Optimal control problems for the equation of motion of membrane with strong viscosity

Hwang, Jin-soo

Nakagiri, Shin-ichi

(Citation)
Journal of Mathematical Analysis and Applications,321(1):327-342

(Issue Date)
2006-09

(Resource Type)
journal article

(Version)
Accepted Manuscript

(URL)
https://hdl.handle.net/20.500.14094/90000455

Optimal control problems for the equation of motion of membrane with strong viscosity

Jin-soo Hwang and Shin-ichi Nakagiri*

Division of Mathematical and Material Science,

The Graduate School of Science and Technology, Kobe University

Department of Applied Mathematics, Faculty of Engineering, Kobe University, Japan

Abstract Optimal control problems are studied for the equation of membrane with strong viscosity. The Gâteaux differentiability of solution mapping on control variables is proved and the various types of necessary optimality conditions corresponding to the distributive and terminal values observations are established.

Key words: Optimal control; Equation of membrane with strong viscosity; Gâteaux differentiability; Necessary optimality conditions

1 Introduction

We consider a freely flexible stretched film which is called a membrane. It is well known that the vibrating of the longitudinal motion of membrane is described by the following nonlinear equation

$$\frac{\partial^2 y}{\partial t^2} - \operatorname{div}\Big(\frac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) = 0, \tag{1.1}$$

where y is the height of the membrane. It seems to be difficult to construct a solution of (1.1) in a Hilbert or reflexive Banach spaces not only for theoretical construction but also for any other applications. So some modified but more realistic model equations are proposed, and among them we consider the following equation with strong viscosity terms

$$\frac{\partial^2 y}{\partial t^2} - \operatorname{div}\Big(\frac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) - \mu\Delta\frac{\partial y}{\partial t} = f, \tag{1.2}$$

*Supported by Grant-in-Aid for Scientific Research (C)(2) 16540194.

where $\mu > 0$ and f is a forcing function. The equation (1.2) is proposed in Kobayashi, Pecher and Shibata [4] and the well posedness of strongly regular solutions are studied by using the resolvent estimates of linearized operators in a modified Banach space. Recently in Hwang and Nakagiri [3] the well posedness of less regular solutions, called weak solutions of (1.2) is proved in the framework of variational method in Dautray and Lions [1] under Dirichlet boundary conditions. The result enables us to study the optimal control problems associated with (1.2) in the standard manner due to the theory of Lions [5]. We also refer to Ha and Nakagiri [2] for the optimal control problems on second order semilinear equations.

In this paper we study the optimal control problems for the controlled equation

$$\frac{\partial^2 y(v)}{\partial t^2} - \mathrm{div}\Big(\frac{\nabla y(v)}{\sqrt{1+|\nabla y(v)|^2}}\Big) - \mu\Delta\frac{\partial y(v)}{\partial t} = f + Bv, \tag{1.3}$$

where B is a controller, v is a control and $y(v)$ denote the state for a given $v \in \mathcal{U}$, $\mathcal{U}$ is a Hilbert space of control variables. Let $\mathcal{U}_{ad} \subset \mathcal{U}$ be an admissible set. We propose the quadratic cost functional $J(v)$ as studied in Lions [5]. The purpose of this paper is to establish the necessary conditions of optimality for various observation cases. For this we prove the Gâteaux differentiability of the nonlinear mapping $v \to y(v)$, which is used to define the associate adjoint system. We want to emphasis that in the velocity's observation case, the first order Volterra integro-differential equation is utilized as a proper adjoint system inspite of the original system being described by the second order equation.

2 Preliminaries

Let Ω be an open bounded set of $\mathbf{R}^n$ with the smooth boundary Γ. We set $Q = (0,T)\times\Omega$, $\Sigma = (0,T)\times\Gamma$ for $T > 0$. We consider the following Dirichlet boundary value problem for the equation of motion of membrane with strong viscosity

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \mathrm{div}\Big(\dfrac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) - \mu\Delta\dfrac{\partial y}{\partial t} = f \quad \text{in } \ Q, \\ y = 0 \quad \text{on } \ \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \quad \text{in } \ \Omega, \end{cases} \tag{2.1}$$

where f is a forcing function, y_0 and y_1 are initial data and $\mu > 0$ is a constant. In (2.1) we suppose $f \in L^2(0,T;H^{-1}(\Omega))$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. The solution space $W(0,T)$ of (2.1) is defined by

$$W(0,T) = \{g | g \in L^2(0,T;H_0^1(\Omega)),\ g' \in L^2(0,T;H_0^1(\Omega)),\ g'' \in L^2(0,T;H^{-1}(\Omega))\}$$

endowed with the norm

$$\|g\|_{W(0,T)} = \Big(\|g\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g'\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g''\|^2_{L^2(0,T;H^{-1}(\Omega))}\Big)^{\frac{1}{2}},$$

where g' and g'' denote the first and second order distributive derivatives of g. We remark that $W(0,T)$ is continuously imbedded in $C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ (cf. Dautray and Lions [1, p.555]). The scalar products and norms on $L^2(\Omega)$ and $H_0^1(\Omega)$ are denoted by (ϕ,ψ), $|\phi|$ and $(\phi,\psi)_{H_0^1(\Omega)}$, $\|\phi\|$, respectively. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by (ϕ,ψ) and $|\phi|$. Then the scalar product $(\phi,\psi)_{H_0^1(\Omega)}$ and the norm $\|\phi\|$ of $H_0^1(\Omega)$ are given by $(\nabla\phi,\nabla\psi)$ and $\|\phi\| = |(\nabla\phi,\nabla\phi)|^{\frac{1}{2}}$, respectively. The duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle\phi,\psi\rangle$. Related to the nonlinear term in (2.1), we define the function $G : \mathbf{R}^n \to \mathbf{R}^n$ by $G(x) = \dfrac{x}{\sqrt{1+|x|^2}}, x \in \mathbf{R}^n$. Then it is verified that

$$|G(x) - G(y)| \le 2|x-y|, \quad \forall x, y \in \mathbf{R}^n. \tag{2.2}$$

The nonlinear operator $G(\nabla\cdot) : H_0^1(\Omega) \to [L^2(\Omega)]^n$ is introduced by

$$G(\nabla\phi)(x) = \frac{\nabla\phi(x)}{\sqrt{1+|\nabla\phi(x)|^2}}, \quad \text{a.e. } x \in \Omega, \quad \forall\phi \in H_0^1(\Omega). \tag{2.3}$$

