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ASYMPTOTICALLY EXACT CONFIDENCE INTERVALS OF CUSUM AND CUSUMSQ TESTS: A Numerical Derivation Using Simulation Technique

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ABSTRACT: In testing a structural change, the approximated confidence intervals are conventionally used for CUSUM and CUSUMSQ tests. This paper numerically derives the asymptotically exact confidence intervals of CUSUM and CUSUMSQ tests. It can be easily extended to nonnormal and/or nonlinear models.

KEY WORDS: CUSUM test, CUSUMSQ test, Monte-Carlo simulation, Asymptotically exact confidence interval

1 INTRODUCTION

There is a great amount of literature on use of recursive residuals, e.g., Brown, Durbin and Evans (1975), Galpin and Hawkins (1984), Harvey (1989), Johnston (1984), Ploberger (1989), Ploberger, Krämer and Alt (1989), and Westlund and Törnkvist (1989). Especially, Brown, Durbin and Evans (1975) described an important application of recursive residuals in testing for structural change over time. The technique is appropriate for time series data, and might be used if one is uncertain about when a structural change might have taken place. We have two tests; cumulative sum (CUSUM) test and cumulative sum of squares (CUSUMSQ) test The null hypothesis is that the coefficient vector β is the same in every period; the alternative is simply that it (or the disturbance variance) is not. The test is quite general in that it does not require a prior specification of when the structural change takes place. However, it is known that the power of the test is rather limited compared to that of the Chow test. The test is frequently criticized on this basis. However, the Chow test is based on a rather definite piece of information, namely, when the structural change takes place. If this is not known or must be estimated, the advantage of the Chow test diminishes considerably (see Greene (1990) and Krämer (1989)). See Galpin and Hawkins (1984) for an application.

One of the reasons why the CUSUM test is less powerful is that the confidence interval of the test is approximated. For the CUSUM test statistic, we cannot derive the explicit distribution, and therefore the approximated confidence interval is conventionally used in testing the stability of the coefficient. Although it is known that the CUSUMSQ test statistic is distributed as a beta random variable, the confidence interval of the CUSUMSQ test is also approximated. Therefore, in this paper, an attempt is made to obtain the exact confidence intervals of the CUSUM and CUSUMSQ tests asymptotically using the Monte-Carlo simulation technique. Moreover, the power is compared for both the approximated confidence intervals (the confidence intervals conventionally used) and the simulated ones (the confidence intervals proposed in this paper).

2 OVERVIEW OF CUSUM AND CUSUMSQ TESTS

Consider the following regression model:

$$y_t = x_t\beta + u_t, \qquad u_t \sim N(0,\sigma^2), \qquad t = 1,\cdots,T,$$

where β is a $k\times 1$ unknown parameter vector. y_t is a dependent variable while x_t is a $1\times k$ vector of independent variables. u_t is assumed to be normally distributed with mean zero and variance σ^2. Define X_t and Y_t as follows:

$$Y_t = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{pmatrix}, \qquad X_t = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{pmatrix}.$$

The null hypothesis of no structural change for the model is specified as:

$$H_0 : \beta_1 = \beta_2 = \cdots = \beta_T = \beta \ \text{ and } \ \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_T^2 = \sigma^2,$$

where β_t denotes the vector of coefficients in period t and σ_t^2 the disturbance variance in that period. The null hypothesis would be violated if β remained constant but σ^2 varies.

Let b_t be the ordinary least squares estimate of β_t using the data Y_t and X_t, i.e., $b_t = (X_t'X_t)^{-1}X_t'Y_t$. It is well known that b_t is obtained recursively (see Brown, Durbin and Evans (1975), Johnston (1984) and Riddell (1975) for the recursive algorithm). The recursive residual is defined as:

$$\omega_t = \frac{y_t - x_t b_{t-1}}{\sqrt{1 + x_t(X_{t-1}'X_{t-1})^{-1}x_t'}}, \tag{1}$$

which has mean zero and variance σ^2.

The CUSUM test statistic for testing structural change is given by:

$$W_t = \sum_{i=k+1}^{t} \omega_i \Big/ \widehat{\sigma}, \qquad t = k+1,\cdots,T, \tag{2}$$

where $\widehat{\sigma}^2 = \dfrac{1}{T-k}\displaystyle\sum_{i=k+1}^{T}\omega_i^2$, which implies the unbiased estimate of σ^2. The expected value of W_t is zero and the distribution of W_t is symmetric about zero if the disturbance in the regression model is symmetric. Since we cannot obtain the explicit distribution of W_t, we conventionally test as follows. The null hypothesis is accepted if W_t, $t = k+1,\cdots,T$, stay within a pair of straight lines which pass through the points $(k,\ \pm c_w\sqrt{T-k})$ and $(T,\ \pm 3c_w\sqrt{T-k})$, where c_w is a parameter depending on the significance level α chosen for the test. It is known that we have c_w =1.143 when α =0.01, c_w =0.948 when α =0.05 and c_w =0.850 when α =0.10 (see Brown, Durbin and Evans (1975) and Johnston (1984)).

The CUSUM test is usually used with the CUSUMSQ test, which plays a role of complement to the CUSUM test. Using the CUSUM test, we can see when the structural change occurs. However, the CUSUM test is not very powerful. Even though the structural change clearly takes place in a period, the null hypothesis is often accepted. Conversely, the CUSUMSQ test is too powerful but we cannot know the period when the structure changes. Accordingly, both tests are complementarily used.

The CUSUMSQ test statistic which is the alternative test to the CUSUM test is represented as:

$$S_t = \sum_{i=k+1}^{t} \omega_i^2 \Big/ \sum_{i=k+1}^{T} \omega_i^2, \tag{3}$$

Table 1. α =0.10, 0.05, 0.01 and $T-k$ =10, 20, 30, 40, 50

$T-k$ / α	10	20	30	40	50
0.10	.39075	.31325	.26767	.23781	.21630
0.05	.44641	.35277	.30081	.26685	.24245
0.01	.54210	.43071	.36649	.32459	.29456

which is distributed as a Beta random variable with parameter $(\frac{t-k}{2}, \frac{T-t}{2})$. The expected value of S_t is $\frac{t-k}{T-k}$. Also, the confidence interval is conventionally approximated, which is given by a pair of straight lines $c_s \pm \frac{t-k}{T-k}$, where c_s depends on both the sample size $T-k$ and the significance level α. For α =0.10, 0.05, 0.01 and $T-k$ =10, 20, 30, 40, 50, c_s is given by Table 1, which represents a both-sided test. See Durbin (1969) and Johnston (1984) for c_s. Durbin (1969) pointed out that the test statistic gives us a good approximation when $T-k$ is large (see Harvey (1981)).

An example of $T-k=50$ and $\alpha = 0.01,\ 0.05,\ 0.10$ is taken in Figures 1a and 1b. 90%, 95% and 99% denote the confidence intervals for each significance level, i.e., $\alpha = 0.10,\ 0.05,\ 0.01$.

The confidence intervals displayed in Figures 1a and 1b are conventionally used for CUSUM and CUSUMSQ tests.

