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Cartel Stability in a Delivered Pricing Oligopoly

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Abstract

Using the spatial price discrimination framework, the relationship between the locations of firms and their ability to collude is investigated. Gupta and Venkatu (2002) show that in a duopoly model agglomeration at one point is the most stable location. We find that agglomeration stabilizes the cartel when there are three firms, too. When there are more than three firms, however, agglomeration of all firms is never the most stable location. With four firms, the following location pattern produced the most stable cartel: two firms at one point and the other two at the farthest point from the first two.

JEL classification numbers: D43, L41, R32

Key words: tacit collusion, spatial price discrimination, anti-trust

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1 Introduction

Issues related to cartel stability have been intensively discussed for many years. One concern is determining under what conditions the firms can easily collude. If these conditions become clear, anti-monopoly departments can more effectively monitor anti-competitive behavior. In this paper, we investigate the relationship between the locations of the firms and their ability to collude.

As a result of this analysis, the policy implications are presented. In many countries, a merger of firms having a large domestic market share was not allowed. Recently, the restrictions on such mergers have become quite weak. For example, in Japan, a merger of firms with a large share in a district is permissible, when strong competitors exist outside. The Japanese government assumes that a firm that has merged with others does not exercise its monopoly power when the potential competition against the outside firms exists, regardless of the size of the market share of the outside firm.[1] For this reason, the Japanese government is changing its anti-monopoly policy. However, the current anti-monopoly policy by the Japanese government might be very harmful. On the one hand, if a firm easily sustains collusion with a potential rival that is distant from the firm (the firm locating outside the district) but not with another firm that is closer (within the same district), the outside competitors cannot be considered to be real competitors. Therefore, a substantial welfare loss of the district would occur when inside firms merge. On the other hand, if it is difficult for a firm to sustain collusion with a potential rival that is distant from the firm but not with a nearby firm, the anti-monopoly policy described above would be entirely plausible. Regarding competition within a country, it would be valuable to know whether or not collusion is more likely occur when firms are in close proximity to one another.

Chang (1991) have already investigated the relationship between the locations of firms and their ability to collude.[2] He uses a Hotelling-type mill-pricing model, in which consumers pay the transport costs. Surprisingly, in his model, the more distance between duopolists, the more stable the cartel. In other words, the maximum distance between duopolists stabilizes the cartel.

[1] On May 31, 2004, the Japan Fair Trade Commission (JFTC) published new guidelines on mergers. The guidelines reflect the mitigation of merger restrictions and consider the potential and actual competitive pressures: import and entry, the presence of adjacent product and geographic markets, and competition in vertically related markets.

[2] See also Ross (1992).

His findings imply that collusive behavior is more likely between a foreign and a domestic firm than between domestic firms. Later, Gupta and Venkatu (2002) show that this result is crucially dependent on Chang's mill pricing model. They use a spatial price discrimination model (delivered pricing model), in which firms rather than consumers pay the transport costs. They derive a result that is opposite to Chang's (1991): the minimal distance between duopolists stabilizes a cartel.

We also use a spatial price discrimination framework, and investigate the relationship between the location of firms and their ability to collude. In contrast to Chang (1991), agglomeration at the center is the most stable location for a cartel when there are two or three firms. When there are more than three firms, the agglomeration of all firms is never the most stable location. With four firms, the following location pattern yields the most stable cartel: two firms at the one point and the other two at the farthest point from the first two. As an example, two market structures will be presented for consideration. In one market, all producers agglomerate at Nagoya District (central position in Japan). In another market, some firms locate at Tokyo District (east in Japan), and other locate at Osaka District (west in Japan). It is important to determine which market structure will effectively induce collusion. At first glance, the results of Gupta and Venkatu (2002) indicate that collusion can be sustained more easily in the former market structure. However, our result suggests that collusion can more easily be sustained in the latter market structure when at least four firms participate.

In the paper, we use a spatial price discrimination model. There are two reasons for investigating the spatial price discrimination model.[3] First, in international trade, firms often set different prices for each country (a phenomenon referred to as "market segmentation"). The spatial price discrimination model describes this situation well.[4] Second, it is not always realistic that consumers directly pay the transport costs (although they may indirectly pay the transport costs through the higher prices of the products).

[3] For an excellent survey of spatial price discrimination models, see Anderson, de Palma, and Thisse (1992, Ch. 8). For the applications of the spatial price discrimination models to the antitrust policies or regulations, see Gupta, Kats, and Pal (1995), Matsushima (2001), Matsushima and Matsumura (2003), and Matsumura (2003, 2004). In this paper we use a circular-city model as a main model. For the discussions of circular-city spatial price discrimination models, see Pal (1998) and Gupta (2004).

[4] Lommerud and Sørgard (2001) consider the relationship between trade liberalization and cartel stability.

The paper is organized as follows. In Section 2, we explain the basic setting. Section 3 discusses tacit collusion. Sections 4 and 5 present the results of circular-city and linear-city models respectively. Section 6 provides alternative analysis. Section 7 concludes the paper.

2 The Basic Setting

Let the firms locate at x_i ($i = 1, 2, ..., n$), with $x_1 \leq x_2 \leq ... \leq x_n$ and $x_i \in [0, 1]$. Let N denote the set of firms. We consider two models: circular city and linear city. In the circular-city model, there is a circular market of length 1 in which infinitely many consumers are uniformly distributed. The points on the circle are identified with numbers in $[0, 1]$, the north most point being 0 and the values increasing in a clockwise direction. Thus the north most point is considered both 0 and 1. In the linear-city model, there is a linear market of length 1 in which infinitely many consumers are uniformly distributed. The left edge is denoted as 0 (the right edge as 1) and each point x is a point on the line located at a distance from 0.

The demand at point $x \in [0, 1]$ is given by $q(p)$, where the inverse demand is given by $p(q)$ (p is the price of the homogeneous products and q is the total quantity sold by the firms). We assume that $p' < 0$. Each firm i produces at a constant marginal cost. Without loss of generality, we assume that the marginal production cost is zero. Let $d(x, x_i)$ denote the distance between x and x_i. To ship a unit of the product from its own location to a consumer at point x, each firm pays a transport cost $T(d(x, x_i))$, in which $T' > 0$ and $T(0) = 0$. In the circular-city model, the norm signifies the shorter distance of the two possible ways to transfer the goods along the perimeter. Let $T^L(x) \equiv \min_{i \in N} T(d(x, x_i))$ (the transport cost of the firm with the lowest transport cost). The monopoly price $p^M(x)$ is derived from the following first-order condition for the monopolist:

$$p + p'q = T^L(x). \tag{1}$$

We assume that the second-order condition is satisfied (marginal revenue is decreasing). We also assume that $p^M(x)$ when $T^L(x) = 0$ is larger than $T(1/2)$ in the circular-city model and $p^M(x)$ when $T^L(x) = 0$ is larger than $T(1)$ in the linear-city model. These conditions ensure that the equilibrium price under Bertrand competition is lower than the monopoly price regardless of the locations of the firms. Firms are able to discriminate among consumers since they control transportation. Consumer arbitrage is assumed to be prohibitively costly. These assumptions are standard in the literature.

