each $j' > j_1$,

$$(q_jf, q_{jj'}f_{j'}p_{f(j')}) < \mathcal{V}. \tag{6.22}$$

Let $j' > j_1$, and take $\mathcal{U} \in \mathrm{Cov}(X_{f(j')})$ such that

$$\mathcal{U} < f_{j'}^{-1}q_{jj'}^{-1}\mathcal{V}. \tag{6.23}$$

For this j', take $i_0 > f(j'), j+m$ as in $(\mathrm{P}_4)_{m,n}$. By (AS), there exists $i > i_0$ such that

$$(p_{f(j')}, p_{f(j')i}p_i) < \mathcal{U}, \tag{6.24}$$

and

$$(p_{j+m}, p_{j+mi}p_i) < \mathcal{U}_{j+m}. \tag{6.25}$$

Now let $U \in \mathcal{U}_{j+m}$. Then, by $(\mathrm{P}_4)_{m,n}$, thre exists $V_U \in \mathcal{V}_j$ such that

$$q_{jj'}f_{j'}p_{f(j')i}(p_{j+mi}^{-1}(U)) \subseteq \mathrm{st}^n(V_U, \mathcal{V}_j). \tag{6.26}$$

But by (6.25),

$$p_{j+m}^{-1}(U) \subseteq p_i^{-1} p_{j+mi}^{-1}(\mathrm{st}(U, \mathcal{U}_{j+m})),$$

which, by the surjectivity of p_i, implies

$$p_i(p_{j+m}^{-1}(U)) \subseteq \mathrm{st}(p_{j+mi}^{-1}(U), p_{j+mi}^{-1}\mathcal{U}_{j+m}), \tag{6.27}$$

and hence

$$p_{f(j')i} p_i(p_{j+m}^{-1}(U)) \subseteq \mathrm{st}(p_{f(j')i}(p_{j+mi}^{-1}(U)), p_{f(j')i} p_{j+mi}^{-1}\mathcal{U}_{j+m}).$$

This together with (6.24) implies that

$$p_{f(j')}(p_{j+m}^{-1}(U)) \subseteq \mathrm{st}(\mathrm{st}(p_{f(j')i}(p_{j+mi}^{-1}(U)), p_{f(j')i} p_{j+mi}^{-1}\mathcal{U}_{j+m}), \mathcal{U}).$$

This together with (6.26), $(\mathrm{P}_4)_{m,n}$ and (6.23) implies

$$\begin{aligned} q_{jj'} f_{j'} p_{f(j')}(p_{j+m}^{-1}(U)) &\subseteq \mathrm{st}(\mathrm{st}(q_{jj'} f_{j'} p_{f(j')i}(p_{j+mi}^{-1}(U)), q_{jj'} f_{j'} p_{f(j')i} p_{j+mi}^{-1}\mathcal{U}_{j+m}), q_{jj'} f_{j'}\mathcal{U}) \\ &\subseteq \mathrm{st}(\mathrm{st}(\mathrm{st}^n(V_U, \mathcal{V}_j), \mathrm{st}^n \mathcal{V}_j), \mathcal{V}) \\ &= \mathrm{st}(\mathrm{st}^{n+1}(V_U, \mathcal{V}_j), \mathcal{V}). \end{aligned} \tag{6.28}$$

Now (6.28), (6.22) and (6.21) yield

$$q_j f(p_{j+m}^{-1}(U)) \subseteq \mathrm{st}(\mathrm{st}(\mathrm{st}^{n+1}(V_U, \mathcal{V}_j), \mathcal{V}), \mathcal{V}) \subseteq \mathrm{st}^{n+2}(V_U, \mathcal{V}_j).$$

Hence $(\mathrm{P}_2)_{m,n+2}$ holds as required. □

Corollary 6.3. *Let X and Y be spaces, and let $f : X \to Y$ be a map. Let $\boldsymbol{p} \in \mathcal{APRES}(X)$, $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} \in \mathcal{APRES}(Y)$, $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$. Then f is a $(\boldsymbol{p}, \boldsymbol{q})$-Lipschitz function if and only if f satisfies any one of the following conditions:*

i) *There exist $m, n \geq 0$ such that $(P_2)_{m,n}$ holds;*

ii) *There exist $m, n \geq 0$ such that $(P_3)_{m,n}$ holds for some (any) $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}) \in \mathcal{APRES}(f)$ such that $\boldsymbol{f} = \{f_i\} : \boldsymbol{X} \to \boldsymbol{Y}$ is a level morphism; and*

iii) *There exist $m, n \geq 0$ such that $(P_4)_{m,n}$ holds for some (any) $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}) \in \mathcal{APRES}(f)$, $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$.*