There is some evidence that the CUSUM test is less powerful than the CUSUMSQ test in small sample. Some Monte-Carlo experiments by Garbade (1977) also suggest that the latter may not be very powerful in comparison with tests based on variable parameter models.

It is known that the confidence intervals of these two tests are the approximated ones. The exact confidence interval cannot be obtained explicitly for the CUSUM test, because its distribution is not known. Also, for the CUSUMSQ test, the exact confidence interval is not utilized, even though its distribution is known. Thus, because the exact distribution is not utilized for these two tests, the CUSUM test is not powerful and the CUSUMSQ test is too powerful in small sample. In the next section, we obtain the asymptotically exact confidence intervals using the simulation technique such as the method of simulated moments (see McFadden (1989)).

3 NUMERICAL DERIVATION OF CONFIDENCE INTERVALS

In this section, we construct both exact confidence intervals of CUSUM and CUSUMSQ tests. The intervals are obtained by generating random numbers.

For the CUSUM test, the confidence interval is simulated as follows. First, generate $T-k$ normal random numbers, which are denoted by ω_t, $t = k+1, \cdots, T$, and compute the CUSUM test statistic (2). Repeat this procedure m times. Denote the CUSUM test statistic in the j-th smallest value of the m simulation runs by $W_t^{(j)}$, which implies that $W_t^{(j)}$, $j = 1, \cdots, m$, are sorted by size for all $t = k+1, \cdots, T$. That is, $W_t^{(1)}$ is the smallest value and $W_t^{(m)}$ the largest one. We take m =20,000 in this paper.

Figure 1a. CUSUM test

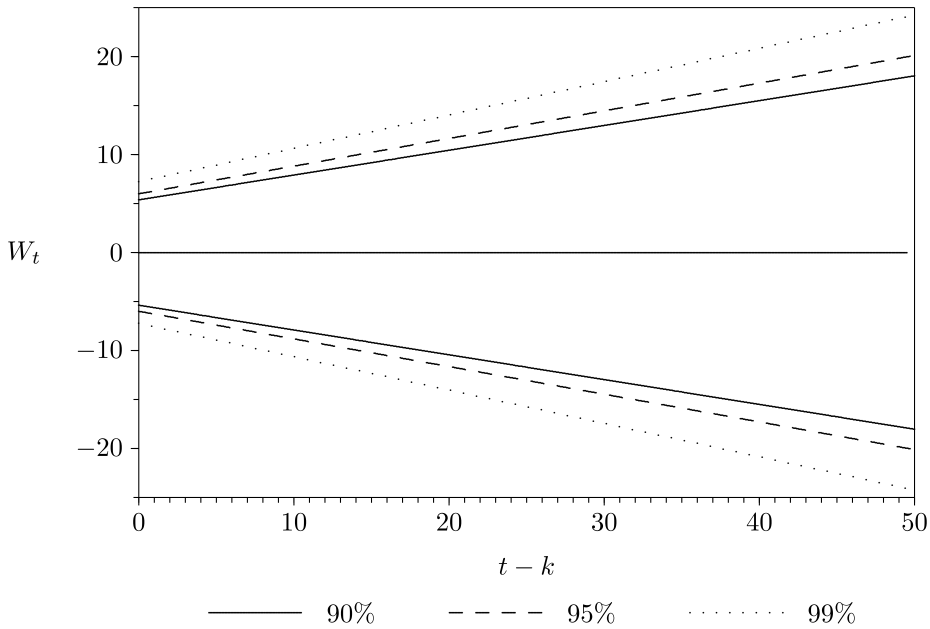
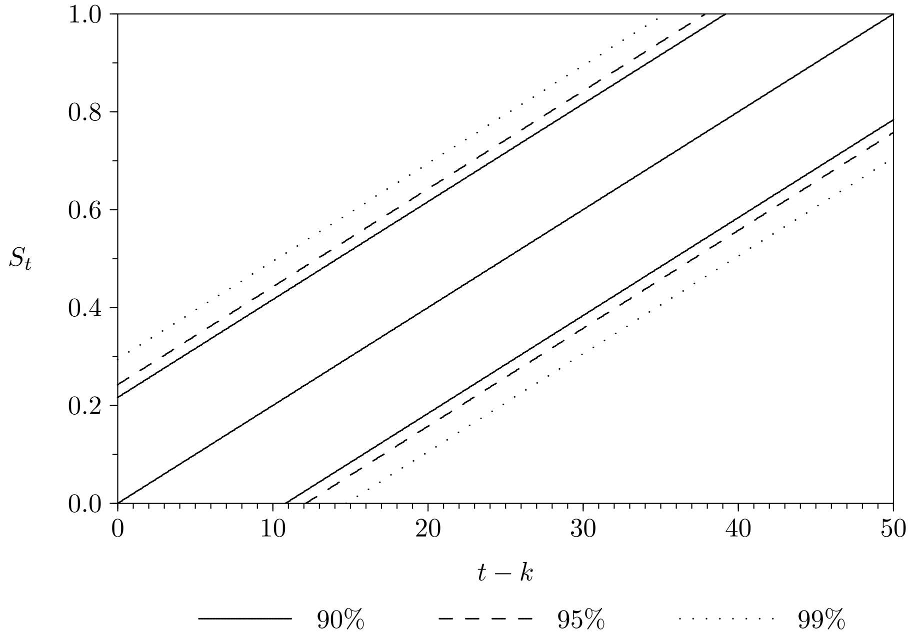


Figure 1b. CUSUMSQ test



Denote the significance level through all periods by α and that in each period by α_w. Let $W_t^{(U)}$ be the upper bound and $W_t^{(L)}$ be the lower one, which implies $(U-L)/m = 1-\alpha_w$ in the case of $100(1-\alpha_w)\%$ confidence interval at time t, where α_w is the significance level at time t for the CUSUM test. If we assume that the distribution of the disturbance u_t is symmetric about zero, that of the CUSUM test statistic is also symmetric, and therefore we have $W_t^{(U)} = -W_t^{(L)}$ for all t. In this procedure, we need to obtain the confidence interval $[W_t^{(L)},\ W_t^{(U)}]$, $t = k+1, \cdots, T$, and the significance level at each time (i.e., α_w), given α. Note that we have the relationship $1-\alpha = (1-\alpha_w)^{T-k}$ if W_s is independent of W_t for $t \neq s$. Since W_t is clearly correlated with W_s for $t \neq s$, we have $1-\alpha \neq (1-\alpha_w)^{T-k}$.

Summarizing above, we obtain $W_t^{(U)}$, $W_t^{(L)}$ and α_w, given α, satisfying the following three conditions:

$$\begin{aligned}\mathrm{Prob}(W_{k+1}^{(U)} > W_{k+1} > W_{k+1}^{(L)}, \cdots, W_t^{(U)} > W_t > W_t^{(L)}, \cdots, \\ W_T^{(U)} > W_T > W_T^{(L)}) = 1-\alpha,\end{aligned} \tag{4}$$

$$\mathrm{Prob}(W_t^{(U)} > W_t > W_t^{(L)}) = 1-\alpha_w, \tag{5}$$

$$W_t^{(U)} = -W_t^{(L)}, \qquad \text{for all } t = k+1, \cdots, T. \tag{6}$$

Equation (4) corresponds to the joint density function of $W_{k+1}, \cdots, W_T$, while Equation (5) is based on the marginal density function of W_t. Here, we apply the simulation technique used in Diebold and Rudebusch (1991) to obtain $W_t^{(U)}$, $W_t^{(L)}$ and α_w, given α.

