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Abstract

We consider the existence of solutions to the semilinear elliptic problem

$$(*)_\kappa \qquad \begin{cases} -\Delta u + u = u^p + \kappa \sum_{i=1}^m c_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N), \\ u > 0 \quad \text{a.e. in } \mathbf{R}^N \quad \text{and} \quad u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

with prescribed given finite points $\{a_i\}_{i=1}^m$ in $\mathbf{R}^N$ and positive numbers $\{c_i\}_{i=1}^m$, where $N \geq 3$, $1 < p < N/(N-2)$, $\kappa \geq 0$ is a parameter, and δ_a is the Dirac delta function supported at $a \in \mathbf{R}^N$. We reduce the problem $(*)_\kappa$ to the problem in $H^1(\mathbf{R}^N) \cap C_0(\mathbf{R}^N)$ in terms of auxiliary functions, and then show the existence of a positive constant $\kappa^* > 0$ such that $(*)_\kappa$ has at least two solutions if $\kappa \in (0, \kappa^*)$, a unique solution if $\kappa = \kappa^*$, and no solution if $\kappa > \kappa^*$.

MSC: 35J60; 35J20

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1. Introduction

We are concerned with the problem of finding positive solutions with prescribed isolated singularities to semilinear elliptic equations. Choosing a finite set of points $\{a_i\}_{i=1}^m$ in $\mathbf{R}^N$ and a set of positive numbers $\{c_i\}_{i=1}^m$, we consider the existence of positive solutions of the problem

$$(1.1)_\kappa \qquad -\Delta u + u = u^p + \kappa \sum_{i=1}^m c_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N),$$

with the condition at infinity

$$(1.2) \qquad u(x) \to 0 \quad \text{as } |x| \to \infty,$$

where $N \geq 3$, $1 < p < N/(N-2)$, $\kappa \geq 0$ is a parameter, and δ_a is the Dirac delta function supported at $a \in \mathbf{R}^N$. We denote the Laplacian on $\mathbf{R}^N$ by Δ and the class of distributions on $\mathbf{R}^N$ by $\mathcal{D}'(\mathbf{R}^N)$.

We recall some known results concerning the singularities of possible solutions of the equation. Let Ω be a bounded domain in $\mathbf{R}^N$ containing 0. By the works due to Lions [23] and Brezis and Lions [9], we obtain the following result.

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Theorem A [23, 9]**.** *Assume that* $u \in C^2(\Omega \setminus \{0\})$ *satisfies*

$$-\Delta u + u = u^q \quad \text{in } \Omega \setminus \{0\} \tag{1.3}$$

with $q > 1$ *and* $u \geq 0$ *a.e. in* Ω*. Then* $u \in L^q_{\rm loc}(\Omega)$ *and*

$$-\Delta u + u = u^q + \kappa\delta_0 \quad \text{in } \mathcal{D}'(\Omega) \tag{1.4}$$

for some $\kappa \geq 0$*. Furthermore, the following* (i) *and* (ii) *hold.*

(i) *In the case* $1 < q < N/(N-2)$*, if* $\kappa = 0$ *in* (1.4) *then* $u \in C^2(\Omega)$*, and if* $\kappa > 0$ *then* u *behaves like a multiple of the fundamental solution* E_0 *for* $-\Delta$ *in* $\mathbf{R}^N$*, i.e.,* $-\Delta E_0 = \delta_0$ *in* $\mathcal{D}'(\mathbf{R}^N)$*.*

(ii) *In the case* $q \geq N/(N-2)$*, there holds* $\kappa = 0$ *in* (1.4)*.*

For the proof, see Theorem 1 in [9] and Corollary 1, Theorem 2, and Remark 2 in [23].

It should be mentioned that Johnson, Pan, and Yi [21] showed the existence and asymptotic behaviour of singular positive radial solution u of (1.3) with $1 < q < (N+2)/(N-2)$. In particular, they showed that, if $N/(N-2) < q < (N+2)/(N-2)$, there exists a positive solution u of (1.3) satisfying $u(x) \sim c|x|^{-2/(p-1)}$ as $|x| \to 0$ for some constant $c > 0$. Then, in this case, the singularity of u at $x = 0$ exists, but is not visible in the sense of distribution.

In this paper, we investigate the existence of positive solutions with prescribed isolated singularities to the equation in $\mathbf{R}^N$. By (ii) of Theorem A, if $p \geq N/(N-2)$ then $(1.1)_\kappa$ with $\kappa > 0$ has no positive solution $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. Hence, the condition $1 < p < N/(N-2)$ is necessary for the existence of positive solutions $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ of $(1.1)_\kappa$ with $\kappa > 0$.

We review some known results concerning related problems. Lions [23] studied the existence of positive solutions of the problem

$$\begin{cases} -\Delta u = u^p + \kappa\delta_0 & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded domain in $\mathbf{R}^N$ containing 0 with smooth boundary $\partial\Omega$. It was shown in [23] that there exists $\kappa^* > 0$ such that (1.5) has at least two positive solutions for each $\kappa \in (0, \kappa^*)$ and no such solution for $\kappa > \kappa^*$. Later, Baras and Pierre [5] studied the existence of positive solutions for the problem

$$\begin{cases} -\Delta u = u^p + \kappa\mu & \text{in } \mathcal{D}'(\Omega), \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where μ is a positive bounded Radon measure in Ω. In [5] they showed that (1.6) has at least one positive solution for each sufficiently small $\kappa > 0$ by investigating the corresponding integral equations. See also Roppongi [25]. Amann and Quittner [3] exhibited the existence of $\kappa^* > 0$ such that (1.6) has at least two positive solutions for $0 < \kappa < \kappa^*$ and no solution for $\kappa > \kappa^*$.

Bidaut-Veron and Yarur [7] gave the existence results and a priori estimates for (1.6) including the case where μ is unbounded. In [3], [7], they also consider the problems involving measures as boundary data. We also refer a survey by Veron [28], [29], and the references therein. In [26] the second author studied the existence of positive solutions for the problem

$$-\Delta u + f(u) = \sum_{i=1}^{m} c_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N)$$

in the cases where f is nonnegative. In [26] he also showed the nonexistence of positive solutions for some f with sign changing.

In order to state our results, we introduce some notations. Let E_1 denote the fundamental solution for $-\Delta + I$ in $\mathbf{R}^N$, that is,

$$E_1(x) = E_1(|x|) = \frac{1}{(2\pi)^{N/2}|x|^{(N-2)/2}} K_{(N-2)/2}(|x|) \quad \text{for } x \in \mathbf{R}^N \setminus \{0\},$$

where K_ν is the modified Bessel function of order ν. We see that E_1 has the following properties:

$$E_1(x) \sim \frac{1}{(N-2)N\omega_N |x|^{N-2}} \quad \text{as } |x| \to 0, \quad \text{and}$$

$$E_1(x) \sim c_1 |x|^{-(N-1)/2} e^{-|x|} \quad \text{as } |x| \to \infty,$$

where ω_N denotes the volume of the unit ball in $\mathbf{R}^N$ and $c_1 > 0$ is a constant depends on N. In particular, $E_1 \in C^\infty(\mathbf{R}^N \setminus \{0\})$ and $E_1 \in L^r(\mathbf{R}^N)$ for all $1 \le r < N/(N-2)$. Define f_0 by

$$f_0(x) = \sum_{i=1}^{m} c_i E_1(x - a_i).$$

Then $f_0 \in C^\infty(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $f_0 \in L^r(\mathbf{R}^N)$ for all $1 \le r < N/(N-2)$, and f_0 satisfies

$$-\Delta f_0 + f_0 = \sum_{i=1}^{m} c_i \delta_{a_i} \quad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

In this paper we refer to u as a positive solution of $(1.1)_\kappa$ if $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ satisfies $(1.1)_\kappa$ in the sense of distribution and $u > 0$ a.e. in $\mathbf{R}^N$.

Proposition 1.1. *Let $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ be a positive solution of $(1.1)_\kappa$ with $\kappa > 0$. Then $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $u(x) > 0$ for $x \in \mathbf{R}^N \setminus \{a_i\}_{i=1}^m$. Assume, in addition, that (1.2) holds. Then $u \in L^q(\mathbf{R}^N)$ for all $q \in [1, N/(N-2))$ and u satisfies*

$$(1.7)_\kappa \qquad u = E_1 * [u^p] + \kappa f_0 \quad \text{a.e. in } \mathbf{R}^N$$

and $u(x) = O(E_1(x))$ as $|x| \to \infty$, where the symbol $*$ denotes the convolution.

For each $\kappa > 0$, we define U_j^κ for $j = 0, 1, 2, \ldots$, inductively, by

$$(1.8) \qquad U_0^\kappa = \kappa f_0 \quad \text{and} \quad U_j^\kappa = E_1 * \left[(U_{j-1}^\kappa)^p\right] + \kappa f_0 \quad \text{for } j = 1, 2, \ldots.$$

Take $q_0 \in (p, N/(N-2))$ arbitrarily, and define $\{q_j\}$ by

$$(1.9) \qquad \frac{1}{q_j} = \frac{1}{q_0} - \left(\frac{2}{N} - \frac{p-1}{q_0}\right) j = \frac{1}{q_{j-1}} - \left(\frac{2}{N} - \frac{p-1}{q_0}\right) \quad \text{for } j = 1, 2, \ldots.$$

From $p < N/(N-2)$ and $q_0 > p$, it follows that $2/N - (p-1)/q_0 > 0$. Then, by choosing suitable q_0 if necessary, there exists an positive integer denoted by j_0 satisfying

$$(1.10) \qquad \frac{1}{q_{j_0-1}} > 0 > \frac{1}{q_{j_0}}.$$

We use the notation $C_0(\mathbf{R}^N) = \{u \in C(\mathbf{R}^N) : u(x) \to 0 \quad \text{as } |x| \to \infty\}$.

Proposition 1.2. *For each $\kappa \in (0, \infty)$, the following* (i) -(iii) *are equivalent to each other* :

(i) *$u = w + U^{\kappa}_{j_0} \in L^p_{\rm loc}(\mathbf{R}^N)$ is a positive solution of* $(1.1)_\kappa$–(1.2);

(ii) *$w \in C_0(\mathbf{R}^N)$ is positive in $\mathbf{R}^N$ and satisfies*

$$(1.11)_\kappa \qquad w = E_1 * \left[(w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p\right] \quad \text{in } \mathbf{R}^N;$$

(iii) *$w \in H^1(\mathbf{R}^N)$ is a weak positive solution of*

$$(1.12)_\kappa \qquad -\Delta w + w = (w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p \quad \text{in } \mathbf{R}^N,$$

that is, $w > 0$ a.e. in $\mathbf{R}^N$ and satisfies

$$(1.13)_\kappa \qquad \int_{\mathbf{R}^N} (\nabla w \cdot \nabla \psi + w\psi)\, dx = \int_{\mathbf{R}^N} \left((w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p\right) \psi dx$$

for any $\psi \in H^1(\mathbf{R}^N)$.

Remark. In $(1.12)_\kappa$ we have $(w + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p \in H^{-1}(\mathbf{R}^N)$ for $w \in H^1(\mathbf{R}^N)$ with $w \geq 0$. (See (ii) of Lemma 2.8 below.)

By Proposition 1.2, the problem $(1.1)_\kappa$–(1.2) can be reduced to the problems $(1.11)_\kappa$ in $C_0(\mathbf{R}^N)$ and $(1.12)_\kappa$ in $H^1(\mathbf{R}^N)$. We will investigate the problems $(1.11)_\kappa$ and $(1.12)_\kappa$ by an approach based on adaptation of the methods by [1, 2, 14, 23].

Our main results are stated in the following theorems.

Theorem 1. *There exists $\kappa^* \in (0, \infty)$ such that*

(i) *if $0 < \kappa < \kappa^*$ then the problem $(1.1)_\kappa$–(1.2) has a positive minimal solution $\underline{u}_\kappa$, that is, $\underline{u}_\kappa \leq u$ a.e. in $\mathbf{R}^N$ for any positive solution u of $(1.1)_\kappa$–(1.2). Furthermore, if $0 < \kappa < \hat{\kappa} < \kappa^*$ then $\underline{u}_\kappa < \underline{u}_{\hat{\kappa}}$ a.e. in $\mathbf{R}^N$;*

(ii) *if $\kappa > \kappa^*$ then the problem $(1.1)_\kappa$–(1.2) has no positive solution.*

Theorem 2. *If $\kappa = \kappa^*$ then the problem $(1.1)_\kappa$–(1.2) has a unique positive solution.*

Theorem 3. *If $0 < \kappa < \kappa^*$ then the problem $(1.1)_\kappa$–(1.2) has a positive solution $\overline{u}_\kappa$ satisfying $\overline{u}_\kappa > \underline{u}_\kappa$.*

In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of $(1.12)_\kappa$ and $(1.11)_\kappa$, respectively, to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of $(1.12)_\kappa$. We will prove Theorem 3 by employing the variational method with the Mountain Pass Lemma. In the proofs of Theorems 2 and 3, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions play a crucial role.

Concerning nonhomogeneous semilinear elliptic problems of the form

$$-\Delta u + u = u^q + \kappa f(x) \quad \text{in } \mathbf{R}^N$$

with $q > 1$ and $f \in H^{-1}(\mathbf{R}^N)$, we refer to Zhu [30], Deng and Li [15], [16], Cao and Zhou [12], and Hirano [20]. They successfully showed the existence of at least two positive solutions of the problems under suitable conditions. See also [27, 13] for closely related problems.

This paper is organized as follows. In Section 2 we investigate the representation of solutions and give the proofs of Propositions 1.1 and 1.2. In Sections 3 we show the existence of minimal solutions and give the proof of Theorem 1. In Section 4 we prove Theorem 2, and in Section 5 we show a priori estimate employed in Section 4. In Section 6 we prove Theorem 3 by applying the variational method, and in Section 7 we give the proof of the proposition stated in Section 6. The basic inequalities, which used in this paper, are listed in Appendix A. The eigenvalue problems with singular coefficient are studied in Appendix B. Proofs of auxiliary lemmas are given in Appendix C.

In the remaining of this paper, we denote $B_R = \{x \in \mathbf{R}^N : |x| < R\}$ for $R > 0$. The norms $L^q(\mathbf{R}^N)$ and $H^1(\mathbf{R}^N)$ are denoted by $\|\cdot\|_{L^q}$ and $\|\cdot\|_{H^1}$, respectively. We define $\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$ for $u \in H^1(\mathbf{R}^N)$. We denote by $H^{-1}(\mathbf{R}^N)$ the dual space of $H^1(\mathbf{R}^N)$. We denote by $\mathcal{S}'(\mathbf{R}^N)$ the class of tempered distributions. We note here that the operator $-\Delta + I : \mathcal{S}'(\mathbf{R}^N) \to \mathcal{S}'(\mathbf{R}^N)$ is invertible and the inverse operator is given by $(-\Delta + I)^{-1} f = E_1 * f$ for $f \in \mathcal{S}'(\mathbf{R}^N)$. The letter C denotes inessential constants which may vary from line to line.

2. Representation of solutions: Proofs of Propositions 1.1 and 1.2

Lemma 2.1. *Assume that $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ satisfies $(1.1)_\kappa$ with $\kappa > 0$ in the sense of distribution. Then $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$.*

Proof. We observe that $u^p \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ and u satisfies

$$-\Delta u + u = u^p \quad \text{in } \mathcal{D}'(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m). \tag{2.1}$$

Then, by the result due to Brezis and Strauss [11] (see also Veron [29]), we have $u \in W^{1,r}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ for all $r \in [1, N/(N-1))$. By the Sobolev embedding, it follows that

$$u \in L^s_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m) \quad \text{for all } s \in [1, N/(N-2)). \tag{2.2}$$

Put $s_0 \in [p, N/(N-2))$ arbitrariry, and define $\{s_k\}_{k=1}^{\infty}$ by

$$\frac{1}{s_k} = \frac{p}{s_{k-1}} - \frac{2}{N} \quad \text{for } k = 1, 2, \dots, \tag{2.3}$$

inductively, that is,

$$\frac{1}{s_k} = \frac{2}{N(p-1)} - \left(\frac{2}{N(p-1)} - \frac{1}{s_0} \right) p^k \quad \text{for } k = 1, 2, \dots.$$

From $s_0 \ge p$ and $p < N/(N-2)$ it follows that $2/(N(p-1)) - 1/s_0 > 0$. Then, by choosing suitable s_0 if necessary, we have $1/s_{k_0} < 0 < 1/s_{k_0-1}$ with some integer k_0. From (2.3) we see that

$$\frac{s_k}{p} < \frac{N}{2} \quad \text{for } k = 1, 2, \dots, k_0 - 2 \quad \text{and} \quad \frac{s_{k_0-1}}{p} > \frac{N}{2}.$$

From (2.2) we have $u \in L^{s_0}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. We will show, by induction, that

$$u \in L^{s_k}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m) \quad \text{for } k = 1, 2, \dots, k_0 - 1. \tag{2.4}$$

Assume that $u \in L^{s_{k-1}}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ with some $k \in \{1, 2, \dots, k_0 - 1\}$. Then $u^p \in L^{s_{k-1}/p}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. By applying the interior L^q estimate in (2.1) and the Sobolev embedding, we obtain

$$u \in W^{2, s_{k-1}/p}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m) \subset L^{s_k}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m).$$

Thus (2.4) holds by induction. In particular, we obtain $u \in L^{s_{k_0-1}}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$, and hence $u^p \in L^{s_{k_0-1}/p}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. By applying the interior L^q estimate in (2.1) and the Sobolev embedding with $s_{k_0-1}/p > N/2$, we obtain

$$u \in W^{2, s_{k_0-1}/p}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m) \subset C^{\alpha}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$$

with some $\alpha \in (0,1)$. By employing the Schauder interior estimate in (2.1), we obtain $u \in C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$, and hence $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. □

The following lemma will be used in the proof of Proposition 1.2.

