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# Cost differentials and mixed strategy equilibria in a Hotelling model

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## Abstract

We introduce heterogeneity of production costs into the location-price Hotelling model discussed by d'Aspremont, Gabszewicz, and Thisse (1979). Maximum differentiation appears if the cost difference between two firms is small, whereas no pure strategy equilibrium exists if it is large. We examine the mixed strategy equilibria when no pure strategy equilibrium exists. We find that the following simple symmetric mixed strategy equilibrium exists, which never becomes an equilibrium if no cost differential exists: each firm chooses to locate at the two edges of the linear city randomly.

**Key words:** Hotelling, mixed strategy, heterogeneity, non-existence of pure strategy equilibrium

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# 1 Introduction

Since the seminal work of Hotelling (1929), the model of spatial competition has been seen by many researchers as an attractive framework for analyzing oligopoly markets. The Hotelling spatial model has become one of the most important methods of analyzing product differentiation. The major advantage of this approach is that it allows the explicit analysis of product selection. Of particular interest is the equilibrium pattern of product locations and the degree of product differentiation. The original result of Hotelling is that firms produce similar products (minimal differentiation).

d'Aspremont, Gabszewicz and Thisse (1979) introduce price competition into Hotelling's model and consider two-stage location-price games. They show that there is no pure strategy equilibrium under the original assumption in Hotelling, which is that transport costs are proportional to the distance between the firm and consumer. They show that a pure strategy equilibrium exists when transport costs are quadratic. In contrast to the original Hotelling result, in equilibrium the products are maximally differentiated.

The existence problem is discussed by many researchers. Most investigate this problem in the price-competition stage by allowing more general transport cost functions (See Gabszewicz and Thisse (1986)). Schulz and Stahl (1985) and Ziss (1993) take a different approach to explore the problem of non-existence of pure strategy equilibrium. They consider heterogeneity of production costs. Ziss (1993) introduces it into d'Aspremont, Gabszewicz and Thisse (1979), and he finds that maximum differentiation is the unique (pure strategy) equilibrium outcome as long as pure strategy equilibria exist, and that pure strategy equilibria exist if the cost difference between the two firms is small. He also finds that, if the heterogeneity exceeds the threshold value, no pure strategy equilibrium exists. Although no pure strategy equilibrium exists under significant cost heterogeneity, mixed strategy equilibria do exist. However, he did not discuss mixed strategy equilibria. In this paper we investigate mixed strategy equilibria.

As mentioned above, the Hotelling model is one of the most important models for analyzing product differentiation and is intensively used in many fields such as industrial organization, public economics, and international economics. Potentially, the lack of knowledge of the equilibrium

location pattern becomes an obstacle for using the Hotelling model. First, we cannot use this important model for analyzing firms with significant cost heterogeneity. Second, it is inconvenient for investigating endogenous production costs. Consider the following model. Two symmetric firms independently make cost-reducing R&D investment and then face the two stage location-price game. On the equilibrium path two firms might choose the same investment level, and so no cost differential appears on the equilibrium path. However, to complete the analysis and to find the subgame perfect equilibrium, researchers must consider every subgame off the equilibrium; thus, they cannot avoid analyzing the situation under cost differentials. In addition, if we consider uncertainty of R&D, *ex post* cost difference between firms may appear in equilibrium. Therefore, even when analyzing *ex ante* symmetric firms, it is important to find at least one mixed strategy equilibrium when there is no pure strategy equilibrium. We find a very simple and tractable mixed strategy equilibrium, which always exists when no pure strategy equilibrium exists.

With few exceptions, the literature of spatial competition has predominantly been concerned with the characterization of pure strategy equilibria. As Bester *et al.* (1996) argue, however, it is not certain how those pure strategy equilibria would be attained in the absence of some coordination device. Depending on the underlying context of the model, there may arise situations in which it is more appropriate to consider mixed strategy equilibria; for instance, it may be more appropriate to consider mixed strategy equilibria when the location decisions are irreversible and the firms cannot communicate with each other.<sup>1</sup>

Along this line, we investigate a mixed strategy equilibrium in a Hotelling model with cost differentials. In the real world, cost differentials between firms are often observed and such differentials are natural. The investigation concerning the mixed strategies of the firms is important for the study of product differentiation. We now emphasize that we discuss the cases where no pure strategy equilibrium exists as well as the cases where pure strategy equilibria exist. As Bester *et*

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<sup>1</sup> Shaked (1982) and Osborne and Pitchik (1986) also consider mixed strategies in the linear city model with the prices exogenously fixed. Ishida and Matsushima (2004) consider mixed strategies in the circular city model. They show that each location of one firm alternates with that of the other firm with equal distance between any two neighboring locations. For the importance of mixed strategy equilibria in the spatial competition, see also Wang and Yang (2001) and Matsumura and Shimizu (2007). For the general rationalization for investing mixed strategy equilibria, see, among others, Amaldoss and Jain (2002) and Camerer (2001, chap.2).

*al.* (1996) argue, investigating mixed strategy equilibrium is important even when pure strategy equilibrium exists. The analysis of mixed strategy equilibrium is more important when no pure strategy equilibrium exists.

We find that the following simple symmetric mixed strategy equilibrium exists when no pure strategy equilibrium exists: each firm chooses an edge of the linear city with probability  $1/2$  and another edge of the linear city with probability  $1/2$ , if the cost differentials of the firms are significant. We also show that the above pair of mixed strategies does not constitute an equilibrium when the cost difference is small. This result indicates that not only the properties of pure strategy equilibria, but also those of mixed strategy equilibria are quite different when significant cost differentials are introduced.

The remainder of this paper is organized as follows. Section 2 briefly presents the model and results of Ziss (1993). Section 3 investigates mixed strategy equilibria. Section 4 extends the analysis. Section 5 concludes the paper.

## 2 The Model and Pure Strategy Equilibria

We formulate a duopoly model. We consider a model (based on Hotelling (1929) and d'Aspremont, Gabszewicz and Thisse (1979)) in which a linear city of length 1 lies on the abscissa of a line and consumers are uniformly distributed with density 1 along this interval. Suppose that firm  $i$  ( $i = 1, 2$ ) is located at point  $x_i \in [0, 1]$ . A consumer living at  $y \in [0, 1]$  incurs a transport cost of  $t(x_i - y)^2$  when he or she purchases the product from firm  $i$ . The consumers have unit demands, i.e., each consumes one or zero unit of the product. Each consumer derives a surplus from consumption (gross of price and transport costs) equal to  $s$ . We assume that  $s$  is so large that every consumer consumes one unit of the product. Firms 1 and 2 produce the same physical product. The unit cost of the product for each firm is  $c_i$ , which is given exogenously. Without loss of generality, we assume that  $c_1 \leq c_2$ . We assume that  $c_2 - c_1 \leq t$ . If  $c_2 - c_1 > t$ , from the viewpoint of total social surplus, monopoly is better than duopoly *regardless of*  $x_1$  and  $x_2$ . Suppose that  $x_1 = 0$  and  $x_2 = 1$  (maximal differentiation). The supply by firm 1 is more efficient than that by firm 2 even for consumers living at 1. Under such extreme cost disadvantage, the government can improve welfare by eliminating firm 2. Thus, we neglect such an extreme cost

heterogeneity by assuming  $c_2 - c_1 \leq t$ .

The game runs as follows. Each firm maximizes its own profits. In the first stage, each firm  $i$  chooses its location  $x_i \in [0, 1]$  simultaneously. In the second stage, each firm  $i$  chooses its price  $p_i \in [c_i, \infty)$  simultaneously.

First, consider the subgame where  $x_1 \neq x_2$ . For a consumer living at

$$x = \frac{x_1 + x_2}{2} + \frac{p_2 - p_1}{2t(x_2 - x_1)}, \quad (1)$$

the total cost is the same at either of the two firms. Thus, the demand of firm 1,  $D_1$ , and that of firm 2,  $D_2$ , are given by

$$D_2(p_1, p_2, x_1, x_2) = 1 - D_1(p_1, p_2, x_1, x_2), \quad D_1(p_1, p_2, x_1, x_2) = \begin{cases} \min\{\max(x, 0), 1\}, & \text{if } x_1 < x_2 \\ \min\{\max(1 - x, 0), 1\}, & \text{otherwise} \end{cases}$$

The profit of firm  $i$  is

$$\pi_i = (p_i - c_i)D_i. \quad (2)$$

Next, consider the subgame where  $x_1 = x_2$  (i.e., no product differentiation exists). From the standard analysis of homogenous products Bertrand competition, we have that firm 2's profit is zero and firm 1's profit is  $(c_2 - c_1)$  in this case.