It might be appropriate to assume that there is a certain relationship between α_w and α, which is represented as:

$$\alpha = f_w(\alpha_w),$$

where $f_w(\cdot)$ is unknown but derived by the simulation technique. $f_w(\alpha_w)$ denotes the probability that $\{W_t\}_{t=k+1}^T$ lies outside the interval $\{(W_t^{(L)}, W_t^{(U)})\}_{t=k+1}^T$, given the significance level at time t (i.e., α_w).[1] In practice, we use the conventional Newton-Raphson nonlinear optimization procedure to obtain α_w, i.e.,

$$\alpha_w^{(i+1)} = \alpha_w^{(i)} + d_w^{(i)}\Big(\alpha - f_w(\alpha_w^{(i)})\Big), \qquad \text{for } i = 1, 2, \cdots, \tag{7}$$

where $d_w^{(i)}$ can be interpreted as $d_w^{(i)} = 1/f_w'(\alpha_w^{(i)})$. The superscript (i) denotes the i-th iteration. We take $\alpha_w^{(1)} = \alpha$, which is the initial value for α_w. However, since the function $f_w(\cdot)$ is unknown and accordingly we cannot obtain the first derivative of $f_w(\cdot)$, we choose $d_w^{(i)} = \delta_w d_w^{(i-1)}$, $d_w^{(0)} = 1$ and $\delta_w = .5$. Moreover, taking into account the convergence speed, at each i we consider choosing either the above Newton-Raphson procedure or the following:

$$\alpha_w^{(i+1)} = \frac{\alpha}{f_w(\alpha_w^{(i)})}\alpha_w^{(i)}, \qquad \text{if } \Big|\frac{\alpha}{f_w(\alpha_w^{(i)})} - 1\Big| \geq 0.1, \tag{8}$$

where $\alpha_w^{(1)} = \alpha$. The convergence criterion is: $|\alpha_w^{(i+1)} - \alpha_w^{(i)}| < 0.0001$. For the significance level at each time (i.e., α_w), we choose $\alpha_w^{(i+1)}$ satisfying the convergence criterion.

Thus, summarizing above, the procedure constructing the confidence interval for the CUSUM test is as follows:

[1] Given α_w, we can compute the probability $f_w(\alpha_w)$ by the simulation. Accordingly, we obtain α_w such that $f_w(\alpha_w)$ is equal to α. That is, $\alpha_w = f_w^{-1}(\alpha)$ is derived. See Figure 2 for relationship between α_w and $f_w(\alpha_w)$.

(i) Generate m sets of $T-k$ standard normal random numbers (i.e., $m\times(T-k)$ standard random draws),[2] which corresponds to m sets of the normalized recursive residuals $\{\omega_t/\sigma\}_{t=k+1}^{T}$. For each set of the $T-k$ random draws, we compute $\{W_t\}_{t=k+1}^{T}$ and $\{S_t\}_{t=k+1}^{T}$.

The CUSUM test statistic (i.e., $\{W_t\}_{t=k+1}^{T}$) and the CUSUMSQ test statistic (i.e., $\{S_t\}_{t=k+1}^{T}$) obtained from the $j-$th set of the $T-k$ random draws are denoted by $\{W_t^{(j)}\}_{t=k+1}^{T}$ and $\{S_t^{(j)}\}_{t=k+1}^{T}$.

(ii) Given the random draws, α and $\alpha_w^{(i)}$, obtain $W_t^{(U_i)}$ and $W_t^{(L_i)}$ for $t=k+1,\cdots,T$ such that $(U_i-L_i)/m = 1-\alpha_w^{(i)}$ and $W_t^{(U_i)} = -W_t^{(L_i)}$. $W_t^{(U_i)}$ and $W_t^{(L_i)}$ denote the U-th and L-th largest values at time t in the i-th iteration.

(iii) Count the number of the series $\{W_t^{(j_i)}\}_{t=k+1}^{T}$, $j=1,\cdots,m$, outside the interval $\{(W_t^{(L_i)},\ W_t^{(U_i)})\}_{t=k+1}^{T}$. Note that

$$f_w(\alpha_w^{(i)}) = 1 - \frac{\begin{array}{c}[\text{ the number of } \{W_t^{(j_i)}\}_{t=k+1}^{T} \\ \text{falling in the interval } \{(W_t^{(L_i)},\ W_t^{(U_i)})\}_{t=k+1}^{T}\ ]\end{array}}{m},$$

which corresponds to the significance level through all periods, i.e., α. Also, note that we have the restriction $W_t^{(U_i)} = -W_t^{(L_i)}$ because the probability density function of the CUSUM test statistic is symmetric.

(iv) Using the following optimization procedure, update from $\alpha_w^{(i)}$ to $\alpha_w^{(i+1)}$.

$$\alpha_w^{(i+1)} = \begin{cases} \dfrac{\alpha}{f_w(\alpha_w^{(i)})}\alpha_w^{(i)}, & \text{if } \left|\dfrac{\alpha}{f_w(\alpha_w^{(i)})} - 1\right| \geq 0.1, \\ \alpha_w^{(i)} + d_w^{(i)}\Big(\alpha - f_w(\alpha_w^{(i)})\Big), & \text{otherwise,} \end{cases}$$

where $\alpha_w^{(1)} = \alpha$, $d_w^{(i)} = \delta_w d_w^{(i-1)}$, $d_w^{(0)}$ =1 and δ_w is a constant value.

(v) Repeat (ii)-(iv) until $\alpha_w^{(i+1)}$ is stable, i.e., $|\alpha_w^{(i+1)} - \alpha_w^{(i)}| < 0.0001$.

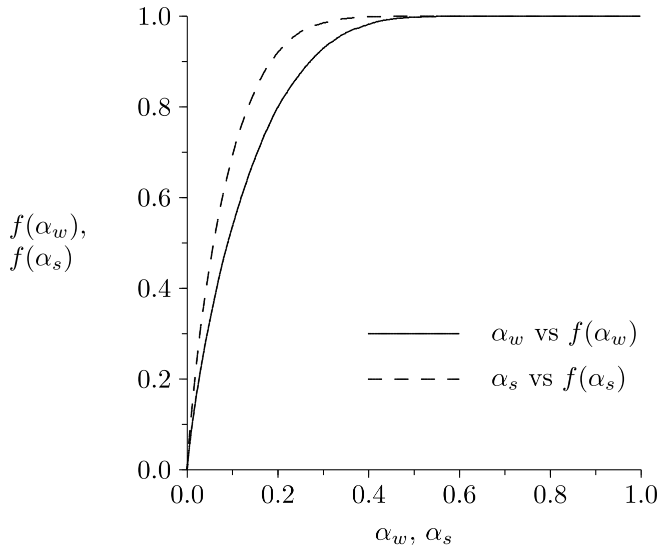
For the CUSUMSQ test, we can take the almost same procedure as above. Similarly, after m simulation runs, denote the j-th largest value of the CUSUMSQ test statistics in m simulations by $S_t^{(j)}$, where $S_t^{(j)}$, $j=1,\cdots,m$, are already sorted by size. That is, for all t, $S_t^{(1)}$ is the smallest value and $S_t^{(m)}$ the largest one. Also, m =20,000 is taken for the CUSUMSQ test.