3 Tacit Collusion

We investigate a model in which the locations of the firms are given exogenously in the beginning of the horizon, and the firms then engage in an infinitely repeated game thereafter.[5] Let δ denote the discount factor between periods. We examine the effect of the firms' locations on the sustainability of the joint-profit maximizing behavior by the firms. Along the punishment path, the firms are assumed to use the "grim trigger strategy" of Friedman (1971).[6]

First, we discuss joint-profit maximization. The joint profits are maximized when $p(x) = p^M(x)$ and the firm having minimum transport cost supplies for market x (recall that $p^M(x)$ is derived by (1)). Let $\Pi^M(x_1, x_2, ..., x_n)$ denote the maximized total joint profit obtained from all market $x \in [0, 1]$, given the location of the firms. We assume that each firm obtains $\Pi_i^M = \Pi^M/n$ when tacit collusion is maintained.[7]

Next, we discuss the deviation from the tacit collusion. Given the cooperative pricing of all other firms, firm 1 can increase its one-shot profit by deviating from the cartel. Firm 1 cuts the price infinitesimally from the monopoly price $p^M(x)$ at each market x and obtains the whole market. Thus, it obtains $\Big(p^M(x) - T(d(x, x_i))\Big)q(p^M(x))$ at this period. Let $\Pi_i^D(x_1, x_2, ..., x_n)$ be this one-shot profit of firm 1.

This deviation induces the competition thereafter. Let $p^C(x)$ denote the equilibrium price in the competitive phase. It is given by the unit transport cost of the firm with the second lowest cost. In the competitive phase, firm i obtains positive profit only from market x, such that $d(x, x_i) < d(x, x_j)$ for all $j \neq i$, and the profit from the market and the margin (price minus cost) is the difference in its transport cost and in the rival whose cost is the second lowest cost. $\Pi_i^C(x_1, x_2, ..., x_n)$ the profit

[5] If relocation costs are assumed to be negligible, the results change completely. For a discussion in the context of horizontal product differentiation, see Chang (1992) and Häckner (1995).

[6] This punishment strategy is not optimal (See Abreu (1988)). We use the grim trigger strategy for simplicity and tractability. We believe that the permanent Nash revision is one of very realistic punishment because of its simplicity. Many works adopt this strategy for analyzing stability of agreement. See, among others, Deneckere (1983), Chang (1991), Häckner (1994, 1995), Lambertini, Poddar and Sasaki (1998), and Maggi (1999).

[7] The firms may use the distribution of the total joint profit so as to stabilize the collusion rather than adopt equal distribution. We discuss this problem in Section 6.

of firm i in this competitive phase.

The tacit collusion is sustainable if and only if

$$\frac{\Pi^M}{n(1-\delta)} \geq \Pi_i^D + \frac{\delta \Pi_i^C}{1-\delta} \quad \forall i \in N.$$

Let δ_i^* be the δ satisfying the above equation with equality for $i \in N$. If $\delta < \delta_i^*$, firm i has an incentive for deviating from the collusion. We have

$$\delta_i^* = \frac{\Pi_i^D - \Pi^M/n}{\Pi_i^D - \Pi_i^C}. \tag{2}$$

Define $\delta^* \equiv \max_{i \in N} \delta_i^*$. The tacit collusion is sustainable if and only if $\delta \geq \delta^*$. Following the tradition of this field, we measure the stability of collusion by this minimum discount factor δ^*.

4 Results in the Circular-city Model

In this section, we present the results of the circular-city model, and, in the next section, we present similar results obtained in the liner-city model.

First, we present a result in the duopoly with linear demand, and a linear unit transport cost is used as a benchmark.

Proposition 1: *Suppose that $n = 2$, $p = 1 - q$ (linear demand function), and $T_i = td(x, x_i)$, where t is a positive constant and $t < 1$. Define $d(x_1, x_2) \equiv z$. (i) If t is sufficiently small, δ^* is increasing in z (monotonicity). (ii) There exists a parameter t, such that δ^* is not maximized when $z = 1/2$. (iii) δ^* is minimized when $z = 0$.*

Proof See Appendix.

Proposition 1(i) is a circular-city version of Gupta and Venkatu (2002). Proposition 1(i) implies that, if t is small enough, the agglomeration of two firms stabilizes the collusion most effectively. Proposition 1(ii) and (iii) state that δ^* can be non-monotonous with respect to the distance between two firms,[8] and Proposition 1 (iii) states that, even if δ^* is non-monotone, agglomeration still stabilizes the collusion most effectively.

[8] Using multi-product demand functions, Deneckere (1983) investigates the relationship between the substitutability of the products of firms and their ability to collude. He shows that a non-monotonous relationship appears in the price-setting model. Wernerfelt (1989) extends it to an oligopoly model and shows that a non-monotonous relationship also appears in the quantity-setting model if optimal penal code is used. Chang (1991) shows that such a non-monotonous relationship does not appear in his location-price model. Our results show that, in the context of spatial competition,

On the one hand, a decrease in the distance between two firms increases the deviation incentive. On the other hand, a decrease in the distance between two firms makes the competition between two firms more severe. As a result, the punishment effect is weakened. Proposition 1 indicates that the latter effect dominates the former effect, so agglomeration stabilizes the cartel. The intuition is similar to that of Gupta and Venkatu (2002) discussing a linear-city model. For more detailed discussions behind these two effects, see Section 3 in their paper.

We present our main results. We show that agglomeration of all firms does not stabilize the collusion if $n \geq 4$.

Proposition 2: *Suppose that $n \geq 4$. δ^* is not minimized when either n or $n-1$ firms locate at the same point.*

Proof See Appendix.

The agglomeration of two firms makes the competition perfect. Thus, the agglomeration of two firms is sufficient, and firms need not agglomerate in one location.