By the definition of $G(\nabla\cdot)$ in (2.3), we have the following useful property on $G(\nabla\cdot)$:

$$|G(\nabla\phi)| \le |\nabla\phi|, \quad |G(\nabla\phi) - G(\nabla\psi)| \le 2|\nabla\phi - \nabla\psi|, \qquad \forall\phi,\psi \in H_0^1(\Omega). \tag{2.4}$$

A function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot),\phi\rangle + (G(\nabla y(\cdot)),\nabla\phi) + \mu(\nabla y'(\cdot),\nabla\phi) = \langle f(\cdot),\phi\rangle \\ \qquad\qquad \text{for all } \ \phi \in H_0^1(\Omega) \ \text{ in the sense of } \ \mathcal{D}'(0,T), \\ y(0) = y_0 \in H_0^1(\Omega), \quad y'(0) = y_1 \in L^2(\Omega). \end{cases} \tag{2.5}$$

The following theorem on existence, uniqueness and regularity of the weak solution of (2.1) is proved in Hwang and Nakagiri [3] by the Galerkin method.

Theorem 2.1 *Assume that $\mu > 0$, $f \in L^2(0,T;H^{-1}(\Omega))$ and $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$. Then the problem* (2.1) *has a unique weak solution y in $W(0,T)$.*

Next we give the result on the continuous dependence of weak solutions of (2.1) on on initial values y_0, y_1 and forcing terms f. Let P be a product space defined by

$$P = H_0^1(\Omega) \times L^2(\Omega) \times L^2(0,T;H^{-1}(\Omega)). \tag{2.6}$$

For each $p = (y_0, y_1, f) \in P$ we have a unique weak solution $y = y(p) \in W(0,T)$ of (2.1) by Theorem 2.1. Hence we can define the solution mapping $p = (y_0,y_1,f) \to y(q)$ of P into $W(0,T)$. The following theorem is also proved in [3].

Theorem 2.2 *The solution mapping $p = (y_0,y_1,f) \to y(q)$ of P into $W(0,T)$ is strongly continuous. Further, for each $p_1 = (y_0^1, y_1^1, f_1) \in P$ and $p_2 =$*

$(y_0^2, y_1^2, f_2) \in P$ *we have the inequality*

$$\begin{aligned}
& |y'(p_1;t) - y'(p_2;t)|^2 + |\nabla y(p_1;t) - \nabla y(p_2;t)|^2 \\
+ & \int_0^t |\nabla y'(p_1;s) - \nabla y(p_2;s)|^2 ds \\
\leq & C(\|y_0^1 - y_0^2\|^2 + |y_1^1 - y_1^2|^2 + \|f_1 - f_2\|^2_{L^2(0,T;H^{-1}(\Omega))}), \ \forall t \in [0,T], \quad (2.7)
\end{aligned}$$

where C *is a constant depending only on* $\mu > 0$.

We will omit writing the integral variables in the definite integral without any confusion. For example, in (2.7) we will write $\int_0^t |\nabla y'(p_1)|^2 ds$ instead of $\int_0^t |\nabla y'(p_1;s)|^2 ds$.

3 Quadratic cost optimal control problems

In this section we study the quadratic cost optimal control problems for the equation of motion of membrane in the framework of Lions [5]. Let $\mathcal{U}$ be a Hilbert space of control variables, and let B be an operator,

$$B \in \mathcal{L}(\mathcal{U}, L^2(0,T;L^2(\Omega))) \subset \mathcal{L}(\mathcal{U}, L^2(0,T;H^{-1}(\Omega))), \quad (3.1)$$

called a controller. We consider the following nonlinear control system:

$$\begin{cases} \dfrac{\partial^2 y(v)}{\partial t^2} - \mathrm{div}\Big(\dfrac{\nabla y(v)}{\sqrt{1+|\nabla y(v)|^2}}\Big) - \mu\Delta \dfrac{\partial y(v)}{\partial t} = f + Bv \quad \text{in} \quad Q, \\ y(v) = 0 \qquad \text{on} \quad \Sigma, \\ y(v;0,x) = y_0(x), \qquad \dfrac{\partial y}{\partial t}(v;0,x) = y_1(x) \qquad \text{in} \quad \Omega, \end{cases} \quad (3.2)$$

where $y_0 \in H_0^1(\Omega)$, $y_1 \in L^2(\Omega)$, $f \in L^2(0,T;H^{-1}(\Omega))$ and $v \in \mathcal{U}$ is a control. By virtue of Theorem 2.1 and (3.1), we can define uniquely the solution map $v \to y(v)$ of $\mathcal{U}$ into $W(0,T)$. We shall call the weak solution $y(v)$ of (3.2) the state of the control system (3.2). The observation of the state is assumed to be given by

$$z(v) = Cy(v), \quad C \in \mathcal{L}(W(0,T), M), \quad (3.3)$$

where C is an operator called the observer, and M is a Hilbert space of observation variables. The quadratic cost function associated with the control system (3.2) is given by

$$J(v) = \|Cy(v) - z_d\|_M^2 + (Rv, v)_{\mathcal{U}} \quad \text{for } v \in \mathcal{U}, \quad (3.4)$$

where $z_d \in M$ is a desired value of $z(v)$ and $R \in \mathcal{L}(\mathcal{U}, \mathcal{U})$ is symmetric and positive, i.e.,

$$(Rv, v)_{\mathcal{U}} = (v, Rv)_{\mathcal{U}} \geq d\|v\|_{\mathcal{U}}^2 \quad (3.5)$$

for some $d > 0$. Let $\mathcal{U}_{ad}$ be a closed convex subset of $\mathcal{U}$, which is called the admissible set. An element $u \in \mathcal{U}_{ad}$ which attains the minimum of $J(v)$ over $\mathcal{U}_{ad}$ is called an optimal control for the cost (3.4).