For the CUSUMSQ test, let α_s be the significance level in each period. Given α, we compute $S_t^{(L)}$, $S_t^{(U)}$ and α_s satisfying:

$$\begin{aligned} \text{Prob}(S_{k+1}^{(U)} > S_{k+1} > S_{k+1}^{(L)}, \cdots, S_t^{(U)} > S_t > S_t^{(L)}, \cdots, \\ S_T^{(U)} > S_T > S_T^{(L)}) = 1-\alpha, \end{aligned} \tag{9}$$

[2] In the case where the error u_t is nonnormal, we generate the random numbers corresponding to the underling assumption of u_t.

Figure 2. α_w **vs** $f(\alpha_w)$ **and** α_s **vs** $f(\alpha_s)$**:** $T-k=50$ **and** $m=20,000$



$$\text{Prob}(S_t^{(U)} > S_t > S_t^{(L)}) = 1-\alpha_s, \tag{10}$$
$$S_t^{(L)} = 1 - S_{T+k-t}^{(U)}, \tag{11}$$
$$1 - S_t^{(U)} = S_{T+k-t}^{(L)}, \qquad \text{for } t = k+1, \cdots, T-1. \tag{12}$$

Given the significance level α, we consider computing $S_t^{(U)}$, $S_t^{(L)}$ and α_s for the CUSUMSQ test. We have to obtain α_s which is constant over time. If ω_t, $t=k+1,\cdots,T$, are mutually independently and identically distributed, then the density function of S_t is equivalent to that of $1-S_{T+k-t}$ for all $t=k+1,\cdots,T-1$. Accordingly, we have the relationships: $S_t^{(L)} = 1-S_{T+k-t}^{(U)}$ and $1-S_t^{(U)} = S_{T+k-t}^{(L)}$. We obtain α_s which satisfies $\alpha = f_s(\alpha_s)$. $f_s(\alpha_s)$ denotes the probability which $\{S_t\}_{t=k+1}^T$ lies outside the interval $\{(S_t^{(L)}, S_t^{(U)})\}_{t=k+1}^T$, given the significance level at time t (i.e., α_s). The procedure constructing the confidence interval for the CUSUMSQ test is as follows.

$$\alpha_s^{(i+1)} = \begin{cases} \dfrac{\alpha}{f_s(\alpha_s^{(i)})}\alpha_s^{(i)}, & \text{if } |\dfrac{\alpha}{f_s(\alpha_s^{(i)})} - 1| \geq 0.1, \\ \alpha_s^{(i)} + d_s^{(i)}\Big(\alpha - f_s(\alpha_s^{(i)})\Big), & \text{otherwise,} \end{cases}$$

where $\alpha_s^{(1)} = \alpha$, $d_s^{(i)} = \delta_s d_s^{(i-1)}$, $d_s^{(0)} = 1$ and δ_s is a constant value. Note that we need to have $S_t^{(U)}$ and $S_t^{(L)}$ such that $S_t^{(U)} - S_t^{(L)}$ is minimized, $S_t^{(L)} = 1 - S_{T+k-t}^{(U)}$ and $1 - S_t^{(U)} = S_{T+k-t}^{(L)}$ for $t=k+1,\cdots,T-1$. In this paper, we take $\delta_s = .5$. Also, the convergence criterion is: $|\alpha_s^{(i+1)} - \alpha_s^{(i)}| < 0.0001$.

Taking an example of the case: $T-k=50$ and $m=20,000$, $\big(\alpha_w,\ f_w(\alpha_w)\big)$ and $\big(\alpha_s,\ f_w(\alpha_s)\big)$ are displayed in Figure 2.

$f_w(\cdot)$ and $f_s(\cdot)$ are monotone and accordingly their inverse functions exist. The proposed approach of constructing the confidence intervals is guaranteed to work in all circumstances, even for nonnormal cases.[3]

In Table 2, we check the convergence speed in simulating the confidence intervals, where $\alpha = 0.10,\ 0.05,\ 0.01$, $T-k$ =50 and m =20,000 are taken. We can see that the convergence is very quick.

We concretely show the computation procedure, taking an example of the case: CUSUM test and $\alpha = 0.10$. Constructing the confidence interval based on $\alpha_w = 0.1000$ and computing the probability which lies outside the interval $(W_t^{(L_1)},\ W_t^{(U_1)})$ for some t, we have $f_w(\alpha_w^{(1)}) = 0.5374$. Using equation (8), $\alpha_w^{(1)}$ is updated to $\alpha_w^{(2)}$, which is 0.0186. Based on $\alpha_w^{(2)} = 0.0186$, again, we compute the confidence interval and the corresponding significance level, i.e., $f_w(\alpha_w^{(2)}) = 0.1527$. For the computation from $f_w(\alpha_w^{(3)})$ to $\alpha_w^{(4)}$, the convergence criterion is switched to equation (7). The procedure is completed by the sixth iteration since $\alpha_w^{(5)}$ is the same value as $\alpha_w^{(6)}$, i.e., $\alpha_w^{(5)}$=$\alpha_w^{(6)}$=0.0112. As it is seen in Table 2, the convergence speed is quite fast. By the results obtained from Table 2, we have the confidence intervals in Figures 3a and 3b, where the case of $T-k = 50$, $m = 20,000$ and $\alpha = 0.10, 0.05, 0.01$ is displayed.

Thus, we have the confidence intervals of CUSUM and CUSUMSQ tests as shown in Figures 3a and 3b. However, the intervals in Figures 3a and 3b are not smooth, which implies the obtained confidence intervals are not reliable. In order to improve this problem, repeating the procedure n times, we take the arithmetic average of n sets of upper bounds and lower bounds for the confidence intervals under each significance level α. As n goes to infinity, by law of large number, the arithmetic average of the intervals becomes the true intervals.[4] $n = 2,000$ is taken in Figures 4a and 4b, where each simulated 90%, 95% and 99% confidence intervals are drawn.

According to the procedure shown in Figures 4a and 4b, we can construct the intervals for any $T-k$ and any α, given m and n.

[3] α_w and α_s denote the significance levels at a single time, while $f_w(\alpha_w)$ and $f_s(\alpha_s)$ denote the significance levels over all time periods. When α_w increases, $f_w(\alpha_w)$ also clearly increases. And similarly, when α_s increases, $f_s(\alpha_s)$ also clearly increases. Therefore, $f_w(\alpha_w)$ and $f_s(\alpha_s)$ are monotone. Accordingly, the inverse functions, i.e., $f_w^{-1}(\alpha)$ and $f_s^{-1}(\alpha)$, exist for all nonnormal errors.