Corollary 6.4. *Let X be a space, and let $\boldsymbol{p}, \boldsymbol{p}' \in \mathcal{APRES}(X)$, $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X}$, $\boldsymbol{p}' = \{p'_i : i \in \mathbb{N}\} : X \to \boldsymbol{X}'$. Then there exists a constant $\alpha > 0$ such that*

$$\mathrm{d}_{\boldsymbol{p}}(x, x') \leq \alpha\, \mathrm{d}_{\boldsymbol{p}'}(x, x') \text{ for all } x, x' \in X$$

if and only if there exist $m, n \geq 0$ and $(\boldsymbol{p}, \boldsymbol{p}', \boldsymbol{f}) \in \mathcal{APRES}(1_X)$, $\boldsymbol{f} = \{f_i, f\} : \boldsymbol{X} \to \boldsymbol{X}'$ for which $(P_4)_{m,n}$ holds, where $1_X : X \to X$ is the identity map.

7. Contraction maps and fixed point theorem

Throughout this section, *a space means a continuum* and *all normal sequences are assumed to have property (B)* unless otherwise stated. A function $f : X \to Y$ between spaces X and Y is called a *contraction* provided there exists a constant α with $0 < \alpha < 1$ such that

$$\mathrm{d}(f(x), f(x')) \le \alpha\, \mathrm{d}(x, x') \text{ for all } x, x' \in X.$$

Let X and Y be any spaces, and let $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$ be normal sequences on X and Y, respectively. Then a map $f : X \to Y$ is called a $(\mathbb{U}, \mathbb{V})$-*contraction* (alternatively, *contraction with respect to* $\mathbb{U}$ *and* $\mathbb{V}$) provided there exists a constant α with $0 < \alpha < 1$ such that

$$\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \le \alpha\, \mathrm{d}_{\mathbb{U}}(x, x') \text{ for all } x, x' \in X,$$

and for any $\boldsymbol{p} \in \mathcal{APRES}(X)$ and $\boldsymbol{q} \in \mathcal{APRES}(Y)$, a map $f : X \to Y$ is said to be a $(\boldsymbol{p}, \boldsymbol{q})$-*contraction* (alternatively, *contraction with respect to* $\boldsymbol{p}$ *and* $\boldsymbol{q}$) provided there exists a constant α with $0 < \alpha < 1$ such that

$$\mathrm{d}_{\boldsymbol{q}}(f(x), f(x')) \le \alpha\, \mathrm{d}_{\boldsymbol{p}}(x, x') \text{ for all } x, x' \in X.$$

Theorem 7.1. *Let X and Y be spaces with normal sequences $\mathbb{U} = \{\mathcal{U}_i : i = 1, 2, ...\}$ and $\mathbb{V} = \{\mathcal{V}_i : i = 1, 2, ...\}$, respectively, and let $f : X \to Y$ be a map. For $m, n \ge 0$, consider the following statements:*

(M)$_m$: $\mathrm{d}_{\mathbb{V}}(f(x), f(x')) \le 3^{-m}\, \mathrm{d}_{\mathbb{U}}(x, x')$ *for all $x, x' \in X$; and*
(N)$_{m,n}$: $\Sigma^n\mathbb{U} < f^{-1}\Sigma^m\mathbb{V}$.

Then, for each $m \ge 0$, the following implications hold:

i) *(M)$_m$ $\Rightarrow$ (N)$_{m,n+2}$ for any $n \ge 0$; and*

ii) *(N)$_{m,n}$ $\Rightarrow$ (M)$_{n-m}$ if $n \ge m$, and (N)$_{m,n}$ $\Rightarrow$ (L)$_{m-n}$ (see Theorem 5.2) if $m > n$.*

Proof. For i), suppose (M)$_m$. Then

$$3^m\, \mathrm{d}_{\mathbb{V}}(f(x), f(x')) \le \mathrm{d}_{\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

By Proposition 3.4 iii),

$$\mathrm{d}_{\Sigma^m\mathbb{V}}(f(x), f(x')) = 3^m\, \mathrm{d}_{\mathbb{V}}(f(x), f(x')) \text{ for } x, x' \in X.$$

Theorem 5.1 now implies

$$\Sigma^2\mathbb{U} < f^{-1}\Sigma^m\mathbb{V}.$$

Since $\Sigma^{n+2}\mathbb{U} < \Sigma^2\mathbb{U}$ for any $n \ge 0$,

$$\Sigma^{n+2}\mathbb{U} < f^{-1}\Sigma^m\mathbb{V},$$

which means (N)$_{m,n+2}$.