We now reconstruct the results on pure strategy equilibria, which were originally presented by Ziss (1993), to present the minimal information needed for our discussion of mixed strategy equilibria in the next section.

First, we present a result about the second stage subgames given  $x_1$  and  $x_2$ . It is possible that firm 1 is the monopolist (i.e.,  $D_1 = 1$ ) by limit pricing, while firm 2 never becomes the monopolist regardless of  $x_1$  and  $x_2$ . Needless to say, if the required limit price is too low, firm 1 gives up the position of monopolist. The following Result 1 (i) and (ii) present the conditions under which firm 1 becomes the monopolist.

**Result 1:** (i) Suppose that  $x_1 \leq x_2$ .  $D_1 = 1$  if and only if  $c_2 - c_1 \geq t(x_2 - x_1)(4 - x_1 - x_2)$ . In this case, firm 1's profit is  $c_2 - c_1 - t(x_2 - x_1)(2 - x_1 - x_2)$ . (ii) Suppose that  $x_1 > x_2$ .  $D_1 = 1$  if and only if  $c_2 - c_1 \geq t(x_1 - x_2)(2 + x_1 + x_2)$ . In this case, firm 1's profit is  $c_2 - c_1 - t(x_1 - x_2)(x_1 + x_2)$ .

We now discuss the case where  $D_1 < 1$ . Suppose that  $x_1 \leq x_2$ . From Result 1 (i), we consider the cases where  $c_2 - c_1 < t(x_2 - x_1)(4 - x_1 - x_2)$ . The first order conditions are

$$\frac{\partial \pi_1}{\partial p_1} = 0 \Leftrightarrow \frac{c_1 + p_2 - 2p_1 + t(x_2 - x_1)(x_2 + x_1)}{2t(x_2 - x_1)} = 0, \quad (3)$$

$$\frac{\partial \pi_2}{\partial p_2} = 0 \Leftrightarrow \frac{c_2 + p_1 - 2p_2 + t(x_2 - x_1)(2 - x_1 - x_2)}{2t(x_2 - x_1)} = 0. \quad (4)$$

The second order conditions are satisfied. These equations yield

$$p_1 = \frac{2c_1 + c_2 + t(x_2 - x_1)(2 + x_1 + x_2)}{3}, \quad p_2 = \frac{c_1 + 2c_2 + t(x_2 - x_1)(4 - x_1 - x_2)}{3}.$$

The quantity supplied by firm 1 is

$$D_1 = \frac{t(x_2 - x_1)(2 + x_1 + x_2) + c_2 - c_1}{6t(x_2 - x_1)}.$$

The profit functions are

$$\pi_1 = \frac{(c_2 - c_1 + t(x_2 - x_1)(2 + x_1 + x_2))^2}{18t(x_2 - x_1)}, \quad \pi_2 = \frac{(c_1 - c_2 + t(x_2 - x_1)(4 - x_1 - x_2))^2}{18t(x_2 - x_1)}.$$

Suppose that  $x_2 \leq x_1$ . From Result 1 (ii), we consider the cases where  $c_2 - c_1 < t(x_1 - x_2)(2 + x_1 + x_2)$ . From the symmetry of the linear city, we obtain

$$p_1 = \frac{2c_1 + c_2 + t(x_1 - x_2)(4 - x_1 - x_2)}{3}, \quad p_2 = \frac{c_1 + 2c_2 + t(x_1 - x_2)(2 + x_1 + x_2)}{3}.$$

The quantity supplied by firm 1 is

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Next, we discuss the location choice. Ziss (1993) shows that the best reply of firm 2 is  $x_2 = 0$  if  $x_1 \geq 1/2$  and  $x_2 = 1$  if  $x_1 \leq 1/2$ . In other words, firm 2 wants to maximize the distance between two firms. The best reply of firm 1 is slightly complicated. The best reply of firm 1 is

$$x_1 = \begin{cases} 0 & \text{if } c_2 - c_1 \leq tx_2(7 - x_2 - 3\sqrt{5 - 2x_2}) \text{ and } x_2 \geq 1/2, \\ x_2 & \text{if } tx_2(7 - x_2 - 3\sqrt{5 - 2x_2}) < c_2 - c_1 \text{ and } x_2 \geq 1/2. \end{cases} \quad (5)$$

$$x_1 = \begin{cases} 1 & \text{if } c_2 - c_1 \leq t(1 - x_2)(6 + x_2 - 3\sqrt{3 + 2x_2}) \text{ and } x_2 < 1/2, \\ x_2 & \text{if } t(1 - x_2)(6 + x_2 - 3\sqrt{3 + 2x_2}) < c_2 - c_1 \text{ and } x_2 < 1/2. \end{cases} \quad (6)$$

As long as it does limit pricing, the minimal distance is the best for it, and the maximal distance is the best otherwise.

Finally, we present a result for the pure strategy equilibrium.

**Result 2:** (i) **(Type 0 equilibrium: pure strategy equilibrium)** If  $c_2 - c_1 \leq t(6 - 3\sqrt{3}) (\sim 0.804t)$ , the (pure strategy) equilibrium location pattern is:  $x_1 = 0$  and  $x_2 = 1$ , or  $x_1 = 1$  and  $x_2 = 0$ . (ii) If  $c_2 - c_1 > t(6 - 3\sqrt{3})$ , no pure strategy equilibrium exists.

Henceforth, we call this pure strategy equilibrium ‘Type 0’. Each firm prefers maximal differentiation so as to soften the price competition in the next stage as long as the cost differentiation is insignificant. However, if the cost differentiation exceeds  $t(6 - 3\sqrt{3})$ , the equilibrium pattern changes drastically.<sup>2</sup>

### 3 Mixed Strategy Equilibrium

In this section, we investigate mixed strategy equilibria. We find the following three types of mixed strategy equilibria. Which type appears in equilibrium depends on the degree of cost difference  $c_2 - c_1$ .

#### Definition

**Type 1:** Firm 1 chooses the central point certainly and firm 2 chooses two edges of the linear city with equal probability.

**Type 2:** Firm 2 chooses the central point certainly and firm 1 chooses two edges of the linear city with equal probability.

**Type 3:** Two firms independently choose two edges of the linear city with equal probability.

**Proposition 1:** (i) Type 1 is an equilibrium location pattern if and only if  $0 \leq c_2 - c_1 \leq t(10 - \sqrt{61})/6 (\sim 0.365t)$ ; (ii) Type 2 is an equilibrium location pattern if and only if  $0 \leq c_2 - c_1 \leq t/4$ ; (iii) Type 3 is an equilibrium location pattern if and only if  $t/2 \leq c_2 - c_1$ .

**Proof:** See Appendix.

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<sup>2</sup> In our setting, market share of firm 1 is at largest 56.1% at pure strategy equilibrium. If the cost advantage of firm 1 is so significant that it yields market share larger than 56.1% market share, no pure strategy equilibrium exists.



Types 1 and 2 are the same as those in Proposition 2 of Bester *et al.* (1996), which discusses the case of homogenous firms. Propositions 1 (i) and (ii) indicate that their result holds true if we allow small cost differentials. If  $(c_2 - c_1) \leq t/4$ , both Type 1 and Type 2 become equilibria (see Figure 2). When  $t(10 - \sqrt{61})/6 \geq (c_2 - c_1) > t/4$ , Type 2 is not an equilibrium while Type 1 is an equilibrium. In this sense Type 1 is more robust, i.e., the more efficient firm locates at the center more likely, resulting in a further large market share of the more efficient firm.