Lemma 2.2. *Assume that $u \in L^p_{\mathrm{loc}}(\mathbf{R}^N)$ is a solution of $(1.1)_\kappa$ satisfying $u \in L^q(\mathbf{R}^N \setminus B_R)$ for some $q \ge p$ and $R > 0$. Then $u(x) \to 0$ as $|x| \to \infty$.*

Proof. Take $R_0 \ge R$ such that

$$R_0 > |a_i| \quad \text{for } i = 1, 2, \dots, m. \tag{2.5}$$

Lemma 2.1 implies that $u \in C^2(\mathbf{R}^N \setminus B_{R_0})$. Let $\zeta \in C^\infty(\mathbf{R}^N)$ be a cut-off function such that $\zeta(x) \equiv 1$ for $|x| \ge 2R_0$ and $\zeta(x) \equiv 0$ for $|x| \le R_0$. Then $\zeta u \in C^2(\mathbf{R}^N)$, and ζu satisfies

$$-\Delta(\zeta u) + \zeta u = \eta[u] \quad \text{in } \mathbf{R}^N \quad \text{with} \quad \eta[u] = \zeta u^p - 2\nabla\zeta \cdot \nabla u - u\Delta\zeta.$$

From $u \in L^q(\mathbf{R}^N \setminus B_{R_0})$, we have $\zeta u \in L^q(\mathbf{R}^N)$, and then $\eta[u] \in L^{q/p}(\mathbf{R}^N)$. Note that ζu satisfies

$$\zeta u = E_1 * [\eta[u]] \quad \text{in } \mathbf{R}^N.$$

Put $s_0 = q$ and define $\{s_k\}_{k=1}^{\infty}$ by (2.3). Then we observe that $1/s_{k_0} \le 0 < 1/s_{k_0-1}$ with some integer k_0. We will show, by induction, that

$$\zeta u \in L^{s_k}(\mathbf{R}^N) \quad \text{for } k = 1, 2, \dots, k_0 - 1. \tag{2.6}$$

Note that $\zeta u \in L^{s_0}(\mathbf{R}^N)$. Assume that $u \in L^{s_{k-1}}(\mathbf{R}^N)$ with some $k \in \{1, 2, \dots, k_0 - 1\}$. Then $\eta[u] \in L^{s_{k-1}/p}(\mathbf{R}^N)$. By (ii) of Lemma A.3 in Appendix A, we obtain $\zeta u \in L^{s_k}(\mathbf{R}^N)$. Thus (2.6) holds by induction. In particular, we obtain $\zeta u \in L^{s_{k_0-1}}(\mathbf{R}^N)$, and hence $\eta[u] \in L^{s_{k_0-1}/p}(\mathbf{R}^N)$. Note here that $s_{k_0-1}/p \ge N/2$. In the case $s_{k_0-1}/p > N/2$, by applying (iii) of Lemma A.3, we obtain $\zeta u \in C_0(\mathbf{R}^N)$. This implies that $u(x) \to 0$ as $|x| \to \infty$. In the case $s_{k_0-1}/p = N/2$, we have $\eta[u] \in L^r(\mathbf{R}^N)$ for all $r \in [s_{k_0-2}/p, N/2]$. Applying (ii) of Lemma A.3, we obtain $\zeta u \in L^s(\mathbf{R}^N)$ for all $s \ge s_{k_0-1}$. In particular, we have $\eta[u] \in L^s(\mathbf{R}^N)$ with $s > N/2$. By (iii) of Lemma A.3 we obtain $\zeta u \in C_0(\mathbf{R}^N)$, and hence $u(x) \to 0$ as $|x| \to \infty$. □

Lemma 2.3. *Let $u \in L^p_{\rm loc}(\mathbf{R}^N)$ be a positive solution of* (1.1)$_\kappa$–(1.2). *Then $u(x) = O(e^{-\alpha|x|})$ as $|x| \to \infty$ for any $\alpha \in (0,1)$.*

Proof. Lemma 2.1 implies that $u \in C^2(\mathbf{R} \setminus B_{R_0})$ with $R_0 > 0$ satisfying (2.5). Take $\alpha \in (0,1)$ arbitrarily. From (1.2) there exists $R_1 \ge R_0$ such that $|u(x)|^{p-1} \le 1 - \alpha^2$ for $|x| \ge R_1$. Then u satisfies $-\Delta u + u \le (1-\alpha^2)u$ for $|x| \ge R_1$, or $-\Delta u + \alpha^2 u \le 0$ for $|x| \ge R_1$. Put $v(x) = C_1 E_1(\alpha x)$, where $C_1 > 0$ is a constant so large that $C_1 E_1(\alpha x) > u(x)$ at $|x| = R_1$. Then v satisfies

$$-\Delta v + \alpha^2 v = 0 \quad \text{for } |x| \ge R_1 \quad \text{and} \quad v > u \quad \text{on } |x| = R_1.$$

Since $u(x), v(x) \to 0$ as $|x| \to \infty$, by the maximum principle we obtain $v(x) \ge u(x)$ for $|x| \ge R_1$. This implies that $u(x) = O(E_1(\alpha x)) = O(e^{-\alpha|x|})$ as $|x| \to \infty$. □

We recall the asymptotic result by Gidas, Ni, and Nirenberg [18].

Lemma 2.4 ([18, Proposition 4.2])**.** *Assume that $\psi \in L^1(\mathbf{R}^N)$ satisfies $\psi(x) = O(e^{-\beta|x|})$ as $|x| \to \infty$ for some $\beta > 1$. Then $E_1 * \psi(x) = O(E_1(x))$ as $|x| \to \infty$.*

Lemma 2.5. *Let $u \in L^p_{\rm loc}(\mathbf{R}^N)$ be a positive solution of* (1.1)$_\kappa$–(1.2) *with $\kappa > 0$, and put $v = u - \kappa f_0$. Then $v > 0$ a.e. in $\mathbf{R}^N$, $v \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$, and v satisfies*

$$v = E_1 * [(v + \kappa f_0)^p] \quad \textit{a.e. in } \mathbf{R}^N \tag{2.7}$$

and

$$v(x) = O(E_1(x)) \quad \textit{as } |x| \to \infty. \tag{2.8}$$

Proof. Lemma 2.3 implies that

$$u(x) = (e^{-\alpha|x|}) \quad \text{as } |x| \to \infty \tag{2.9}$$

for any $\alpha \in (0,1)$, which implies $u \in L^p(\mathbf{R}^N)$. Let $v = u - \kappa f_0$. Then v satisfies $-\Delta v + v = u^p$ in $\mathcal{S}'(\mathbf{R}^N)$, and hence

$$v = E_1 * [u^p] \quad \text{a.e. in } \mathbf{R}^N. \tag{2.10}$$

Note that $u^p \in L^1(\mathbf{R}^N)$. Then, by (i) of Lemma A.3 in Appendix A, we obtain $v \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$. From (2.10) it is clear that $v > 0$ a.e. in $\mathbf{R}^N$ and (2.7) holds. From (2.9) we have $u^p(x) = O(e^{-\beta|x|})$ as $|x| \to \infty$ for some $\beta > 1$. By Lemma 2.4 we obtain (2.8). □

Proof of Proposition 1.1. For a positive solution u of $(1.1)_\kappa$, Lemma 2.1 implies that $u \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. By the strong maximum principle, we have $u > 0$ in $\mathbf{R}^N \setminus \{a_i\}_{i=1}^m$. Let u be a positive solution of $(1.1)_\kappa$–(1.2), and put $v = u - \kappa f_0$. Recall that $f_0 \in L^q(\mathbf{R}^N)$ for all $q \in [1, N/(N-2))$ and f_0 satisfies $f_0(x) = O(E_1(x))$ as $|x| \to \infty$. Then, from Lemma 2.5 we have $u = v + \kappa f_0 \in L^q(\mathbf{R}^N)$ for all $q \in [1, N/(N-2))$ and u satisfies $(1.7)_\kappa$ and $u(x) = O(E_1(x))$ as $|x| \to \infty$. □

In order to prove Proposition 1.2 we need some lemmas.

Lemma 2.6. *For each $j \in \mathbf{N}$ and $\kappa > 0$, U_j^κ defined by* (1.8) *satisfies the following*:
(i) $U_j^\kappa \in L^r(\mathbf{R}^N)$ *for all* $r \in [1, N/(N-2))$ *and* $U_{j-1}^\kappa \le U_j^\kappa$ *a.e. in* $\mathbf{R}^N$;
(ii) $U_j^\kappa \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ *and* $U_j^\kappa(x) = O(E_1(x))$ *as* $|x| \to \infty$.
(iii) *Assume that* $u \in L^p(\mathbf{R}^N)$ *is positive a.e. in* $\mathbf{R}^N$ *and satisfies* $(1.7)_\kappa$. *Then* $u > U_j^\kappa$ *a.e. in* $\mathbf{R}^N$ *for each* $j \in \{0\} \cup \mathbf{N}$.

Proof. (i) By the definition, we have $U_0^\kappa \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$. In particular, $U_0^\kappa \in L^p(\mathbf{R}^N)$ and $(U_0^\kappa)^p \in L^1(\mathbf{R}^N)$. From (i) of Lemma A.3 in Appendix A, we have $E_1 * [(U_0^\kappa)^p] \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$. This implies that $U_1^\kappa \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$. By induction we obtain $U_j^\kappa \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$ with each $j \in \mathbf{N}$.

By the definition, we have $U_1^\kappa \ge U_0^\kappa$ a.e. in $\mathbf{R}^N$. Then, by induction, we obtain $U_j^\kappa \ge U_{j-1}^\kappa$ a.e. in $\mathbf{R}^N$ for each $j \in \mathbf{N}$.

(ii) By the definition, it is clear that $U_0^\kappa \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $U_0^\kappa(x) = O(E_1(x))$ as $|x| \to \infty$. Assume that $U_{j-1}^\kappa \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $U_{j-1}^\kappa = O(E_1(x))$ as $|x| \to \infty$ for some $j \in \mathbf{N}$. Then, by the definition of U_j^κ, we observe that

$$-\Delta U_j^\kappa + U_j^\kappa = (U_{j-1}^\kappa)^p \quad \text{in } \mathcal{D}'(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m).$$

By the Schauder estimate, it follows that $U_j^\kappa \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$. Note that $(U_{j-1}^\kappa(x))^p = O(e^{-\beta x})$ as $|x| \to \infty$ with some $\beta > 1$. Then, from Lemma 2.4, we have $E_1 * [(U_{j-1}^\kappa)^p](x) = O(E_1(x))$ as $|x| \to \infty$, and then $U_j^\kappa(x) = O(E_1(x))$ as $|x| \to \infty$. Thus, by induction, (ii) holds for each $j \in \mathbf{N}$.

(iii) From $(1.7)_\kappa$ and $U_0^\kappa = \kappa f_0$, we have $u - U_0^\kappa = E_1 * [u^p] > 0$ a.e. in $\mathbf{R}^N$. Assume that $u - U_{j-1}^\kappa > 0$ a.e. in $\mathbf{R}^N$ for some $j \in \mathbf{N}$. From $(1.7)_\kappa$ and (1.8) we have

$$u - U_j^\kappa = E_1 * [u^p - (U_{j-1}^\kappa)^p] > 0 \quad \text{a.e. in } \mathbf{R}^N.$$

By induction, we obtain $u > U_j^\kappa$ a.e. in $\mathbf{R}^N$ for each $j = \mathbf{N}$. □

For each $j \in \mathbf{N}$ and $\kappa > 0$, define V_j^κ by

$$V_j^\kappa = U_j^\kappa - U_{j-1}^\kappa. \tag{2.11}$$

From (ii) of Lemma 2.6, it is clear that $V_j^\kappa \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $V_j^\kappa(x) = O(E_1(x))$ as $|x| \to \infty$. From (2.11) and (1.8) we observe that

$$V_1^\kappa = E_1 * [(U_0^\kappa)^p] \quad \text{and} \quad V_j^\kappa = E_1 * [(U_{j-1}^\kappa)^p - (U_{j-2}^\kappa)^p] \quad \text{for } j \geq 2. \tag{2.12}$$

The right-hand side of (2.12) can be written as

$$V_j^\kappa = E_1 * [(U_{j-2}^\kappa + V_{j-1}^\kappa)^p - (U_{j-2}^\kappa)^p] \quad \text{for } j \geq 2. \tag{2.13}$$

Recall that $\{q_j\}$ is the sequence defined by (1.9) with $q_0 \in (p, N/(N-2))$ and j_0 is the integer satisfying (1.10). Define $\{r_j\}$ by

$$\frac{1}{r_j} = \frac{p-1}{q_0} + \frac{1}{q_{j-1}} = \frac{1}{q_j} + \frac{2}{N} \quad \text{for } j = 1, 2, \dots. \tag{2.14}$$

From (1.10) it follows that $r_j < N/2$ for $j = 1, 2, \dots, j_0 - 1$ and $r_{j_0} > N/2$.

Lemma 2.7. (i) *$V_j^\kappa(x)$ is strictly increasing with respect to $\kappa > 0$ for each fixed $j \in \mathbf{N}$ and $x \in \mathbf{R}^N \setminus \{a_i\}_{i=1}^m$.*

(ii) *$V_j^\kappa \in (L^1 \cap L^{q_j})(\mathbf{R}^N)$ for $j = 1, 2, \dots, j_0 - 1$ and $V_{j_0}^\kappa \in (L^1 \cap C_0 \cap H^1)(\mathbf{R}^N)$ for each $\kappa > 0$.*

Proof. (i) From (2.12) it is clear that V_1^κ is strictly increasing in $\kappa > 0$. We note here that $s \mapsto (t+s)^p - s^p$ and $t \mapsto (t+s)^p - s^p$ are increasing for each fixed $t > 0$ and $s > 0$, respectively. Then, from (2.13) we see that V_2^κ is strictly increasing in $\kappa > 0$. By induction, we obtain V_j^κ is strictly increasing with respect to $\kappa > 0$ for each $j \geq 1$.

(ii) Since $U_j^\kappa \in L^1(\mathbf{R}^N)$ for $j \geq 1$, we have $V_j^\kappa \in L^1(\mathbf{R}^N)$ for each $j \in \mathbf{N}$ from (2.11). We will show that

$$V_j^\kappa \in L^{q_j}(\mathbf{R}^N) \quad \text{for } 1 \leq j \leq j_0 - 1. \tag{2.15}$$

Note that $U_0^\kappa \in L^{q_0}(\mathbf{R}^N)$. From (2.12) and (ii) of Lemma A.3 in Appendix A, we obtain $V_1^\kappa \in L^{q_1}(\mathbf{R}^N)$. Assume that $V_{j-1}^\kappa \in L^{q_{j-1}}(\mathbf{R}^N)$ for some $j \in \{2, 3, \dots, j_0 - 1\}$. From (2.13) and the mean value theorem, we observe that

$$V_j^\kappa \le E_1 * [p(U_{j-1}^\kappa)^{p-1} V_{j-1}^\kappa] \quad \text{a.e. in } \ \mathbf{R}^N.$$

From $U_{j-1}^\kappa \in L^{q_0}(\mathbf{R}^N)$ and (i) of Lemma A.1, we obtain

$$(U_{j-1}^\kappa)^{p-1} V_{j-1}^\kappa \in L^{r_j}(\mathbf{R}^N),$$

where r_j is given by (2.14). Then, from (ii) of Lemma A.3, we have $V_j^\kappa \in L^{q_j}(\mathbf{R}^N)$. Thus (2.15) holds by induction. In particular, $V_{j_0-1}^\kappa \in L^{q_0-1}(\mathbf{R}^N)$ and then

$$(U_{j_0-1}^\kappa)^{p-1} V_{j_0-1}^\kappa \in L^{r_{j_0}}(\mathbf{R}^N).$$

Note here that $r_{j_0} > N/2$. From $U_{j_0-1}^\kappa \ge V_{j_0-1}^\kappa$ and $U_{j_0-1}^\kappa \in L^p(\mathbf{R}^N)$, we have $(U_{j_0-1}^\kappa)^{p-1} V_{j_0-1}^\kappa \le (U_{j_0-1}^\kappa)^p \in L^1(\mathbf{R}^N)$. Thus Lemma A.4 implies that $V_{j_0}^\kappa \in (C_0 \cap H^1)(\mathbf{R}^N)$. □

Lemma 2.8. (i) *Let $j \in \mathbf{N}$ and $w \in C_0(\mathbf{R}^N)$ with $w \ge 0$. Then, for each $\kappa > 0$,*

$$p(U_{j-1}^\kappa)^{p-1}(w + V_j^\kappa) \le (w + U_j^\kappa)^p - (U_{j-1}^\kappa)^p \le p(w + U_j^\kappa)^{p-1}(w + V_j^\kappa) \quad \text{a.e. in } \ \mathbf{R}^N. \tag{2.16}$$

Furthermore, for $\kappa_1 > \kappa_2 > 0$,

$$(w + U_j^{\kappa_1})^p - (U_{j-1}^{\kappa_1})^p > (w + U_j^{\kappa_2})^p - (U_{j-1}^{\kappa_2})^p \quad \text{a.e. in } \ \mathbf{R}^N. \tag{2.17}$$

(ii) *Let $w \in H^1(\mathbf{R}^N)$ with $w \ge 0$ a.e. in $\mathbf{R}^N$. Then $(w + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p \in H^{-1}(\mathbf{R}^N)$ for each $\kappa > 0$.*

(iii) *Let $w \in L^\infty(\mathbf{R}^N)$ with $w \ge 0$ a.e. in $\mathbf{R}^N$. Then, for each $\kappa > 0$,*

$$E_1 * [(w + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p] \in C(\mathbf{R}^N). \tag{2.18}$$

Proof. (i) Observe that $V_j^\kappa \ge 0$ and

$$(w + U_j^\kappa)^p - (U_{j-1}^\kappa)^p = (w + V_j^\kappa + U_{j-1}^\kappa)^p - (U_{j-1}^\kappa)^p.$$

By the mean value theorem, (2.16) holds. Recall that V_j^κ and U_j^κ is strictly increasing in $\kappa > 0$ for each $j = 1, 2, \dots$. Then, by the similar argument as in the proof of (i) of Lemma 2.7, $(w + U_j^\kappa)^p - (U_{j-1}^\kappa)^p$ is strictly increasing with respect to $\kappa > 0$. Thus (2.17) holds.

(ii) From (ii) of Lemma 2.7 and (i) of Lemma 2.6, we have $V_{j_0}^\kappa \in H^1(\mathbf{R}^N)$ and $U_{j_0-1}^\kappa \in L^p(\mathbf{R}^N)$. By applying (ii) of Lemma C.1 in Appendix C, we obtain $(w + V_{j_0}^\kappa + U_{j_0-1}^\kappa)^p - (U_{j_0-1}^\kappa)^p \in H^{-1}(\mathbf{R}^N)$. Thus (ii) holds.

(iii) From (2.16) and the fact $V^\kappa_{j_0} \in L^\infty(\mathbf{R}^N)$ and $U^\kappa_{j_0} \in L^{q_0}(\mathbf{R}^N)$ with $q_0 \in (p, N/(N-2))$, we have

$$(w + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \le p(w + U^\kappa_{j_0})^{p-1}(w + V^\kappa_{j_0}) \in (L^\infty + L^{q_0/(p-1)})(\mathbf{R}^N).$$

Note that $q_0/(p-1) > N/2$. Then, from (iv) of Lemma A.3 in Appendix A, we obtain (2.18). □

Lemma 2.9. (i) *Assume that* $w \in L^{q_0}(\mathbf{R}^N)$ *with* $w > 0$ *a.e. in* $\mathbf{R}^N$ *and satisfies*

$$(2.19)_\kappa \qquad w = E_1 * \left[(w + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p\right] \quad \text{a.e. in } \mathbf{R}^N.$$

Then $w \in C_0(\mathbf{R}^N)$ *and* $w > 0$ *in* $\mathbf{R}^N$.

(ii) *Assume that* $w \in C_0(\mathbf{R}^N)$ *is positive in* $\mathbf{R}^N$ *and satisfies* $(1.11)_\kappa$. *Then* $w \in H^1(\mathbf{R}^N)$.

Proof. (i) First we will show, by induction, that

$$(2.20) \qquad w \in L^{q_j}(\mathbf{R}^N) \quad \text{for } 1 \le j \le j_0 - 1.$$

Assume that $w \in L^{q_{j-1}}(\mathbf{R}^N)$ with some $j \in \{1, \dots, j_0 - 1\}$. It follows from (i) of Lemma 2.8 that

$$0 \le (w + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \le p(w + U^\kappa_{j_0})^{p-1}(w + V^\kappa_{j_0}) \quad \text{a.e. in } \mathbf{R}^N.$$

Since $V^\kappa_{j_0} \in (L^1 \cap C_0)(\mathbf{R}^N)$ and $U^\kappa_{j_0-1} \in L^{q_0}(\mathbf{R}^N)$, we have $w + V^\kappa_{j_0} \in L^{q_{j-1}}(\mathbf{R}^N)$ and $w + U^\kappa_{j_0-1} \in L^{q_0}(\mathbf{R}^N)$. From (i) of Lemma A.1 we obtain

$$0 \le (w + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \le p(w + U^\kappa_{j_0-1})^{p-1}(w + V^\kappa_{j_0}) \in L^{r_j}(\mathbf{R}^N),$$

where r_j is given by (2.14). By applying (ii) of Lemma A.3 to $(2.19)_\kappa$, we obtain $w \in L^{q_j}(\mathbf{R}^N)$. Then, by induction (2.20) holds. In particular, we have $w \in L^{q_{j_0-1}}(\mathbf{R}^N)$. By the similar augment as above, we obtain

$$0 \le (w + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \le p(w + U^\kappa_{j_0})^{p-1}(w + V^\kappa_{j_0}) \in L^{r_{j_0}}(\mathbf{R}^N).$$

We note here that $r_{j_0} > N/2$. Then, by applying (iii) of Lemma A.3 we obtain $w \in C_0(\mathbf{R}^N)$. Furthermore, w satisfies $(1.11)_\kappa$, and then $w > 0$ in $\mathbf{R}^N$.