Proposition 2 states that total (all firms) and nearly total agglomeration (all but one firm) never minimize δ^*. This result indicates that no implications should be derived from the results of duopoly, in which the agglomeration of all firms stabilizes the collusion. For example, the following two market structures are considered. In one market, all producers agglomerate in the mid of the U.S.. In another market, some firms locate in the west side of the U.S., and others locate in the east side in the U.S.. The market structure that are most likely to induce collusive behavior are examined. At first glance, Proposition 1 indicates that collusion can be sustained more easily in the former market structure. However, Proposition 2 suggests that deriving such policy implications from Proposition 1 is quite misleading when the number of firms exceed three.

Proposition 2 says nothing about the triopoly case. We discuss a case with three firms in the linear demand and linear transport cost functions. This formulation is quite popular in the literature on delivered pricing models.

Proposition 3: *Suppose that $n = 3$, $p = 1 - q$ (linear demand function), and $T_i = td(x, x_i)$, where*

a result similar to that obtained be Deneckere (1983) (non-monotonousness) can be obtained from the spatial price discrimination models, not from the mill-pricing model.

t is a positive constant and $t < 1$. δ^* is minimized when $x_1 = x_2 = x_3$.

Proof See Appendix.

The result is similar to that of Proposition 1. As mentioned above, in the delivered pricing model, the punishment effect is more important to stabilize the cartel. In the case of a triopoly, when a firm moves to another point, the degree of the punishment for the other two firms remain unchanged, but that for the moving firm is weakened. The movement enhances the incentive of the moving firm to deviate from the cartel and produces instability. Therefore, agglomeration of three firms stabilizes a cartel.

5 Results in the Linear-city Model

In this section, we briefly discuss the results of the linear-city version. First, we present Proposition 4, which is a linear-city version of Proposition 2.

Proposition 4: *Suppose that $n \geq 4$. δ^* is not minimized when either n or $n-1$ firms locate at the same point.*

Proof See Appendix.

The proof of Proposition 4 is much more complicated than that of Proposition 2 in the linear-city version. However, we can derive a similar result in the linear city.

Second, we present the results of two, three, and four-firm cases. We explicitly solve the location pattern among symmetric locations, minimizing δ^* in the cased involving the linear-demand and linear-transport cost.

Proposition 5: *Suppose that $p = 1 - q$ and $T_i = td(x, x_i)$, where t is positive constant and $t < 1/2$. Consider the symmetric locations. (i) If $n = 2$, δ^* is minimized when $x_1 = x_2 = 1/2$. (ii) If $n = 3$, δ^* is minimized when $x_1 = x_2 = x_3 = 1/2$. (iii) If $n = 4$, δ^* is minimized when $x_1 = x_2 = 0$ and $x_3 = x_4 = 1$.*

In Proposition 5, (i) and (ii) are linear-city versions of Propositions 1 and 3. The proofs are similar to those of Propositions 1 and 3 and are, therefore, omitted. Proposition 5(iii) makes a sharp contrast with that presented by Gupta and Venkatu (2002) and our Propositions 1, 3, 5(i), and

5(ii). We present a proof in the Appendix and explain the intuition here.[9]

Consider the case of a duopoly. As is mentioned earlier, a decrease in $d(x_2, x_1)$ makes the punishment more severe and increases the deviation incentive. The former effect dominates the latter, so agglomeration decreases δ^*. Consider the case of four firms. Suppose that $x_1 = x_3$ and $x_2 = x_4$. A decrease in $d(x_2, x_1)$ does not affect the punishment effect because the competition is quite severe regardless of $d(x_2, x_1)$. A decrease in $d(x_2, x_1)$ increases the deviation incentive, for the same reason reported in the case of a duopoly. Thus, a decrease of $d(x_2, x_1)$ increases δ^*, so the maximal distance between firms 1 and 2 stabilizes the collusive behavior.

An example mentioned earlier is used again. The following two market structures are considered. In one product market, all producers agglomerate in the Central Area in Japan. In another product market, some firms locate in Osaka Area, and others locate in Tokyo Area. The market structure that are most likely to induce collusive behavior examined. Contrary to the implications of Propositions 1 and 3, Proposition 5 indicates that collusion can be sustained more easily in the latter market structure.

6 Alternative Approach of Profit Distribution in Collusive Phase

In the previous sections, we assume that all firms obtains the same profit in collusive phase. However, firms can strategically use the distribution of the total joint profit so as to stabilize the cartel (i.e., minimize the critical discount factor).[10] In this section, we assume that firms distribute the total joint profit so as to minimize the critical discount factor, and examine whether or not our main results, Propositions 2 and 4, hold.

Under the assumption, Π_i^M is determined so as to minimize the critical discount factor of firm

[9] In Proposition 5(iii), we restrict our attention to symmetric locations. Thus, we control two locations and consider two critical discount factors, δ_1^* and δ_2^*. When we consider a four-firm case in the circular-city model, we must control three locations and consider four discount factors and the analysis become quite complicated. For this reason, the results of four firms are presented in the linear-city model rather than in the circular-city model. For the same reason we does not solve the optimal location pattern in more than four firm models. We believe that restricting to symmetric locations in the model with four firms with a linear city is quite natural.

[10] We owe a referee for this point. We appreciate his(her) constructive comment.

i, which is given by

$$\frac{\Pi_i^D - \Pi_i^M}{\Pi_i^D - \Pi_i^C}. \tag{3}$$

Each of the critical discount factor becomes the same across all of the firms. Let δ^{**} denote the critical discount factor.

To highlight the relationship between δ^* and δ^{**}, consider a duopoly case. Suppose that δ_1^* (the critical discount factor of firm 1 in Section 3) is strictly larger than δ^{**}. Then $\delta_2^* < \delta^{**} < \delta_1^* = \delta^*$ and $\Pi_1^M > \Pi_2^M$. In other words, the firm having a stronger deviation incentive obtains a higher share of the monopoly profit.

We now discuss whether or not Propositions in the previous sections hold if we replace δ^* with δ^{**}. Consider the duopoly model with circular-city. Because of the symmetry of the circular-city, $\delta_1^* = \delta_2^* = \delta^*$, it is obvious that Proposition 1 holds if we replace δ^* with δ^{**}. We then investigate whether or not Propositions 2–5 hold if we replace δ^* with δ^{**}. We find that Proposition 2 holds and that similar results to Propositions 3, 4, and 5 hold.

Proposition 2': *Consider the circular-city model. Suppose that $n \geq 4$. δ^{**} is not minimized when either n or $n-1$ firms locate at the same point.*

Proof See Appendix.