From Result 2 (ii) and Proposition 1 (iii), we find that, if no pure strategy equilibrium exists, neither Type 1 nor Type 2 is an equilibrium, whereas Type 3 is an equilibrium. Type 3 has a very interesting property. Despite cost heterogeneity, the equilibrium is symmetric between two firms. In the equilibrium, both maximal and minimal differentiations appear with equal probability.

We think that Type 3 sheds light on the strategies of minor firms in the real world. Textbooks of management strategies mention that minor firms should differentiate their products from those of major firms, because the minor firms do not have any advantage over the major firms if they do not differentiate their products. By adopting a niche strategy, some minor firms achieve success. However, other minor firms do not follow the niche strategy and then fail. Whether the latter firms are irrational may depend on whether a coordination failure induces a result in which minor firms do not differentiate their product from those of major firms. That is, the minor firms may be merely unlucky. The analysis in this paper may shed light on such strategies.

We explain the intuition behind these properties in Proposition 1.

First, we explain why the threshold value for Type 1 is larger than that for Type 2 (see Figure 2). In d'Aspremont, Gabszewicz and Thisse (1979), a reduction of the distance between two firms increases the price elasticity of the demand and decreases the rival's price in the second stage. The same mechanism works in our model, unless the rival's market share is zero. When  $|x_2 - x_1| = 1$  (maximal distance), both firms have positive market share, so a slight decrease in the distance between two firms from one (maximal differentiation) reduces both firm's profits. For firm 2 (the firm with a higher cost), a decrease in the distance between two firms is always harmful because firm 1's market share is always positive. However, for firm 1 (the firm with a lower cost), it is possible that a decrease in the distance between the two firms increases its profits. We explain the

mechanism. If  $x_1 = x_2$ , firm 1 becomes the monopolist and the equilibrium price is  $c_2$  (standard Bertrand equilibrium with homogeneous goods). If  $x_1 \neq x_2$  and firm 1 still wants to obtain the whole market, it must charge a price strictly lower than  $c_2$ . Thus, it is possible that a decrease in  $|x_2 - x_1|$  increases firm 1's profit. In fact, if the cost difference is large, firm 1's profit is decreasing in  $|x_2 - x_1|$  when  $|x_2 - x_1|$  is close to 0 (minimal differentiation), and that minimal differentiation is firm 1's best reply.

We compare Type 1 and Type 2. First, we consider Type 2 equilibrium. Suppose that firm 2 chooses the central point with probability 1. Suppose that  $c_2 - c_1 = t/4 + \varepsilon$ , where  $\varepsilon$  is a positive and sufficiently small constant. If firm 1 also chooses the central point, it enjoys 100 % market share by the minimal differentiation, and the profit exceeds firm 1's profit when the distance is  $1/2$ . Thus, Type 2 fails to be an equilibrium outcome in such a situation. Second, we consider Type 1 equilibrium. Suppose that firm 2 chooses the two edges with equal probability. Suppose that  $c_2 - c_1 = t/4 + \varepsilon$ . On the contrary to Type 2 case, firm 1 can enjoy 100 % market share by the minimal differentiation only with probability  $1/2$  even if it deviates from the equilibrium strategy. Thus, it has a weaker incentive to deviate from the equilibrium strategy in Type 1 than in Type 2. This makes the range of cost difference for Type 1 be broader than that for Type 2.

Next, we explain why a pure strategy equilibrium does not exist under significant cost heterogeneity and Type 3 is an equilibrium in such a situation. If the cost advantage of firm 1 is significant, it is optimal for firm 1 to set a price inducing monopoly by firm 1. If firm 1 chooses  $x_1 = x_2$ , firm 1 obtains the whole market by setting  $p_1 = c_2$ . If it chooses  $x_1 \neq x_2$ , firm 1 must charge a price strictly lower than  $c_2$  in order to obtain the whole market. Thus, firm 1 prefers minimal differentiation. On the other hand, if  $x_1 = x_2$ , firm 2's profit becomes zero. Thus, firm 2 has a strong incentive to avoid agglomeration and wants to be far away from firm 1 so as to avoid severe competition against the strong rival. Since the strong firm prefers minimal differentiation and the weak firm prefers maximal differentiation, pure strategies never constitute an equilibrium.

Under such a significant cost difference, firm 1 has a strong incentive to minimize product differentiation. Suppose that firm 2 randomizes its locations at the two edges of the linear city. As explained above, firm 1's profit is increasing in  $|x_2 - x_1|$  when  $|x_2 - x_1|$  is close to 1 (maximal

differentiation), and it is decreasing in  $|x_2 - x_1|$  when  $|x_2 - x_1|$  is close to 0 (minimal differentiation). In short, both maximal and minimal differentiations become local optimum for firm 1. Thus, it is natural that firm 1 prefers to locate at an edge of the linear city rather than to locate at the central point.

In contrast, firm 2 has a strong incentive to maximize differentiation. Suppose that firm 1 randomizes its locations at the two edges of the linear city. Firm 2 has an incentive to choose to locate at the opposite edge so as to maximize differentiation. Under significant cost heterogeneity, firm 2's profit is zero when  $|x_2 - x_1| = 0$ , and firm 2's profit when  $|x_2 - x_1| = 1/2$  becomes negligibly small, while firm 2's profit when  $|x_2 - x_1| = 1$  is non-negligible. Thus, firm 2 dares to choose to locate at an edge rather than the central point so as to obtain non-negligible profit with probability  $1/2$ .

Proposition 1 discusses mixed strategy equilibria when  $c_2 - c_1 \leq t(10 - \sqrt{61})/6 (\sim 0.365t)$  and when  $c_2 - c_1 \geq t/2$ , but says nothing when  $t(10 - \sqrt{61})/6 (\sim 0.365t) \leq c_2 - c_1 \leq t/2$ . Proposition 2 presents a mixed strategy equilibrium in such cases.

**Proposition 2: (Type 4 equilibrium)** *If  $t(10 - \sqrt{61})/6 \leq c_2 - c_1 \leq t/2$ , there is a mixed strategy equilibrium in which firm 1 randomizes among three locations  $(0, 1/2, 1)$ , with probability  $(p, 1 - 2p, p)$ , and firm 2 randomizes three locations  $(0, 1/2, 1)$ , with probability  $(q, 1 - 2q, q)$ , where*

$$p = \frac{(5t - 4d)^2}{4(19t^2 - 36dt + 14d^2)}, \quad q = \frac{25t^2 - 104dt + 16d^2}{4(19t^2 - 72dt + 14d^2)},$$

where  $d \equiv c_2 - c_1$ .

**Proof:** See Appendix.

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Insert Figure 1 here

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We present a figure summarizing Result 2 and Propositions 1–2. Figure 2 indicates the ranges for the existence of type 0–4 equilibria.

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Insert Figure 2 here

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Finally, we make a remark on the property of Type 3 equilibrium.

**Proposition 3:** *If  $c_2 - c_1 > t(6 - 3\sqrt{3})$ , both players choose at least two locations with a positive probability in every equilibrium.*

**Proof:** See Appendix.

Proposition 3 indicates that Type 3 equilibrium is the easiest way for firms to find an equilibrium pattern when no pure strategy equilibrium exists.

## 4 Extension

In this section, we extend the analysis along the line of Tabuchi and Thisse (1995) and Lambertini (1997). We allow firms to locate outside the city boundaries. Except the allowance, the structure of the game is the same as that in the former sections.

The game runs as follows. Each firm maximizes its own profits. In the first stage, each firm  $i$  chooses its location  $x_i \in (-\infty, \infty)$  simultaneously. In the second stage, each firm  $i$  chooses its price  $p_i \in [c_i, \infty)$  simultaneously.

We first show that there exist a pure strategy equilibrium. After some calculus, we have the following proposition:

**Proposition 4:** *For  $c_2 - c_1 \leq t$ , the (pure strategy) equilibrium location pattern is:  $x_1 = (4(c_2 - c_1) - 3t)/12t$  and  $x_2 = (4(c_2 - c_1) + 15t)/12t$ , or  $x_1 = (15t - 4(c_2 - c_1))/12t$  and  $x_2 = -(4(c_2 - c_1) + 3t)/12t$ .*

In this case, the pure strategy equilibrium always exists for any  $c_2 - c_1 \leq t$ . The firms are able to avoid the tough competition between them because they can locate outside the city boundaries. The inefficient firm gets away from the center as the cost difference increases. Given the location of the inefficient firm, the efficient firm is able to locate near to the center and takes a large amount of demand. Therefore, it does not have the incentive to locate at the location of the inefficient firm.