(ii) First we will show that $w \in L^r(\mathbf{R}^N)$ for all $r \in [1, \infty]$. From $w \in C_0(\mathbf{R}^N)$ it suffices to show that $w \in L^1(\mathbf{R}^N)$. Put $u = w + U^\kappa_{j_0}$. Then, from (1.8) and $(1.11)_\kappa$, u satisfies $(1.7)_\kappa$. This implies that u satisfies $(1.1)_\kappa$ in $\mathcal{S}'(\mathbf{R}^N)$, and hence in $\mathcal{D}'(\mathbf{R}^N)$. Since $U^\kappa_{j_0}(x) \to 0$ as $|x| \to \infty$, u satisfies (1.2). Then, by Proposition 1.1 we have $w + U^\kappa_{j_0} \in L^1(\mathbf{R}^N)$. From $U^\kappa_{j_0} \in L^1(\mathbf{R}^N)$ we obtain $w \in L^1(\mathbf{R}^N)$, and hence $w \in L^r(\mathbf{R}^N)$ for all $r \in [1, \infty]$.

From $w + V^\kappa_{j_0} \in C_0(\mathbf{R}^N)$ and $w + U^\kappa_{j_0-1} \in L^{q_0}(\mathbf{R}^N)$ it follows that

$$p(w + U^\kappa_{j_0})^{p-1}(w + V^\kappa_{j_0}) \in L^{q_0/(p-1)}(\mathbf{R}^N).$$

Note here that $q_0/(p-1) > p/(p-1) > N/2$. From $V^\kappa_{j_0} \le U^\kappa_{j_0}$ we have

$$p(w+U^\kappa_{j_0})^{p-1}(w+V^\kappa_{j_0}) \le p(w+U^\kappa_{j_0})^p \in L^1(\mathbf{R}^N).$$

Thus $(w+U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \in (L^1 \cap L^q)(\mathbf{R}^N)$ with some $q > N/2$. By Lemma A.4 we obtain $w \in H^1(\mathbf{R}^N)$. □

Proof of Proposition 1.2. (i) ⇒ (ii). Let $u = w + U^\kappa_{j_0} \in L^p_{\rm loc}(\mathbf{R}^N)$ be a positive solution of $(1.1)_\kappa$–(1.2). Proposition 1.1 implies that $u \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$ and u satisfies $(1.7)_\kappa$. Then, from (iii) of Lemma 2.6 we have $w > 0$ a.e. in $\mathbf{R}^N$. From $u, U^\kappa_{j_0} \in L^{q_0}(\mathbf{R}^N)$ it follows that $w \in L^{q_0}(\mathbf{R}^N)$. From $(1.7)_\kappa$ and (1.8), w satisfies

$$w = E_1 * [u^p - (U^\kappa_{j_0-1})^p] = E_1 * [(w+U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p] \quad \text{a.e. in } \mathbf{R}^N.$$

Thus w satisfies $(2.19)_\kappa$. From (i) of Lemma 2.9 we obtain $w \in C_0(\mathbf{R}^N)$ and $w > 0$ in $\mathbf{R}^N$. Hence (ii) holds.

(ii) ⇒ (iii). From (ii) of Lemma 2.9 we have $w \in H^1(\mathbf{R}^N)$. Lemma 2.8 implies that $(w + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \in H^{-1}(\mathbf{R}^N) \subset \mathcal{S}'(\mathbf{R}^N)$. From $(1.11)_\kappa$, w satisfies

$$-\Delta w + w = (w+U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \quad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

Since $\mathcal{D}(\mathbf{R}^N)$ is dense in $H^1(\mathbf{R}^N)$, we have $(1.13)_\kappa$ for any $\psi \in H^1(\mathbf{R}^N)$.

(iii) ⇒ (i). It is clear that $u = w + U^\kappa_{j_0} \in L^p_{\rm loc}(\mathbf{R}^N)$ and $u > 0$ a.e. in $\mathbf{R}^N$. From (iii) of Lemma 2.8 we have $(w + U^\kappa_{j_0})^p - (U^\kappa_{j_0-1})^p \in H^{-1}(\mathbf{R}^N) \subset \mathcal{S}'(\mathbf{R}^N)$. Then w satisfies $(1.11)_\kappa$. From (1.8), $u = w + U^\kappa_{j_0}$ satisfies $(1.7)_\kappa$, and hence u satisfies $(1.1)_\kappa$. Take $R_0 > 0$ such that (2.5) holds. From (ii) of Lemma 2.6, we have $U^\kappa_{j_0} \in L^r(\mathbf{R}^N \setminus B_{R_0})$ for all $r \in [1, \infty]$. Then $u = w + U^\kappa_{j_0} \in L^{2N/(N-2)}(\mathbf{R}^N \setminus B_{R_0})$. Lemma 2.2 implies that $u(x) \to 0$ as $|x| \to \infty$. Thus (i) holds. □

3. Existence of minimal solutions and linearized eigenvalue problems: Proof of Theorem 1

First we will show the following lemma.

Lemma 3.1. *Let $w \in H^1(\mathbf{R}^N)$ be a solution of*

$$-\Delta w + w = (w+U^\kappa_{j_0})^p_+ - (U^\kappa_{j_0-1})^p \quad \textit{in } \mathbf{R}^N, \tag{3.1$_\kappa$}$$

where $(s)_+ = \max\{s, 0\}$. Then $w > 0$ a.e. in $\mathbf{R}^N$.

Proof. Note that $U^\kappa_{j_0} = V^\kappa_{j_0} + U^\kappa_{j_0-1}$ and $V^\kappa_{j_0} \in H^1(\mathbf{R}^N)$. Then, from (ii) of Lemma C.1, we have $(w + U^\kappa_{j_0})^p_+ - (U^\kappa_{j_0-1})^p \in H^{-1}(\mathbf{R}^N) \subset \mathcal{S}'(\mathbf{R}^N)$. It follows that

$$w = E_1 * [(w+U^\kappa_{j_0})^p_+ - (U^\kappa_{j_0-1})^p] \quad \text{a.e. in } \mathbf{R}^N.$$

From (1.8) we obtain

$$w + U_{j_0}^\kappa = E_1 * [(w + U_{j_0}^\kappa)_+^p] + \kappa f_0 > 0 \quad \text{a.e. in} \ \ \mathbf{R}^N.$$

Put $u = w + U_{j_0}^\kappa$. Then $u > 0$ a.e. in $\mathbf{R}^N$, and u satisfies $(1.7)_\kappa$. From (iii) of Lemma 2.6 we obtain $w = u - U_{j_0}^\kappa > 0$ a.e. in $\mathbf{R}^N$. □

Lemma 3.2. *There exists $\kappa_0 > 0$ such that $(1.12)_\kappa$ has a positive solution $w \in H^1(\mathbf{R}^N)$ for $\kappa \in (0, \kappa_0]$.*

Proof. Define $\Phi : (0, \infty) \times H^1(\mathbf{R}^N) \to H^{-1}(\mathbf{R}^N)$ by

$$\Phi(\kappa, w) = -\Delta w + w - (w + U_{j_0}^\kappa)_+^p + (U_{j_0-1}^\kappa)^p. \tag{3.2}$$

Then it follows that, for $u \in H^1(\mathbf{R}^N)$,

$$\Phi_w(\kappa, w)u = -\Delta u + u - p(w + U_{j_0}^\kappa)_+^{p-1} u. \tag{3.3}$$

In particular, $\Phi_w(0, 0)u = -\Delta u + u$. It is clear that $\Phi_w(0, 0) : H^1(\mathbf{R}^N) \to H^{-1}(\mathbf{R}^N)$ is invertible. Then, by the implicit function theorem (see, e.g., Berger [6]), there exists a solution $w \in H^1(\mathbf{R}^N)$ of $(3.1)_\kappa$ for $\kappa \in (0, \kappa_0]$ with some $\kappa_0 > 0$. Lemma 3.1 implies that $w > 0$ a.e. in $\mathbf{R}^N$. Thus we obtain a positive solution of $(1.12)_\kappa$ for $\kappa \in (0, \kappa_0]$. □

We will show comparison results for the solutions $w \in C_0(\mathbf{R}^N)$ of the problem $(1.11)_\kappa$.

Lemma 3.3. *Assume that there exists a positive function $\hat{w} \in C_0(\mathbf{R}^N)$ satisfying*

$$\hat{w} \geq E_1 * [(\hat{w} + U_{j_0}^{\hat{\kappa}})^p - (U_{j_0-1}^{\hat{\kappa}})^p] \quad \text{in } \mathbf{R}^N$$

for some $\hat{\kappa} > 0$. Then, for any $\kappa \in (0, \hat{\kappa}]$, there exists a positive solution $w \in C_0(\mathbf{R}^N)$ of $(1.11)_\kappa$ satisfying $0 < w(x) \leq \hat{w}(x)$ for $x \in \mathbf{R}^N$. Furthermore, for any positive function $\tilde{w} \in C_0(\mathbf{R}^N)$ satisfying

$$\tilde{w} \geq E_1 * [(\tilde{w} + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p] \quad \text{in } \mathbf{R}^N, \tag{3.4}$$

we have $w(x) \leq \tilde{w}(x)$ for $x \in \mathbf{R}^N$.

Proof. Let $\kappa \in (0, \hat{\kappa}]$. Define $\{w_n\}$, inductively, by $w_0 \equiv 0$ and

$$w_n = E_1 * [(w_{n-1} + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p] \quad \text{for } n = 1, 2, \ldots. \tag{3.5}$$

From (iii) of Lemma 2.8 and (2.17), we see that $w_1 \in C(\mathbf{R}^N)$ and $0 < w_1(x) < \hat{w}(x)$ for $x \in \mathbf{R}^N$. By induction, we obtain $w_n \in C(\mathbf{R}^N)$ and

$$0 < w_1(x) \leq w_2(x) \leq \cdots \leq \hat{w}(x) \quad \text{for } x \in \mathbf{R}^N.$$

Put $w(x) = \lim_{n\to\infty} w_n(x)$ for each $x \in \mathbf{R}^N$. Then $0 < w(x) \le \hat{w}(x)$ for $x \in \mathbf{R}^N$. Letting $n \to \infty$ in (3.5), by the Lebesgue convergence theorem, we find that w satisfies $(1.11)_\kappa$. From (iii) of Lemma 2.8 we obtain $w \in C(\mathbf{R}^N)$, and hence $w \in C_0(\mathbf{R}^N)$.

Let $\tilde{w} \in C_0(\mathbf{R}^N)$ be a positive function satisfying (3.4). By induction, we have $w_n \le \tilde{w}$ in $\mathbf{R}^N$ for $n = 1, 2, \ldots$. Thus $w \le \tilde{w}$ in $\mathbf{R}^N$. □

For each $\kappa > 0$ we define the solution set S_κ by

$$S_\kappa = \{w \in H^1(\mathbf{R}^N) : w \text{ is a positive solution of } (1.12)_\kappa.\}.$$

By Proposition 1.2, $w \in S_\kappa$ if and only if $w \in C_0(\mathbf{R}^N)$ is positive in $\mathbf{R}^N$ and satisfies $(1.11)_\kappa$. We call a minimal solution $\underline{w}_\kappa \in S_\kappa$, if $\underline{w}_\kappa$ satisfies $\underline{w}_\kappa \le w$ in $\mathbf{R}^N$ for all $w \in S_\kappa$. Lemma 3.2 implies that $S_\kappa \ne \emptyset$ for sufficient small $\kappa > 0$.

Lemma 3.4. (i) *Assume that $S_{\kappa_0} \ne \emptyset$ for some $\kappa_0 > 0$. Then $S_\kappa \ne \emptyset$ for all $\kappa \in (0, \kappa_0)$.*

(ii) *If $S_\kappa \ne \emptyset$ then there exists a minimal solution $\underline{w}_\kappa \in S_\kappa$. Moreover, $\underline{w}_\kappa \le \tilde{w}$ for any positive function $\tilde{w}$ satisfying* (3.4).

(iii) *Assume that $\underline{w}_\kappa \in S_\kappa$ and $\underline{w}_{\hat\kappa} \in S_{\hat\kappa}$ are minimal solutions with $0 < \kappa < \hat\kappa$. Then $\underline{w}_\kappa < \underline{w}_{\hat\kappa}$ in $\mathbf{R}^N$.*

Proof. (i) Let $\kappa \in (0, \kappa_0)$ and $w_0 \in S_{\kappa_0}$. Applying Lemma 3.3 with $\hat{w} = w_0$ and $\hat\kappa = \kappa_0$, we obtain a positive solution of $(1.11)_\kappa$. This implies that $S_\kappa \ne \emptyset$ for all $\kappa \in (0, \kappa_0)$.

(ii) Assume that $w \in S_\kappa$. Applying Lemma 3.3 with $\hat{w} = w$ and $\hat\kappa = \kappa$, there exists $\underline{w}_\kappa \in S_\kappa$ such that $\underline{w}_\kappa \le w$ in $\mathbf{R}^N$. Furthermore, $\underline{w}_\kappa \le \tilde{w}$ for any positive function $\tilde{w}$ satisfying (3.4). In particular, we obtain $\underline{w}_\kappa \le w$ for all $w \in S_\kappa$. Thus $\underline{w}_\kappa$ is the minimal solution of S_κ.

(iii) From (2.17) we have

$$\underline{w}_{\hat\kappa} = E_1 * [(\underline{w}_{\hat\kappa} + U^{\hat\kappa}_{j_0})^p - (U^{\hat\kappa}_{j_0-1})^p] > E_1 * [(\underline{w}_{\hat\kappa} + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p].$$

In particular, (3.4) holds with $\tilde{w} = \underline{w}_{\hat\kappa}$. From (ii) of this lemma, we have $\underline{w}_{\hat\kappa} \ge \underline{w}_\kappa$ in $\mathbf{R}^N$. Then it follows that

$$\underline{w}_{\hat\kappa} > E_1 * [(\underline{w}_{\hat\kappa} + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p] \ge E_1 * [(\underline{w}_{\kappa} + U^{\kappa}_{j_0})^p - (U^{\kappa}_{j_0-1})^p] = \underline{w}_\kappa.$$

Thus we obtain $\underline{w}_{\hat\kappa} > \underline{w}_\kappa$ in $\mathbf{R}^N$. □

Lemma 3.5. *Let $\kappa^* = \sup\{\kappa > 0 : S_\kappa \ne \emptyset\}$. Then $0 < \kappa^* < \infty$.*

Proof. Lemma 3.2 implies that $\kappa^* > 0$. Then it suffices to show that $\kappa^* < \infty$. Take $R > 0$ arbitrarily, and consider the eigenvalue problem

$$\begin{cases} -\Delta\phi + \phi = \lambda\phi & \text{in } B_R, \\ \phi = 0 & \text{on } \partial B_R. \end{cases} \tag{3.6}$$

We denote by (λ_1, ϕ_1) the first eigenvalue and the corresponding eigenfunction of the problem. It follows that $\lambda_1 > 0$, $\phi_1 > 0$ in B_R, and that $\partial\phi_1/\partial\nu < 0$ on ∂B_R, where ν is the outer unit normal vector on ∂B_R. Let $\kappa > 0$ such that $S_\kappa \neq \emptyset$, and let $w \in S_\kappa$. Then, from (2.16), w satisfies

$$-\Delta w + w = (w + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p \geq p(U_{j_0-1}^\kappa)^{p-1}(w + V_{j_0}^\kappa) \quad \text{in} \quad B_R.$$

Note that $U_{j_0-1}^\kappa \geq \kappa f_0$ and $V_{j_0}^\kappa > 0$. Put $m = \inf\{f_0(x) : x \in \overline{B_R}\} > 0$. Then w satisfies

$$-\Delta w + w \geq p\kappa^{p-1}m^{p-1}w \quad \text{in} \quad B_R.$$

Multiplying this by ϕ_1 and integrating it on B_R, we have

$$\int_{B_R} (\nabla w \cdot \nabla\phi_1 + w\phi_1)\, dx \geq p\kappa^{p-1}m^{p-1} \int_{B_R} w\phi_1 dx. \tag{3.7}$$

On the other hand, multiplying the equation in (3.6) by w and integrating it on B_R, we have

$$-\int_{B_R} (\Delta\phi_1)w dx + \int_{B_R} w\phi_1 dx = \lambda_1 \int_{B_R} w\phi_1 dx.$$

Since $\partial\phi_1/\partial\nu < 0$ and $w > 0$ on ∂B_R, we have

$$-\int_{B_R} (\Delta\phi_1)w dx > \int_{B_R} \nabla\phi_1 \cdot \nabla w dx.$$

Thus

$$\int_{B_R} (\nabla\phi_1 \cdot \nabla w + w\phi_1)\, dx < \lambda_1 \int_{B_R} w\phi_1 dx. \tag{3.8}$$

From (3.7) and (3.8) it follows that $p\kappa^{p-1}m^{p-1} < \lambda_1$, and hence $\kappa < m(\lambda_1/p)^{1/(p-1)}$. This implies that $\kappa^* \leq m(\lambda_1/p)^{1/(p-1)} < \infty$. □

Proof of Theorem 1. (i) Let $\kappa^* = \sup\{\kappa > 0 : S_\kappa \neq \emptyset\}$. Lemma 3.5 implies that $0 < \kappa^* < \infty$. By (i) and (ii) of Lemma 3.4, there exists a minimal solution $\underline{w}_\kappa \in S_\kappa$ for $\kappa \in (0, \kappa^*)$. From (iii) of Lemma 3.4, $\underline{w}_\kappa$ is strictly increasing in $\kappa \in (0, \kappa^*)$. Put $\underline{u}_\kappa = \underline{w}_\kappa + U_{j_0}^\kappa$. Then, from Proposition 1.2, $\underline{u}_\kappa$ is the minimal solution of $(1.1)_\kappa$–(1.2), and it is clear that $\underline{u}_\kappa$ is increasing in $\kappa \in (0, \kappa^*)$.

(ii) By the definition of κ^*, $(1.12)_\kappa$ has no positive solution $w \in H^1(\mathbf{R}^N)$ for $\kappa > \kappa^*$. This implies, from Proposition 1.2, that $(1.1)_\kappa$–(1.2) has no positive solution for $\kappa > \kappa^*$. □

For $\kappa \in (0, \kappa^*)$, let $\underline{u}_\kappa$ be the minimal solution of $(1.1)_\kappa$–(1.2) obtained in Theorem 1. Let us consider the following linearized eigenvalue problem

$$\begin{cases} -\Delta\phi + \phi = \lambda p(\underline{u}_\kappa)^{p-1}\phi & \text{in } \mathbf{R}^N, \\ \phi \in H^1(\mathbf{R}^N). \end{cases} \tag{3.9}$$

From Proposition 1.1, we have $\underline{u}_\kappa \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$. Then, from $p < N/(N-2)$, it follows that $(\underline{u}_\kappa)^{p-1} \in L^r(\mathbf{R}^N)$ for all $r \in [1, q)$ with some $q > N/2$. By Lemma

B.2 in Appendix B, there exists a first eigenvalue $\lambda_1 = \lambda_1(\kappa) > 0$ of the problem (3.9). The following lemma plays an important role in the proofs of Theorems 2 and 3.