For ii), suppose (N)$_{m,n}$. Then Theorem 5.1 implies

$$\mathrm{d}_{\Sigma^m\mathbb{V}}(f(x), f(x')) \le \mathrm{d}_{\Sigma^n\mathbb{U}}(x, x') \text{ for } x, x' \in X.$$

By Proposition 3.4 iii),

$$\mathrm{d}_{\Sigma^n\mathbb{U}}(x,x') = 3^n\,\mathrm{d}_{\mathbb{U}}(x,x') \text{ and } \mathrm{d}_{\Sigma^m\mathbb{V}}(f(x),f(x')) = 3^m\,\mathrm{d}_{\mathbb{V}}(f(x),f(x')) \text{ for } x,x'\in X.$$

So,

$$\mathrm{d}_{\mathbb{V}}(f(x),f(x')) \le 3^{n-m}\,\mathrm{d}_{\mathbb{U}}(x,x') \text{ for } x,x'\in X,$$

which means $(\mathrm{L})_{n-m}$ if $n \ge m$, and $(\mathrm{M})_{m-n}$ if $m > n$. □

Theorem 7.2. *Let X and Y be spaces, and let $f : X \to Y$ be a map. Let $\boldsymbol{p} \in \mathcal{APRES}(X)$, $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} \in \mathcal{APRES}(Y)$, $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$, and let $(\boldsymbol{p},\boldsymbol{q},\boldsymbol{f}) \in \mathcal{APRES}(f)$ such that $\boldsymbol{f} = \{f_i\} : \boldsymbol{X} \to \boldsymbol{Y}$ is a level morphism. For $k,m,n \ge 0$, consider the following statements:*

$(\mathbf{Q}_1)_m$: $\mathrm{d}_{\boldsymbol{q}}(f(x),f(x')) \le 3^{-m}\,\mathrm{d}_{\boldsymbol{p}}(x,x')$ *for all $x,x' \in X$;*
$(\mathbf{Q}_2)_{k,m,n}$: $p_{i+k}^{-1}\mathcal{U}_{i+k} < f^{-1}q_{i+m}^{-1}(\mathrm{st}^n\,\mathcal{V}_{i+m})$ *for each i; and*
$(\mathbf{Q}_3)_{k,m,n}$: *For each $i \in \mathbb{N}$, there exists $j_0 > i+m, i+k$ such that*

$$p_{i+kj}^{-1}\mathcal{U}_{i+k} < f_j^{-1}q_{i+mj}^{-1}(\mathrm{st}^n\,\mathcal{V}_{i+m}) \text{ for } j > j_0.$$

Then, for each $k,m,n \ge 0$, the following implications hold:

i) *$(Q_1)_m \Rightarrow (Q_2)_{2,m,0}$;*

ii) *$(Q_2)_{k,m,n} \Rightarrow (Q_3)_{k,m,n+2}$;*

iii) *$(Q_3)_{k,m,n} \Rightarrow (Q_2)_{k,m,n+2}$; and*

iv) *$(Q_2)_{k,m,n} \Rightarrow (Q_1)_{m-n-k}$ for $m \ge n+k$.*

Proof. i) follows by the following implications (see Theorem 6.1 i)):

$$\begin{aligned} & (\mathrm{Q}_1)_m \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{P}_1)_0 \text{ for } \boldsymbol{p} \text{ and } \Sigma^m\boldsymbol{q} \\ \Longrightarrow\ & (\mathrm{P}_2)_{2,0} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m\boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{Q}_2)_{2,m,0} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \end{aligned}$$

ii) follows by the following implications (see Theorem 6.1 ii)):

$$\begin{aligned} & (\mathrm{Q}_2)_{k,m,n} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{P}_2)_{k,n} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m\boldsymbol{q} \\ \Longrightarrow\ & (\mathrm{P}_3)_{k,n+2} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m\boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{Q}_3)_{k,m,n+2} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \end{aligned}$$