We now consider mixed strategy equilibria. We concentrate on the following two types of location patterns.

**Definition**

**Type 1’:** Firm 1 chooses the central point certainly and firm 2 chooses  $x_2$  and  $1 - x_2$  with equal probability, where  $x_2 = -\sqrt{3t(3t + 4(c_2 - c_1))}/6t$ .

**Type 2’:** Firm 2 chooses the central point certainly and firm 1 chooses  $x_1$  and  $1 - x_2$  with equal probability, where  $x_1 = -\sqrt{3t(3t - 4(c_2 - c_1))}/6t$ .

After some calculus, we have the following proposition:<sup>3</sup>

**Proposition 5:** (i) *Type 1’ is an equilibrium location pattern if  $0 \leq c_2 - c_1 \leq t$ ; (ii) Type 2’ is an equilibrium location pattern if and only if  $0 \leq c_2 - c_1 \leq 9(11\sqrt{33} - 59)t/128 (\sim 0.295t)$ .*

The parameter range of  $c_2 - c_1$  in Type 1’ is wider than that in Type 2’. This property is similar to the former section. The efficient firm tends to locate at the center. It is efficient from the viewpoint of social welfare.

Type 1’ location pattern is the equilibrium location pattern when firm 1 is the leader and firm 2 is the follower. Suppose that firm chooses its location and then firm 2 chooses its location. Given firm 1 locates at the center, firm 2 locates at  $-\sqrt{3t(3t + 4(c_2 - c_1))}/6t$  or  $1 - \sqrt{3t(3t + 4(c_2 - c_1))}/6t$ . Anticipating the location choice by firm 2, firm 1 chooses the advantageous position (the center). This location is the same as Type 1’.

## 5 Concluding Remarks

In this paper, we investigate a new approach to the Hotelling model. We show that a simple symmetric mixed strategy equilibrium exists when cost differentials between two firms cause the non-existence of the pure strategy equilibrium. In equilibrium, both maximal and minimal differentiations appear with probability 1/2. The mixed strategy equilibrium we present is very simple. The applied analysis using a Hotelling model with cost heterogeneity is tractable even when it has no pure strategy equilibrium. Thus, the Hotelling spatial model still serves as a

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<sup>3</sup> The proof is available upon request.

tractable model for analyzing oligopoly markets even after introducing heterogeneity between firms.

The above mixed strategy equilibrium exhibits the symmetric location pattern, the incentive structure between two firms is contrasting. A smaller firm (less efficient firm) has a strong incentive to maximize the differentiation, while a larger firm has a strong incentive to minimize the differentiation. Our discussion might suggest a cycle that minor firms first make differentiated products, then major firms mimic them, which is repeatedly observed in the household electrical appliance and software industries.

Some readers might criticize discussing mixed strategy equilibria in the Hotelling model. Martin (2001) points out that real world decisions about the locations of plants are not made randomly. Myerson (1991) discusses in a general context that the pure strategy equilibrium may be more likely to become the focal point than the mixed strategy equilibrium. As mentioned above, Hotelling's linear city can be interpreted as the utility space; therefore, we believe that mixed strategy makes sense in the context. More importantly, we discuss the case where no pure strategy equilibrium exists. Even if we accept that pure strategy equilibria are more natural than mixed strategy equilibria, discussing mixed strategy equilibria is important because there is a range of the cost difference where no pure strategy equilibrium exists in our model.

We can interpret a model with cost differences as a model with two-dimensional product differentiation as discussed by Ma and Burgess (1993) and Ishibashi (2001). In their model, each firm supplies a product that is differentiated both horizontally (represented by the linear city) and vertically. Instead of assuming that firm 1's production cost is lower than firm 2's, we can assume that firm 1's product has higher quality than firm 2's and firm 1 can charge a higher price even if both firms choose the same location. In their paper, the quality of the product is endogenous, but the locations of the firms are exogenous. If we endogenize the quality of the product in their model, we face the same difficulty when we endogenize production cost, as discussed in our Introduction. Since our result can apply to the models with endogenous quality, the applicability of our result is broad.<sup>4</sup>

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<sup>4</sup> Other applications of location models with cost differentials, see Ashiya (2000) and Matsumura and Matsushima (2004).

In this paper, we assume that duopolists choose their actions simultaneously in each stage. Under cost heterogeneity, it might be reasonable to consider sequential location choice or sequential price choice (price leadership). Extending our analysis in this direction remains for future research.<sup>5</sup>

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<sup>5</sup> For related works on sequential location choice or price leadership, see Ono (1978, 1982), Deneckere and Kovenock (1992), Furth and Kovenock (1993), and Ishibashi (2003).

## Appendix

In this Appendix we prove Propositions 1–3. First, we shall introduce four supplementary lemmata in order to prove Propositions.

**Lemma 1:** *Suppose that firm 1 chooses  $x_1 = 0$  with probability  $1/2$  and chooses  $x_1 = 1$  with probability  $1/2$ . Then (i)  $x_2 = 0$  is a best reply of firm 2 if and only if  $c_2 - c_1 \geq t/2$ , (ii)  $x_2 = 1/2$  is a best reply of firm 2 if and only if  $c_2 - c_1 \leq t/2$ , and (iii)  $x_2 = y$  is a best reply of firm 2 only if  $y \in \{0, 1/2, 1\}$  for any  $c_2 - c_1 \leq t$ .*

**Proof:** Suppose that firm 1 chooses  $x_1 = 0$ . From Result 1, we have that  $D_2 = 0$  if and only if

$$x_2 \leq 2 - \sqrt{4 - \frac{c_2 - c_1}{t}}. \quad (7)$$

If the inequality in (7) does not hold, firm 2's payoff is

$$\frac{(c_1 - c_2 + tx_2(4 - x_2))^2}{18tx_2}. \quad (8)$$

Suppose that firm 1 chooses  $x_1 = 1$ . From Result 1 we have that  $D_2 = 0$  if and only if

$$x_2 \geq \sqrt{4 - \frac{c_2 - c_1}{t}} - 1. \quad (9)$$

If the inequality in (9) does not hold, firm 2's payoff is

$$\frac{(c_1 - c_2 + t(1 - x_2)(3 + x_2))^2}{18t(1 - x_2)}. \quad (10)$$

From (10), firm 2's expected payoff from choosing  $x_2 \in [0, 2 - \sqrt{4 - \frac{c_2 - c_1}{t}}]$  is

$$\phi_a(x_2) \equiv \frac{1}{2} \times \frac{(c_1 - c_2 + t(1 - x_2)(3 + x_2))^2}{18t(1 - x_2)}. \quad (11)$$

In other words, firm 2 obtains positive profit only if  $x_1 = 1$ .

As we discuss in Section 2, given  $x_1 = 1$ , the best reply of firm 2 is choosing  $x_2 = 0$ . Thus, among  $x_2 \in [0, 2 - \sqrt{4 - \frac{c_2 - c_1}{t}}]$ ,  $x_2 = 0$  is best for firm 2.

From (8) and (10), firm 2's expected payoff from choosing  $x_2 \in [2 - \sqrt{4 - \frac{c_2 - c_1}{t}}, 1/2]$  is

$$\phi_b(x_2) \equiv \frac{1}{2} \times \frac{(c_1 - c_2 + tx_2(4 - x_2))^2}{18tx_2} + \frac{1}{2} \times \frac{(c_1 - c_2 + t(1 - x_2)(3 + x_2))^2}{18t(1 - x_2)}. \quad (12)$$



Differentiating  $\phi_b(x_2)$  with respect to  $x_2$ , we have:

$$\phi'_b(x_2) = \frac{(1 - 2x_2)(13t^2(1 - x_2)^2x_2^2 - (c_2 - c_1)^2)}{18t(1 - x_2)^2x_2^2}. \quad (13)$$

The denominator of (13) is positive.  $(1 - 2x_2)$  in the numerator of (13) is non-negative and positive except for  $x_2 = 1/2$ . Thus  $\phi'_b(x_2) = 0$  when  $x_2 = 1/2$  and otherwise the sign of  $\phi'_b(x_2)$  is the same as that of  $(13t^2(1 - x_2)^2x_2^2 - (c_2 - c_1)^2)$ . Define  $f(x_2) \equiv (13t^2(1 - x_2)^2x_2^2 - (c_2 - c_1)^2)$ .