[4] As m increases, the simulated confidence intervals become smooth. However, an increase in m yields an increase in data storage. We cannot take an extremely large m in practice because of a computer and/or compiler limitation. Therefore, we consider repeating the procedure (i)–(v) n times, which implies obtaining n sets of simulated confidence intervals.

Table 2. Convergence of Simulated Confidence Intervals ($T-k=50$)

	CUSUM Test		CUSUMSQ Test	
i	$\alpha_w^{(i)}$	$f_w(\alpha_w^{(i)})$	$\alpha_s^{(i)}$	$f_w(\alpha_s^{(i)})$
		$\alpha=0.10$		
1	.1000	.5374	.1000	.6933
2	.0186	.1527	.0144	.1737
3	.0122	.1064	.0083	.1101
4	.0106	.0954	.0058	.0819
5	.0112	.1001	.0071	.0956
6	.0112	–	.0073	.0988
7	–	–	.0074	–
		$\alpha=0.05$		
1	.0500	.3285	.0500	.4507
2	.0076	.0713	.0055	.0783
3	.0053	.0532	.0035	.0546
4	.0045	.0459	.0024	.0393
5	.0051	.0518	.0031	.0474
6	.0049	.0498	.0032	.0503
7	.0050	–	.0032	–
		$\alpha=0.01$		
1	.0100	.0914	.0100	.1285
2	.0011	.0138	.0008	.0143
3	.0008	.0107	.0005	.0104
4	.0006	.0083	.0004	.0090
5	.0007	.0096	.0005	.0090
6	.0008	–	.0005	–

Figure 3a. CUSUM test

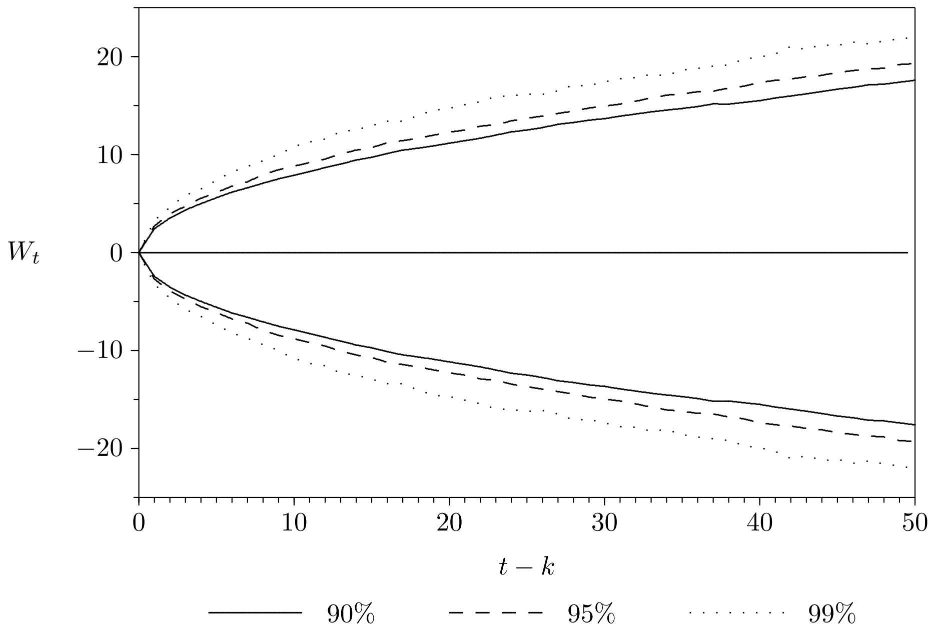


Figure 3b. CUSUMSQ test

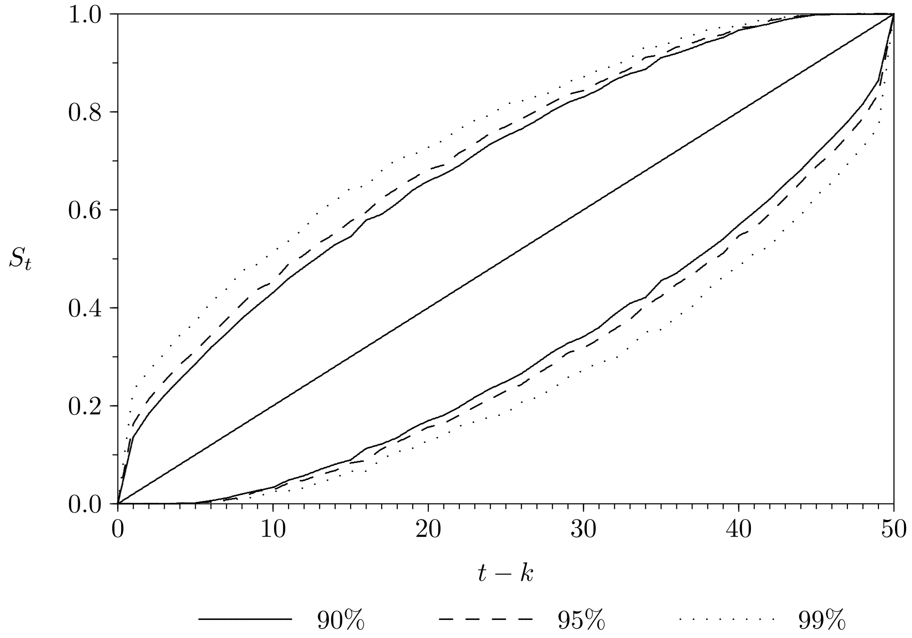


Table 3. Maximum Standard Errors of Simulated Confidence Intervals
($m = 20,000$ and $n = 2,000$)

α	$T-k$	10	20	30	40	50
0.10	CUSUM	.0391	.0755	.1027	.1259	.1510
	CUSUMSQ	.0038	.0041	.0040	.0039	.0038
0.05	CUSUM	.0474	.1010	.1351	.1739	.2037
	CUSUMSQ	.0049	.0044	.0046	.0046	.0044
0.01	CUSUM	.0758	.1847	.2745	.3465	.4337
	CUSUMSQ	.0084	.0097	.0090	.0083	.0079

Each value in Table 3 denotes the maximum value of the $T-k$ arithmetic standard errors obtained from n sets of confidence intervals.

For the intervals of CUSUM test, the standard error is large when $T-k$ increases and/or α is small. For the CUSUMSQ test, the standard error increases as α is small. In any case, it is seen from Table 3 that all the standard errors are enough small compared with length of the confidence intervals. It is clear that the standard errors in Table 3 approach zero as n increases.

When comparing the conventional confidence intervals and the simulated confidence intervals, the findings are as follows. For the CUSUM test (see Figures 1a and 4a), the simulated confidence interval is larger than the conventional one at the middle of the period, while the former is smaller than the latter otherwise. As for the CUSUMSQ test (see Figures 1b and 4b), similarly, the simulated one is larger than the conventional one in the middle, while the former is smaller than the latter in the tails.

Finally, note as follows. For any underlying distribution of the disturbance u_t in the regression model, the computational procedure can be applied in the exactly same fashion.