iii) follows by the following implications (see Theorem 6.1 iii)):

$$\begin{aligned} & (\mathrm{Q}_3)_{k,m,n} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{P}_3)_{k,n} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m\boldsymbol{q} \\ \Longrightarrow\ & (\mathrm{P}_2)_{k,n+2} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m\boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{Q}_2)_{k,m,n+2} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \end{aligned}$$

iv) follows by the following implications (see Theorem 6.1 iv)):

$$\begin{aligned} & (\mathrm{Q}_2)_{k,m,n} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{P}_2)_{k,n} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m \boldsymbol{q} \\ \Longrightarrow\ & (\mathrm{P}_1)_{k+n} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{Q}_1)_{m-k-n} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \end{aligned}$$

□

Theorem 7.3. *Let X and Y be spaces, and let $f : X \to Y$ be a map. Let $\boldsymbol{p} \in \mathcal{APRES}(X)$, $\boldsymbol{p} = \{p_i\} : X \to \boldsymbol{X} = \{X_i, \mathcal{U}_i, p_{ii'}\}$ and $\boldsymbol{q} \in \mathcal{APRES}(Y)$, $\boldsymbol{q} = \{q_j\} : Y \to \boldsymbol{Y} = \{Y_j, \mathcal{V}_j, q_{jj'}\}$, and let $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}) \in \mathcal{APRES}(f)$, $\boldsymbol{f} = \{f_j, f\} : \boldsymbol{X} \to \boldsymbol{Y}$. For $m, n \geq 0$, consider statement $(Q_2)_{m,n}$ and the following statement:*

$(\mathrm{Q}_4)_{k,m,n}$: *For each $j \in \mathbb{N}$, there exists $j_0 > j + m$ with the property that each $j' > j_0$ admits $i_0 > f(j'), j + k$ such that for each $i > i_0$,*

$$p^{-1}_{j+ki} \mathcal{U}_{j+k} < p^{-1}_{f(j')i} f^{-1}_{j'} q^{-1}_{j+mj'} (\mathrm{st}^n\, \mathcal{V}_{j+m}).$$

Then, for $k, m, n \geq 0$, the following implications hold:

i) *$(Q_2)_{k,m,n} \Rightarrow (Q_4)_{k,m,n+2}$; and*

ii) *$(Q_4)_{k,m,n} \Rightarrow (Q_2)_{k,m,n+2}$.*

Proof. i) follows by the following implications (see Theorem 6.2 i)):

$$\begin{aligned} & (\mathrm{Q}_2)_{k,m,n} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{P}_2)_{k,n} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m \boldsymbol{q} \\ \Longrightarrow\ & (\mathrm{P}_4)_{k,n+2} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{Q}_4)_{k,m,n+2} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \end{aligned}$$

ii) follows by the following implications (see Theorem 6.2 ii)):

$$\begin{aligned} & (\mathrm{Q}_4)_{k,m,n} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{P}_4)_{k,n} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m \boldsymbol{q} \\ \Longrightarrow\ & (\mathrm{P}_2)_{k,n+2} \text{ for } \boldsymbol{p} \text{ and } \Sigma^m \boldsymbol{q} \\ \Longleftrightarrow\ & (\mathrm{Q}_2)_{k,m,n+2} \text{ for } \boldsymbol{p} \text{ and } \boldsymbol{q} \end{aligned}$$

□

Theorem 7.4. *Let X be a space. Then a map $f : X \to X$ has a unique fixed point if f satisfies any one of the following conditions:*

i) *There exist $k, m, n \geq 0$ with $m > k + n$ and $\boldsymbol{p} \in \mathcal{APRES}(X)$ such that $(Q_2)_{m,n}$ holds;*

ii) *There exist $m, n \geq 0$ with $m > k + n + 2$ and $\boldsymbol{p} \in \mathcal{APRES}(X)$ such that $(Q_3)_{k,m,n}$ holds for some (any) $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}) \in \mathcal{APRES}(f)$ with $\boldsymbol{f}$ being a level morphism; and*

iii) *There exist $m, n \geq 0$ with $m > k + n + 2$ and $\boldsymbol{p} \in \mathcal{APRES}(X)$ such that $(Q_4)_{k,m,n}$ holds for some (any) $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{f}) \in \mathcal{APRES}(f)$.*

Proof. Any one of the conditions implies that the map f is a contraction for some metric on X. □

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