Since  $f' = 26t^2(1 - x_2)x_2(1 - 2x_2)$ , it is increasing in  $x_2$  for  $x_2 < 1/2$ . This implies that for  $x_2 < 1/2$ , one of the following three statements is true:  $\phi_b(x_2)$  is always positive,  $\phi'_b(x_2)$  is always negative, or  $\phi'_b(x_2) \geq 0$  if and only if  $x_2 > \tilde{l}$ , where  $\tilde{l} \in [2 - \sqrt{4 - \frac{c_2 - c_1}{t}}, 1/2]$ . Under these conditions, for  $x_2 \in [2 - \sqrt{4 - \frac{c_2 - c_1}{t}}, 1/2]$ ,  $\phi_b(x_2)$  is non-decreasing,  $\phi_b(x_2)$  is non-increasing, or  $\phi_b(x_2)$  is non-increasing for  $x_2 \leq \tilde{l}$  and non-decreasing for  $x_2 \geq \tilde{l}$ . In any case,  $x_2 = 2 - \sqrt{4 - \frac{c_2 - c_1}{t}}$  or  $x_2 = 1/2$  maximizes  $\phi_b(x_2)$ .

If  $x_2 = 2 - \sqrt{4 - \frac{c_2 - c_1}{t}}$  maximizes  $\phi_b(x_2)$ ,  $\phi_a(0)$  is larger than  $\phi_b(x_2)$ , because  $\phi_a(2 - \sqrt{4 - \frac{c_2 - c_1}{t}}) = \phi_b(2 - \sqrt{4 - \frac{c_2 - c_1}{t}})$ . These discussions show that the possible best responses are either  $x_2 = 0$ ,  $x_2 = 1/2$ , or  $x_2 = 1$  (by the symmetry of the linear city and strategy of firm 1). It implies Lemma 1 (iii).

We compare  $\phi_a(0)$  with  $\phi_b(1/2)$ .

$$\begin{aligned} \phi_a(0) - \phi_b(1/2) &= \frac{(3t - (c_2 - c_1))^2}{36t} - \frac{(7t - 4(c_2 - c_1))^2}{144t} \\ &= \frac{(2(c_2 - c_1) - t)(13t - 6(c_2 - c_1))}{144t}. \end{aligned}$$

Since we assume that  $c_2 - c_1 \leq t$ , we obtain that  $x_2 = 0$  is best reply of firm 2 if and only if  $2(c_2 - c_1) - t \geq 0$ , and that  $x_2 = 1/2$  is best reply of firm 2 if and only if  $2(c_2 - c_1) - t \leq 0$ .

**Q.E.D.**

**Lemma 2:** Suppose that firm 2 chooses  $x_2 = 0$  with probability  $1/2$  and chooses  $x_2 = 1$  with probability  $1/2$ . Then (i)  $x_1 = 0$  is a best reply of firm 1 if and only if  $c_2 - c_1 \geq t(10 - \sqrt{61})/6$ , (ii)  $x_1 = 1/2$  is a best reply of firm 1 if and only if  $c_2 - c_1 \leq t(10 - \sqrt{61})/6$ , and  $x_2 = y$  is a best reply of firm 1 only if  $y \in \{0, 1/2, 1\}$  for any  $c_2 - c_1 \leq t$ .

**Proof:** Suppose that firm 2 chooses  $x_2 = 1$ . From Result 1, we have that  $D_1 = 1$  if and only if

the following inequality is satisfied:

$$x_1 \geq 2 - \sqrt{1 + \frac{c_2 - c_1}{t}}. \quad (14)$$

If the inequality in (14) does not hold, firm 1's payoff is

$$\frac{(c_2 - c_1 + t(1 - x_1)(3 + x_1))^2}{18t(1 - x_1)}. \quad (15)$$

Otherwise, firm 1's payoff is

$$(c_2 - c_1) - t(1 - x_1)^2. \quad (16)$$

Suppose that firm 2 chooses  $x_2 = 0$ . From Result 1, we have that  $D_1 = 1$  if and only if the following inequality is satisfied:

$$x_1 \leq \sqrt{1 + \frac{c_2 - c_1}{t}} - 1. \quad (17)$$

If the inequality in (17) does not holds, firm 1's payoff is

$$\frac{(c_2 - c_1 + tx_1(4 - x_1))^2}{18tx_1}. \quad (18)$$

Otherwise, firm 1's payoff is

$$(c_2 - c_1) - tx_1^2. \quad (19)$$

From (15) and (19), firm 1's expected payoff from choosing  $x_1 \in [0, \sqrt{1 + \frac{c_2 - c_1}{t}} - 1]$  is

$$\psi_a(x_1) \equiv \frac{1}{2} \times \frac{(c_2 - c_1 + t(1 - x_1)(3 + x_1))^2}{18t(1 - x_1)} + \frac{1}{2} \times [(c_2 - c_1) - tx_1^2]. \quad (20)$$

The second term in (20) is maximized when  $x_1 = 0$ . We then show that the first term in (20) is also maximized when  $x_1 = 0$ . We show that it is non-increasing in  $x_1$ . Define

$$g \equiv \frac{(c_2 - c_1 + t(1 - x_1)(3 + x_1))^2}{18t(1 - x_1)}$$

. Differentiating  $g$  with respect to  $x_1$ , we have

$$\frac{\partial g}{\partial x_1} = - \frac{(t(3 - 2x_1 - x_1^2) + (c_2 - c_1))(t(1 + 2x_1 - 3x_1^2) - (c_2 - c_1))}{18t(1 - x_1)^2}. \quad (21)$$

The denominator in (21) is positive and  $t(3 - 2x_1 - x_1^2) + (c_2 - c_1)$  in (21) is also positive. Thus the sign of (21) is the same as that of  $-t(1 + 2x_1 - 3x_1^2) + (c_2 - c_1)$ . Define  $h(x_1) \equiv -t(1 + 2x_1 - 3x_1^2) + (c_2 - c_1)$ . Since  $h(x_1)$  is convex, it is maximized at either of the boundaries.  $h(0) = c_2 - c_1 - t$  and  $h(\sqrt{1 + \frac{c_2 - c_1}{t}} - 1) = -8t\sqrt{1 + \frac{c_2 - c_1}{t}} + 7t + 4(c_2 - c_1)$ . Both are non-positive since  $c_2 - c_1 \leq t$ . Therefore, for  $x_1 \in [0, \sqrt{1 + \frac{c_2 - c_1}{t}} - 1]$ ,  $\partial g / \partial x_1 < 0$  and then  $\psi'_a(x_1) < 0$ .

From (15) and (18), firm 1's expected payoff from choosing  $x_1 \in [\sqrt{1 + \frac{c_2 - c_1}{t}} - 1, 1/2]$  is

$$\psi_b(x_1) \equiv \frac{1}{2} \times \frac{(c_2 - c_1 + tx_1(4 - x_1))^2}{18tx_1} + \frac{1}{2} \times \frac{(c_2 - c_1 + t(1 - x_1)(3 + x_1))^2}{18t(1 - x_1)}. \quad (22)$$

Differentiating it with respect to  $x_1$ , we have:

$$\psi'_b(x_1) = \frac{(1 - 2x_1)(13t^2(1 - x_1)^2x_1^2 - (c_2 - c_1)^2)}{18t(1 - x_1)^2x_1^2}. \quad (23)$$

Note that  $\psi'_b = \phi'_b$ , where  $\phi_b$  is defined in the Proof of Lemma 1. For the same reason showed in the Proof of Lemma 1,  $\psi_b$  is maximized at boundaries, i.e., either  $x_1 = \sqrt{1 + \frac{c_2 - c_1}{t}} - 1$  or  $x_1 = 1/2$  maximizes  $\psi_b(x_1)$ . If  $x_1 = \sqrt{1 + \frac{c_2 - c_1}{t}} - 1$  maximizes  $\psi_b(x_1)$ ,  $\psi_a(0)$  is larger than  $\psi_b(x_1)$ , because  $\psi_a(\sqrt{1 + \frac{c_2 - c_1}{t}} - 1) = \psi_b(\sqrt{1 + \frac{c_2 - c_1}{t}} - 1)$  and  $\psi'_a < 0$ . These discussions show that the possible best responses are either  $x_1 = 0$ ,  $x_1 = 1/2$ , or  $x_1 = 1$  (by the symmetry of the linear city and strategy of firm 2). It implies Lemma 2 (iii).