Proposition 3': *Suppose that $n = 3$, $p = 1 - q$ (linear demand function), and $T_i = td(x, x_i)$, where t is a positive constant. If $t < 12/17$, δ^{**} is minimized when $x_1 = x_2 = x_3$.*

Proof The procedure to prove the proposition is similar to that of Proposition 3. Because it has much math, we omit it. The proof is available upon a request.

Proposition 4': *Consider the linear-city model. Suppose that $n \geq 4$. (i) δ^{**} is not minimized when n firms locate at the same point; and (ii) in addition if $\delta^{**} \geq 1/2$, δ^{**} is not minimized when $n-1$ firms locate at the same point.*

Proof See Appendix.

We discuss the additional condition $\delta^{**} \geq 1/2$ in Proposition 4'(ii). First, we emphasize that it is a sufficient but not a necessarily condition. Second, we believe that this condition is not so restrictive. Consider a duopoly model with a non-spatial homogenous goods market. In this case

$\delta^* = \delta^{**} = 1/2$. The condition that $\delta^{**} \geq 1/2$ implies that the market with more than two firms in our spatial model is less or equally stable than the non-spatial duopoly, and we believe it is a plausible assumption.

Proposition 5': *Suppose that $p = 1-q$ and $T_i = td(x, x_i)$, where t is positive constant and $t < 1/2$. Consider the symmetric locations. (i) When $n = 2$, δ^{**} is minimized when $x_1 = x_2 = 1/2$. (ii) When $n = 3$, δ^{**} is minimized when $x_1 = x_2 = x_3 = 1/2$, if and only if $t \leq 12/31$. δ^* is minimized when $x_1 = 0$, $x_2 = 1/2$, and $x_3 = 1/2$, if and only if $t \geq 12/31$. (iii) When $n = 4$, δ^{**} is minimized when $x_1 = x_2 = 0$ and $x_3 = x_4 = 1$.*

Proof The first case is similar to that of Proposition 5(i). The procedure to prove Proposition 5'(ii) and (iii) is similar to that of Proposition 5(ii) and (iii). Because it has much math, we omit it. The proof is available upon a request.

Proposition 5(ii)' is different from Proposition 5(ii). In the symmetric triopoly with linear city, δ^* is always minimized by the central agglomeration, while δ^{**} is minimized by the central agglomeration only when t is small. We explain the intuition.

Consider the symmetric triopoly with the linear-city, where $x_1 = 1 - x_3$ and $x_2 = 1/2$. We can show that δ^* is decreasing in x_1 when x_1 is close to 1/2 or 0. On the other hand, we can show that δ^{**} is decreasing in x_1 when x_1 is close to 1/2, and is increasing when x_1 is close to 0. In other words, δ^{**} is locally minimized when $x_1 = 0$ and $x_1 = 1/2$. First, we explain it.

When x_1 is close to 1/2, a decrease in x_1 increases monopoly profits and weakens the punishment effect. The latter effect is stronger than the former one. This is why both δ^{**} and δ^* is (at least locally) minimized when $x_1 = 1/2$.

When x_1 is close to 0, an increase in x_1 yields the different impacts on δ^{**} and δ^*. When x_1 is close to 0, $\delta_1^* = \delta_3^* < \delta_2^* (= \delta^*)$. An increase in x_1 strengthens the punishment effect for firm 2 and it effectively reduces $\delta_2^* (= \delta^*)$. However, an increase in x_1 weakens the punishment effect for firms 1 and 3. In the second model (with strategic distribution of monopoly profit), firms 1 and 3 must be compensated and the compensation increases δ^{**}. In short, in the first model, only δ_2^* affects δ^*, while in the second model, all of δ_1^*, δ_2^*, and δ_3^* affect δ^{**}. This is why δ^{**} is increasing in x_1 while δ^* is decreasing in x_1 when x_1 is close to 0.

Next, we explain why $x_1 = 0$ yields the smaller δ^{**} than $x_1 = 1/2$ does, when t is large. $x_1 = 1/2$ yields the strongest punishment effect. Regardless of $t > 0$, $x_1 = 0$ yields larger total joint profits than $x_1 = 1/2$. As t increases, the difference of the total joint profits between the two situation ($x_1 = 0$ and $x_1 = 1/2$) increases. Thus, if t is sufficiently large, this effect dominates the punishment effect. This is the reason why $x_1 = 0$ yields the smaller δ^{**} than $x_1 = 1/2$ does when t is large.

7 Concluding Remarks

Using the spatial price discrimination framework of Hamilton, Thisse, and Weskamp (1989), we investigate the relationship between the locations of firms and their ability to collude. First, we investigate duopoly and triopoly models. We find that firms can sustain collusive pricing most effectively when they agglomerate in one location. In the case of more than three firms, however, agglomeration never yields the most stable cartel. In the case of four firms, the following location pattern makes the collusive pricing most effectively: two firms locate at one point, and the other two locate at the farthest distance from the first two.

The results suggest that collusive pricing should be monitored in markets with multi-point agglomeration as well as one-point agglomeration. They also indicate that the location pattern that adds the most stability to a cartel depends on the number of the firms. If the number of firms in an industry is quite small, agglomeration in one country induces the most stable cartel; however, this is not true when there are many firms.

We believe that the extension of a duopoly to an oligopoly has important implications. The critical discount factor depends on the number of firms, and it is usually (although it is not always) on the increase in the number of firms. Thus, if the number of firms is large, it is possible that the critical discount factor will exceed the discount factor of each firm, which is usually independent from the number of the firms (and is dependent on a real interest rate, a time interval, and so on). In such a case, it is possible that one location pattern could yield a monopoly outcome by the cartel and another might not. Thus, from an antitrust viewpoint, the location pattern has significant importance.