Assume the existence of optimal control u for the cost (3.4). In this section we shall characterize the optimal controls by giving necessary conditions for optimality. For this it is necessary to write down the necessary optimality condition

$$DJ(u)(v-u) \geq 0 \qquad \text{for all } \ v \in \mathcal{U}_{ad} \tag{3.6}$$

and to analyze (3.6) in view of the proper adjoint state system, where $DJ(u)$ denote the Gâteaux derivative of $J(v)$ at $v = u$. That is, we have to prove that the mapping $v \to y(v)$ of $\mathcal{U} \to W(0,T)$ is Gâteaux differentiable at $v = u$. At first we can see the continuity of the mapping. The following lemma follows immediately from Theorem 2.2.

Lemma 3.1 *Let $w \in \mathcal{U}$ be arbitrarily fixed. Then*

$$\lim_{\lambda \to 0} y(u + \lambda w) = y(u) \quad \textit{strongly in} \;\; W(0,T). \tag{3.7}$$

Especially,

$$\lim_{\lambda \to 0} \nabla y(u + \lambda w) = \nabla y(u) \quad \textit{in} \;\; C([0,T];[L^2(\Omega)]^n). \tag{3.8}$$

The solution map $v \to y(v)$ of $\mathcal{U}$ into $W(0,T)$ is said to be Gâteaux differentiable at $v = u$ if for any $w \in \mathcal{U}$ there exists a $Dy(u) \in \mathcal{L}(\mathcal{U}, W(0,T))$ such that

$$\left\| \frac{1}{\lambda}(y(u+\lambda w) - y(u)) - Dy(u)w \right\|_{W(0,T)} \to 0 \;\; \text{as} \;\; \lambda \to 0.$$

The operator $Dy(u)$ denotes the Gâteaux derivative of $y(u)$ at $v = u$ and the function $Dy(u)w \in W(0,T)$ is called the Gâteaux derivative in the direction $w \in \mathcal{U}$, which is a crucial tool to investigate the nonlinear optimal control problem.

Theorem 3.1 *The map $v \to y(v)$ of $\mathcal{U}$ into $W(0,T)$ is Gâteaux differentiable at $v = u$ and such the Gâteaux derivative of $y(v)$ at $v = u$ in the direction $v - u \in \mathcal{U}$, say $z = Dy(u)(v-u)$, is a unique weak solution of the following problem*

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - \operatorname{div}\left(\dfrac{\nabla z}{\sqrt{1 + |\nabla y(u)|^2}} - \nabla y(u) \dfrac{\nabla y(u) \cdot \nabla z}{(1 + |\nabla y(u)|^2)^{\frac{3}{2}}} \right) - \mu \Delta \dfrac{\partial z}{\partial t} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = B(v-u) \quad \text{in} \;\; Q, \\ z = 0 \qquad \text{on} \;\; \Sigma, \\ z(0,x) = 0, \quad \dfrac{\partial z}{\partial t}(0,x) = 0 \quad \text{in} \;\; \Omega. \end{cases} \tag{3.9}$$

Proof. Let $\lambda \in (-1,1)$, $\lambda \neq 0$. We set

$$z_\lambda = \lambda^{-1} y_\lambda = \lambda^{-1}(y(u + \lambda(u-v)) - y(u)).$$

Then z_λ satisfies

$$\begin{cases} \dfrac{\partial^2 z_\lambda}{\partial t^2} - \mathrm{div}\dfrac{1}{\lambda}\Big(G(\nabla y(u+\lambda(v-u))) - G(\nabla y(u))\Big) - \mu\Delta\dfrac{\partial z_\lambda}{\partial t} \\ \qquad\qquad\qquad\qquad\qquad\qquad = B(v-u) \quad \text{in} \ \ Q, \\ z_\lambda = 0 \quad \text{on} \ \ \Sigma, \\ z_\lambda(0,x) = 0, \quad \dfrac{\partial z_\lambda}{\partial t}(0,x) = 0 \quad \text{in} \ \ \Omega \end{cases} \tag{3.10}$$

in the weak sense. Set $w = v - u$. Multiply the weak form of (3.10) by z'_λ and z_λ, integrate them over $[0,t]$ and add the integrals. Then as similarly as in [1, p.567] (cf. [3]) we have that z_λ satisfies

$$\begin{aligned} & |z'_\lambda(t)|^2 + 2\int_0^t \mu|\nabla z'_\lambda|^2 ds + \mu|\nabla z_\lambda(t)|^2 \\ = \ & -2(z'_\lambda(t), z_\lambda(t)) + \ 2\int_0^t |z'_\lambda|^2 ds + 2\int_0^t \langle (Bw), z'_\lambda + z_\lambda\rangle ds \\ & - \frac{2}{\lambda}\int_0^t (G(\nabla y(u+\lambda w)) - G(\nabla y(u)), \nabla z'_\lambda) ds \\ & - \frac{2}{\lambda}\int_0^t (G(\nabla y(u+\lambda w)) - G(\nabla y(u)), \nabla z_\lambda) ds. \end{aligned} \tag{3.11}$$