We compare  $\psi_a(0)$  with  $\psi_b(1/2)$ .

$$\begin{aligned} \psi_a(0) - \psi_b(1/2) &= \frac{(c_2 - c_1)^2 + 24t(c_2 - c_1) + 9t^2}{36t} - \frac{(7t + 4(c_2 - c_1))^2}{144t} \\ &= \frac{4(c_2 - c_1)(10t - 3(c_2 - c_1)) - 13t^2}{144t}. \end{aligned} \quad (24)$$

We have that  $\psi_a(0) - \psi_b(1/2) \geq 0$  if and only if  $c_2 - c_1 \geq t(10 - \sqrt{61})/6$  (note that we assume that  $c_2 - c_1 \leq t$ ). This implies Lemma 2 (i) and (ii). **Q.E.D.**

**Lemma 3:** Suppose that firm 1 chooses  $x_1 = 1/2$  with probability 1. Then  $x_2 = 0$  and  $x_2 = 1$  are always best replies of firm 2 and no other choice becomes its best reply.

**Proof:** As is discussed in Section 2, firm 2 maximizes the distance between firm 1 and firm 2. It yields Lemma 3. **Q.E.D.**

**Lemma 4:** Suppose that firm 2 chooses  $x_2 = 1/2$  with probability 1. Then  $x_1 = 0$  and  $x_1 = 1$  are best replies of firm 1 if and only if  $c_2 - c_1 \leq t/4$ .

**Proof:** Substituting  $x_2 = 1/2$  into the condition  $c_2 - c_1 \leq tx_2(7 - x_2 - 3\sqrt{5 - 2x_2})$  in (5), we obtain Lemma 4. **Q.E.D.**

**Proof of Proposition 1:** Lemmas 2 and 3 imply Proposition 1 (i). Lemmas 1 and 4 imply Proposition 1 (ii), and Lemmas 1 and 2 imply Proposition 1 (iii). **Q.E.D.**

**Proof of Proposition 2:** Given that firm 1 randomizes three locations,  $\mathbf{x}_1 \equiv (0, 1/2, 1)$ , with probability  $\mathbf{p} \equiv (p, 1 - 2p, p)$ , if firm 2 locates at  $x_2 = 0$  with probability 1 ( $x_2 = 1$  with probability 1), its payoff,  $\pi_2(x_2, \mathbf{x}_1, \mathbf{p})$ , is:

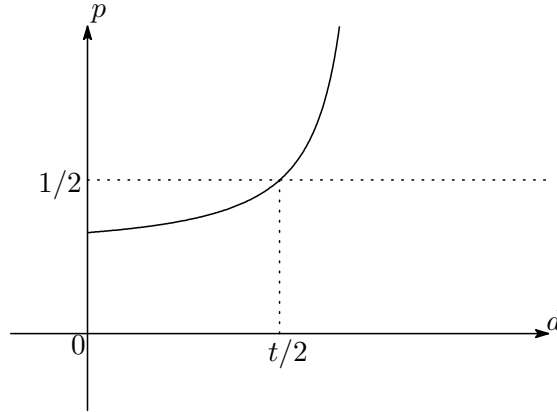
$$\pi_2(0, \mathbf{x}_1, \mathbf{p}) = (1 - 2p) \frac{(-d + t(1/2 - 0)(2 + 0 + 1/2))^2}{18t(1/2 - 0)} + p \frac{(-d + t(1 - 0)(2 + 0 + 1))^2}{18t(1 - 0)}, \quad (25)$$

where  $d \equiv c_2 - c_1$ . If firm 2 locates at  $x_2 = 1/2$  with probability 1, its payoff,  $\pi_2(x_2, \mathbf{x}_1, \mathbf{p})$ , is:

$$\pi_2(1/2, \mathbf{x}_1, \mathbf{p}) = p \frac{(-d + t(1/2 - 0)(4 - 0 - 1/2))^2}{18t(1/2 - 0)} + p \frac{(-d + t(1 - 1/2)(2 + 1 + 1/2))^2}{18t(1 - 1/2)}. \quad (26)$$

If  $\pi_2(0, \mathbf{x}_1, \mathbf{p}) = \pi_2(1/2, \mathbf{x}_1, \mathbf{p})$ , firm 2 may randomize three locations,  $(0, 1/2, 1)$ .

$$\pi_2(0, \mathbf{x}_1, \mathbf{p}) = \pi_2(1/2, \mathbf{x}_1, \mathbf{p}) \Leftrightarrow p = \frac{(5t - 4d)^2}{4(19t^2 - 36dt + 14d^2)}. \quad (27)$$



Given that firm 2 randomizes three locations,  $\mathbf{x}_2 \equiv (0, 1/2, 1)$ , with probability  $\mathbf{q} \equiv (q, 1 - 2q, q)$ , if firm 1 locates at  $x_1 = 0$  with probability 1 ( $x_1 = 1$  with probability 1), its payoff,  $\pi_1(x_1, \mathbf{x}_2, \mathbf{q})$ , is:

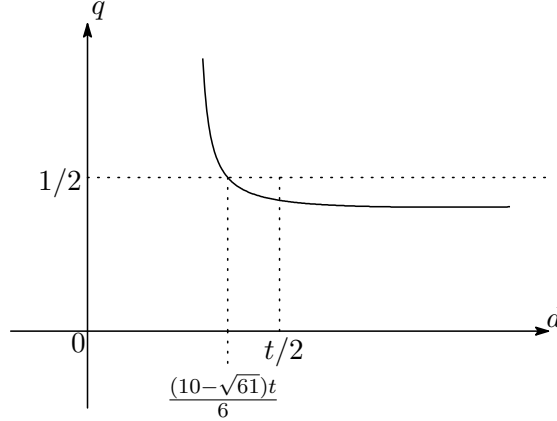
$$\pi_1(0, \mathbf{x}_2, \mathbf{q}) = qd + (1 - 2q) \frac{(d + t(1/2 - 0)(2 + 0 + 1/2))^2}{18t(1/2 - 0)} + q \frac{(d + t(1 - 0)(2 + 0 + 1))^2}{18t(1 - 0)}. \quad (28)$$

If firm 1 locates at  $x_1 = 1/2$  with probability 1, its payoff,  $\pi_1(x_1, \mathbf{x}_2, \mathbf{q})$ , is:

$$\pi_1(1/2, \mathbf{x}_2, \mathbf{q}) = q \frac{(d + t(1/2 - 0)(4 - 0 - 1/2))^2}{18t(1/2 - 0)} + (1 - 2q)d + q \frac{(d + t(1 - 1/2)(2 + 1 + 1/2))^2}{18t(1 - 1/2)}. \quad (29)$$

If  $\pi_1(0, \mathbf{x}_2, \mathbf{q}) = \pi_1(1/2, \mathbf{x}_2, \mathbf{q})$ , firm 1 may randomize three locations,  $(0, 1/2, 1)$ .

$$\pi_1(0, \mathbf{x}_2, \mathbf{q}) = \pi_1(1/2, \mathbf{x}_2, \mathbf{q}) \Leftrightarrow q = \frac{25t^2 - 104dt + 16d^2}{4(19t^2 - 72dt + 14d^2)}. \quad (30)$$



Given that firm 1 randomizes three locations,  $\mathbf{x}_1 = (0, 1/2, 1)$ , with probability  $\mathbf{p} = (p, 1 - 2p, p)$  (where  $p$  is in (27)), we show that the best responses of firm 2 are  $x_2 = 0$ ,  $x_2 = 1/2$ , and  $x_2 = 1$ .