Appendix

Proof of Proposition 1: Without loss of generality, we assume that $x_1 = 0$ and $x_2 = h$ ($h \in [0, 1/2]$). The monopoly profit of the firms is

$$\begin{aligned}
\Pi^M &= \int_0^{\frac{h}{2}} \left(\frac{1+tm}{2} - tm\right) \frac{1-tm}{2} dm + \int_{\frac{h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - t(h-m)\right) \frac{1-t(h-m)}{2} dm \\
&\quad + \int_h^{\frac{1+h}{2}} \left(\frac{1+t(m-h)}{2} - t(m-h)\right) \frac{1-t(m-h)}{2} dm \\
&\quad + \int_{\frac{1+h}{2}}^{1} \left(\frac{1+t(1-m)}{2} - t(1-m)\right) \frac{1-t(1-m)}{2} dm \\
&= \frac{12 - 6t + t^2 + 3t(4-t)(h-h^2)}{96}.
\end{aligned}$$

When one of the firms deviates from the cartel, it earns

$$\begin{aligned}
\Pi_i^D &= \int_0^{\frac{h}{2}} \left(\frac{1+tm}{2} - tm\right) \frac{1-tm}{2} dm + \int_{\frac{h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - tm\right) \frac{1-t(h-m)}{2} dm \\
&\quad + \int_h^{\frac{1}{2}} \left(\frac{1+t(m-h)}{2} - tm\right) \frac{1-t(m-h)}{2} dm \\
&\quad + \int_{\frac{1}{2}}^{\frac{1+h}{2}} \left(\frac{1+t(m-h)}{2} - t(1-m)\right) \frac{1-t(m-h)}{2} dm \\
&\quad + \int_{\frac{1+h}{2}}^{1} \left(\frac{1+t(1-m)}{2} - t(1-m)\right) \frac{1-t(1-m)}{2} dm \\
&= \frac{12 - 6t + t^2 - 2t^2(3-4h)h^2}{48}.
\end{aligned}$$

When the competition phase arises, each firm earns

$$\begin{aligned}
\Pi_i^C &= \int_0^{\frac{h}{2}} (t(h-m) - tm)(a - t(h-m)) dm \\
&\quad + \int_{\frac{1+h}{2}}^{\frac{1+2h}{2}} (t(m-h) - t(1-m))(1 - t(m-h)) dm \\
&\quad + \int_{\frac{1+2h}{2}}^{1} (t(1+h-m) - t(1-m))(1 - t(1+h-m)) dm \\
&= \frac{th(3(4-t) - 3(4+t)h + 8th^2)}{24}.
\end{aligned}$$

The critical discount factor is

$$\delta^* = \frac{12 - 6t + t^2 - 3t(4-t)h + 3t(4-5t)h^2 + 16t^2h^3}{2(12 - 6t + t^2 - 6t(4-t)h + 24th^2 - 8t^2h^3)}. \tag{4}$$

Differentiating δ^* with respect to h, we have

$$\frac{\partial \delta^*}{\partial h} = \frac{3(1-2h)tf(t,h)}{2(12-6t+t^2-6t(4-t)h+24th^2-8t^2h^3)^2}. \tag{5}$$

where $f(t,h) \equiv (4-(1+12h)t)(12-6t+t^2)+10(3-2h)h^2(4-t)t^2$. Since $3(1-2h)t \geq 0$ and the denominator in (5) is positive, we consider the sign of $f(t,h)$. If t is sufficiently small, $f(t,h) > 0$, thus $\partial\delta^*/\partial h$ in (5) is positive. This implies Proposition 1(i).

$f(t,h) > 0$ regardless of $t < 1$, $\partial\delta^*/\partial h$ in (5) is positive when $h = 0$. Since $\lim_{(t,h)\to(1,1/2)} f(t,h) < 0$, $\partial\delta^*/\partial h$ in (5) can be negative. These imply Proposition 1(ii).

Since $\partial f/\partial h < 0$, either $h = 1/2$ or $h = 0$ minimizes δ^*. From (4) we have

$$\delta^*(1/2) - \delta^*(0) = \frac{t(12-11t)}{8(12-6t+t^2-3t(4-t)+6t-t^2)} > 0. \tag{6}$$

This implies Proposition 1(iii). **Q.E.D.**

Proof of Proposition 2: First, we show that the location pattern $x_1 = x_2 =, ..., = x_n$ does not minimize δ^*. We prove it by contradiction. Suppose that the location $x_1 = x_2 =, ..., = x_n$ minimizes δ^*. Without loss of generality, we assume that $x_1 = 0$. Suppose that firm $n-1$ and firm n relocate and choose $x_{n-1} = x_n = 1/2$. Π_i^C is still zero for all firms. This relocation increases Π^M, and it decreases Π_i^D for $i \leq n-2$. By the symmetry of the circular city, after the relocation, $\Pi_1^D = \Pi_n^D$. Thus $\delta_1^* = \delta_n^*$. Since the above relocation reduces δ_1, it reduces δ^*, which is a contradiction.

Next, we show that the location pattern $x_1 < x_2 =, ..., = x_n$ does not minimize δ^*. We prove it by contradiction. Without loss of generality, we assume that $x_1 = 0$. Suppose that the location $x_1 < x_2 =, ..., = x_n$ minimizes δ^*. Under the locations, $\Pi_1^C > 0 = \Pi_2^C =, ..., = \Pi_n^D$. By the symmetry of the circular city, $\Pi_1^D = \Pi_2^D =, ..., = \Pi_n^D$. Thus, $\delta^* = \delta_1^* > \delta_2^* =, ..., = \delta_n^*$. Suppose that firm 2 relocates to 0. After the relocation, $\Pi_1^C = 0$. This relocation does not affect Π^M and Π_i^D for all $i \in N$. Thus, this relocation reduces δ_1^* without affecting δ_i^* for all $i \neq 1$, which is a contradiction. **Q.E.D.**

Proof of Proposition 3: Without loss of generality, we assume that $x_1 \leq h$, $x_2 = h$, and $x_3 = 1-h$, $(h \in [0, 1/3])$.[11] We consider two cases: (i) $h \in [0, 1/4]$ and (ii) $h \in [1/4, 1/3]$.

[11] We explain the reason why we can assume $x_1 \leq h$, $x_2 = h$, and $x_3 = 1-h$, $(h \in [0, 1/3])$ without loss of generality. Let us consider three points on the circle. Choose a pair of points whose distance is maximum among three possible