Lemma 3.6. *We have $\lambda_1(\kappa) > 1$ for each $\kappa \in (0, \kappa^*)$. In particular, there holds*

$$\|\psi\|_{H^1}^2 \geq \lambda_1(\kappa) \int_{\mathbf{R}^N} p(\underline{u}_\kappa)^{p-1}\psi^2 dx > \int_{\mathbf{R}^N} p(\underline{u}_\kappa)^{p-1}\psi^2 dx \tag{3.10}$$

for any $\psi \in H^1(\mathbf{R}^N) \setminus \{0\}$.

Proof. For $\kappa \in (0, \kappa^*)$, let $\underline{w}_\kappa$ be the minimal solutions of S_κ. Then $\underline{u}_\kappa = \underline{w}_\kappa + U_{j_0}^\kappa$. Let $\kappa, \hat{\kappa} \in (0, \kappa^*)$ with $\kappa < \hat{\kappa}$, and put $z(x) = \underline{w}_{\hat{\kappa}}(x) - \underline{w}_\kappa(x)$. From (iii) of Lemma 3.3 we have $z > 0$ in $\mathbf{R}^N$. From (2.17), $\underline{w}_{\hat{\kappa}}$ satisfies

$$-\Delta \underline{w}_{\hat{\kappa}} + \underline{w}_{\hat{\kappa}} = (\underline{w}_{\hat{\kappa}} + U_{j_0}^{\hat{\kappa}})^p - (U_{j_0-1}^{\hat{\kappa}})^p > (\underline{w}_{\hat{\kappa}} + U_{j_0}^{\kappa})^p - (U_{j_0-1}^{\kappa})^p \quad \text{in } \mathbf{R}^N.$$

Then z satisfies

$$-\Delta z + z > p(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} z = p(\underline{u}_\kappa)^{p-1} z \quad \text{in } \mathbf{R}^N. \tag{3.11}$$

Let $\phi_1 \in H^1(\mathbf{R}^N)$ be the eigenfunction of the problem (3.9) corresponding to $\lambda_1 > 0$, that is,

$$-\Delta \phi_1 + \phi_1 = \lambda_1 p(\underline{u}_\kappa)^{p-1}\phi_1 \quad \text{in } \mathbf{R}^N. \tag{3.12}$$

We note here that $\phi_1 > 0$ a.e. in $\mathbf{R}^N$. Multiplying (3.11) and (3.12) by ϕ_1 and z, respectively, and integrating them on $\mathbf{R}^N$, we obtain

$$\lambda_1 \int_{\mathbf{R}^N} p(\underline{u}_\kappa)^{p-1} z\phi_1 dx = \int_{\mathbf{R}^N} (\nabla z \cdot \nabla \phi_1 + z\phi_1) dx > \int_{\mathbf{R}^N} p(\underline{u}_\kappa)^{p-1} z\phi_1 dx.$$

Thus $\lambda_1 > 1$. From Lemma B.2 in Appendix B, we obtain (3.10). □

4. Existence and uniqueness of the extremal solution: Proof of Theorem 2

For $\kappa \in (0, \kappa^*)$, let $\underline{w}_\kappa \in S_\kappa$ be the minimal solution obtained in Lemma 3.4.

Proposition 4.1. *For any $R > 0$, there exists a constant $M_0 = M_0(R) > 0$ independent of κ such that $\sup_{x \in \overline{B}_R} \underline{w}_\kappa(x) \leq M_0$ for $\kappa \in (0, \kappa^*)$.*

We will give the proof of Proposition 4.1 in the next section.

Lemma 4.1. *For any small $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) \geq 0$ such that*

$$(t+s)^p - s^p \leq (1+\varepsilon)(t+s)^{p-1} t + Cs^p \quad \textit{for } s, t \geq 0. \tag{4.1}$$

Proof. Take $\tilde{\varepsilon} > 0$ so that $\tilde{\varepsilon}/(1-\tilde{\varepsilon}) = \varepsilon$. By using the Young inequality, we have

$$(t+s)^{p-1}s \le \tilde{\varepsilon}(t+s)^p + C(\tilde{\varepsilon})s^p = \tilde{\varepsilon}(t+s)^{p-1}t + \tilde{\varepsilon}(t+s)^{p-1}s + C(\tilde{\varepsilon})s^p.$$

This implies that

$$(t+s)^{p-1}s \le \varepsilon(t+s)^{p-1}t + Cs^p$$

with $C = C(\tilde{\varepsilon})/(1-\tilde{\varepsilon})$. From $(t+s)^p = (t+s)^{p-1}t + (t+s)^{p-1}s$, we obtain (4.1). □

Lemma 4.2. *There exists a constant $M_1 > 0$ which is independent of κ such that $\|\underline{w}_\kappa\|_{H^1} \le M_1$ for all $\kappa \in (0, \kappa^*)$.*

Proof. Putting $\psi = \underline{w}_\kappa$ in $(1.13)_\kappa$, we obtain

$$\|\underline{w}_\kappa\|_{H^1}^2 = \int_{\mathbf{R}^N} \left((\underline{w}_\kappa + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p\right) \underline{w}_\kappa dx.$$

We note here that $U_{j_0}^\kappa = V_{j_0}^\kappa + U_{j_0-1}^\kappa$. From Lemma 4.1 we have, for any $\varepsilon > 0$,

$$\|\underline{w}_\kappa\|_{H^1}^2 \le (1+\varepsilon)\int_{\mathbf{R}^N} (\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}(\underline{w}_\kappa + V_{j_0}^\kappa)\underline{w}_\kappa dx + C\int_{\mathbf{R}^N} (U_{j_0-1}^\kappa)^p \underline{w}_\kappa dx. \tag{4.2}$$

Recall that $\underline{u}_\kappa = \underline{w}_\kappa + U_{j_0}^\kappa$. Then it follows from (3.10) that

$$\int_{\mathbf{R}^N} (\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}\underline{w}_\kappa^2 dx < \frac{1}{p}\|\underline{w}_\kappa\|_{H^1}^2. \tag{4.3}$$

We note, from (ii) of Lemma 2.7, that $V_{j_0}^\kappa \in H^1(\mathbf{R}^N)$. By using the Young inequality and (3.10), we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^N} (\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} V_{j_0}^\kappa \underline{w}_\kappa dx \\
\le\ & \varepsilon\int_{\mathbf{R}^N} (\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}\underline{w}_\kappa^2 dx + C(\varepsilon)\int_{\mathbf{R}^N} (\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}(V_{j_0}^\kappa)^2 dx \\
<\ & \frac{\varepsilon}{p}\|\underline{w}_\kappa\|_{H^1}^2 + \frac{C(\varepsilon)}{p}\|V_{j_0}^\kappa\|_{H^1}^2.
\end{aligned} \tag{4.4}$$

From (4.2)–(4.4), there exists a constant $C_1 > 0$ such that

$$\left(1 - \frac{(1+\varepsilon)^2}{p}\right)\|\underline{w}_\kappa\|_{H^1}^2 \le C_1\left(\|V_{j_0}^\kappa\|_{H^1}^2 + \int_{\mathbf{R}^N} (U_{j_0-1}^\kappa)^p \underline{w}_\kappa dx\right)$$

for sufficient small $\varepsilon > 0$. Take $R_0 > 0$ such that (2.5) holds, and put $R \ge R_0$. From (ii) of Lemma 2.6, we see that $U_{j_0-1}^\kappa \in L^r(\mathbf{R}^N \setminus B_R)$ for all $r \in [1, \infty]$. By Proposition 4.1 we obtain

$$\begin{aligned}
\int_{\mathbf{R}^N} (U_{j_0-1}^\kappa)^p \underline{w}_\kappa dx &= \int_{B_R} (U_{j_0-1}^\kappa)^p \underline{w}_\kappa dx + \int_{\mathbf{R}^N \setminus B_R} (U_{j_0-1}^\kappa)^p \underline{w}_\kappa dx \\
&\le M_0\int_{B_R} (U_{j_0-1}^\kappa)^p dx + \int_{\mathbf{R}^N \setminus B_R} (U_{j_0-1}^\kappa)^p \underline{w}_\kappa dx.
\end{aligned}$$

By the Young inequality, for any $\tilde{\varepsilon} > 0$ there exists a constant $C(\tilde{\varepsilon}) > 0$ such that

$$\int_{\mathbf{R}^N \setminus B_R} (U_{j_0-1}^\kappa)^p \underline{w}_\kappa dx \le \tilde{\varepsilon}\int_{\mathbf{R}^N \setminus B_R} \underline{w}_\kappa^2 dx + C(\tilde{\varepsilon})\int_{\mathbf{R}^N \setminus B_R} (U_{j_0-1}^\kappa)^{2p} dx.$$

Therefore, we have

$$\left(1-\frac{(1+\varepsilon)^2}{p}-\tilde{\varepsilon}C_1\right)\|\underline{w}_\kappa\|_{H^1}^2 \le C\left(\|V_{j_0}^\kappa\|_{H^1}^2+\int_{B_R}(U_{j_0-1}^\kappa)^p dx+\int_{\mathbf{R}^N\setminus B_R}(U_{j_0-1}^\kappa)^{2p}dx\right)$$

for sufficient small $\tilde{\varepsilon}>0$. We note here that $V_{j_0}^\kappa \le V_{j_0}^{\kappa^*}$ and $U_{j_0-1}^\kappa \le U_{j_0-1}^{\kappa^*}$ for $\kappa\in(0,\kappa^*)$. Thus we obtain $\|\underline{w}_\kappa\|_{H^1}\le M_1$ with a constant $M_1>0$ independent of $\kappa\in(0,\kappa^*)$. □

Lemma 4.3. *There exists a minimal solution $\underline{w}_{\kappa^*}\in S_{\kappa^*}$. Furthermore, $\lambda_1(\kappa^*)=1$, where $\lambda_1(\kappa^*)$ is the first eigenvalue of the linearized problem* (3.9) *with $\underline{u}_\kappa=\underline{w}_{\kappa^*}+U_{j_0}^{\kappa^*}$.*

Proof. Let $\{\kappa_n\}$ be a sequence such that $\kappa_n<\kappa_{n+1}$ and $\kappa_n\to\kappa^*$ as $n\to\infty$. Since $\underline{w}_\kappa$ is increasing in $\kappa\in(0,\kappa^*)$, we have $\underline{w}_{\kappa_n}<\underline{w}_{\kappa_{n+1}}$ in $\mathbf{R}^N$. Then, from Lemma 4.2, there exists a positive function $w^*\in H^1(\mathbf{R}^N)$ such that

$$\underline{w}_{\kappa_n}<w^* \quad \text{a.e. in } \mathbf{R}^N \quad \text{and} \quad \underline{w}_{\kappa_n}\rightharpoonup w^* \text{ weakly in } H^1(\mathbf{R}^N) \text{ as } n\to\infty.$$

We note here that $\underline{w}_{\kappa_n}$ satisfies

$$\int_{\mathbf{R}^N}(\nabla\underline{w}_{\kappa_n}\cdot\nabla\psi+\underline{w}_{\kappa_n}\psi)\,dx=\int_{\mathbf{R}^N}\left((\underline{w}_{\kappa_n}+U_{j_0}^{\kappa_n})^p-(U_{j_0-1}^{\kappa_n})^p\right)\psi dx \tag{4.5}$$

for any $\psi\in H^1(\mathbf{R}^N)$. From (ii) of Lemma 2.8 we have

$$\int_{\mathbf{R}^N}\left((w^*+U_{j_0}^{\kappa^*})^p-(U_{j_0-1}^{\kappa^*})^p\right)|\psi|dx<\infty$$

for any $\psi\in H^1(\mathbf{R}^N)$. Letting $n\to\infty$ in (4.5), by the Lebesgue convergence theorem we obtain

$$\int_{\mathbf{R}^N}(\nabla w^*\cdot\nabla\psi+w^*\psi)\,dx=\int_{\mathbf{R}^N}\left((w^*+U_{j_0}^{\kappa^*})^p-(U_{j_0-1}^{\kappa^*})^p\right)\psi dx.$$

Thus $w^*\in H^1(\mathbf{R}^N)$ is a positive solution of $(1.12)_{\kappa^*}$, i.e., $w^*\in S_{\kappa^*}$. From (ii) of Lemma 3.4 there exists a minimal solution $\underline{w}_{\kappa^*}\in S_{\kappa^*}$. We verify that $\underline{w}_{\kappa^*}\equiv w^*$. In fact, from (iii) of Lemma 3.4 we see that $\underline{w}_{\kappa_n}<\underline{w}_{\kappa^*}$ in $\mathbf{R}^N$ for $n=1,2\ldots$. Then $w^*\le\underline{w}_{\kappa^*}$, and hence $w^*\equiv\underline{w}_{\kappa^*}$.

We will show that $\lambda_1(\kappa^*)=1$. Since $\lambda_1(\kappa_n)>1$ from Lemma 3.6, we have

$$\int_{\mathbf{R}^N}p(\underline{u}_{\kappa_n})^{p-1}\psi^2dx<\lambda_1(\kappa_n)\int_{\mathbf{R}^N}p(\underline{u}_{\kappa_n})^{p-1}\psi^2dx\le\|\psi\|_{H^1}^2 \tag{4.6}$$

for any $\psi\in H^1(\mathbf{R}^N)$. Let $n\to\infty$ in (4.6). Then, by the monotone convergence theorem, we obtain

$$\int_{\mathbf{R}^N}p(\underline{u}_{\kappa^*})^{p-1}\psi^2dx\le\|\psi\|_{H^1}^2$$

for any $\psi\in H^1(\mathbf{R}^N)$. Let $\phi_1\in H^1(\mathbf{R}^N)$ be the eigenfunction corresponding to $\lambda_1(\kappa^*)$. Then it follows that

$$\lambda_1(\kappa^*)\int_{\mathbf{R}^N}p(\underline{u}_{\kappa^*})^{p-1}\phi_1^2dx=\|\phi_1\|_{H^1}^2.$$

This implies that $\lambda_1(\kappa^*)\ge 1$.

Assume to the contrary that $\lambda_1(\kappa^*) > 1$. Define $\Phi : (0,\infty) \times H^1(\mathbf{R}^N) \to H^{-1}(\mathbf{R}^N)$ by (3.2). Then (3.3) holds for $u \in H^1(\mathbf{R}^N)$. In particular,

$$\Phi_w(\kappa^*, \underline{w}_{\kappa^*})u = -\Delta u + u - p(\underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*})_+^{p-1} u.$$

By Lemma B.3 in Appendix B, we find that, for every $f \in H^{-1}(\mathbf{R}^N)$, there exists a unique solution $u \in H^1(\mathbf{R}^N)$ of $\Phi_w(\kappa^*, \underline{w}_{\kappa^*})u = f$, that is, $\Phi_w : H^1(\mathbf{R}^N) \to H^{-1}(\mathbf{R}^N)$ is invertible at $\kappa = \kappa^*$ and $w = \underline{w}_{\kappa^*}$. Then, by the implicit function theorem, there exists $\varepsilon > 0$ such that $\Phi(\kappa, w) = 0$ has a solution $w_\kappa \in H^1(\mathbf{R}^N)$ for $\kappa \in (\kappa^* - \varepsilon, \kappa^* + \varepsilon)$. From Lemma 3.1 we obtain a positive solution w_κ of $(1.12)_\kappa$ for $\kappa \in (\kappa^* - \varepsilon, \kappa^* + \varepsilon)$. This contradicts the definition of κ^*. Then $\lambda_1(\kappa^*) \le 1$, and hence $\lambda_1(\kappa^*) = 1$. □

Proof of Theorem 2. Let $\underline{w}_{\kappa^*} \in S_{\kappa^*}$ be the minimal solution obtained in Lemma 4.3. We will show that $\underline{w}_{\kappa^*}$ is a unique positive solution of $(1.12)_{\kappa^*}$ in $H^1(\mathbf{R}^N)$. Assume that $w \in H^1(\mathbf{R}^N)$ is a positive solution of $(1.12)_{\kappa^*}$. Since $\underline{w}_{\kappa^*}$ is the minimal solution, we have $w \ge \underline{w}_{\kappa^*}$ in $\mathbf{R}^N$. Put $z = w - \underline{w}_{\kappa^*}$. Then $z \ge 0$ and satisfies

$$-\Delta z + z = (w + U_{j_0}^{\kappa^*})^p - (\underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*})^p \quad \text{in } \mathbf{R}^N. \tag{4.7}$$

Let $\phi_1 \in H^1(\mathbf{R}^N)$ be an eigenfunction of the linealized problem of (3.9) with $\underline{u}_\kappa = \underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*}$. Since $\lambda_1(\kappa^*) = 1$ from Lemma 4.3, we have

$$-\Delta\phi_1 + \phi_1 = p(\underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*})^{p-1}\phi_1 \quad \text{in } \mathbf{R}^N.$$

Multiplying this equation by $z = w - \underline{w}_{\kappa^*} \in H^1(\mathbf{R}^N)$, and integrating on $\mathbf{R}^N$, we have

$$\int_{\mathbf{R}^N} (\nabla\phi_1 \cdot \nabla z + \phi_1 z)\, dx = \int_{\mathbf{R}^N} p(\underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*})^{p-1}(w - \underline{w}_{\kappa^*})\phi_1 dx. \tag{4.8}$$

Multiplying (4.7) by ϕ_1 and integrating on $\mathbf{R}^N$, we obtain

$$\int_{\mathbf{R}^N} (\nabla z \cdot \nabla\phi_1 + z\phi_1)\, dx = \int_{\mathbf{R}^N} \left((w + U_{j_0}^{\kappa^*})^p - (\underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*})^p\right)\phi_1 dx. \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$\int_{\mathbf{R}^N} F(w + U_{j_0}^{\kappa^*}, \underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*})\phi_1 dx = 0,$$

where $F(t,s) = t^p - s^p - ps^{p-1}(t-s)$. We note here that, for $t \ge s \ge 0$, $F(t,s) \ge 0$ and $F(t,s) = 0$ if and only if $t = s$. Then, from $\phi_1 > 0$, we conclude that

$$F(w + U_{j_0}^{\kappa^*}, \underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*}) = 0 \quad \text{a.e. in } \mathbf{R}^N \quad \text{and} \quad w \equiv \underline{w}_{\kappa^*} \text{ in } \mathbf{R}^N.$$

Thus $(1.12)_{\kappa^*}$ has a unique positive solution $\underline{w}_{\kappa^*}$ in $H^1(\mathbf{R}^N)$. Put $\underline{u}_{\kappa^*} = \underline{w}_{\kappa^*} + U_{j_0}^{\kappa^*}$. Then, by Proposition 1.2, $\underline{u}_{\kappa^*}$ is a unique positive solution of $(1.1)_\kappa$–(1.2) with $\kappa = \kappa^*$. □

5. A priori bound of minimal solutions: Proof of Proposition 4.1

For $\mu > 0$, we define $E_\mu(x) = \mu^{N-2}E_1(\mu x)$ for $x \in \mathbf{R}^N$. Then E_μ is the fundamental solution for $-\Delta + \mu^2 I$ in $\mathbf{R}^N$. We see that E_μ satisfies

$$E_\mu(x) \sim c_\mu |x|^{-(N-1)/2} e^{-\mu|x|} \quad \text{as } |x| \to \infty,$$

where $c_\mu > 0$ is the constant depends on N and μ. We denote by E_0 the fundamental solution for $-\Delta$ in $\mathbf{R}^N$, that is,

$$E_0(x) = \frac{1}{(N-2)N\omega_N |x|^{N-2}} \quad \text{for } x \in \mathbf{R}^N \setminus \{0\},$$

where ω_N denotes the volume of the unit ball in $\mathbf{R}^N$. We see that $E_0(x) > E_\mu(x)$ for $x \in \mathbf{R}^N \setminus \{0\}$ and $E_\mu(x) \sim E_0(x)$ as $|x| \to 0$.