We now show that the best responses of firm 2 are  $x_2 = 0$ ,  $x_2 = 1/2$ , and  $x_2 = 1$ .  $\pi_2(x_2, \mathbf{x}_1, \mathbf{p})$  is decreasing function on  $x_2 \in [0, 2 - \sqrt{4 - d/t}]$ .  $\pi_2(x_2, \mathbf{x}_1, \mathbf{p})$  is increasing function on  $x_2 \in [\sqrt{4 - d/t} - 3/2, 1/2]$ . If  $\pi_2(x_2, \mathbf{x}_1, \mathbf{p}) < \pi_2(0, \mathbf{x}_1, \mathbf{p})$  for any  $x_2 \in [2 - \sqrt{4 - d/t}, \sqrt{4 - d/t} - 3/2]$ , the best responses of firm 2 are  $x_2 = 0$ ,  $x_2 = 1/2$ , and  $x_2 = 1$ .

For any  $x_2 \in [2 - \sqrt{4 - d/t}, \sqrt{4 - d/t} - 3/2]$ , the profit function of firm 2 is

$$\begin{aligned} \pi_2(x_2, \mathbf{x}_1, \mathbf{p}) &= p \frac{(-d + tx_2(4 - x_2))^2}{18tx_2} + p \frac{(-d + t(1 - x_2)(3 + x_2))^2}{18t(1 - x_2)} \\ &\quad + (1 - 2p) \frac{(-d + t(1/2 - x_2)(5/2 + x_2))^2}{18t(1/2 - x_2)}. \end{aligned} \quad (31)$$

We define:

$$\phi(x_2) \equiv p \frac{(-d + tx_2(4 - x_2))^2}{18tx_2} + p \frac{(-d + t(1 - x_2)(3 + x_2))^2}{18t(1 - x_2)}. \quad (32)$$

The third term of the profit function in (31) is decreasing with respect to  $x_2$ . Therefore,

$$\partial \pi_2(x_2, \mathbf{x}_1, \mathbf{p}) / \partial x_2 < \phi'(x_2).$$

Differentiating  $\phi(x_2)$  three times, we have:

$$\phi'(x_2) = \frac{(5t - 4d)^2(1 - 2x_2)(13t^2x_2^2(1 - x_2)^2 - d^2)}{72t(14d^2 - 36dt + 19t^2)(1 - x_2)^2x_2^2}, \quad (33)$$

$$\phi''(x_2) = \frac{(5t - 4d)^2((1 - 3x_2 + 3x_2^2)d^2 - 13t^2x_2^3(1 - x_2)^3)}{36t(14d^2 - 36dt + 19t^2)(1 - x_2)^3x_2^3}, \quad (34)$$

$$\phi'''(x_2) = -\frac{d^2(5t - 4d)^2(1 - 2x_2)(1 - 2x_2 + 2x_2^2)}{12t(14d^2 - 36dt + 19t^2)(1 - x_2)^4x_2^4} < 0. \quad (35)$$

From (33),  $\phi'(x_2) < 0$  if  $x_2 < (1 - \sqrt{1 - 4\sqrt{d/13}})/2 \equiv u$ . Since  $d > (10 - \sqrt{61})t/6$ , we can show that  $u$  is larger than  $1/5$ . Therefore,  $\pi_2(x_2, \mathbf{x}_1, \mathbf{p}) < \pi_2(0, \mathbf{x}_1, \mathbf{p})$  for any  $x_2 \in [2 - \sqrt{4 - d/t}, 1/5]$ .

Because  $\phi'''(x_2) < 0$ , for any  $x$  and  $k$  ( $x > k$ ), the inequality holds:

$$\phi'(x) < \phi'(k) + (x - k)\phi''(k). \quad (36)$$

We integrate both sides:

$$\begin{aligned} \int_k^{x_2} \phi'(x)dx &< (x_2 - k)\phi'(k) + \int_k^{x_2} (x - k)\phi''(k)dx \\ &= (x_2 - k)\phi'(k) + \frac{(x_2 - k)^2\phi''(k)}{2}. \end{aligned} \quad (37)$$

We have the following inequality:

$$\begin{aligned} \pi_2(x_2, \mathbf{x}_1, \mathbf{p}) &= \pi_2(k, \mathbf{x}_1, \mathbf{p}) + \int_k^{x_2} \frac{\partial \pi_2(m, \mathbf{x}_1, \mathbf{p})}{\partial m} dm < \pi_2(k, \mathbf{x}_1, \mathbf{p}) + \int_k^{x_2} \phi'(x)dx \\ &< \pi_2(k, \mathbf{x}_1, \mathbf{p}) + (x_2 - k)\phi'(k) + \frac{(x_2 - k)^2\phi''(k)}{2}. \end{aligned} \quad (38)$$

We then consider 5 segments:  $x_2 \in [1/5 + (i - 1)/20, 1/5 + i/20]$  ( $i \in \{1, 2, \dots, 5\}$ ) (note that  $\sqrt{4 - d/t} - 3/2$  is smaller than  $9/20$  since  $d > (10 - \sqrt{61})t/6$ ). For each segment, if the following inequality holds, the best responses of firm 2 are  $x_2 = 0$ ,  $x_2 = 1/2$ , and  $x_2 = 1$ :

$$\pi_2(k, \mathbf{x}_1, \mathbf{p}) + (x_2 - k)\phi'(k) + \frac{(x_2 - k)^2\phi''(k)}{2} < \pi_2(0, \mathbf{x}_1, \mathbf{p}). \quad (39)$$

Using (25), (31), (33), and (34), we check whether the inequality holds or not for any  $x_2 \in [1/5 + (i - 1)/20, 1/5 + i/20]$  ( $i \in \{1, 2, \dots, 5\}$ ) and  $d \in [(10 - \sqrt{61})t/6, t/2]$ . After tedious calculus, we can show that the inequality is satisfied for the 5 segments. Therefore,  $\pi_2(x_2, \mathbf{x}_1, \mathbf{p}) < \pi_2(0, \mathbf{x}_1, \mathbf{p})$  for any  $x_2 \in [2 - \sqrt{4 - d/t}, \sqrt{4 - d/t} - 3/2]$  and then the best responses of firm 2 are  $x_2 = 0$ ,  $x_2 = 1/2$ , and  $x_2 = 1$ .

We now show that the best responses of firm 1 are  $x_1 = 0$ ,  $x_1 = 1/2$ , and  $x_1 = 1$ .  $\pi_1(x_1, \mathbf{x}_2, \mathbf{q})$  is decreasing function on  $x_1 \in [0, \sqrt{1 + d/t} - 1]$ .  $\pi_1(x_1, \mathbf{x}_2, \mathbf{q})$  is increasing function on  $x_1 \in [3/2 - \sqrt{1 + d/t}, 1/2]$ . If  $\pi_1(x_1, \mathbf{x}_2, \mathbf{q}) < \pi_1(0, \mathbf{x}_2, \mathbf{q})$  for any  $x_1 \in [\sqrt{1 + d/t} - 1, 3/2 - \sqrt{1 + d/t}]$ , the best responses of firm 1 are  $x_2 = 0$ ,  $x_2 = 1/2$ , and  $x_2 = 1$ . The profit function of firm 1 is

$$\begin{aligned} \pi_1(x_1, \mathbf{x}_2, \mathbf{q}) = & q \frac{(d + tx_1(4 - x_1))^2}{18tx_1} + q \frac{(d + t(1 - x_1)(3 + x_1))^2}{18t(1 - x_1)} \\ & + (1 - 2q) \frac{(d + t(1/2 - x_1)(5/2 + x_1))^2}{18t(1/2 - x_1)}. \end{aligned} \quad (40)$$

We define the following functions:

$$\psi_a(x_1) \equiv q \frac{(d + tx_1(4 - x_1))^2}{18tx_1} + q \frac{(d + t(1 - x_1)(3 + x_1))^2}{18t(1 - x_1)}, \quad (41)$$

$$\psi_b(x_1) \equiv (1 - 2q) \frac{(d + t(1/2 - x_1)(5/2 + x_1))^2}{18t(1/2 - x_1)}. \quad (42)$$