In the following section, we compare the power for the numerical confidence intervals and the conventional ones under the normal disturbance.

4 POWER OF CUSUM AND CUSUMSQ TESTS

Using the simulated confidence intervals obtained in the last section, we perform Monte-Carlo experiments to examine the power of CUSUM and CUSUMSQ tests. Since the recursive residuals are identically, independently and normally distributed, we generate normal random draws for ω_t, $t = k+1, \cdots, T$. Suppose that at the middle point of $T-k$ the structural change takes place. Thus, we can assume that ω_t, $t = k+1, \cdots, T$, are distributed as follows:

$$\omega_t \sim N(0,\ 1), \qquad \text{for } t = k+1, \cdots, (T-k)/2,$$

$$\omega_t \sim N(\mu,\ \sigma^2), \qquad \text{for } t = (T-k)/2+1, \cdots, T,$$

Figure 4a. CUSUM test

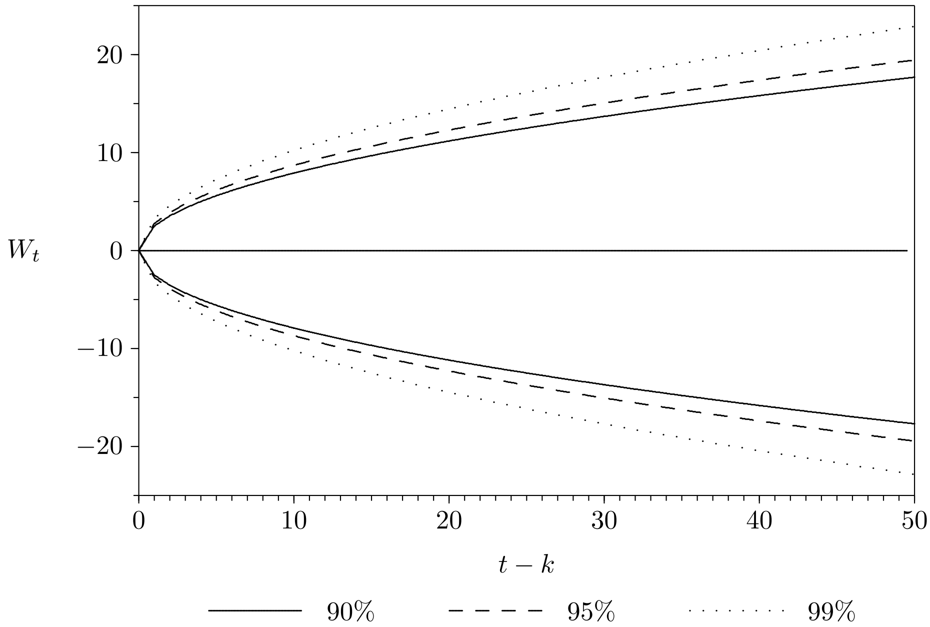
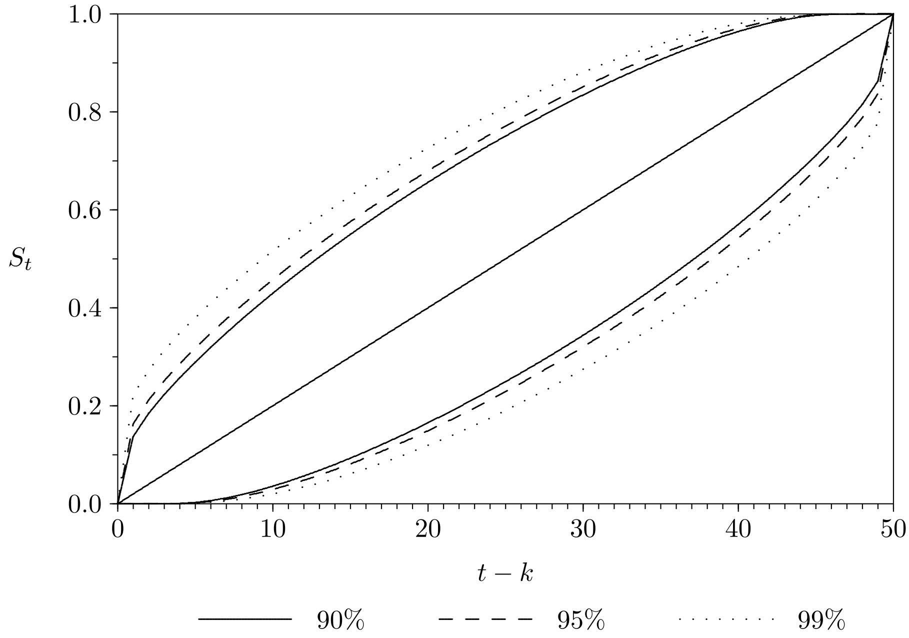


Figure 4b. CUSUMSQ test



where we examine testing not only a shift in location but also that in scale. Then we compute each of the test statistics, and record whether or not rejection occurs for tests with each size. The experiment is repeated M times, and then compute the sample powers (for each statistic and the test size) as the relative rejection frequency. We can obtain the estimated standard errors for the sample powers as $\sqrt{\widehat{p}(1-\widehat{p})/M}$, where $\widehat{p}$ denotes the sample power. Since we perform the experiment 10,000 times, i.e., M =10,000, the estimated standard errors are enough small (at most $\sqrt{.5(1-.5)/10,000} \approx 0.005$).

α, $T-k$, μ and σ are chosen as follows:

$$\begin{aligned}
&\alpha = 0.10,\ 0.05,\ 0.01,\\
&T-k = 10,\ 20,\ 30,\ 40,\ 50,\\
&\mu = 0.0,\ 0.5,\ 1.0,\ 1.5,\\
&\sigma = 1.0,\ 1.5,\ 2.0,\ 2.5.
\end{aligned}$$

In order to obtain the simulated confidence intervals, m sets of random draws are generated, i.e., m =20,000 as in Section 3. In this section, different M sets of normal random draws are generated to examine the power for both CUSUM and CUSUMSQ tests. The results are in Tables 4a – 4c and 5a – 5c. In the case of μ =0.0 and σ =1.0, theoretically the sample power should be equal to the significance level α. However, for the conventional 90% confidence interval of the CUSUM test, the sample power is .0236 when $T-k$ =10, .0474 when $T-k$ =20, .0572 when $T-k$ =30, .0633 when $T-k$ =40, and .0690 when $T-k$ =50 (see Table 4a, Conventional 90% Confidence Interval, Case: μ =0.0 and σ =1.0). This implies that the conventional confidence interval of the CUSUM test gives us a poor approximation even if $T-k$ is large. For the conventional 90% confidence interval of the CUSUMSQ test, the power is .1445 when $T-k$ =10, .1253 when $T-k$ =20, .1205 when $T-k$ =30, .1133 when $T-k$ =40, and .1133 when $T-k$ =50 (see Table 5a, Conventional 90% Confidence Interval, Case: μ =0.0 and σ =1.0), which implies that the power of the confidence interval of the CUSUMSQ test is too large.