The nonlinear term in (3.11) can be estimated by (2.4) as

$$\frac{1}{\lambda}\Big|G(\nabla y(u+\lambda(v-u))) - G(\nabla y(u))\Big| \le 2|\nabla z_\lambda|. \tag{3.12}$$

Then, by using the Schwartz's inequality we can verify from (3.11) and (3.12) that

$$\begin{aligned} & |\nabla z_\lambda(t)|^2 + |z'_\lambda(t)|^2 + \int_0^t |\nabla z'_\lambda|^2 ds \\ \le \ & K\int_0^t \{|\nabla z_\lambda|^2 + |z'_\lambda|^2\} ds + K\|Bw\|^2_{L^2(0,T;H^{-1}(\Omega))} \end{aligned} \tag{3.13}$$

for some $K > 0$. Hence by applying the Gronwall's inequality to (3.13), we have

$$|\nabla z_\lambda(t)|^2 + |z'_\lambda(t)|^2 + \int_0^t |\nabla z'_\lambda|^2 ds \le K_1\|Bw\|^2_{L^2(0,T;H^{-1}(\Omega))}. \tag{3.14}$$

Therefore there exists a $z \in W(0,T)$ and a sequence $\{\lambda_k\} \subset (-1,1)$ tending to 0 such that

$$\begin{cases} z_{\lambda_k} \to z \ \text{ weakly star in } \ L^\infty(0,T;H_0^1(\Omega)) \\ \qquad\qquad \text{and weakly in } \ L^2(0,T;H_0^1(\Omega)) \ \text{ as } \ k\to\infty, \\ z'_{\lambda_k} \to z' \ \text{ weakly star in } \ L^\infty(L^2(\Omega)) \\ \qquad\qquad \text{and weakly in } \ L^2(0,T;H_0^1(\Omega)) \ \text{ as } \ k\to\infty, \\ z''_{\lambda_k} \to z'' \ \text{ weakly in } \ L^2(0,T;H^{-1}(\Omega)) \ \text{ as } \ k\to\infty, \\ z(0) = 0, \quad z'(0) = 0. \end{cases} \tag{3.15}$$

Let us prove that

$$\begin{aligned}
&\frac{1}{\lambda_k}\mathrm{div}\Big(G(\nabla y(u+\lambda_k w)) - G(\nabla y(u))\Big)\\
\to\quad &\mathrm{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla z}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\right)\\
&\qquad\qquad\text{weakly in } \ L^2(0,T;H^{-1}(\Omega)). \qquad (3.16)
\end{aligned}$$

For brevity of notations we set $y_k(u) = y(u+\lambda_k w)$ and $z_k = z_{\lambda_k}$. We can deduce from (2.3) that for all $\phi \in L^2(0,T;H_0^1(\Omega))$,

$$\begin{aligned}
&\frac{1}{\lambda_k}\int_0^T \Big(G(\nabla y_k(u)) - G(\nabla y(u)), \nabla\phi\Big)dt\\
&= \int_0^T \Big((1+|\nabla y_k(u)|^2)^{-\frac{1}{2}}\nabla z_k, \nabla\phi\Big)dt\\
&+\frac{1}{\lambda_k}\int_0^T \Big(\Big((1+|\nabla y_k(u)|^2)^{-\frac{1}{2}}\\
&\qquad\qquad -(1+|\nabla y(u)|^2)^{-\frac{1}{2}}\Big)\nabla y(u), \nabla\phi\Big)dt. \qquad (3.17)
\end{aligned}$$

Moreover the right hand side of (3.17) can be written as

$$\begin{aligned}
&\int_0^T \Big((1+|\nabla y_k(u)|^2)^{-\frac{1}{2}}\nabla(z_k - z), \nabla\phi\Big)dt\\
&+\int_0^T \Big(\Big((1+|\nabla y_k(u)|^2)^{-\frac{1}{2}} - (1+|\nabla y(u)|^2)^{-\frac{1}{2}}\Big)\nabla z, \nabla\phi\Big)dt\\
&+\int_0^T \Big((1+|\nabla y(u)|^2)^{-\frac{1}{2}}\nabla z, \nabla\phi\Big)dt\\
&+\frac{1}{\lambda_k}\int_0^T \Big(\Big((1+|\nabla y_k(u)|^2)^{-\frac{1}{2}}\\
&\qquad\qquad -(1+|\nabla y(u)|^2)^{-\frac{1}{2}}\Big)\nabla y(u), \nabla\phi\Big)dt. \qquad (3.18)
\end{aligned}$$

We can easily know that the first term of (3.18) tends to 0 by (3.15) and Lemma 3.1, that is the strong convergency of $\{y_k(u)\}$. Also the second term of (3.18) tends to 0 by Lemma 3.1 and Lebesgue dominated convergence theorem. To have the limit of the fourth term of (3.18), we will employ the following notation

$$\mathcal{G}_i(\phi,\psi) = \frac{\phi_{x_i} + \psi_{x_i}}{\sqrt{1+|\nabla\phi|^2}\sqrt{1+|\nabla\psi|^2}(\sqrt{1+|\nabla\phi|^2}+\sqrt{1+|\nabla\psi|^2})}, \qquad (3.19)$$

where $\phi,\psi \in H_0^1(\Omega)$. Using the above notation, the fourth term of (3.18) can be rewritten by

$$-\int_0^T ((\sum_{i=1}^n \mathcal{G}_i(y(u), y_k(u)) z_{kx_i})\nabla y(u), \nabla\phi)dt$$

$$
\begin{aligned}
= & -\int_0^T ((\sum_{i=1}^n \mathcal{G}_i(y(u),y_k(u))(z_{kx_i}-z_{x_i}))\nabla y(u),\nabla\phi)dt \\
& -\int_0^T ((\sum_{i=1}^n (\mathcal{G}_i(y(u),y_k(u)) - \frac{y_{x_i}(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}})z_{x_i})\nabla y(u),\nabla\phi)dt \\
& -\int_0^T (\frac{\nabla y(u)\cdot\nabla z}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\nabla y(u),\nabla\phi)dt. \qquad (3.20)
\end{aligned}
$$