For each $\mu > 0$, we define $e_\mu \in C^\infty(\mathbf{R}^N) \cap H^1(\mathbf{R}^N)$ such that $0 < e_\mu \le 1$ in $\mathbf{R}^N$ and $e_\mu(x) \equiv E_\mu(x)$ for $|x| \ge R_\mu$ with some $R_\mu > 0$. Then it is clear that

$$e_\mu(x) \sim c_\mu |x|^{-(N-1)/2} e^{-\mu|x|} \quad \text{as } |x| \to \infty \tag{5.1}$$

with the constant $c_\mu > 0$. By the definition, e_μ satisfies $-\Delta e_\mu + \mu^2 e_\mu \equiv 0$ for $|x| > R_\mu$. Then there exists a constant $C = C(\mu) > 0$ such that

$$-\Delta e_\mu + \mu^2 e_\mu \le C e_\mu \quad \text{in } \mathbf{R}^N.$$

Hence, for $\mu \in (0,1)$ we have

$$-\Delta e_\mu + e_\mu \le C_1 e_\mu \quad \text{in } \mathbf{R}^N \tag{5.2}$$

with $C_1 = C(\mu) + 1 - \mu^2 > 0$.

From (5.1) and (5.2), we easily obtain the following results.

Lemma 5.1. (i) *There exists a constant $C = C(\mu) > 0$ such that*

$$0 < \frac{e_\mu(x)e_\mu(x-y)}{e_\mu(y)} \le C \quad \text{for } x, y \in \mathbf{R}^N;$$

(ii) *Let $0 < \nu < 1$. Then, for $\tilde{\mu} \in (0, \nu\mu)$, there exists $C = C(\tilde{\mu}, \nu) > 0$ such that $e_\mu^\nu(x) \le Ce_{\tilde{\mu}}(x)$ for $x \in \mathbf{R}^N$;*

(iii) *Let $0 < \mu < 1$. Then, for any $\psi \in H^1(\mathbf{R}^N)$ with $\psi \ge 0$,*

$$\int_{\mathbf{R}^N} (\nabla e_\mu \cdot \nabla \psi + e_\mu \psi)\, dx \le C_1 \int_{\mathbf{R}^N} e_\mu \psi dx$$

for some constant $C_1 = C_1(\mu) > 0$.

For $\mu \in (0,1)$, we define $E_{1,\mu}(x) = E_1(x)/e_\mu(x)$ for $x \in \mathbf{R}^N \setminus \{0\}$.

Lemma 5.2. *For $\mu \in (0,1)$, the following* (i)–(iv) *hold.*

(i) *There exists a constant* $C = C(\mu) > 0$ *such that, for* $u \in \mathcal{S}'$ *with* $u \geq 0$,

$$e_\mu E_1 * u \leq C E_{1,\mu} * [u e_\mu] \quad \textit{a.e. in} \ \ \mathbf{R}^N.$$

(ii) *Let* $q \in (1, N/2)$. *Then there exists a constant* $C = C(q, \mu) > 0$ *such that, for* $u \in L^q(\mathbf{R}^N)$,

$$\|E_{1,\mu} * u\|_{L^r} \leq C \|u\|_{L^q} \quad \textit{with} \ \frac{1}{r} = \frac{1}{q} - \frac{2}{N}.$$

(iii) *Let* $r \in [1, N/(N-2))$. *Then there exists a constant* $C = C(r, \mu) > 0$ *such that*

$$\|E_{1,\mu} * u\|_{L^r} \leq C \|u\|_{L^1} \quad \textit{for} \ u \in L^1(\mathbf{R}^N).$$

(iv) *Let* $r > N/2$. *Then there exists a constant* $C = C(r, \mu) > 0$ *such that*

$$\|E_{1,\mu} * u\|_{L^\infty} \leq C \|u\|_{L^r} \quad \textit{for} \ u \in L^r(\mathbf{R}^N).$$

Proof. For $u \in \mathcal{S}'$ with $u \geq 0$, we have

$$e_\mu(x) E_1 * u(x) = \int_{\mathbf{R}^N} E_{1,\mu}(x-y) \frac{e_\mu(x) e_\mu(x-y)}{e_\mu(y)} u(y) e_\mu(y) dy \leq C E_{1,\mu} * [u e_\mu](x)$$

for some constant $C > 0$. Here we have used the property (i) of Lemma 5.1. Then (i) holds. By the definition of $E_{1,\mu}$ we see that $E_{1,\mu} \leq C E_0$ for some constant $C > 0$. From (iii) of Lemma A.1 in Appendix A, (ii) holds. We find that $E_{1,\mu} \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$. Then, from (ii) of Lemma A.1, we obtain (iii) and (iv). □

For $\kappa \in (0, \kappa^*)$, let $\underline{w}_\kappa \in S_\kappa$ be the minimal solution obtained in Lemma 3.4. Multiplying $(1.11)_\kappa$ by $e_\mu \in H^1(\mathbf{R}^N)$ and putting $\psi = e_\mu$ in $(1.13)_\kappa$, we obtain

(5.3) $$\underline{w}_\kappa(x) e_\mu(x) = e_\mu(x) E_1 * [(\underline{w}_\kappa + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p](x) \quad \text{for } x \in \mathbf{R}^N,$$

and

(5.4) $$\int_{\mathbf{R}^N} (\nabla \underline{w}_\kappa \cdot \nabla e_\mu + \underline{w}_\kappa e_\mu) dx = \int_{\mathbf{R}^N} \left((\underline{w}_\kappa + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p \right) e_\mu dx,$$

respectively. Recall that $q_0 \in (p, N/(N-2))$. We obtain the following lemma.

Lemma 5.3. *Put* $p_0 = q_0/(p-1)$. *For any* $\mu \in (0,1)$, *the following* (i) *and* (ii) *hold.*

(i) $\|\underline{w}_\kappa e_\mu\|_{L^{q_0}} \leq C$ *for all* $\kappa \in (0, \kappa^*)$, *where* $C = C(\mu) > 0$ *is a constant independent of* κ.

(ii) $\|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} e_\mu\|_{L^{p_0}} \leq C$ *for all* $\kappa \in (0, \kappa^*)$, *where* $C = C(\mu) > 0$ *is a constant independent of* κ.

Proof. We will first show that

(5.5) $$\|(\underline{w}_\kappa + U_{j_0}^\kappa)^p e_\mu\|_{L^1} \leq C \quad \text{for all } \kappa \in (0, \kappa^*),$$

where $C = C(\mu) > 0$ is a constant independent of κ. From (5.4) and (iii) of Lemma 5.1, we have

$$\|((\underline{w}_\kappa + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p)e_\mu\|_{L^1} \le C_1 \|\underline{w}_\kappa e_\mu\|_{L^1} \tag{5.6}$$

for the constant $C_1 > 0$ which is independent of κ. By the Young inequality, for any $\varepsilon > 0$, we have

$$\|\underline{w}_\kappa e_\mu\|_{L^1} \le \varepsilon \|\underline{w}_\kappa^p e_\mu\|_{L^1} + C(\varepsilon)\|e_\mu\|_{L^1} \le \varepsilon \|(\underline{w}_\kappa + U_{j_0}^\kappa)^p e_\mu\|_{L^1} + C(\varepsilon)\|e_\mu\|_{L^1}.$$

From (5.6) we obtain

$$(1 - C_1\varepsilon)\|(\underline{w}_\kappa + U_{j_0}^\kappa)^p e_\mu\|_{L^1} \le C_1 C(\varepsilon)\|e_\mu\|_{L^1} + \|(U_{j_0-1}^\kappa)^p e_\mu\|_{L^1}$$

for sufficient small $\varepsilon > 0$. We note here that $U_{j_0-1}^\kappa \le U_{j_0-1}^{\kappa^*}$ for $\kappa \in (0, \kappa^*)$. Then we obtain (5.5) for some constant C independent of κ.

From (5.3) we have

$$\|\underline{w}_\kappa e_\mu\|_{L^{q_0}} = \|e_\mu E_1 * [(\underline{w}_\kappa + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p]\|_{L^{q_0}} \le \|e_\mu E_1 * [(\underline{w}_\kappa + U_{j_0}^\kappa)^p]\|_{L^{q_0}}.$$

Then it follows from (i) and (iii) of Lemma 5.2 that

$$\|\underline{w}_\kappa e_\mu\|_{L^{q_0}} \le C\|E_{1,\mu} * [(\underline{w}_\kappa + U_{j_0}^\kappa)^p e_\mu]\|_{L^{q_0}} \le \tilde{C}\|(\underline{w}_\kappa + U_{j_0}^\kappa)^p e_\mu\|_{L^1}$$

for some positive constants C and $\tilde{C}$. From (5.5) we obtain (i).

From $p_0 = q_0/(p-1)$ it follows that

$$\|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} e_\mu\|_{L^{p_0}} = \|(\underline{w}_\kappa + U_{j_0}^\kappa) e_\mu^{1/(p-1)}\|_{L^{q_0}}^{p-1}.$$

In the case $1 < p \le 2$, from $\|e_\mu\|_{L^\infty} \le 1$, we have

$$\|(\underline{w}_\kappa + U_{j_0}^\kappa) e_\mu^{1/(p-1)}\|_{L^{q_0}} \le \|(\underline{w}_\kappa + U_{j_0}^\kappa) e_\mu\|_{L^{q_0}}.$$

In the case $p > 2$, put $\tilde{\mu} \in (0, \mu/(p-1))$. Then, from (ii) of Lemma 5.1, there exists a constant $C = C(\tilde{\mu}, \kappa) > 0$ such that

$$\|(\underline{w}_\kappa + U_{j_0}^\kappa) e_\mu^{1/(p-1)}\|_{L^{q_0}} \le C\|(\underline{w}_\kappa + U_{j_0}^\kappa) e_{\tilde{\mu}}\|_{L^{q_0}}.$$

From (i) of this lemma, for any $\mu \in (0,1)$ there exists $C = C(\mu) > 0$ such that

$$\|(\underline{w}_\kappa + U_{j_0}^\kappa) e_\mu\|_{L^{q_0}} \le \|\underline{w}_\kappa e_\mu\|_{L^{q_0}} + \|U_{j_0}^\kappa e_\mu\|_{L^{q_0}} \le C + \|U_{j_0}^{\kappa^*} e_\mu\|_{L^{q_0}}.$$

Thus, both in cases $1 < p \le 2$ and $p > 2$, we obtain (ii). □

Proof of Proposition 4.1. Recall that $\{q_j\}$ is the sequence defined by (1.9) with $q_0 \in (p, N/(N-2))$ and j_0 is the integer satisfying (1.10). Define $\{r_j\}$ by (2.14). Take $\{\mu_j\}_{j=0}^{j_0}$ such that

$\mu_{j_0} \in (0,1)$ and $\mu_{j-1} < \mu_j/2$ for $j = 1, 2, \ldots, j_0$. Applying (ii) of Lemma 5.1 with $\nu = 1/2$, there exist positive constants $\tilde{C}_j$, $j = 1, 2, \ldots, j_0$ such that

$$e_{\mu_j}^{1/2} \le \tilde{C}_j e_{\mu_{j-1}} \quad \text{in } \mathbf{R}^N \quad \text{for } j = 1, 2, \ldots, j_0. \tag{5.7}$$

First we will show

$$\|\underline{w}_\kappa e_{\mu_j}\|_{L^{q_j}} \le C_j \quad \text{for } j = 1, 2, \ldots, j_0 - 1, \tag{5.8}$$

where $C_j = C_j(q_j, \mu_j) > 0$ is a constant independent of κ. From (i) of Lemma 5.3 we have $\|\underline{w}_\kappa e_{\mu_0}\|_{L^{q_0}} \le C_0$. Assume that $\|\underline{w}_\kappa e_{\mu_{j-1}}\|_{L^{q_{j-1}}} \le C_{j-1}$ with some $j \in \{1, \ldots, j_0 - 2\}$. From (5.3) and (2.16) we have

$$\underline{w}_\kappa e_{\mu_j} \le e_{\mu_j} E_1 * [p(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}(\underline{w}_\kappa + V_{j_0}^\kappa)] \quad \text{in } \mathbf{R}^N. \tag{5.9}$$

By using (i) and (ii) of Lemma 5.2, we obtain

$$\begin{aligned} \|\underline{w}_\kappa e_{\mu_j}\|_{L^{q_j}} &\le C\|E_{1,\mu_j} * [p(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}(\underline{w}_\kappa + V_{j_0}^\kappa)e_{\mu_j}]\|_{L^{q_j}} \\ &\le C\|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}(\underline{w}_\kappa + V_{j_0}^\kappa)e_{\mu_j}\|_{L^{r_j}}. \end{aligned}$$

From (i) of Lemma A.1 in Appendix A and (5.7), we obtain

$$\begin{aligned} \|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}\underline{w}_\kappa e_{\mu_j}\|_{L^{r_j}} &\le \|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} e_{\mu_j}^{1/2}\|_{L^{p_0}} \|\underline{w}_\kappa e_{\mu_j}^{1/2}\|_{L^{q_{j-1}}} \\ &\le \tilde{C}_j^2 \|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} e_{\mu_{j-1}}\|_{L^{p_0}} \|\underline{w}_\kappa e_{\mu_{j-1}}\|_{L^{q_{j-1}}} \end{aligned} \tag{5.10}$$

and

$$\|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} V_{j_0}^\kappa e_{\mu_j}\|_{L^{r_j}} \le \|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1} e_{\mu_j}\|_{L^{p_0}} \|V_{j_0}^\kappa\|_{L^{q_{j-1}}}. \tag{5.11}$$

From (ii) of Lemma 5.3 and the fact that $V_{j_0}^\kappa \le V_{j_0}^{\kappa^*}$ for $\kappa \in (0, \kappa^*)$, we obtain $\|\underline{w}_\kappa e_{\mu_j}\|_{L^{q_j}} \le C_j$. Thus, by induction, we obtain (5.8).

Note here that $r_{j_0} > N/2$. Then, from (5.9) and (i) and (iv) of Lemma 5.2, we have

$$\begin{aligned} \|\underline{w}_\kappa e_{\mu_{j_0}}\|_{L^\infty} &\le C\|E_{1,\mu} * [(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}(\underline{w}_\kappa + V_{j_0}^\kappa)e_{\mu_{j_0}}\|_{L^\infty} \\ &\le C\|(\underline{w}_\kappa + U_{j_0}^\kappa)^{p-1}(\underline{w}_\kappa + V_{j_0}^\kappa)e_{\mu_{j_0}}\|_{L^{r_{j_0}}}. \end{aligned}$$

By the similar argument as above, we obtain (5.10) and (5.11) with $j = j_0$. By using (5.8) with $j = j_0 - 1$, we obtain $\|\underline{w}_\kappa e_{\mu_{j_0}}\|_{L^\infty} \le C_{j_0}$ for some constant $C_{j_0} > 0$ which is independent of κ. By the fact that $\inf_{x \in \overline{B}_R} e_{\mu_{j_0}} > 0$, we obtain the conclusion of Proposition 4.1. □

6. Existence of second solutions: Proof of Theorem 3

Let $\underline{u}_\kappa$ be the minimal solution of $(1.1)_\kappa$–(1.2) for $\kappa \in (0, \kappa^*)$ obtained in Theorem 1. In order to find a second solution of $(1.1)_\kappa$–(1.2), we introduce the problem

$$\begin{cases} -\Delta v + v = (v + \underline{u}_\kappa)^p - \underline{u}_\kappa^p & \text{in } \mathbf{R}^N, \\ v \in H^1(\mathbf{R}^N) \quad \text{and} \quad v > 0 \text{ a.e. in } \mathbf{R}^N. \end{cases} \tag*{$(6.1)_\kappa$}$$

By Proposition 1.2, $\underline{w}_\kappa = \underline{u}_\kappa - U_{j_0}^\kappa \in H^1(\mathbf{R}^N)$ is positive in $\mathbf{R}^N$ and satisfies $(1.12)_\kappa$. Assume that $(6.1)_\kappa$ has a solution v_κ, and put $\overline{w}_\kappa = v_\kappa + \underline{w}_\kappa$. Then $\overline{w}_\kappa \in H^1(\mathbf{R}^N)$ is positive and satisfies

$$-\Delta \overline{w}_\kappa + \overline{w}_\kappa = (\overline{w}_\kappa + U_{j_0}^\kappa)^p - (U_{j_0-1}^\kappa)^p \quad \text{in } \mathbf{R}^N.$$

Thus, by Proposition 1.2, we can get another positive solution $\overline{u}_\kappa = \overline{w}_\kappa + U_{j_0}^\kappa$ of $(1.1)_\kappa$–(1.2) satisfying $\overline{u}_\kappa > \underline{u}_\kappa$ a.e. in $\mathbf{R}^N$.

We will show the existence of solutions of $(6.1)_\kappa$ by using a variational method. To this end we define the corresponding variational functional of $(6.1)_\kappa$ by

$$I_\kappa(v) = \frac{1}{2}\int_{\mathbf{R}^N} \left(|\nabla v|^2 + v^2\right) dx - \int_{\mathbf{R}^N} G(v, \underline{u}_\kappa) dx$$

with $v \in H^1(\mathbf{R}^N)$, where

$$G(t,s) = \frac{1}{p+1}(t_+ + s)^{p+1} - \frac{1}{p+1}s^{p+1} - s^p t_+. \tag{6.2}$$

From (i) and (ii) of Lemma C.3 in Appendix C, $I_\kappa : H^1(\mathbf{R}^N) \to \mathbf{R}$ is C^1 and the critical point $v_0 \in H^1(\mathbf{R}^N)$ satisfies

$$\int_{\mathbf{R}^N} \left(\nabla v_0 \cdot \nabla \psi + v_0 \psi + g(v_0, \underline{u}_\kappa)\psi\right) dx = 0$$

for any $\psi \in H^1(\mathbf{R}^N)$, where

$$g(t,s) = (t_+ + s)^p - s^p. \tag{6.3}$$

Furthermore, if $v_0 \not\equiv 0$ then $v_0 > 0$ in $\mathbf{R}^N$.

Define $h(t,s)$ and $H(t,s)$ by

$$h(t,s) = g(t,s) - t_+^p \quad \text{and} \quad H(t,s) = G(t,s) - \frac{1}{p+1}t_+^{p+1}, \tag{6.4}$$

respectively. The proof of the following proposition will be given in the next section.

Proposition 6.1. *Let $\{v_n\} \subset H^1(\mathbf{R}^N)$ be a sequence satisfying $v_n \rightharpoonup v_0$ weakly in $H^1(\mathbf{R}^N)$ as $n \to \infty$ for some $v_0 \in H^1(\mathbf{R}^N)$. Then there exists a subsequence, still denoted by $\{v_n\}$, such that the following* (i) *and* (ii) *hold.*

(i) *For any $\psi \in H^1(\mathbf{R}^N)$,*

$$\int_{\mathbf{R}^N} g(v_n, \underline{u}_\kappa)\psi dx \to \int_{\mathbf{R}^N} g(v_0, \underline{u}_\kappa)\psi dx \quad \text{as } n \to \infty. \tag{6.5}$$

(ii) *As $n \to \infty$,*

$$\int_{\mathbf{R}^N} H(v_n, \underline{u}_\kappa) dx \to \int_{\mathbf{R}^N} H(v_0, \underline{u}_\kappa) dx \tag{6.6}$$

and

$$\int_{\mathbf{R}^N} h(v_n, \underline{u}_\kappa) v_n dx \to \int_{\mathbf{R}^N} h(v_0, \underline{u}_\kappa) v_0 dx. \tag{6.7}$$

We will verify the existence of nontrivial solution of $(6.1)_\kappa$ by means of the Mountain Pass Lemma.