First, we consider the case in which $h \in [0, 1/4]$. The monopoly profit of the firms is

$$\begin{aligned}
\Pi^M &= \Bigg[\int_0^{x_1} \left(\frac{1+t(x_1-m)}{2} - t(x_1-m)\right)\frac{1-t(x_1-m)}{2}dm \\
&+ \int_{x_1}^{\frac{x_1+h}{2}} \left(\frac{1+t(m-x_1)}{2} - t(m-x_1)\right)\frac{1-t(m-x_1)}{2}dm \\
&+ \int_{\frac{x_1+h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - t(h-m)\right)\frac{1-t(h-m)}{2}dm \\
&+ \int_{h}^{\frac{1}{2}} \left(\frac{1+t(m-h)}{2} - t(m-h)\right)\frac{1-t(m-h)}{2}dm \\
&+ \int_{\frac{1}{2}}^{1-h} \left(\frac{1+t(1-h-m)}{2} - t(1-h-m)\right)\frac{1-t(1-h-m)}{2}dm \\
&+ \int_{1-h}^{\frac{2-h+x_1}{2}} \left(\frac{1+t(m-(1-h))}{2} - t(m-(1-h))\right)\frac{1-t(m-(1-h))}{2}dm \\
&+ \int_{\frac{2-h+x_1}{2}}^{1} \left(\frac{1+t(1+x_1-m)}{2} - t(1+x_1-m)\right)\frac{1-t(1+x_1-m)}{2}dm\Bigg] \\
&= \frac{12-6t+t^2-12tx_1^2+6t(4-t+tx_1^2)h-12(3-t)th^2-6t^2h^3}{48}.
\end{aligned}$$

If firm 3 deviates from the cartel, it earns

$$\begin{aligned}
\Pi_3^D &= \int_0^{x_1} \left(\frac{1+t(x_1-m)}{2} - t(m+h)\right)\frac{1-t(x_1-m)}{2}dm \\
&+ \int_{x_1}^{\frac{x_1+h}{2}} \left(\frac{1+t(m-x_1)}{2} - t(m+h)\right)\frac{1-t(m-x_1)}{2}dm \\
&+ \int_{\frac{x_1+h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - t(m+h)\right)\frac{1-t(h-m)}{2}dm \\
&+ \int_{h}^{\frac{1-2h}{2}} \left(\frac{1+t(m-h)}{2} - t(m+h)\right)\frac{1-t(m-h)}{2}dm \\
&+ \int_{\frac{1-2h}{2}}^{\frac{1}{2}} \left(\frac{1+t(m-h)}{2} - t(1-h-m)\right)\frac{1-t(m-h)}{2}dm \\
&+ \int_{\frac{1}{2}}^{1-h} \left(\frac{1+t(1-h-m)}{2} - t(1-h-m)\right)\frac{1-t(1-h-m)}{2}dm \\
&+ \int_{1-h}^{\frac{2-h+x_1}{2}} \left(\frac{1+t(m-(1-h))}{2} - t(m-(1-h))\right)\frac{1-t(m-(1-h))}{2}dm
\end{aligned}$$

pairs. Denote two points of the pair x_2 and x_3, and denote the remaining one point x_1 (Choose x_2 so as to satisfy $d(x_3, x_1) \geq d(x_2, x_1)$). Normalize the midpoint between x_2 and x_3 as zero (Choose the midpoint from the side in which x_1 exists). Then, denote $x_2 = h$ and $x_3 = 1-h$, where $h \leq 1/2$. h is never larger than $1/3$. Suppose that $h > 1/3$. Then, $d(x_3, x_2) = 1-2h$. $d(x_3, x_1) \geq h$. By definition $d(x_3, x_2) = 1-2h \geq d(x_3, x_1) \geq h$, and it contradicts $h > 1/3$.

$$
\begin{aligned}
&+\int_{\frac{2-h+x_1}{2}}^{1}\left(\frac{1+t(1+x_1-m)}{2}-t(m-(1-h))\right)\frac{1-t(1+x_1-m)}{2}dm \\
=&\ \frac{12-6t+t^2+6t^2x_1^3+6t^2x_1^2h-6t^2(4+x_1)h^2+58t^2h^3}{48}.
\end{aligned}
$$

When the competition phase arises, firm 3 earns

$$
\begin{aligned}
\Pi_3^C &= \int_{\frac{1}{2}}^{\frac{1+x_1+h}{2}}(t(m-h)-t(1-h-m))(1-t(m-h))dm \\
&+\int_{\frac{1+x_1+h}{2}}^{1-h}(t(1+x_1-m)-t(1-h-m))(1-t(1+x_1-m))dm \\
&+\int_{1-h}^{\frac{1-h+1+x_1}{2}}(t(1+x_1-m)-t(m-(1-h)))(1-t(1+x_1-m))dm \\
&= \frac{t(h+x_1)(12-3t-9tx_1+2tx_1^2-(24-3t-22tx_1)h+8th^2)}{24}.
\end{aligned}
$$

The critical discount factor of firm 3, δ_3^* is

$$
\delta_3^* = \frac{2[12-6((1-h)(1+3h)-x_1^2)t+(1+3h-42h^2+90h^3-3x_1(3h^2-2hx_1-3x_1^2))t^2]}{J},
$$

where $J = 3[12-6((1+4h-8h^2)+4(1-2h)x_1)t+(1+6h-30h^2+42h^3+6x_1(1+2h-11h^2)+6(3-7h)x_1^2+2x_1^3)t^2] > 0$.

Substituting $x_1 = h = 0$ into it yields $\delta_3^* = 2/3$. By the symmetry, $\delta_1^* = \delta_2^* = \delta_3^* = 2/3$ when $x_1 = h = 0$.

We prove that $x_1 = h = 0$ minimize δ^* by showing that δ_3^* is never smaller than $2/3$ for any h and x_1.

$$
\delta_3^* - \frac{2}{3} = \frac{2t}{J}(f(h,x_1)-g(h,x_1)t),
$$

where $f(h,x_1) \equiv 6((2-5h)h+4(1-2h)x+x_1^2)$ and $g(h,x_1) \equiv 3h(1+4h-16h^2)+3(2+4h-19h^2)x_1+6(3-8h)x_1^2-7x_1^3$. t, J and $f(h,x_1)$ are positive. If $g(h,x_1)$ is non-positive, $\delta_3^*-2/3$ is positive. If $g(h,x_1)$ is positive, $f(h,x_1)-g(h,x_1)t$ is minimized when $t=1$. Thus, $\delta_3^* geq 2/3$ for any $t<1$ if it holds when $t=1$.

$$
\left[\delta_3^* - \frac{2}{3}\right]_{t=1} = \frac{3h(1-2h)(3-8h)+3(6-20h+19h^2)x_1-12(1-4h)x_1^2+7x_1^3}{2J}.
$$

This is non-negative for any h and x_1 and strictly positive unless $h = x_1 = 0$.

Second, we consider the case in which $h \in [1/4, 1/3]$.