By virtue of Lemma 3.1,

$$
\mathcal{G}_i(y(u),y_k(u)) \ \to \ \frac{y_{x_i}(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}} \qquad \text{strongly in } \ L^2(0,T;L^2(\Omega)). \qquad (3.21)
$$

Therefore the first term of the right hand side of (3.20) tends to 0. And it is clear that

$$
\begin{aligned}
& \Big|(\sum_{i=1}^n \mathcal{G}_i(y(u),y_k(u)) - \frac{y_{x_i}(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}})z_{x_i})\nabla y(u)\cdot\nabla\phi\Big| \\
\leq \ & c|\nabla z||\nabla\phi| \ \in \ L^1(Q) \qquad (3.22)
\end{aligned}
$$

for some $c>0$ by $\nabla z, \nabla\phi \in L^2(0,T;[L^2(\Omega)]^n) = [L^2(Q)]^n$. The strong convergence (3.21) in $L^2(0,T;L^2(\Omega)) = L^2(Q)$ implies, by choosing subsequence of $\{\lambda_k\}$ if necessary, that

$$
\mathcal{G}_i(y(u),y_k(u)) - \frac{y_{x_i}(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}} \ \to \ 0 \qquad \text{a.e. in } \ Q. \qquad (3.23)
$$

Hence, (3.22), (3.23) and Lebesgue dominated convergence theorem imply that the second term of the right hand side of (3.20) tends to 0. Thus, we have proved (3.16). This means that z is a weak solution of (3.9). Hence by (3.15), (3.16) and (3.9) we see that $z_\lambda \to z = Dy(u)w$ weakly in $W(0,T)$ as $\lambda \to 0$. It remains now to show the strong convergency of $\{z_\lambda\}$. Using (3.19), the equation (3.10) becomes

$$
\begin{cases}
\dfrac{\partial^2 z_\lambda}{\partial t^2} - \mathrm{div}\left(\dfrac{\nabla z_\lambda}{\sqrt{1+|\nabla y(u+\lambda w)|^2}} - \displaystyle\sum_{i=1}^n \mathcal{G}_i(y(u),y(u+\lambda w))z_{\lambda x_i}\nabla y(u)\right) \\
\qquad\qquad\qquad\qquad\qquad -\mu\Delta\dfrac{\partial z_\lambda}{\partial t} \ = Bw \quad \text{in } \ Q, \\
z_\lambda = 0 \quad \text{on } \ \Sigma, \\
z_\lambda(0,x) = 0, \quad \dfrac{\partial z_\lambda}{\partial t}(0,x) = 0 \quad \text{in } \ \Omega.
\end{cases}
\qquad (3.24)
$$

And let z be a unique weak solution of (3.9). Subtracting (3.24) from (3.9) and denoting $z_\lambda - z$ by ϕ_λ, we can obtain

$$\begin{cases} \dfrac{\partial^2 \phi_\lambda}{\partial t^2} - \mathrm{div}\left(\dfrac{\nabla z_\lambda}{\sqrt{1+|\nabla y(u+\lambda w)|^2}} - \dfrac{\nabla z}{\sqrt{1+|\nabla y(u)|^2}} \right) - \mu \Delta \dfrac{\partial \phi_\lambda}{\partial t} \\ = -\mathrm{div}\Big(\displaystyle\sum_{i=1}^n \mathcal{G}_i(y(u), y(u+\lambda w)) \phi_{\lambda x_i}(s) \nabla y(u) \\ \qquad + \displaystyle\sum_{i=1}^n \Big\{ \dfrac{y_{x_i}(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}} - \mathcal{G}_i(y(u), y(u+\lambda w)) \Big\} z_{x_i} \nabla y(u) \Big) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad Q, \\ \phi_\lambda = 0 \quad \text{on} \quad \Sigma, \\ \phi_\lambda(0,x) = 0, \quad \dfrac{\partial \phi_\lambda}{\partial t}(0,x) = 0 \;\; \text{in} \;\; \Omega. \end{cases} \tag{3.25}$$

For simplicity, we set

$$\begin{aligned} \mathcal{G}^i_\lambda(t) &= \mathcal{G}^i_\lambda(y(u;t), y(u+\lambda w;t)), \\ K_\lambda(t) &= \frac{1}{\sqrt{1+|\nabla y(u+\lambda w;t)|^2}}, \quad \lambda \in (-1,1). \end{aligned}$$

From the weak form of (3.25), as in the derivation of (3.11), we can deduce

$$\begin{aligned} & |\phi'_\lambda(t)|^2 + \mu|\nabla\phi_\lambda(t)|^2 + 2\mu\int_0^t |\nabla\phi'_\lambda|^2 ds \\ & + 2\int_0^t \Big(K_\lambda \nabla z_\lambda - K_0 \nabla z, \nabla\phi_\lambda + \nabla\phi'_\lambda \Big) ds \\ & + 2(\phi'_\lambda(t), \phi_\lambda(t)) - 2\int_0^t |\phi'_\lambda|^2 ds \\ = {} & 2\int_0^t (\sum_{i=1}^n \mathcal{G}^i_\lambda \phi_{\lambda x_i} \nabla y(u), \nabla\phi_\lambda + \nabla\phi'_\lambda) ds \\ & + 2\int_0^t \Big(\sum_{i=1}^n \Big\{ \mathcal{G}^i_\lambda - (K_0)^3 y_{x_i}(u) \Big\} z_{x_i} \nabla y(u), \nabla\phi_\lambda + \nabla\phi'_\lambda \Big) ds. \end{aligned} \tag{3.26}$$