Lemma 6.1. *Let* $\{v_n\} \subset H^1(\mathbf{R}^N)$ *be a sequence satisfying* $I'_\kappa(v_n) \to 0$ *as* $n \to \infty$ *in* $H^{-1}(\mathbf{R}^N)$. (i) *Then, for any* $\varepsilon > 0$,

$$\left| \|v_n\|_{H^1}^2 - \int_{\mathbf{R}^N} g(v_n, \underline{u}_\kappa) v_n dx \right| \le \varepsilon \|v_n\|_{H^1} \tag{6.8}$$

with sufficient large n.

(ii) *Assume, in addition, that* $v_n \rightharpoonup v_0$ *weakly in* $H^1(\mathbf{R}^N)$ *as* $n \to \infty$ *for some* $v_0 \in H^1(\mathbf{R}^N)$. *Then*

$$\int_{\mathbf{R}^N} (\nabla v_0 \cdot \nabla \psi + v_0 \psi)\, dx - \int_{\mathbf{R}^N} g(v_0, \underline{u}_\kappa) \psi dx = 0 \tag{6.9}$$

for any $\psi \in H^1(\mathbf{R}^N)$. *Furthermore,* $v_0 \ge 0$ *a.e. in* $\mathbf{R}^N$, *and if* $v_0 \not\equiv 0$ *then* $v_0 > 0$ *a.e. in* $\mathbf{R}^N$.

Proof. (i) Let $\varepsilon > 0$. From $I'_\kappa(v_n) \to 0$ as $n \to \infty$, we have, for sufficient large n,

$$\left| \int_{\mathbf{R}^N} (\nabla v_n \cdot \nabla \psi + v_n \psi)\, dx - \int_{\mathbf{R}^N} g(v_n, \underline{u}_\kappa) \psi dx \right| \le \varepsilon \|\psi\|_{H^1} \tag{6.10}$$

for any $\psi \in H^1(\mathbf{R}^N)$. Putting $\psi = v_n$, we obtain (6.8).

(ii) Let $n \to \infty$ in (6.10). Since (i) of Proposition 6.1 holds and $\varepsilon > 0$ is arbitrarily, we obtain (6.9). From (ii) of Lemma C.3 in Appendix C, we obtain $v_0 \ge 0$ a.e. in $\mathbf{R}^N$ and, if $v_0 \not\equiv 0$, then $v_0 > 0$ a.e. in $\mathbf{R}^N$. □

Lemma 6.2. *Assume that* $\{v_n\}$ *is the Palais-Smale sequence for* $I_\kappa(v)$, *that is,*

$$v_n \in H^1(\mathbf{R}^N), \quad \{I_\kappa(v_n)\} \text{ is bounded,} \quad \text{and} \quad I'_\kappa(v_n) \to 0 \text{ as } n \to \infty \text{ in } H^{-1}(\mathbf{R}^N).$$

Then $\{v_n\}$ *is bounded in* $H^1(\mathbf{R}^N)$.

Proof. Since $\{I_\kappa(v_n)\}$ is bounded, we have

$$\frac{1}{2}\|v_n\|_{H^1}^2 - \int_{\mathbf{R}^N} G(v_n, \underline{u}_\kappa) dx \le M \tag{6.11}$$

for some $M > 0$. Take $\varepsilon > 0$ arbitrarily. From (i) of Lemma 6.1 we obtain

$$\|v_n\|_{H^1}^2 \ge \int_{\mathbf{R}^N} g(v_n, \underline{u}_\kappa) v_n dx - \varepsilon \|v_n\|_{H^1} \tag{6.12}$$

for sufficient large n. Multiply (6.11) by $2 + c_p$ with $c_p = \min\{1, p-1\}$. Then, from (6.12) we obtain

$$\begin{aligned}(2 + c_p)M &\ge \left(1 + \frac{c_p}{2}\right) \|v_n\|_{H^1(\mathbf{R}^N)}^2 - (2 + c_p) \int_{\mathbf{R}^N} G(v_n, \underline{u}_\kappa) dx \\ &\ge \frac{c_p}{2}\|v_n\|_{H^1}^2 + \int_{\mathbf{R}^N} (g(v_n, \underline{u}_\kappa) v_n - (2 + c_p) G(v_n, \underline{u}_\kappa))\, dx - \varepsilon \|v_n\|_{H^1}.\end{aligned}$$

From (iv) of Lemma C.2 in Appendix C and Lemma 3.6, it follows that

$$\begin{aligned}(2+c_p)M &\geq \frac{c_p}{2}\left(\|v_n\|_{H^1}^2 - \int_{\mathbf{R}^N} p\underline{u}_\kappa^{p-1} v_n^2 dx\right) - \varepsilon\|v_n\|_{H^1} \\ &\geq \frac{c_p}{2}\left(1 - \frac{1}{\lambda_1(\kappa)}\right)\|v_n\|_{H^1}^2 - \varepsilon\|v_n\|_{H^1}\end{aligned}$$

with $\lambda_1(\kappa) > 1$. Thus, $\{v_n\}$ is bounded in $H^1(\mathbf{R}^N)$. □

Let us denote by S_{p+1} the best Sobolev constant of the embedding $H^1(\mathbf{R}^N) \subset L^{p+1}(\mathbf{R}^N)$ for $1 \leq p \leq (N+2)/(N-2)$, which is given by

$$S_{p+1} = \inf_{u \in H^1(\mathbf{R}^N)\setminus\{0\}} \frac{\|u\|_{H^1}^2}{\|u\|_{L^{p+1}}^2}. \tag{6.13}$$

It is known that the infimum is achieved by some positive function $u_0 \in H^1(\mathbf{R}^N)$ for $1 < p < (N+2)/(N-2)$, namely, $S_{p+1}\|u_0\|_{L^{p+1}}^2 = \|u_0\|_{H^1}^2$. For simplicity, put

$$S_{p+1}^* = \left(\frac{1}{2} - \frac{1}{p+1}\right) S_{p+1}^{(p+1)/(p-1)}.$$

Lemma 6.3. *The functional I_κ satisfies the $(PS)_c$ condition for $c \in (0, S_{p+1}^*)$, that is, any sequence $\{v_n\} \subset H^1(\mathbf{R}^N)$ such that*

$$I_\kappa(v_n) \to c \in (0, S_{p+1}^*) \quad \textit{and} \quad I_\kappa'(v_n) \to 0 \quad \textit{as } n \to \infty$$

contains a convergent subsequence.

Proof. Lemma 6.2 implies that $\{v_n\}$ is bounded in $H^1(\mathbf{R}^N)$. Then, there exist a subsequence, still denoted by $\{v_n\}$, and some $v_0 \in H^1(\mathbf{R}^N)$ such that

$$\begin{aligned} &v_n \rightharpoonup v_0 && \text{weakly in } H^1(\mathbf{R}^N) \text{ as } n \to \infty, \\ &v_n \to v_0 && \text{strongly in } L^2_{\text{loc}}(\mathbf{R}^N) \text{ as } n \to \infty, \\ &v_n \to v_0 && \text{a.e. in } \mathbf{R}^N \text{ as } n \to \infty. \end{aligned}$$

From (i) of Lemma 6.1 we have

$$\|v_n\|_{H^1}^2 - \int_{\mathbf{R}^N} g(v_n, \underline{u}_\kappa) v_n dx \to 0 \quad \text{as } n \to \infty. \tag{6.14}$$

Furthermore, from (ii) of Lemma 6.1, we obtain (6.9) and $v_0 \geq 0$ a.e. in $\mathbf{R}^N$. Putting $\psi = v_0$ in (6.9), we have

$$\|v_0\|_{H^1}^2 - \int_{\mathbf{R}^N} g(v_0, \underline{u}_\kappa) v_0 dx = 0. \tag{6.15}$$

Then, from (ii) of Lemma C.2, we obtain

$$I_\kappa(v_0) = \frac{1}{2}\int_{\mathbf{R}^N} g(v_0, \underline{u}_\kappa) v_0 dx - \int_{\mathbf{R}^N} G(v_0, \underline{u}_\kappa) dx \geq 0. \tag{6.16}$$

It should be mentioned that (6.14) and $I_\kappa(v_n) \to c$ can be written by, respectively,

$$\|v_n\|_{H^1}^2 - \|(v_n)_+\|_{L^{p+1}}^{p+1} - \int_{\mathbf{R}^N} h(v_n, \underline{u}_\kappa) v_n dx = o(1) \tag{6.17}$$

and

$$\frac{1}{2}\|v_n\|_{H^1}^2 - \frac{1}{p+1}\|(v_n)_+\|_{L^{p+1}}^{p+1} - \int_{\mathbf{R}^N} H(v_n, \underline{u}_\kappa) dx = c + o(1). \tag{6.18}$$

Now we will show that $v_n \to v_0$ strongly in $H^1(\mathbf{R}^N)$. Set $w_n = v_n - v_0$. Then

$$\begin{aligned} w_n &\rightharpoonup 0 \qquad \text{weakly in } H^1(\mathbf{R}^N) \text{ as } n \to \infty, \\ w_n &\to 0 \qquad \text{strongly in } L^2_{\text{loc}}(\mathbf{R}^N) \text{ as } n \to \infty, \\ w_n &\to 0 \qquad \text{a.e. in } \mathbf{R}^N \text{ as } n \to \infty. \end{aligned}$$

Then it follows that

$$\|v_n\|_{H^1}^2 = \|v_0\|_{H^1}^2 + \|w_n\|_{H^1}^2 + o(1). \tag{6.19}$$

Put $\tilde{w}_n = (v_n)_+ - v_0$. From $v_0 \ge 0$ we have $w_n \le \tilde{w}_n \le (w_n)_+$, and then $\tilde{w}_n \to 0$ a.e. in $\mathbf{R}^N$. By the Brezis-Lieb lemma [8] we have

$$\|(v_n)_+\|_{L^{p+1}}^{p+1} = \|v_0\|_{L^{p+1}}^{p+1} + \|\tilde{w}_n\|_{L^{p+1}}^{p+1} + o(1). \tag{6.20}$$

Substituting (6.19) and (6.20) into (6.18) and using (ii) of Proposition 6.1, we deduce that

$$I_\kappa(v_0) + \frac{1}{2}\|w_n\|_{H^1}^2 - \frac{1}{p+1}\|\tilde{w}_n\|_{L^{p+1}}^{p+1} = c + o(1). \tag{6.21}$$

Substituting (6.19) and (6.20) into (6.17) and using (ii) of Proposition 6.1, we have

$$\|v_0\|_{H^1}^2 - \int_{\mathbf{R}^N} g(v_0, \underline{u}_\kappa) v_0 dx + \|w_n\|_{H^1}^2 - \|\tilde{w}_n\|_{L^{p+1}}^{p+1} = o(1).$$

From (6.15) we obtain $\|w_n\|_{H^1}^2 - \|\tilde{w}_n\|_{L^{p+1}}^{p+1} = o(1)$. Since $\{w_n\}$ is bounded in $H^1(\mathbf{R}^N)$, we may assume that

$$\|w_n\|_{H^1}^2 \to \ell \quad \text{and} \quad \|\tilde{w}_n\|_{L^{p+1}}^{p+1} \to \ell$$

for some $\ell \ge 0$. By the definition of S_{p+1} and the fact that $\|\tilde{w}_n\|_{L^{p+1}}^{p+1} \le \|w_n\|_{L^{p+1}}^{p+1}$, we obtain $S_{p+1}\ell^{2/(p+1)} \le \ell$. We assume here that $\ell > 0$. Then $\ell \ge S_{p+1}^{(p+1)/(p-1)}$. Letting $n \to \infty$ in (6.21), we have

$$I_\kappa(v_0) = c - \left(\frac{1}{2} - \frac{1}{p+1}\right)\ell \le c - S_{p+1}^* < 0.$$

This contradicts (6.16). We therefore obtain $\ell = 0$. Thus $w_n \to 0$ in $H^1(\mathbf{R}^N)$, which implies that $v_n \to v_0$ strongly in $H^1(\mathbf{R}^N)$. This completes the proof. □

Lemma 6.4. *There exist some constants $\rho > 0$ and $\theta > 0$ such that $I_\kappa(v) \ge \theta > 0$ for all $v \in H^1(\mathbf{R}^N)$ satisfying $\|v\|_{H^1} = \rho$.*

Proof. For any $v \in H^1(\mathbf{R}^N)$ we have

$$I_\kappa(v) = \frac{1}{2}\left(\|v\|_{H^1}^2 - \int_{\mathbf{R}^N} p\underline{u}_\kappa^{p-1}v^2dx\right) - \int_{\mathbf{R}^N}\left(G(v,\underline{u}_\kappa) - \frac{p}{2}\underline{u}_\kappa^{p-1}v^2\right)dx \equiv J_1 - J_2.$$

From (iii) of Lemma C.2, for any $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that

$$J_2 \le \varepsilon \int_{\mathbf{R}^N} \underline{u}_\kappa^{p-1}v^2dx + C\|v\|_{L^{p+1}}^{p+1}.$$

From Lemma 3.6 and the Sobolev inequality, we have

$$J_1 \ge \frac{1}{2}\left(1 - \frac{1}{\lambda_1(\kappa)}\right)\|v\|_{H^1}^2 \quad \text{and} \quad J_2 \le \frac{\varepsilon}{p\lambda_1(\kappa)}\|v\|_{H^1}^2 + C\|v\|_{H^1}^{p+1}$$

with $\lambda_1(\kappa) > 1$. Thus, for $\varepsilon > 0$ sufficient small, we obtain

$$I_\kappa(v) \ge C_1\|v\|_{H^1}^2 - C_2\|v\|_{H^1}^{p+1}$$

with some constants $C_1, C_2 > 0$. This implies that there are positive constants ρ and θ such that $I_\kappa(v) \ge \theta$ holds for all $v \in H^1(\mathbf{R}^N)$ with $\|v\|_{H^1} = \rho$. □

Lemma 6.5. *Assume that $u_0 \in H^1(\mathbf{R}^N)$ attains the infimum* (6.13)*, that is, $S_{p+1}\|u_0\|_{L^2}^2 = \|u_0\|_{H^1}^2$. Then $I_\kappa(tu_0) \to -\infty$ as $t \to \infty$ and* $\sup_{t>0} I_\kappa(tu_0) < S_{p+1}^*$.

Proof. From (ii) of Lemma C.2, we have

$$I_\kappa(tu_0) < \frac{t^2}{2}\|u_0\|_{H^1}^2 - \frac{t^{p+1}}{p+1}\|u_0\|_{L^{p+1}}^{p+1} \quad \text{for any } t > 0.$$

Here we use the properties $\underline{u}_\kappa, u_0 > 0$ in $\mathbf{R}^N$. Then it is clear that $I_\kappa(tu_0) \to -\infty$ as $t \to \infty$. We observe that

$$\sup_{t>0}\left(\frac{t^2}{2}\|u_0\|_{H^1}^2 - \frac{t^{p+1}}{p+1}\|u_0\|_{L^{p+1}}^{p+1}\right) = \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{\|u_0\|_{H^1}^2}{\|u_0\|_{L^{p+1}}^2}\right)^{(p+1)/(p-1)} = S_{p+1}^*.$$

Then we obtain $\sup_{t>0} I_\kappa(tu_0) < S_{p+1}^*$. □

Proof of Theorem 3. Let $u_0 \in H^1(\mathbf{R}^N)$ be the function which attains the infimum (6.13). From Lemma 6.5 there is a constant $T > 0$ such that $e = Tu_0$ satisfies $\|e\|_{H^1} > \rho$ and $I_\kappa(e) \le 0$, where ρ is the constant in Lemma 6.4. Denote

$$c = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I_\kappa(\gamma(s)),$$

where $\Gamma = \{\gamma \in C([0,1]; H^1(\mathbf{R}^N)) : \gamma(0) = 0,\ \gamma(1) = e\}$. Then, from Lemmas 6.4 and 6.5, it follows that $0 < \theta \le c < S_{p+1}^*$. By the Mountain Pass Lemma without the (PS) condition ([4, 10]), there exists a sequence $\{v_n\} \subset H^1(\mathbf{R}^N)$ such that $I_\kappa(v_n) \to c$ and $I'_\kappa(v_n) \to 0$ as $n \to \infty$. By Lemma 6.3 there exists a subsequence which converges to some v_κ in $H^1(\mathbf{R}^N)$. Then v_κ is a nontrivial critical point of I_κ, and hence v_κ solves the problem $(6.1)_\kappa$. As has been mentioned above, we obtain a positive solution $\overline{u}_\kappa$ of $(1.1)_\kappa$–(1.2) satisfying $\overline{u}_\kappa > \underline{u}_\kappa$ a.e. in $\mathbf{R}^N$. □

7. Proof of Proposition 6.1

In this section we will show Proposition 6.1. By the definitions of $g(t,s)$, $h(t,s)$, and $H(t,s)$, we may assume that $v_n \geq 0$ and $v_0 \geq 0$ a.e. in $\mathbf{R}^N$ throughout this section.

By Proposition 1.1 we see that $\underline{u}_\kappa \in C^2(\mathbf{R}^N \setminus \{a_i\}_{i=1}^m)$ and $\underline{u}_\kappa(x) = O(E_1(x))$ as $|x| \to \infty$. Take $R_0 > 0$ such that (2.5) holds. Then $\underline{u}_\kappa \in L^r(\mathbf{R}^N \setminus B_{R_0})$ for all $r \in [1,\infty]$ and there exists a constant $C_0 > 0$ such that

$$\underline{u}_\kappa(x) \leq C_0 \quad \text{for } |x| \geq R_0. \tag{7.1}$$

In this section we denote by C_r, for $r \in [2, 2N/(N-2)]$, the constant in the Sobolev inequality (A.1) in Appendix A.

First, we show the following lemma.