The monopoly profit of the firms is

$$\begin{aligned}
\Pi^M \;=\; & \Big[\int_0^{x_1} \left(\frac{1+t(x_1-m)}{2} - t(x_1-m)\right)\frac{1-t(x_1-m)}{2}dm \\
& + \int_{x_1}^{\frac{x_1+h}{2}} \left(\frac{1+t(m-x_1)}{2} - t(m-x_1)\right)\frac{1-t(m-x_1)}{2}dm \\
& + \int_{\frac{x_1+h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - t(h-m)\right)\frac{1-t(h-m)}{2}dm \\
& + \int_{h}^{\frac{1}{2}} \left(\frac{1+t(m-h)}{2} - t(m-h)\right)\frac{1-t(m-h)}{2}dm \\
& + \int_{\frac{1}{2}}^{1-h} \left(\frac{1+t(1-h-m)}{2} - t(1-h-m)\right)\frac{1-t(1-h-m)}{2}dm \\
& + \int_{1-h}^{\frac{2-h+x_1}{2}} \left(\frac{1+t(m-(1-h))}{2} - t(m-(1-h))\right)\frac{1-t(m-(1-h))}{2}dm \\
& + \int_{\frac{2-h+x_1}{2}}^{1} \left(\frac{1+t(1+x_1-m)}{2} - t(1+x_1-m)\right)\frac{1-t(1+x_1-m)}{2}dm\Big] \\
= \; & \frac{12-6t+t^2-12tx_1^2+6t(4-t+tx_1^2)h-12(3-t)th^2-6t^2h^3}{48}.
\end{aligned}$$

When $x_1 < 1-3h$, if firm 3 deviates from the cartel, it earns

$$\begin{aligned}
\Pi_3^D \;=\; & \int_0^{x_1} \left(\frac{1+t(x_1-m)}{2} - t(m+h)\right)\frac{1-t(x_1-m)}{2}dm \\
& + \int_{x_1}^{\frac{x_1+h}{2}} \left(\frac{1+t(m-x_1)}{2} - t(m+h)\right)\frac{1-t(m-x_1)}{2}dm \\
& + \int_{\frac{x_1+h}{2}}^{\frac{1-2h}{2}} \left(\frac{1+t(h-m)}{2} - t(m+h)\right)\frac{1-t(h-m)}{2}dm \\
& + \int_{\frac{1-2h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - t(1-h-m)\right)\frac{1-t(h-m)}{2}dm \\
& + \int_{h}^{\frac{1}{2}} \left(\frac{1+t(m-h)}{2} - t(1-h-m)\right)\frac{1-t(m-h)}{2}dm \\
& + \int_{\frac{1}{2}}^{1-h} \left(\frac{1+t(1-h-m)}{2} - t(1-h-m)\right)\frac{1-t(1-h-m)}{2}dm \\
& + \int_{1-h}^{\frac{2-h+x_1}{2}} \left(\frac{1+t(m-(1-h))}{2} - t(m-(1-h))\right)\frac{1-t(m-(1-h))}{2}dm \\
& + \int_{\frac{2-h+x_1}{2}}^{1} \left(\frac{1+t(1+x_1-m)}{2} - t(m-(1-h))\right)\frac{1-t(1+x_1-m)}{2}dm \\
= \; & \frac{12-6t+3t^2+6t^2x_1^3-6t^2(4-x_1^2)h+6t^2(12-x_1)h^2-70t^2h^3}{48}.
\end{aligned}$$

When $1-3h < x_1 < 1/2-h$, if firm 3 deviates from the cartel, it earns

$$\begin{aligned}
\Pi_3^D &= \int_0^{x_1} \left(\frac{1+t(x_1-m)}{2} - t(m+h)\right) \frac{1-t(x_1-m)}{2} dm \\
&+ \int_{x_1}^{\frac{1-2h}{2}} \left(\frac{1+t(m-x_1)}{2} - t(m+h)\right) \frac{1-t(m-x_1)}{2} dm \\
&+ \int_{\frac{1-2h}{2}}^{\frac{x_1+h}{2}} \left(\frac{1+t(m-x_1)}{2} - t(1-h-m)\right) \frac{1-t(m-x_1)}{2} dm \\
&+ \int_{\frac{x_1+h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - t(1-h-m)\right) \frac{1-t(h-m)}{2} dm \\
&+ \int_{h}^{\frac{1}{2}} \left(\frac{1+t(m-h)}{2} - t(1-h-m)\right) \frac{1-t(m-h)}{2} dm \\
&+ \int_{\frac{1}{2}}^{1-h} \left(\frac{1+t(1-h-m)}{2} - t(1-h-m)\right) \frac{1-t(1-h-m)}{2} dm \\
&+ \int_{1-h}^{\frac{2-h+x_1}{2}} \left(\frac{1+t(m-(1-h))}{2} - t(m-(1-h))\right) \frac{1-t(m-(1-h))}{2} dm \\
&+ \int_{\frac{2-h+x_1}{2}}^{1} \left(\frac{1+t(1+x_1-m)}{2} - t(m-(1-h))\right) \frac{1-t(1+x_1-m)}{2} dm \\
&= \frac{12-6t+t^2+2t^2x_1(3-3x_1+4x_1^2)-6t^2(1+6x_1-4x_1^2)h-6t^2(3+8x_1)h^2-16t^2h^3}{48}.
\end{aligned}$$

When $1/2-h < x_1$, if firm 3 deviates from the cartel, it earns

$$\begin{aligned}
\Pi_3^D &= \int_0^{\frac{1-2h}{2}} \left(\frac{1+t(x_1-m)}{2} - t(m+h)\right) \frac{1-t(x_1-m)}{2} dm \\
&+ \int_{\frac{1-2h}{2}}^{x_1} \left(\frac{1+t(x_1-m)}{2} - t(1-h-m)\right) \frac{1-t(x_1-m)}{2} dm \\
&+ \int_{x_1}^{\frac{x_1+h}{2}} \left(\frac{1+t(m-x_1)}{2} - t(1-h-m)\right) \frac{1-t(m-x_1)}{2} dm \\
&+ \int_{\frac{x_1+h}{2}}^{h} \left(\frac{1+t(h-m)}{2} - t(1-h-m)\right) \frac{1-t(h-m)}{2} dm \\
&+ \int_{h}^{\frac{1}{2}} \left(\frac{1+t(m-h)}{2} - t(1-h-m)\right) \frac{1-t(m-h)}{2} dm \\
&+ \int_{\frac{1}{2}}^{1-h} \left(\frac{1+t(1-h-m)}{2} - t(1-h-m)\right) \frac{1-t(1-h-m)}{2} dm \\
&+ \int_{1-h}^{\frac{2-h+x_1}{2}} \left(\frac{1+t(m-(1-h))}{2} - t(m-(1-h))\right) \frac{1-t(m-(1-h))}{2} dm \\
&+ \int_{\frac{2-h+x_1}{2}}^{1} \left(\frac{1+t(1+x_1-m)}{2} - t(m-(1-h))\right) \frac{1-t(1+x_1-m)}{2} dm \\
&= \frac{12-6t+3t^2-2t^2x_1(3-9x_1+4x_1^2)-6t^2(3-2x_1+4x_1^2)h+42t^2h^2-32t^2h^3}{48}.
\end{aligned}$$