Indeed, (3.26) can be rewritten as

$$\begin{aligned} & |\phi'_\lambda(t)|^2 + \mu|\nabla\phi_\lambda(t)|^2 + 2\mu\int_0^t |\nabla\phi'_\lambda|^2 ds \\ = {} & \sum_{i=1}^2 \Phi^i_\lambda(t) - 2\int_0^t \Big(K_\lambda \nabla\phi_\lambda, \nabla\phi_\lambda + \nabla\phi'_\lambda \Big) ds - 2(\phi'_\lambda(t), \phi_\lambda(t)) \\ & + 2\int_0^t |\phi'_\lambda|^2 ds + 2\int_0^t ((\sum_{i=1}^n \mathcal{G}^i_\lambda \phi_{\lambda x_i}) \nabla y(u), \nabla\phi_\lambda + \nabla\phi'_\lambda) ds, \end{aligned} \tag{3.27}$$

where

$$\Phi^1_\lambda(t) = -2\int_0^t ((K_\lambda - K_0)\nabla z, \nabla\phi_\lambda + \nabla\phi'_\lambda) ds,$$

$$\Phi^2_\lambda(t) \;=\; 2\int_0^t \Big(\sum_{i=1}^n \Big\{\mathcal{G}^i_\lambda - (K_0)^3 y_{x_i}(u)\Big\} z_{x_i}\nabla y(u), \nabla\phi_\lambda + \nabla\phi'_\lambda)\Big) ds.$$

We can estimate the integral of the last term of (3.27), for some $c > 0$, as following

$$\Big|(\sum_{i=1}^n \mathcal{G}^i_\lambda(s)\phi_{\lambda x_i}(s))\nabla y(u;s), \nabla\phi_\lambda(s))\Big| \;\le\; c|\nabla\phi_\lambda(s)|^2, \tag{3.28}$$

$$\Big|(\sum_{i=1}^n \mathcal{G}^i_\lambda(s)\phi_{\lambda x_i}(s)\nabla y(u;s), \nabla\phi'_\lambda(s))\Big| \;\le\; c|\nabla\phi_\lambda(s)||\nabla\phi'_\lambda(s)|. \tag{3.29}$$

If we put

$$S_\lambda(t) = \sum_{i=1}^2 \Phi^i_\lambda(t),$$

then from (3.27), (3.29) and (3.28), we can derive the inequality

$$\begin{aligned} &|\phi'_\lambda(t)|^2 + |\nabla\phi_\lambda(t)|^2 + \int_0^t |\nabla\phi'_\lambda|^2 ds \\ \le\;& C_1 S_\lambda(t) + C_2\int_0^t (|\phi'_\lambda|^2 + |\nabla\phi_\lambda|^2) ds, \end{aligned} \tag{3.30}$$

where C_1 and C_2 are constants. Hence by the Bellman Gronwell's inequality, it is followed that

$$\begin{aligned} &|\phi'_\lambda(t)|^2 + |\nabla\phi_\lambda(t)|^2 + \int_0^t |\nabla\phi'_\lambda|^2 ds \\ \le\;& C_1 S_\lambda(t) + C_1\exp(C_2 T)\int_0^t S_\lambda(s) ds. \end{aligned} \tag{3.31}$$

From (3.15), there exists a sequence $\{\lambda_k\} \subset (-1, 1)$ tending to 0 and $C > 0$ such that

$$S_{\lambda_k}(t) \;\to\; 0, \qquad \text{as } \;\lambda_k \;\to\; 0, \tag{3.32}$$

$$|S_{\lambda_k}(t)| \le C < \infty, \qquad \text{on } \;[0, T]. \tag{3.33}$$

Therefore (3.31), (3.32) and (3.33) imply that

$$\begin{aligned} \phi_{\lambda_k} \;&\to\; 0 \quad \text{in } \; C([0,T]; H^1_0(\Omega)), \\ \phi'_{\lambda_k} \;&\to\; 0 \quad \text{in } \; C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1_0(\Omega)), \end{aligned}$$

so that

$$z_{\lambda_k}(\cdot) \;\to\; z(\cdot) \quad \text{strongly in } \; W(0,T).$$

This completes the proof.

Theorem 3.1 means that the cost $J(v)$ is Gâteaux differentiable at u in the direction $v-u$ and the optimality condition (3.6) is rewritten by

$$\begin{aligned} & (Cy(u)-z_d, C(Dy(u)(v-u)))_M + (Ru, v-u)_{\mathcal{U}} \\ = \;& \langle C^*\Lambda_M(Cy(u)-z_d), Dy(u)(v-u)\rangle_{W(0,T)',W(0,T)} \\ & +(Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \end{aligned} \tag{3.34}$$

where Λ_M is the canonical isomorphism M onto M'. For simplicity, we consider the following two types of identity observations $C = I_d$ of distributive and terminal values.

1. We take $M = L^2(Q) \times L^2(\Omega)$ and $C = I_d \in \mathcal{L}(M, L^2(Q))$ and observe $z(v) = (y(v), y(v;T)) \in L^2(Q) \times L^2(\Omega)$;
2. We take $M = L^2(Q)$ and $C = I_d \in \mathcal{L}(M, L^2(Q))$ and observe $z(v) = y'(v) \in L^2(Q)$.

Since $y \in W(0,T) \subset C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ by Theorem 2.1, the above observations are meaningful.