Lemma 7.1. *Assume that $\{v_n\}$ satisfies the assumption in Proposition 6.1, and let $\psi \in H^1(\mathbf{R}^N)$.*

(i) *For any $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{\mathbf{R}^N \setminus B_R} g(v_n, \underline{u}_\kappa)|\psi| dx \leq \varepsilon. \tag{7.2}$$

(ii) *For any $R > 0$, we have*

$$\int_{B_R} g(v_n, \underline{u}_\kappa)\psi dx \to \int_{B_R} g(v_0, \underline{u}_\kappa)\psi dx \quad \text{as } n \to \infty. \tag{7.3}$$

Proof. (i) For any $\tilde{\varepsilon} > 0$ we can take $\widetilde{R} \geq R_0$ such that

$$\left(\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} |\psi|^2 dx \right)^{1/2} \leq \tilde{\varepsilon}.$$

From (i) of Lemma C.2 in Appendix C, we have

$$\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} g(v_n, \underline{u}_\kappa)\psi dx \leq C \int_{\mathbf{R}^N \setminus B_{\tilde{R}}} \left(v_n^p + \underline{u}_\kappa^{p-1} v_n \right) \psi dx$$

with some constant $C > 0$. Note that $2p \in (2, 2N/(N-2))$. By the Hölder inequality and the Sobolev inequality, we obtain

$$\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} v_n^p \psi dx \leq \left(\int_{\mathbf{R}^N} v_n^{2p} dx \right)^{1/2} \left(\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} \psi^2 dx \right)^{1/2} \leq C_{2p}^p \tilde{\varepsilon} \|v_n\|_{H^1}^p.$$

From (7.1) and the Hölder inequality, it follows that

$$\int_{\mathbf{R}^N\setminus B_{\tilde R}} \underline{u}_\kappa^{p-1} v_n\psi dx \le C_0^{p-1}\left(\int_{\mathbf{R}^N} v_n^2 dx\right)^{1/2}\left(\int_{\mathbf{R}^N\setminus B_{\tilde R}} \psi^2 dx\right)^{1/2} \le C_0^{p-1}\tilde\varepsilon\|v_n\|_{H^1}.$$

Since $\{v_n\}$ is bounded in $H^1(\mathbf{R}^N)$ by the assumption, we obtain

$$\int_{\mathbf{R}^N\setminus B_{\tilde R}} g(v_n,\underline{u}_\kappa)|\psi|dx \le \widetilde{C}\tilde\varepsilon,$$

for some constant $\widetilde{C}>0$ which is independent on n and $\tilde\varepsilon$. For any $\varepsilon>0$, take $\tilde\varepsilon=\varepsilon/\widetilde{C}$. Then (7.2) holds.

(ii) By the assumption, $\{v_n\}$ is bounded in $H^1(B_R)$. Then, there exists a subsequence, still denoted by $\{v_n\}$, and $V\in L^{2p}(B_R)$ such that

$$\begin{aligned} &v_n\to v_0 \quad \text{strongly in } L^{2p}(B_R),\\ &v_n\to v_0 \quad \text{a.e. in } B_R, \quad \text{and} \quad v_n\le V \quad \text{a.e. in } B_R. \end{aligned}$$

From (i) of Lemma C.2 it follows that

$$g(v_n,\underline{u}_\kappa)|\psi| \le C(v_n^p+\underline{u}_\kappa^{p-1}v_n)|\psi| \le C(V^p+\underline{u}_\kappa^{p-1}V)|\psi|.$$

We observe that

$$\int_{B_R} V^p|\psi|dx \le \left(\int_{B_R} V^{2p}dx\right)^{1/2}\left(\int_{B_R}\psi^2dx\right)^{1/2}<\infty,$$

$$\int_{B_R}\underline{u}_\kappa^{p-1}V|\psi|dx \le \left(\int_{B_R}\underline{u}_\kappa^p dx\right)^{(p-1)/p}\left(\int_{B_R}V^{2p}dx\right)^{1/2p}\left(\int_{B_R}\psi^{2p}dx\right)^{1/2p}<\infty,$$

and that $g(v_n,\underline{u}_\kappa)\psi \to g(v_0,\underline{u}_\kappa)\psi$ for a.e. in B_R as $n\to\infty$. By the Lebesgue convergence theorem, we obtain (7.3). □

Proof of (i) *of Proposition* 6.1. For any $\varepsilon>0$ there exists $R>0$ such that (7.2) holds. Observe that

$$\begin{aligned} &\left|\int_{\mathbf{R}^N} g(v_n,\underline{u}_\kappa)\psi dx - \int_{\mathbf{R}^N} g(v_0,\underline{u}_\kappa)\psi dx\right|\\ \le\ &\int_{\mathbf{R}^N\setminus B_R} g(v_n,\underline{u}_\kappa)|\psi|dx + \int_{\mathbf{R}^N\setminus B_R} g(v_0,\underline{u}_\kappa)|\psi|dx + \left|\int_{B_R}(g(v_n,\underline{u}_\kappa)-g(v_0,\underline{u}_\kappa))\psi dx\right|. \end{aligned}$$

From (i) and (ii) of Lemma 7.1 we have

$$\limsup_{n\to\infty}\left|\int_{\mathbf{R}^N} g(v_n,\underline{u}_\kappa)\psi dx - \int_{\mathbf{R}^N} g(v_0,\underline{u}_\kappa)\psi dx\right| \le 2\varepsilon.$$

Since $\varepsilon>0$ is arbitrarily, we obtain (6.5). □

To prove (ii) of Proposition 6.1, we need the following lemma.

Lemma 7.2. *Assume that $\{v_n\}$ satisfies the assumption in Proposition* 6.1*.*

(i) *For any $\varepsilon > 0$, there exists $R > 0$ such that*

$$\int_{\mathbf{R}^N \setminus B_R} H(v_n, \underline{u}_\kappa)dx \le \varepsilon \quad \textit{and} \quad \int_{\mathbf{R}^N \setminus B_R} h(v_n, \underline{u}_\kappa)v_n dx \le \varepsilon. \tag{7.4}$$

(ii) *For any $R > 0$, we have*

$$\int_{B_R} H(v_n, \underline{u}_\kappa)dx \to \int_{B_R} H(v_0, \underline{u}_\kappa)dx \tag{7.5}$$

and

$$\int_{B_R} h(v_n, \underline{u}_\kappa)v_n dx \to \int_{B_R} h(v_0, \underline{u}_\kappa)v_0 dx \tag{7.6}$$

as $n \to \infty$.

Proof. (i) Recall that $\underline{u}_\kappa \in L^r(\mathbf{R}^N \setminus B_{R_0})$ for all $r \in [1, \infty]$. Then, for any $\tilde{\varepsilon} > 0$ there exists $\widetilde{R} \ge R_0$ such that

$$\left(\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} \underline{u}_\kappa^2 dx\right)^{1/2} \le \tilde{\varepsilon} \quad \text{and} \quad \left(\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} \underline{u}_\kappa^{2p} dx\right)^{1/2} \le \tilde{\varepsilon}. \tag{7.7}$$

From (C.7) and (C.9) in Appendix C, we have

$$\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} H(v_n, \underline{u}_\kappa)dx \le C \int_{\mathbf{R}^N \setminus B_{\tilde{R}}} (v_n^p \underline{u}_\kappa + v_n \underline{u}_\kappa^p)\, dx$$

and

$$\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} h(v_n, \underline{u}_\kappa)v_n dx \le C \int_{\mathbf{R}^N \setminus B_{\tilde{R}}} (v_n^p \underline{u}_\kappa + v_n \underline{u}_\kappa^p)\, dx,$$

respectively. It follows from the Hölder inequality, the Sobolev inequality, and (7.7) that

$$\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} v_n^p \underline{u}_\kappa dx \le \left(\int_{\mathbf{R}^N} v_n^{2p} dx\right)^{1/2} \left(\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} \underline{u}_\kappa^2 dx\right)^{1/2} \le C_{2p}^p \tilde{\varepsilon} \|v_n\|_{H^1}^p,$$

$$\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} v_n \underline{u}_\kappa^p dx \le \left(\int_{\mathbf{R}^N} v_n^2 dx\right)^{1/2} \left(\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} \underline{u}_\kappa^{2p} dx\right)^{1/2} \le \tilde{\varepsilon} \|v_n\|_{H^1(\mathbf{R}^N)}.$$

Since $\{v_n\}$ is bounded in $H^1(\mathbf{R}^N)$ by the assumption, we obtain

$$\int_{\mathbf{R}^N \setminus B_{\tilde{R}}} H(v_n, \underline{u}_\kappa)dx \le \widetilde{C}\tilde{\varepsilon} \quad \text{and} \quad \int_{\mathbf{R}^N \setminus B_{\tilde{R}}} h(v_n, \underline{u}_\kappa)v_n dx \le \widetilde{C}\tilde{\varepsilon}$$

for some positive constant $\widetilde{C}$ which is independent of n and $\tilde{\varepsilon}$. For any $\varepsilon > 0$, put $\tilde{\varepsilon} = \varepsilon/\widetilde{C}$. Then (7.4) holds.

(ii) There exist a subsequence, still denoted by $\{v_n\}$, and $V \in L^{p+1}(B_R) \cap L^{2p}(B_R)$ such that

$$\begin{aligned} &v_n \to v_0 \quad \text{strongly in } L^{p+1}(B_R) \cap L^{2p}(B_R), \\ &v_n \to v_0 \quad \text{a.e. in } \ B_R, \quad \text{and} \quad v_n \le V \quad \text{a.e. in } \ B_R. \end{aligned}$$

Then $H(v_n, \underline{u}_\kappa) \to H(v_0, \underline{u}_\kappa)$ and $h(v_n, \underline{u}_\kappa)v_n \to h(v_0, \underline{u}_\kappa)v_0$ for a.e. in B_R as $n \to \infty$. From (C.6) and (C.8) in Appendix C, we have

$$H(v_n, \underline{u}_\kappa) \le C(v_n^{p+1} + \underline{u}_\kappa^{p-1} v_n^2) \le C(V^{p+1} + \underline{u}_\kappa^{p-1} V^2)$$

and

$$h(v_n, \underline{u}_\kappa)v_n \le C(v_n^{p+1} + \underline{u}_\kappa^{p-1} v_n^2) \le C(V^{p+1} + \underline{u}_\kappa^{p-1} V^2),$$

respectively. By the Hölder inequality, we obtain

$$\int_{B_R} \underline{u}_\kappa^{p-1} V^2 dx \le \left(\int_{B_R} \underline{u}_\kappa^p dx \right)^{(p-1)/p} \left(\int_{B_R} V^{2p} dx \right)^{1/p} < \infty.$$

By the Lebesgue convergence theorem, we obtain (7.5) and (7.6). □

Proof of (ii) *of Proposition* 6.1. For any $\varepsilon > 0$ there exists $R > 0$ such that (7.4) holds. Observe that

$$\begin{aligned} &\left| \int_{\mathbf{R}^N} H(v_n, \underline{u}_\kappa) dx - \int_{\mathbf{R}^N} H(v_0, \underline{u}_\kappa) dx \right| \\ \le\ & \int_{\mathbf{R}^N \setminus B_R} H(v_n, \underline{u}_\kappa) dx + \int_{\mathbf{R}^N \setminus B_R} H(v_0, \underline{u}_\kappa) dx + \left| \int_{B_R} (H(v_n, \underline{u}_\kappa) - H(v_0, \underline{u}_\kappa)) dx \right|. \end{aligned}$$

As a consequence of (i) and (ii) of Lemma 7.2, we obtain

$$\limsup_{n\to\infty} \left| \int_{\mathbf{R}^N} H(v_n, \underline{u}_\kappa) dx - \int_{\mathbf{R}^N} H(v_0, \underline{u}_\kappa) dx \right| \le 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily, we obtain (6.6). By a similar argument, we obtain (6.7). □

Appendix A: Inequalities

For convenience, we list below the basic inequalities, which will be used in this paper. We assume $N \ge 3$ and denote by E_0 the fundamental solution for $-\Delta$ in $\mathbf{R}^N$. For the proof, see, e.g., [19], [22], [17].

Lemma A.1. (i) (Hölder's inequality) *For $v \in L^p(\mathbf{R}^N)$ and $w \in L^q(\mathbf{R}^N)$ with $p, q \in [1, \infty]$,*

$$\|vw\|_{L^r} \le \|v\|_{L^p} \|w\|_{L^q}, \quad \textit{where } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

(ii) (Hausdorff-Young's inequality) *For $v \in L^p(\mathbf{R}^N)$ and $w \in L^q(\mathbf{R}^N)$ with $p, q \in [1, \infty]$,*

$$\|v * w\|_{L^r} \le \|v\|_{L^p} \|w\|_{L^q}, \quad \textit{where } \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

In particular, for $v \in L^p(\mathbf{R}^N)$ and $w \in L^q(\mathbf{R}^N)$ with $p, q \in [1, \infty]$, $1/p + 1/q = 1$,

$$\|v * w\|_{L^\infty} \le \|v\|_{L^p} \|w\|_{L^q}.$$

(iii) (Hardy-Littlewood Sobolev's inequality) *For* $q \in (1, N/2)$ *there exists a constant* $C_q > 0$ *such that, for* $v \in L^q(\mathbf{R}^N)$,

$$\|E_0 * v\|_{L^r} \le C_q \|v\|_{L^q} \quad \text{where } \frac{1}{r} = \frac{1}{q} - \frac{2}{N}.$$

(iv) (Sobolev inequality) *For all* $r \in [2, 2N/(N-2)]$ *there exists a constant* $C_r > 0$ *such that*

$$\|v\|_{L^r} \le C_r \|v\|_{H^1} \quad \text{for } v \in H^1(\mathbf{R}^N). \tag{A.1}$$

In this paper, we will use the following result. For reader's convenience, we include a proof.

Lemma A.2. (i) *Assume that* $v \in L^p(\mathbf{R}^N)$ *and* $w \in L^q(\mathbf{R}^N)$ *with* $p, q \in (1, \infty)$, $1/p + 1/q = 1$. *Then* $v * w \in C_0(\mathbf{R}^N)$.

(ii) *Assume that* $v \in L^1(\mathbf{R}^N)$ *and* $w \in L^\infty(\mathbf{R}^N)$. *Then* $v * w \in C(\mathbf{R}^N)$. *Assume, in addition, that* $\|w\|_{L^\infty(\mathbf{R}^N \setminus B_R)} \to 0$ *as* $R \to \infty$. *Then* $v * w \in C_0(\mathbf{R}^N)$.

Proof. (i) From $v \in L^p(\mathbf{R}^N)$ we have $\|\tau_h v - v\|_{L^p} \to 0$ as $|h| \to 0$, where $\tau_h v(x) = v(x+h)$. By the Hölder inequality,

$$\begin{aligned} |v * w(x+h) - v * w(x)| &\le \int_{\mathbf{R}^N} |(v(x+h-y) - v(x-y))w(y)|\, dy \\ &\le \|\tau_h v - v\|_{L^p} \|w\|_{L^q} \to 0 \end{aligned}$$

as $|h| \to 0$. This implies that $v * w \in C(\mathbf{R}^N)$. For $|x| > 2R$ with $R > 0$, we have

$$|v * w(x)| \le \int_{\mathbf{R}^N \setminus B_R} |v(x-y)w(y)| dy + \int_{B_R} |v(x-y)w(y)| dy.$$

By the Hölder inequality, it follows that

$$\int_{\mathbf{R}^N \setminus B_R} |v(x-y)w(y)| dy \le \|v\|_{L^p(\mathbf{R}^N)} \|w\|_{L^q(\mathbf{R}^N \setminus B_R)} \to 0 \quad \text{as } R \to \infty.$$

From $|x| > 2R$, we have $x - y \in \mathbf{R}^N \setminus B_R$ for $y \in B_R$. Then

$$\int_{B_R} |v(x-y)w(y)| dy \le \|v\|_{L^p(\mathbf{R}^N \setminus B_R)} \|w\|_{L^q(\mathbf{R}^N)} \to 0 \quad \text{as } R \to \infty.$$

Thus $v * w(x) \to 0$ as $|x| \to \infty$, and hence $v * w \in C_0(\mathbf{R}^N)$.

(ii) Following the argument in the proof of (i), we obtain (ii). □

Let E_1 denote the fundamental solution for $-\Delta + I$ in $\mathbf{R}^N$. We see that $E_1 \in L^r(\mathbf{R}^N)$ for all $r \in [1, N/(N-2))$ and $E_1 \le E_0$. As a consequence of Lemmas A.1 and A.2 we obtain the following lemma.

Lemma A.3. (i) For $r \in [1, N/(N-2))$ there exists a constant $C_r > 0$ such that

$$\|E_1 * v\|_{L^r} \le C_r \|v\|_{L^1} \quad \text{for } v \in L^1(\mathbf{R}^N).$$

(ii) For $q \in (1, N/2)$ there exists a constant $C_q > 0$ such that, for $v \in L^q(\mathbf{R}^N)$,

$$\|E_1 * v\|_{L^r} \le C_q \|v\|_{L^q} \quad \text{where } \frac{1}{r} = \frac{1}{q} - \frac{2}{N}.$$

(iii) Let $v \in L^q(\mathbf{R}^N)$ with some $q > N/2$. Then $E_1 * v \in C_0(\mathbf{R}^N)$.

(iv) Let $v \in (L^\infty + L^q)(\mathbf{R}^N)$ with some $q > N/2$. Then $E_1 * v \in C(\mathbf{R}^N)$.

Lemma A.4. *Assume that $v \in (L^1 \cap L^q)(\mathbf{R}^N)$ with some $q > N/2$. Then $E_1 * v \in (C_0 \cap L^1 \cap H^1)(\mathbf{R}^N)$.*

Proof. Put $w = E_1 * v$. From (i) and (iii) of Lemma A.3, we have $w \in (L^1 \cap C_0)(\mathbf{R}^N)$, and hence $w \in L^2(\mathbf{R}^N)$. By the assumption, we obtain $v \in L^r(\mathbf{R}^N)$ for all $r \in [1, q]$. In particular, from $N/2 > 2N/(N+2)$ for $N > 2$, we have $v \in L^{2N/(N+2)}(\mathbf{R}^N)$. Put $w = E_1 * v$. Then

$$-\Delta w + w = v \quad \text{in } \mathcal{D}'(\mathbf{R}^N).$$

By employing the L^q-estimate and the Sobolev embedding, we obtain $w \in W^{2,2N/(N+2)}(\mathbf{R}^N)$ and $\nabla w \in L^2(\mathbf{R}^N)$. Thus $w = E_1 * v \in H^1(\mathbf{R}^N)$. □

Appendix B: Eigenvalue problems

Let us consider the eigenvalue problem

$$\text{(B.1)} \qquad \begin{cases} -\Delta\phi + \phi = \lambda a(x)\phi & \text{in } \mathbf{R}^N, \\ \phi \in H^1(\mathbf{R}^N), \end{cases}$$

where $a \in L^{N/2}(\mathbf{R}^N) \cap L^q(\mathbf{R}^N)$ with some $q > N/2$ and $a > 0$ a.e. in $\mathbf{R}^N$.

First we show the following lemma.

Lemma B.1. *Assume that $\{\psi_n\} \subset H^1(\mathbf{R}^N)$ is a sequence such that $\psi_n \rightharpoonup \psi_0$ weakly in $H^1(\mathbf{R}^N)$ as $n \to \infty$. Then*

$$\text{(B.2)} \qquad \int_{\mathbf{R}^N} a\psi_n^2 dx \to \int_{\mathbf{R}^N} a\psi_0^2 dx \quad \text{as } n \to \infty.$$

Proof. From $a \in L^{N/2}(\mathbf{R}^N)$, for any $\varepsilon > 0$ there exists $R > 0$ such that

$$\left(\int_{|x|\ge R} a^{N/2} dx\right)^{2/N} < \varepsilon.$$

By the Hölder inequality and the Sobolev inequality, we obtain

$$\int_{|x|\ge R} a\psi_n^2 dx \le \left(\int_{|x|\ge R} a^{N/2} dx\right)^{2/N} \|\psi_n\|^2_{L^{2N/(N-2)}} \le \varepsilon C \|\psi_n\|^2_{H^1},$$

where $C > 0$ is the constant independent of n. By the assumption, $\{\psi_n\}$ is bounded in $H^1(\mathbf{R}^N)$. Then

$$\int_{|x|\geq R} a\psi_n^2 dx \leq C\varepsilon \quad \text{and} \quad \int_{|x|\geq R} a\psi_0^2 dx \leq C\varepsilon, \tag{B.3}$$

where the constant $C > 0$ is independent of n and $\varepsilon > 0$.