The other findings are as follows. The CUSUM test is more powerful in the location parameter μ while the CUSUMSQ test is more powerful in the scale parameter σ. Both tests are powerful as $T-k$ is large for almost all the cases. Given μ, the CUSUM test is less powerful as σ is large. Therefore, the CUSUM test is sensitive to the structural change of the shifting parameter and the CUSUMSQ test should be used in testing heteroscedasticity.

5 SUMMARY

In this paper, the asymptotically exact confidence intervals for the CUSUM and CUSUMSQ tests were derived using the simulation technique. As the number of simulation (i.e., m and/or n) increases, the approach proposed in this paper clearly gives us the asymptotically exact confidence interval. The main results are as follows: (i) for the conventional confidence intervals, the power of the CUSUM test is too low. The confidence interval of the test gives us a poor approximation even when $T-k$ goes to large (see $T-k = 50$), and the power of the CUSUMSQ test is too large, (ii) the CUSUM test is more sensitive to the location parameter μ than the CUSUMSQ test while the latter is more sensitive to the scale parameter σ than the former, and (iii) given μ, the CUSUM test is poor as σ increases.

Finally, even in the case where the recursive residuals are not normal, the simulation technique to obtain the confidence intervals of the CUSUM and CUSUMSQ tests can be easily applied. That is, the asymptotically exact confidence intervals proposed in this paper can be derived for any regression model with the nonnormal disturbance. However, note that the proposed confidence intervals have larger simulation errors for large sample size $T-k$ and small significance level α than for small $T-k$ and large α.

Table 4a. CUSUM Test ($\alpha = 0.10$)

$\mu \setminus \sigma$	Conventional Interval 1.0	1.5	2.0	2.5	Exact Interval 1.0	1.5	2.0	2.5
				$T-k=10$				
0.0	.0236	.0109	.0057	.0032	.0991	.0509	.0340	.0241
0.5	.0244	.0119	.0066	.0038	.1047	.0578	.0382	.0267
1.0	.0298	.0141	.0074	.0038	.1367	.0806	.0517	.0357
1.5	.0411	.0185	.0092	.0049	.1866	.1177	.0722	.0505
				$T-k=20$				
0.0	.0474	.0229	.0151	.0109	.0948	.0440	.0285	.0217
0.5	.0713	.0405	.0246	.0176	.1330	.0717	.0486	.0357
1.0	.1840	.1034	.0612	.0405	.2779	.1779	.1120	.0777
1.5	.3722	.2266	.1374	.0870	.5124	.3523	.2338	.1606
				$T-k=30$				
0.0	.0572	.0297	.0207	.0168	.0991	.0414	.0267	.0226
0.5	.1220	.0686	.0462	.0349	.1710	.0913	.0618	.0473
1.0	.3762	.2301	.1435	.0966	.4542	.2847	.1833	.1264
1.5	.7118	.5153	.3409	.2281	.7812	.5958	.4162	.2878
				$T-k=40$				
0.0	.0633	.0325	.0248	.0208	.0955	.0406	.0282	.0240
0.5	.1697	.1013	.0651	.0482	.2061	.1172	.0740	.0548
1.0	.5670	.3764	.2436	.1635	.6118	.4119	.2733	.1863
1.5	.8931	.7301	.5398	.3850	.9138	.7661	.5811	.4202
				$T-k=50$				
0.0	.0690	.0337	.0250	.0227	.1008	.0385	.0254	.0226
0.5	.2176	.1289	.0839	.0593	.2496	.1386	.0896	.0619
1.0	.7096	.4931	.3237	.2205	.7338	.5126	.3388	.2332
1.5	.9699	.8700	.6907	.5123	.9747	.8815	.7105	.5311

Table 4b. CUSUM Test ($\alpha = 0.05$)

$\mu \setminus \sigma$	Conventional Interval 1.0	1.5	2.0	2.5	Exact Interval 1.0	1.5	2.0	2.5
				$T-k=10$				
0.0	.0037	.0017	.0008	.0003	.0494	.0225	.0140	.0092
0.5	.0044	.0010	.0003	.0002	.0501	.0262	.0169	.0103
1.0	.0039	.0012	.0005	.0002	.0686	.0344	.0207	.0145
1.5	.0034	.0012	.0003	.0001	.0952	.0502	.0292	.0178
				$T-k=20$				
0.0	.0190	.0074	.0034	.0028	.0482	.0198	.0117	.0083
0.5	.0296	.0131	.0068	.0040	.0689	.0343	.0200	.0140
1.0	.0877	.0406	.0198	.0102	.1750	.0924	.0539	.0346
1.5	.1914	.0930	.0464	.0247	.3571	.2125	.1250	.0772
				$T-k=30$				
0.0	.0250	.0103	.0066	.0045	.0477	.0172	.0120	.0099
0.5	.0602	.0310	.0176	.0115	.0919	.0470	.0305	.0215
1.0	.2346	.1235	.0686	.0429	.3165	.1781	.1046	.0699
1.5	.5337	.3244	.1842	.1102	.6437	.4352	.2714	.1667
				$T-k=40$				
0.0	.0294	.0131	.0087	.0076	.0463	.0177	.0124	.0107
0.5	.0934	.0483	.0312	.0212	.1217	.0606	.0390	.0273
1.0	.4121	.2367	.1331	.0787	.4736	.2874	.1688	.1025
1.5	.7901	.5752	.3735	.2341	.8386	.6428	.4359	.2893
				$T-k=50$				
0.0	.0336	.0142	.0105	.0086	.0510	.0168	.0114	.0098
0.5	.1260	.0677	.0395	.0282	.1498	.0792	.0467	.0311
1.0	.5661	.3434	.2035	.1275	.6074	.3785	.2267	.1445
1.5	.9182	.7561	.5370	.3532	.9356	.7926	.5765	.3899

Table 4c. CUSUM Test ($\alpha = 0.01$)

$\mu \setminus \sigma$	Conventional Interval 1.0	1.5	2.0	2.5	Exact Interval 1.0	1.5	2.0	2.5
				$T-k=10$				
0.0	.0000	.0000	.0000	.0000	.0093	.0044	.0021	.0013
0.5	.0000	.0000	.0000	.0000	.0117	.0046	.0027	.0014
1.0	.0000	.0000	.0000	.0000	.0142	.0064	.0027	.0013
1.5	.0000	.0000	.0000	.0000	.0177	.0079	.0031	.0016
				$T-k=20$				
0.0	.0009	.0001	.0000	.0000	.0097	.0022	.0012	.0010
0.5	.0021	.0005	.0002	.0000	.0146	.0053	.0027	.0016
1.0	.0049	.0016	.0004	.0002	.0511	.0190	.0075	.0037
1.5	.0136	.0031	.0013	.0003	.1218	.0505	.0215	.0094
				$T-k=30$				
0.0	.0024	.0003	.0002	.0001	.0090	.0022	.0011	.0007
0.5	.0085	.0032	.0012	.0004	.0231	.0103	.0057	.0036
1.0	.0448	.0148	.0059	.0027	.1235	.0504	.0240	.0116
1.5	.1491	.0530	.0190	.0083	.3361	.1650	.0784	.0408
				$T-k=40$				
0.0	.0046	.0013	.0008	.0006	.0104	.0033	.0019	.0015
0.5	.0160	.0059	.0027	.0014	.0343	.0147	.0075	.0046
1.0	.1299	.0461	.0194	.0087	.2282	.0997	.0431	.0255
1.5	.4152	.2012	.0820	.0370	.5874	.3428	.1738	.0869
				$T-k=50$				
0.0	.0042	.0014	.0007	.0007	.0097	.0025	.0017	.0012
0.5	.0278	.0117	.0064	.0035	.0471	.0198	.0105	.0075
1.0	.2438	.1032	.0416	.0203	.3370	.1592	.0765	.0378
1.5	.6672	.3892	.1919	.0944	.7717	.5175	.2825	.1563