3.1 Case of distributive and terminal values observations

In this subsection we consider the cost functional expressed by

$$\begin{aligned} J(v) \;=\; & \int_Q |y(v;t,x) - z_d(t,x)|^2 dxdt \\ & + \int_\Omega |y(v;T,x) - z_d(x)|^2 dx + (Rv, v)_{\mathcal{U}}, \quad \forall v \in \mathcal{U}_{ad} \subset \mathcal{U}, \end{aligned} \tag{3.35}$$

where $z_d \in L^2(Q)$ and $z_d^T \in L^2(\Omega)$ are desired values. Let u be the optimal control subject to (3.2) and (3.35). Then the optimality condition (3.34) is represented by

$$\begin{aligned} & \int_Q (y(u;t,x) - z_d(t,x)) z(t,x) dxdt \\ & + \int_\Omega (y(u;T,x) - z_d^T(x)) z(T,x) dx + (Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \end{aligned} \tag{3.36}$$

where z is the weak solution of the equation (3.9). Now we will formulate the adjoint system to describe the optimality condition. Since $y(u) - z_d \in L^2(Q) = L^2(0,T; L^2(\Omega)) \subset L^2(0,T; H^{-1}(\Omega))$ and Since $y(u;T) - z_d^T \in L^2(\Omega)$, there exists a weak solution $p(u) \in W(0,T)$ satisfying

$$\begin{cases} \dfrac{\partial^2 p(u)}{\partial t^2} - \mathrm{div}\left(\dfrac{\nabla p(u)}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u) \dfrac{\nabla y(u)\cdot\nabla p(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}} \right) \\ \qquad\qquad\qquad\qquad + \mu\Delta \dfrac{\partial p(u)}{\partial t} = y(u) - z_d \quad \text{in} \quad Q, \\ p(u) = 0 \quad \text{on} \quad \Sigma, \\ p(u;T,x) = 0, \quad \dfrac{\partial p}{\partial t}(u;T,x) = -y(u;T,x) + z_d(x) \quad \text{in} \quad \Omega. \end{cases} \tag{3.37}$$

Now we proceed the calculations. We multiply both sides of the weak form of equation (3.37) by z and integrate it over Q. Then we have

$$\begin{aligned}
& -\int_Q \frac{\partial p(u)}{\partial t}\frac{\partial z}{\partial t}dxdt \\
& + \int_Q \left(\frac{\nabla p(u)}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla p(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}} \right)\cdot\nabla z(t,x)dxdt \\
& - \mu\int_Q \nabla\frac{\partial p(u)}{\partial t}\cdot\nabla z dxdt \\
= & \int_Q (y(u)-z_d)z dxdt. \qquad (3.38)
\end{aligned}$$

By integration by parts and by the terminal value conditions of p, the left hand side of (3.38) can be given by

$$\begin{aligned}
& \int_\Omega \frac{\partial p}{\partial t}(u;T,x)z(T,x)dx - \int_Q \frac{\partial p(u)}{\partial t}\frac{\partial z}{\partial t}dxdt \\
& + \int_Q \left(\frac{\nabla z}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla z}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}} \right)\cdot\nabla p(t,x)dxdt \\
& + \mu\int_Q \nabla p(u)\cdot\nabla\frac{\partial z}{\partial t}dxdt \\
= & -\int_\Omega (y(u;T,x)-z_d(x))z(T,x)dx + \int_Q p(u)B(v-u)dxdt. \qquad (3.39)
\end{aligned}$$

Therefore, by (3.38), (3.39) and z is a weak solution of (3.9), the optimality condition (3.36) is equivalent to

$$\int_Q p(u;t,x)B(v-u)(t,x)dxdt + (Ru, v-u)_{\mathcal{U}} \geq 0, \qquad \forall v \in \mathcal{U}_{ad}.$$

Hence, we show the following theorem.

Theorem 3.2 *The optimal control* u *for* (3.35) *is characterized by the following system of equations and inequality:*

$$\begin{cases}
\dfrac{\partial^2 y(u)}{\partial t^2} - div\Big(\dfrac{\nabla y(u)}{\sqrt{1+|\nabla y(u)|^2}}\Big) - \mu\Delta\dfrac{\partial y(u)}{\partial t} = f + Bu \quad in \quad Q, \\
y(u) = 0 \quad on \quad \Sigma, \\
y(u;0,x) = y_0(x), \qquad \dfrac{\partial y}{\partial t}(u;0,x) = y_1(x) \quad in \quad \Omega.
\end{cases}$$

$$\begin{cases}
\dfrac{\partial^2 p(u)}{\partial t^2} - div\left(\dfrac{\nabla p(u)}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\dfrac{\nabla y(u)\cdot\nabla p(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}} \right) \\
\qquad\qquad + \mu\Delta\dfrac{\partial p(u)}{\partial t} = y(u) - z_d \quad in \quad Q, \\
p(u) = 0 \quad on \quad \Sigma, \\
p(u;T,x) = 0, \qquad \dfrac{\partial p}{\partial t}(u;T,x) = -y(u;T,x) + z_d(x) \quad in \quad \Omega.
\end{cases}$$

$$\int_Q p(u;t,x)B(v-u)(t,x)dxdt + (Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}.$$

3.2 Case of velosity observations

We consider the following velocity cost functional expressed by

$$J(v) = \int_Q \left|\frac{\partial y}{\partial t}(v;t,x) - z_d(t,x)\right|^2 dxdt + (Rv, v)_{\mathcal{U}}, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.40}$$

where $z_d \in L^2(Q)$. Let u be the optimal control subject to (3.2) and (3.40). Then the optimality condition (3.34) is rewritten as

$$\int_Q \Big(\frac{\partial y(u)}{\partial t} - z_d\Big)\frac{\partial z}{\partial t} dxdt + (Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.41}$$

where z is the weak solution of the equation (3.9). Now we will formulate the following adjoint system to describe the optimality condition. Especially, in this case, an adjoint equation can be represented by the following first order integro-differential equation

$$\begin{cases} \dfrac{\partial p(u)}{\partial t} + \displaystyle\int_t^T \mathrm{div}\left(\frac{\nabla p(u)}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla p(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\right)ds \\ \qquad\qquad\qquad\qquad + \mu\Delta p(u) = \dfrac{\partial y}{\partial t}(u) - z_d \quad \text{in} \quad Q, \\ p(u) = 0 \quad \text{on} \quad \Sigma, \\ p(u;T,x) = 0 \quad \text{in} \quad \Omega. \end{cases} \tag{3.42}$$

Remark 3.1 It is a common sence that the adjoint systems of second order problems are also second order in general. In this observation case we can also construct a second order adjoint system formally. However we can not guarantee the well-posedness of the second order system with the present theory. But the system (3.42) is well-posed by the results of Dautray and Lions [1, p. 656], and is adopted as a better adjoint system.