Put $q' = q/(q-1)$. Note that $q' \in (1, N/(N-2))$. Then we can take a subsequence, still denoted by $\{\psi_n\}$, and $\Psi \in L^{2q'}(B_R)$ such that

$$\psi_n \to \psi_0 \quad \text{strongly in } L^{2q'}(B_R) \quad \text{and} \quad |\psi_n| \leq \Psi \text{ a.e. in } B_R.$$

By the Hölder inequality, it follows that

$$\int_{B_R} a\psi_n^2 dx \leq \int_{B_R} a\Psi^2 dx \leq \left(\int_{B_R} a^q dx\right)^{1/q} \left(\int_{B_R} \Psi^{2q'} dx\right)^{1/q'} < \infty.$$

By the Lebesgue convergence theorem we have

$$\int_{B_R} a\psi_n^2 dx \to \int_{B_R} a\psi_0^2 dx \quad \text{as } n \to \infty. \tag{B.4}$$

Observe that

$$\left|\int_{\mathbf{R}^N} a\psi_n^2 dx - \int_{\mathbf{R}^N} a\psi_0^2 dx\right| \leq \int_{|x|\geq R} a\psi_n^2 dx + \int_{|x|\geq R} a\psi_0^2 dx + \left|\int_{B_R} (a\psi_n^2 - a\psi_0^2) dx\right|.$$

As a consequence of (B.3) and (B.4), we obtain

$$\limsup_{n\to\infty} \left|\int_{\mathbf{R}^N} a\psi_n^2 dx - \int_{\mathbf{R}^N} a\psi_0^2 dx\right| \leq C\varepsilon$$

with some constant $C > 0$. Since $\varepsilon > 0$ is arbitrarily, we obtain (B.2). □

Lemma B.2. *The problem* (B.1) *has the first eigenvalue* $\lambda_1 > 0$ *and the corresponding eigenfunction* $\phi_1 > 0$ *a.e. in* $\mathbf{R}^N$. *Furthermore, there holds*

$$\|\psi\|_{H^1}^2 \geq \lambda_1 \int_{\mathbf{R}^N} a\psi^2 dx \tag{B.5}$$

for any $\psi \in H^1(\mathbf{R}^N) \setminus \{0\}$.

Proof. We consider the minimization problem

$$\lambda_1 = \inf\left\{\|\psi\|_{H^1}^2 : \psi \in H^1(\mathbf{R}^N),\ \int_{\mathbf{R}^N} a\psi^2 dx = 1\right\}.$$

Let $\{\psi_n\} \subset H^1(\mathbf{R}^N)$ be a minimizing sequence of λ_1, that is,

$$\int_{\mathbf{R}^N} a\psi_n^2 dx = 1 \quad \text{and} \quad \|\psi_n\|_{H^1}^2 \to \lambda_1 \quad \text{as } n \to \infty.$$

Since $\{\psi_n\}$ is bounded in $H^1(\mathbf{R}^N)$, there exists a subsequence, still denoted by $\{\psi_n\}$, and a function $\phi_1 \in H^1(\mathbf{R}^N)$ such that $\psi_n \rightharpoonup \phi_1$ weakly in $H^1(\mathbf{R}^N)$ as $n \to \infty$. Then it follows that $\lambda_1 = \liminf_{n\to\infty} \|\psi_n\|_{H^1}^2 \ge \|\phi_1\|_{H^1}^2$. Lemma B.1 implies that

$$\int_{\mathbf{R}^N} a\phi_1^2 dx = 1.$$

Hence $\phi_1 \not\equiv 0$ achieves the infimum $\lambda_1 > 0$. Clearly, $|\phi_1|$ also achieves λ_1. Then we may assume that $\phi_1 \ge 0$ a.e. in $\mathbf{R}^N$. Furthermore, ϕ_1 satisfies

$$-\Delta\phi_1 + \phi_1 = \lambda_1 a\phi_1 \quad \text{in } \mathbf{R}^N,$$

and hence $\phi_1 = \lambda_1 E_1 * [a\phi_1] > 0$ a.e. in $\mathbf{R}^N$. By the definition of λ_1, (B.5) holds. □

Lemma B.3. *Assume that the problem* (B.1) *has the first eigenvalue* $\lambda_1 > 1$*. Then, for any* $f \in H^{-1}(\mathbf{R}^N)$*, the problem*

$$\text{(B.6)} \qquad -\Delta u + u = a(x)u + f \quad \textit{in } \mathbf{R}^N$$

has a unique solution $u \in H^1(\mathbf{R}^N)$*.*

Proof. For $v, w \in H^1(\mathbf{R}^N)$ define $B[v,w]$ by

$$B[v,w] = \int_{\mathbf{R}^N} (\nabla v \cdot \nabla w + vw - avw)\, dx.$$

By the Hölder and the Sobolev inequalities, we have

$$\int_{\mathbf{R}^N} avwdx \le \|a\|_{L^{N/2}}\|v\|_{L^{2N/(N-2)}}\|w\|_{L^{2N/(N-2)}} \le C\|a\|_{L^{N/2}}\|v\|_{H^1}\|w\|_{H^1}$$

with some constant $C > 0$. Then there exists a constant $C > 0$ such that

$$|B[v,w]| \le C\|v\|_{H^1}\|w\|_{H^1} \quad \text{for all } v, w \in H^1(\mathbf{R}^N).$$

From (B.5) we have

$$B[v,v] = \|v\|_{H^1}^2 - \int_{\mathbf{R}^N} av^2 dx \ge \left(1 - \frac{1}{\lambda_1}\right)\|v\|_{H^1}^2$$

for all $v \in H^1(\mathbf{R}^N)$. According to the Lax-Milgram theorem, for every $f \in H^{-1}(\mathbf{R}^N)$ there exists a unique $u \in H^1(\mathbf{R}^N)$ such that

$$B[u,\psi] =< f, \psi > \quad \text{for all } \psi \in H^1(\mathbf{R}^N),$$

where $< \cdot, \cdot >$ is the pairing between $H^{-1}(\mathbf{R}^N)$ and $H^1(\mathbf{R}^N)$. This implies that (B.6) has a unique solution $u \in H^1(\mathbf{R}^N)$. □

Appendix C: Auxiliary lemmas

In this appendix we assume $p \in (1, N/(N-2))$.

Lemma C.1. (i) *There exists a constant* $C = C(p) > 0$ *such that*

$$\big|(t+s)_+^p - s^p\big| \le C(|t|^p + s^{p-1}|t|) \quad \textit{for } s \ge 0 \textit{ and } t \in \mathbf{R}.$$

(ii) *Assume that* $u \in H^1(\mathbf{R}^N)$ *and* $v \in L^p(\mathbf{R}^N)$ *with* $v \ge 0$ *a.e. in* $\mathbf{R}^N$. *Then* $(u+v)_+^p - v^p \in H^{-1}(\mathbf{R}^N)$.

Proof. (i) Note that $|(t+s)_+^p - s^p| \le \big||t+s|^{p-1}(t+s) - |s|^{p-1}s\big|$ for $s \ge 0$ and $t \in \mathbf{R}$. Then, by the mean value theorem, we have

$$|(t+s)_+^p - s^p| \le p(|t|+s)^{p-1}|t| \le \begin{cases} p(|t|^p + s^{p-1}|t|) & \text{if } p \le 2, \\ 2^{p-2}p(|t|^p + s^{p-1}|t|) & \text{if } p > 2. \end{cases}$$

Thus (i) holds with $C = p$ if $p \le 2$ and $C = 2^{p-2}p$ if $p > 2$.

(ii) It suffices to show that there exists a constant $C = C(u,v) > 0$ such that

$$\left|\int_{\mathbf{R}^N} \left((u+v)_+^p - v^p\right)\psi dx\right| < C\|\psi\|_{H^1} \tag{C.1}$$

for any $\psi \in H^1(\mathbf{R}^N)$. It follows from (i) that

$$\big|(u+v)_+^p - v^p\big|\,|\psi| \le C(|u|^p + v^{p-1}|u|)|\psi| \quad \text{in } \mathbf{R}^N.$$

By the Hölder and the Sobelev inequalities, we obtain

$$\int_{\mathbf{R}^N} |u|^p|\psi| dx \le \|u\|_{L^{2p}}^p \|\psi\|_{L^2} \le C\|u\|_{H^1}^p\|\psi\|_{H^1},$$

$$\int_{\mathbf{R}^N} v^{p-1}|u||\psi| dx \le \|v\|_{L^p}^{p-1}\|u\|_{L^{2p}}\|\psi\|_{L^{2p}} \le C\|v\|_{L^p}^{p-1}\|u\|_{H^1}\|\psi\|_{H^1}.$$

Then we obtain (C.1), and hence $(u+v)_+^p - v^p \in H^{-1}(\mathbf{R}^N)$. □

Let G and g be the functions defined by (6.2) and (6.3), respectively.

Lemma C.2. (i) *There exists a constant* $C = C(p) > 0$ *such that*

$$g(t,s) \le C(t^p + s^{p-1}t) \quad \textit{for } s, t \ge 0.$$

(ii) *For* $s, t \ge 0$,

$$\frac{1}{p+1}t^{p+1} \le G(t,s) \le \frac{1}{2}g(t,s)t.$$

In particular, $G(t,s) > t^{p+1}/(p+1)$ *for* $s, t > 0$.

(iii) *For any* $\varepsilon > 0$, *there is a constant* $C = C(\varepsilon) > 0$ *such that*

$$G(t,s) - \frac{p}{2}s^{p-1}t^2 \le \varepsilon s^{p-1}t^2 + Ct^{p+1} \quad \textit{for } s, t \ge 0. \tag{C.2}$$

(iv) *Put* $c_p = \min\{1, p-1\}$. *Then*

$$g(t,s)t - (2+c_p)G(t,s) \geq -\frac{c_p p}{2} s^{p-1} t^2 \quad \textit{for } s,t \geq 0.$$

(v) *There exists a constant* $C = C(p) > 0$ *such that*

$$|G(t+\tau,s) - G(t,s) - g(t,s)\tau| \leq C(|t|^{p-1} + |\tau|^{p-1} + s^{p-1})|\tau|^2 \tag{C.3}$$

for $s \geq 0$ *and* $t, \tau \in \mathbf{R}$.

Proof. (i) Note that $(t+s)_+ = (t+s)$ for $s,t \geq 0$. Then, from (i) of Lemma C.1, (i) holds .

(ii) Put $y(t,s) = G(t,s) - t^{p+1}/(p+1)$ and $\tilde{y}(t,s) = (1/2)g(t,s)t - G(t,s)$. We have $y(0,s) = y_t(0,s) = \tilde{y}(0,s) = \tilde{y}_t(0,s) = 0$ and

$$y_{tt}(t,s) = p((t+s)^{p-1} - t^p) \geq 0 \quad \text{and} \quad \tilde{y}_{tt}(t,s) = \frac{p(p-1)}{2}(t+s)^{p-2}t \geq 0.$$

Integrating them on $[0,t]$ twice with respect t, we obtain $y(t,s) \geq 0$ and $\tilde{y}(t,s) \geq 0$ for $s,t \geq 0$. In particular, if $t, s > 0$, we have $y(t,s) > 0$. Thus (ii) holds.

(iii) Define $Y(t,s) = G(t,s) - (p/2)s^{p-1}t^2$. Then we have $Y(0,s) = Y_t(0,s) = 0$ and

$$Y_{tt}(t,s) = p(t+s)^{p-1} - ps^{p-1}.$$

If $p \leq 2$ then

$$Y_{tt}(t,s) \leq p(t^{p-1} + s^{p-1}) - ps^{p-1} = pt^{p-1}.$$

Integrating on $[0,t]$ twice with respect t, we have $Y(t,s) \leq t^{p+1}/(p+1)$. Thus we obtain (C.2) with $C = 1/(p+1)$ for any $\varepsilon > 0$. Let $p > 2$. By the mean value theorem, we have

$$Y_{tt}(t,s) \leq p(p-1)(t+s)^{p-2}t \leq C(t^{p-1} + s^{p-2}t)$$

with some $C > 0$. Integrating on $[0,t]$ twice with respect t, we have

$$Y(t,s) \leq C(t^{p+1} + s^{p-2}t^3).$$

By the Young inequality, for any $\varepsilon > 0$ we have $s^{p-2}t^3 \leq \varepsilon s^{p-1}t^2 + C(\varepsilon)t^{p+1}$. Thus we obtain (C.2).

(iv) See (v) of Lemma 5.1 in [24].

(v) We will show (C.3) by dividing into four cases according to the signs of $t+\tau$ and t. Put $Z(t,\tau,s) = G(t+\tau,s) - G(t,s) - g(t,s)\tau$. We note here that $G(t,s) = g(t,s) = 0$ if $t \leq 0$, and that $G(t,s)$ and $g(t,s)$ are nondecreasing in $t \geq 0$ for each fixed $s \geq 0$.

(a) *The case where* $t+\tau > 0$ *and* $t > 0$. In this case, we have $Z(t,0,s) = Z_\tau(t,0,s) = 0$ and

$$Z_{\tau\tau}(t,\tau,s) = p(t+\tau+s)^{p-1} \leq C(|t|^{p-1} + |\tau|^{p-1} + s^{p-1})$$

with some constant $C = C(p) > 0$. Integrating on $[0,\tau]$ twice with respect to τ, we obtain (C.3).

(b) *The case where* $t+\tau>0$ *and* $t\le 0$. In this case we have $t+\tau\le\tau$ and $Z(t,s,\tau)=G(t+\tau,s)$. From (i) and (ii) of this lemma, it follows that

$$G(t+\tau,s)\le G(\tau,s)\le \frac{1}{2}g(\tau,s)\tau\le C(\tau^{p+1}+s^{p-1}\tau^2)$$

with some constant $C=C(p)>0$. In particular, (C.3) holds.

(c) *The case where* $t+\tau\le 0$ *and* $t>0$. In this case we have $t\le|\tau|$ and $Z(t,\tau,s)=-G(t,s)-g(t,s)\tau$. Then

$$|Z(t,\tau,s)|\le G(t,s)+g(t,s)|\tau|\le G(|\tau|,s)+g(|\tau|,s)|\tau|.$$

From (i) and (ii) of this lemma, we obtain

$$G(|\tau|,s)+g(|\tau|,s)|\tau|\le\frac{3}{2}g(|\tau|,s)|\tau|\le C(|\tau|^{p+1}+s^{p-1}|\tau|^2).$$

Thus (C.3) holds.

(d) *The case where* $t+\tau\le 0$ *and* $t\le 0$. In this case $Z(t,s,\tau)\equiv 0$. Thus (C.3) holds. □

Lemma C.3. *For* $v\in H^1(\mathbf{R}^N)$ *and* $w\in L^p(\mathbf{R}^N)$ *with* $w\ge 0$ *a.e. in* $\mathbf{R}^N$*, define*

$$I(v)=\frac{1}{2}\int_{\mathbf{R}^N}\left(|\nabla v|^2+v^2\right)dx-\int_{\mathbf{R}^N}G(v,w)dx.$$

Then the following (i) *and* (ii) *hold.*

(i) $I:H^1(\mathbf{R}^N)\to\mathbf{R}$ *is* C^1 *and satisfies*

(C.4) $$I'(v)\psi=\int_{\mathbf{R}^N}\left(\nabla v\cdot\nabla\psi+v\psi-g(v,w)\psi\right)dx$$

for $v,\psi\in H^1(\mathbf{R}^N)$.

(ii) *Let* $v_0\in H^1(\mathbf{R}^N)$ *be a critical point of* I*, that is,* $I'(v_0)\psi=0$ *for any* $\psi\in H^1(\mathbf{R}^N)$*. Then* $v_0\ge 0$ *a.e. in* $\mathbf{R}^N$*. Furthermore, if* $v_0\not\equiv 0$ *then* $v_0>0$ *a.e. in* $\mathbf{R}^N$.

Proof. (i) We see that

$$\begin{aligned}&\left|I(v+\psi)-I(v)-\int_{\mathbf{R}^N}\left(\nabla v\cdot\nabla\psi+v\psi-g(v,w)\psi\right)dx\right|\\ \le\quad&\|\psi\|_{H^1}^2+\int_{\mathbf{R}^N}|G(v+\psi,w)-G(v,w)-g(v,w)\psi|\,dx.\end{aligned}$$

Then, in order to show (C.4), it suffices to show that

(C.5) $$J\equiv\int_{\mathbf{R}^N}|G(v+\psi,w)-G(v,w)-g(v,w)\psi|\,dx=o(\|\psi\|_{H^1})\quad\text{as } \|\psi\|_{H^1}\to 0.$$

From (v) of Lemma C.2, it follows that

$$J\le C\int_{\mathbf{R}^N}\left(|v|^{p-1}\psi^2+|\psi|^{p+1}+|w|^{p-1}\psi^2\right)dx$$

for some constant $C > 0$. By the Hölder and the Sobolev inequalities, we have

$$\begin{aligned} J &\le C(\|v\|_{L^p}\|\psi\|_{L^{2p}}^2 + \|\psi\|_{L^{p+1}}^{p+1} + \|w\|_{L^p}\|\psi\|_{L^{2p}}^2) \\ &\le C(\|v\|_{L^p}\|\psi\|_{H^1}^2 + \|\psi\|_{H^1}^{p+1} + \|w\|_{L^p}\|\psi\|_{H^1}^2). \end{aligned}$$

This implies that (C.5) holds. Thus (C.4) holds. From (i) of Proposition 6.1, it is clear that $I'(v)\psi$ is continuous in $v \in H^1(\mathbf{R}^N)$ for any $\psi \in H^1(\mathbf{R}^N)$. Thus $I : H^1(\mathbf{R}^N) \to \mathbf{R}$ is C^1.

(ii) Note that $((v)_+ + w)^p = (v + w)^p$. Then, from (ii) of Lemma C.1 we have $g(v_0, w) \in H^{-1}(\mathbf{R}^N)$, and hence v_0 satisfies

$$v_0 = E_1 * [((v_0)_+ + w)^p - w^p] \quad \text{a.e. in } \mathbf{R}^N.$$

Then $v_0 \ge 0$ a.e. in $\mathbf{R}^N$ and, if $v_0 \not\equiv 0$, then $v_0 > 0$ a.e. in $\mathbf{R}^N$. □

Let h and H be functions defined by (6.4).

Lemma C.4. *There exists a constant $C > 0$ such that, for $s, t \ge 0$,*

$$H(t,s) \le C(t^{p+1} + s^{p-1}t^2), \tag{C.6}$$

$$H(t,s) \le C(t^p s + s^p t), \tag{C.7}$$

$$h(t,s)t \le C(t^{p+1} + s^{p-1}t^2). \tag{C.8}$$

$$h(t,s)t \le C(t^p s + s^p t), \tag{C.9}$$

Proof. We see that $H_t(t,s) = (t+s)^p - s^p - t^p$. Then $H_t(t,s) \le (t+s)^p - s^p$ and $H_t(t,s) \le (t+s)^p - t^p$. By the mean value theorem,

$$(t+s)^p - s^p \le p(t+s)^{p-1}t \le C(t^p + s^{p-1}t)$$

and

$$(t+s)^p - t^p \le p(t+s)^{p-1}s \le C(t^{p-1}s + s^p).$$

Then it follows that

$$H_t(t,s) \le C(t^p + s^{p-1}t) \quad \text{and} \quad H_t(t,s) \le C(t^{p-1}s + s^p).$$

Integrating on $[0,t]$ with respect to t, we obtain (C.6) and (C.7), respectively.

Observe that

$$\begin{aligned} \frac{\partial}{\partial t}(h(t,s)t) &= (p+1)(t+s)^p - (p+1)t^p - s^p - ps(t+s)^{p-1} \\ &\le (p+1)\{(t+s)^p - t^p - s^p\}. \end{aligned}$$

By the similar argument above, we obtain (C.8) and (C.9). □

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