Differentiating the functions with respect to  $x_1$ , we have:

$$\psi'_a(x_1) = q \frac{(1 - 2x_1)(13t^2x_1^2(1 - x_1)^2 - d^2)}{18t(1 - x_1)^2x_1^2} > 0, \quad (43)$$

$$\psi'_b(x_1) = -(1 - 2q) \frac{(5t + 4d - 8tx_1 - 4tx_1^2)(3t - 4d - 12tx_1^2)}{72t(1 - 2x_1)^2} < 0. \quad (44)$$

From (41), (42), (43), and (44), for any  $x_1 \in [\sqrt{1 + d/t} - 1, 3/2 - \sqrt{1 + d/t}]$  the following inequality holds:

$$\begin{aligned} \pi_1(x_1, \mathbf{x}_2, \mathbf{q}) &= \psi_a(x_1) + \psi_b(x_1) \\ &< \psi_a\left(3/2 - \sqrt{1 + d/t}\right) + \psi_b\left(\sqrt{1 + d/t} - 1\right) \\ &= \frac{(20475t^4 - 72420dt^3 - 53084d^2t^2 + 87728d^3t - 12288d^4)\sqrt{1 + d/t}}{144(5t - 4d)(3t + 4d)(19t^2 - 72dt + 14d^2)} \\ &\quad + \frac{2(-4425t^4 + 17800dt^3 + 10655d^2t^2 - 4096d^4)}{144(5t - 4d)(3t + 4d)(19t^2 - 72dt + 14d^2)}. \end{aligned} \quad (45)$$

Firm 1's profit in which it chooses  $x_1 = 0$  is:

$$\pi_1(0, \mathbf{x}_2, \mathbf{q}) = \frac{256d^4 + 960d^3t - 10400d^2t^2 - 1824dt^3 + 1225t^4}{288t(14d^2 - 72dt + 19t^2)}. \quad (46)$$

If the following inequality satisfies,  $\pi_1(x_1, \mathbf{x}_2, \mathbf{q}) < \pi_1(0, \mathbf{x}_2, \mathbf{q})$  for any  $x_1 \in [\sqrt{1 + d/t} - 1, 3/2 - \sqrt{1 + d/t}]$ .

$$\psi_a\left(3/2 - \sqrt{1 + d/t}\right) + \psi_b\left(\sqrt{1 + d/t} - 1\right) < \pi_1(0, \mathbf{x}_2, \mathbf{q}). \quad (47)$$

From (45) and (46), after tedious calculus, we obtain that this inequality is satisfied for any  $d \in [(10 - \sqrt{61})t/6, t/2]$ . Therefore,  $\pi_1(x_1, \mathbf{x}_2, \mathbf{q}) < \pi_1(0, \mathbf{x}_2, \mathbf{q})$  for any  $x_1 \in [\sqrt{1 + d/t} - 1, 3/2 - \sqrt{1 + d/t}]$  and then the best responses of firm 1 are  $x_1 = 0$ ,  $x_1 = 1/2$ , and  $x_1 = 1$ . **Q.E.D**

**Proof of Proposition 3:** First, we show that there is no equilibrium where firm 1 chooses a pure strategy. We prove it by contradiction. Suppose that firm 1 chooses  $x_1 = y$  with probability 1. As is discussed in Section 2, firm 2's best reply is unique unless  $y = 1/2$ . Since no pure strategy equilibrium exists,  $y$  must be  $1/2$ . Suppose that firm 1 chooses  $x_1 = 1/2$  with probability 1. From Lemma 2 we have that firm 2's best reply is  $x_2 = 0$  and  $x_2 = 1$ . Suppose that firm 2 chooses  $x_2 = 0$  with probability  $p_1$  and  $x_2 = 1$  with probability  $p_2$ . Without loss of generality we assume  $p_1 \geq 1/2$ . Given firm 2's strategy above, the expected firm 1's profit when it chooses  $x_1 = 0$  is increasing in  $p_1$  since  $x_1 = 0$  is the best response to  $x_2 = 0$ . In the Proof of Lemma 2 we show that if  $p_1 = 1/2$ , the expected firm 1's profit when it chooses  $x_1 = 0$  is larger than that when it chooses  $x_1 = 1/2$ . Thus, given firm 2's strategy above,  $x_1 = 1/2$  is never best response for firm 1. Under these conditions, we have that there is no equilibrium where firm 1 takes a pure strategy.

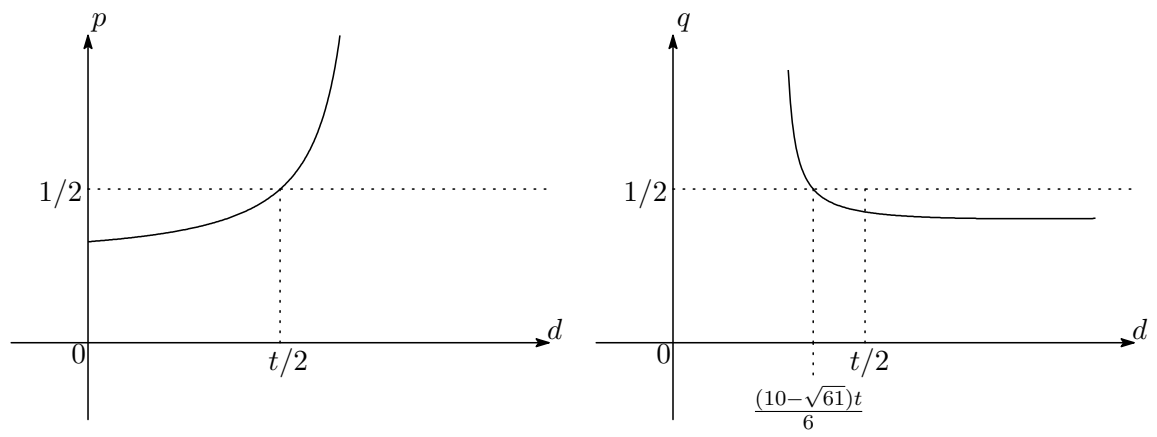
Next, we show that there is no equilibrium where firm 2 chooses a pure strategy. We prove it by contradiction. Suppose that firm 2 chooses  $x_2 = y$  with probability 1. As is discussed in Section 2, firm 2's best reply is unique and it is  $x_1 = x_2$ . Since no pure strategy equilibrium exists, it implies that no equilibrium exists if firm 2 takes a pure strategy. **Q.E.D.**



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**Figure 1:** The values of  $p$  and  $q$ .

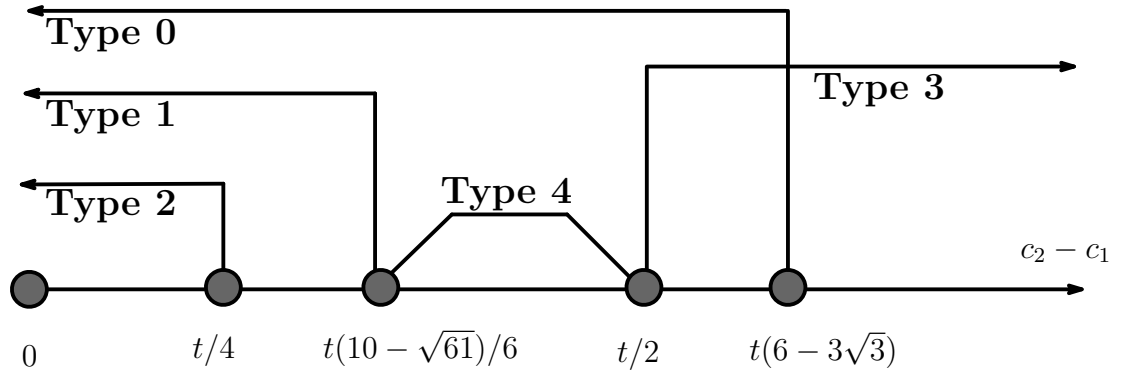


Figure 2: The degree of cost difference and the existence of each equilibrium type