By reversing the direction of time $t \to T-t$ and applying the results [1, pp. 656-662] to the system (3.42), there exists a unique weak solution satisfying

$$p(u) \;\in\; W(H_0^1(\Omega), L^2(\Omega)) \cap C([0,T]; H_0^1(\Omega)). \tag{3.43}$$

Now we also proceed the calculations in the Gelfand triple space

$$(H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega)).$$

We multiply both sides of the weak form of equation (3.42) by z' and integrate it on $[0,T]$. Then we have

$$\int_0^T (p'(u;t), z'(t))dt$$

$$
\begin{aligned}
&-\int_0^T \Big(\int_t^T \Big(\frac{\nabla p(u)}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\frac{\nabla y(u)\cdot\nabla p(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds, \nabla z'(t)\Big)dt \\
&-\mu\int_0^T (\nabla p(u;t), \nabla z'(t))dt \\
= &\int_0^T ((y'(u)-z_d)(t), z'(t))dt, \qquad (3.44)
\end{aligned}
$$

where $z_d(t) = z_d(t,\cdot)$. By integration by parts and by the terminal value conditions of p, the left hand side of (3.44) can be given by

$$
\begin{aligned}
&-\int_0^T \langle p(u;t), z''(t)\rangle dt \\
&+\int_0^T \Big\langle p(u;t), \mathrm{div}\Big(\frac{\nabla z(t)}{\sqrt{1+|\nabla y(u;t)|^2}} - \nabla y(u;t)\frac{\nabla y(u;t)\cdot\nabla z(t)}{(1+|\nabla y(u;t)|^2)^{\frac{3}{2}}}\Big)\Big\rangle dt \\
&+\mu\int_0^T \langle p(u;t), \Delta z'(t)\rangle dt \\
= &-\int_0^T \Big\langle p(u;t), z''(t) \\
&\qquad -\mathrm{div}\Big(\frac{\nabla z(t)}{\sqrt{1+|\nabla y(u;t)|^2}} - \nabla y(u;t)\frac{\nabla y(u;t)\cdot\nabla z(t)}{(1+|\nabla y(u;t)|^2)^{\frac{3}{2}}}\Big) - \Delta z'(t)\Big\rangle dt \\
= &-\int_0^T (p(u;t), B(v-u)(t))dt. \qquad (3.45)
\end{aligned}
$$

Therefore, by (3.44) and (3.45), the optimality condition (3.41) is equivalent to

$$
-\int_0^T (p(u;t), B(v-u)(t))dt + (Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}.
$$

Hence, we show the following theorem.

Theorem 3.3 *The optimal control u for (3.40) is characterized by the following system of equations and inequality:*

$$
\begin{cases}
\dfrac{\partial^2 y(u)}{\partial t^2} - div\Big(\dfrac{\nabla y(u)}{\sqrt{1+|\nabla y(u)|^2}}\Big) - \mu\Delta\dfrac{\partial y(u)}{\partial t} = f + Bu \quad in \quad Q, \\
y(u) = 0 \quad on \quad \Sigma, \\
y(u;0,x) = y_0(x), \qquad \dfrac{\partial y}{\partial t}(u;0,x) = y_1(x) \quad in \quad \Omega.
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial p(u)}{\partial t} + \displaystyle\int_t^T div\Big(\dfrac{\nabla p(u)}{\sqrt{1+|\nabla y(u)|^2}} - \nabla y(u)\dfrac{\nabla y(u)\cdot\nabla p(u)}{(1+|\nabla y(u)|^2)^{\frac{3}{2}}}\Big)ds \\
\qquad\qquad\qquad\qquad + \mu\Delta p(u) = \dfrac{\partial y(u)}{\partial t} - z_d \quad in \quad Q, \\
p(u) = 0 \quad on \quad \Sigma, \\
p(u;T,x) = 0, \qquad \dfrac{\partial p}{\partial t}(u;T,x) = 0 \quad in \quad \Omega.
\end{cases}
$$

$$- \int_Q p(u;t,x) B(v-u)(t,x) dxdt + (Ru, v-u)_{\mathcal{U}} \geq 0, \quad \forall v \in \mathcal{U}_{ad}.$$

References

[1] R. Dautary and J. L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology, Vol. 5, Evolution Problems I, Springer-Verlag, 1992.

[2] J. Ha and S. Nakagiri, Optimal control problems for nonlinear hyperbolic disributed parameter systems with damping terms, Funkcial. Ekvac., **47** (2004) 23.

[3] J. Hwang and S. Nakagiri, Weak solutions of the equation of motion of membrane with strong viscosity, submitted in J. Math. Anal. Appl..

[4] T. Kobayashi, H. Pecher and Y. Shibata, On a global in time existence theorem of smooth solutions to nonlinear wave equation with viscosity, Math. Ann., **296** (1993) 215-234.

[5] J. L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, Springer-Verlag Berlin Heidelberg New York, 1971.

[6] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications I, II, Springer-Verlag Berlin Heidelberg New York, 1972.

[7] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physis, Second Edition, Applied Mathematica Sciences. Vol. 68, Springer-Verlag, Berlin-Heidelberg-New York, 1997.