Table 5a. CUSUMSQ Test ($\alpha = 0.10$)

	Conventional Interval				Exact Interval			
$\mu \setminus \sigma$	1.0	1.5	2.0	2.5	1.0	1.5	2.0	2.5
				$T-k=10$				
0.0	.1445	.2436	.4153	.5808	.0977	.1700	.3067	.4549
0.5	.1522	.2651	.4348	.5886	.1003	.1891	.3215	.4642
1.0	.1909	.3389	.4873	.6238	.1256	.2418	.3726	.5013
1.5	.2858	.4363	.5692	.6832	.1891	.3220	.4439	.5573
				$T-k=20$				
0.0	.1253	.3250	.6239	.8289	.0989	.2430	.5120	.7324
0.5	.1343	.3775	.6578	.8417	.1037	.2856	.5467	.7505
1.0	.2398	.5133	.7414	.8754	.1697	.3998	.6303	.8018
1.5	.4715	.6905	.8365	.9200	.3304	.5617	.7421	.8580
				$T-k=30$				
0.0	.1205	.4190	.7825	.9374	.0993	.3234	.6770	.8878
0.5	.1443	.4904	.8113	.9452	.1114	.3868	.7103	.8955
1.0	.3163	.6584	.8773	.9635	.2249	.5386	.7967	.9258
1.5	.6366	.8505	.9421	.9836	.4884	.7384	.8921	.9594
				$T-k=40$				
0.0	.1133	.4993	.8723	.9800	.1007	.3904	.7897	.9542
0.5	.1397	.5825	.8984	.9831	.1171	.4597	.8166	.9574
1.0	.3729	.7669	.9466	.9898	.2629	.6485	.8924	.9764
1.5	.7673	.9301	.9835	.9960	.6130	.8553	.9586	.9900
				$T-k=50$				
0.0	.1133	.5712	.9343	.9945	.1039	.4464	.8661	.9852
0.5	.1474	.6600	.9478	.9961	.1188	.5311	.8967	.9871
1.0	.4417	.8483	.9777	.9973	.3089	.7423	.9481	.9942
1.5	.8602	.9675	.9949	.9991	.7291	.9235	.9847	.9973

Table 5b. CUSUMSQ Test ($\alpha = 0.05$)

$\mu \setminus \sigma$	Conventional Interval 1.0	1.5	2.0	2.5	Exact Interval 1.0	1.5	2.0	2.5
				$T-k=10$				
0.0	.0722	.1312	.2445	.3723	.0490	.0943	.1965	.3161
0.5	.0728	.1480	.2581	.3770	.0483	.1109	.2077	.3281
1.0	.0940	.1898	.2990	.4099	.0646	.1466	.2488	.3620
1.5	.1339	.2501	.3591	.4563	.0989	.2047	.3094	.4133
				$T-k=20$				
0.0	.0621	.2058	.4746	.7086	.0487	.1526	.3785	.6188
0.5	.0717	.2459	.5118	.7291	.0563	.1823	.4181	.6387
1.0	.1398	.3639	.5977	.7783	.0974	.2723	.5006	.6981
1.5	.2979	.5288	.7163	.8410	.2060	.4214	.6181	.7749
				$T-k=30$				
0.0	.0610	.2933	.6655	.8862	.0493	.2198	.5589	.8170
0.5	.0742	.3584	.7001	.8959	.0571	.2705	.5969	.8324
1.0	.1988	.5218	.7927	.9254	.1329	.4142	.6984	.8762
1.5	.4768	.7359	.8885	.9597	.3442	.6220	.8226	.9269
				$T-k=40$				
0.0	.0609	.3730	.7900	.9580	.0504	.2743	.6897	.9167
0.5	.0797	.4490	.8222	.9634	.0615	.3407	.7308	.9268
1.0	.2463	.6511	.8986	.9784	.1684	.5285	.8236	.9541
1.5	.6243	.8632	.9630	.9912	.4735	.7684	.9248	.9787
				$T-k=50$				
0.0	.0578	.4405	.8786	.9862	.0551	.3287	.7910	.9684
0.5	.0766	.5331	.9040	.9898	.0632	.4077	.8306	.9739
1.0	.3072	.7572	.9528	.9952	.2040	.6351	.9071	.9860
1.5	.7513	.9335	.9874	.9978	.6043	.8705	.9683	.9952

Table 5c. CUSUMSQ Test ($\alpha = 0.01$)

$\mu \setminus \sigma$	Conventional Interval 1.0	1.5	2.0	2.5	Exact Interval 1.0	1.5	2.0	2.5
				$T-k=10$				
0.0	.0175	.0328	.0695	.1121	.0112	.0224	.0607	.1151
0.5	.0175	.0374	.0743	.1154	.0102	.0258	.0657	.1214
1.0	.0169	.0478	.0857	.1296	.0125	.0386	.0806	.1388
1.5	.0205	.0592	.1003	.1426	.0231	.0592	.1084	.1678
				$T-k=20$				
0.0	.0126	.0589	.1782	.3451	.0086	.0472	.1691	.3523
0.5	.0154	.0745	.2005	.3672	.0087	.0616	.1877	.3773
1.0	.0295	.1195	.2524	.4164	.0233	.1012	.2482	.4373
1.5	.0627	.1812	.3331	.4944	.0650	.1784	.3442	.5249
				$T-k=30$				
0.0	.0137	.1021	.3666	.6543	.0106	.0737	.3123	.6057
0.5	.0150	.1387	.4051	.6769	.0107	.1014	.3496	.6294
1.0	.0516	.2385	.5039	.7372	.0347	.1900	.4500	.6969
1.5	.1649	.4102	.6446	.8201	.1344	.3539	.5984	.7917
				$T-k=40$				
0.0	.0146	.1487	.5369	.8343	.0103	.1032	.4498	.7747
0.5	.0193	.1984	.5829	.8538	.0138	.1436	.4960	.7991
1.0	.0772	.3632	.7035	.8984	.0539	.2815	.6233	.8575
1.5	.2897	.6127	.8393	.9488	.2208	.5212	.7826	.9216
				$T-k=50$				
0.0	.0115	.2049	.6757	.9321	.0101	.1403	.5790	.8923
0.5	.0185	.2713	.7250	.9426	.0154	.1957	.6294	.9091
1.0	.1043	.4901	.8360	.9667	.0675	.3803	.7654	.9408
1.5	.4392	.7732	.9372	.9874	.3309	.6798	.8957	.9742

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