When $x_1 < 1-3h$, if the competition phase arises, firm 3 earns

$$
\begin{aligned}
\Pi_3^C &= \int_{\frac{1}{2}}^{\frac{1+x_1+h}{2}} (t(m-h)-t(1-h-m))(1-t(m-h))dm \\
&\quad + \int_{\frac{1+x_1+h}{2}}^{1-h} (t(1+x_1-m)-t(1-h-m))(1-t(1+x_1-m))dm \\
&\quad + \int_{1-h}^{\frac{1-h+1+x_1}{2}} (t(1+x_1-m)-t(m-(1-h)))(1-t(1+x_1-m))dm \\
&= \frac{t(h+x_1)(12-3t-9tx_1+2tx_1^2-(24-3t-22tx_1)h+8th^2)}{24}.
\end{aligned}
$$

When $x_1 > 1-3h$, if the competition phase arises, firm 3 earns

$$
\begin{aligned}
\Pi_3^C &= \int_{\frac{1}{2}}^{1-h} (t(m-h)-t(1-h-m))(1-t(m-h))dm \\
&\quad + \int_{1-h}^{\frac{1+x_1+h}{2}} (t(m-h)-t(m-(1-h)))(1-t(m-h))dm \\
&\quad + \int_{\frac{1+x_1+h}{2}}^{\frac{1-h+1+x_1}{2}} (t(1+x_1-m)-t(m-(1-h)))(1-t(1+x_1-m))dm \\
&= \frac{t(1-2h)(2t+3(4-3t)x_1-3tx_1^2+(12-17t+12tx_1)h+23th^2)}{24}.
\end{aligned}
$$

We show that the critical discount factor of firm 3 is larger than 2/3. The critical discount factor of firm 3 is

$$
\begin{aligned}
\delta_3^* &= \frac{2[12-6((1-h)(1+3h)-x_1^2)t]}{K} \\
&\quad + \frac{2[(4-33h+102h^2-102h^3-3x_1(3h^2-2hx_1-3x_1^2))t^2]}{K}, \\
&\quad \text{if } x_1 < 1-3h, \\
&= \frac{2[12-6((1-h)(1+3h)-x_1^2)t]}{L} \\
&\quad + \frac{2[(1-6h+21h^2-21h^3+9(1-2h)(1-4h)x_1-3(3-11h)x_1^2+12x_1^3)t^2]}{L}, \\
&\quad \text{if } 1-3h \le x_1 < 1/2-h, \\
&= \frac{2[12-6((1-h)(1+3h)-x_1^2)t]}{M} \\
&\quad + \frac{2[(4-24h+57h^2-45h^3-9(1-2h)x_1+3(9-13h)x_1^2-12x_1^3)t^2]}{M}, \\
&\quad \text{if } 1/2-h \le x_1.
\end{aligned}
$$

$K = 3[12-6(1+4h-8h^2+4(1-2h)x_1)t+(3-18h+66h^2-86h^3-6(1+2h-11h^2)x_1+2(9-21h+x_1)x_1^2)t^2]$, $L = 3[12-6(1+4h-8h^2+4(1-2h)x_1)t+(-3+36h-96h^2+76h^3+24(1-$

$2h)^2x_1+4(3h+2x_1)x_1^2)t^2]$, $M=3[12-6(1+4h-8h^2+4(1-2h)x_1)t+(-1+24h-72h^2+60h^3+12(1-2h)^2x_1+12(2-3h)x_1^2-8x_1^3)t^2]$.

$$
\begin{aligned}
\delta_3^*-\frac{2}{3} &= \frac{12[2h-5h^2+4(1-2h)x_1+x_1^2]t}{K} \\
&\quad +\frac{2[1-15h+36h^2-16h^3-3(2+4h-19h^2)x_1-6(3-8h)x_1^2+7x_1^3]t^2}{K}, \\
&\quad \text{if } x_1<1-3h, \\
&= \frac{12[2h-5h^2+4(1-2h)x_1+x_1^2]t}{L} \\
&\quad +\frac{2[4-42h+117h^2-97h^3-3(5-14h+8h^2)x_1-3(3-7h)x_1^2+4x_1^3]t^2}{L}, \\
&\quad \text{if } 1-3h\le x_1<1/2-h, \\
&= \frac{12[2h-5h^2+4(1-2h)x_1+x_1^2]t}{M} \\
&\quad +\frac{2[5-48h+129h^2-105h^3-3(7-8h)(1-2h)x_1+3(1-h)x_1^2-4x_1^3]t^2}{M}, \\
&\quad \text{if } 1/2-h\le x_1.
\end{aligned}
$$

If $\delta_3^*-\frac{2}{3}$ is positive when $t=1$, δ_3^* is larger than 2/3 for any h and x_1 (for the same reason in case (i)).

$$
\begin{aligned}
\left[\delta_3^*-\frac{2}{3}\right]_{t=1} &= \frac{[1-3h+6h^2-16h^3+3(6-20h+19h^2)x_1+12(4h-1)x_1^2+7x_1^3]}{K}, \\
&\quad \text{if } x_1<1-3h, \\
&= \frac{[4-30h+87h^2-97h^3+3(1-2h)(3+4h)x_1+3(7h-1)x_1^2+4x_1^3]}{L}, \\
&\quad \text{if } 1-3h\le x_1<1/2-h, \\
&= \frac{[5-36h+99h^2-105h^3+3(1-2h)(1+8h)x_1+3(3-h)x_1^2-4x_1^3]}{2M}, \\
&\quad \text{if } 1/2-h\le x_1.
\end{aligned}
$$

This is positive for any x_1 and h. **Q.E.D.**

Proof of Proposition 4: First, we show that the location pattern $x_1=x_2=,...,=x_n$ does not minimize δ^*. We prove it by contradiction. Suppose that the location $x_1=x_2=,...,=x_n$ minimizes δ^*. Without loss of generality, we assume that $x_n\ge 1/2$. Suppose that firm 1 and firm 2 relocate and choose $x_1 = x_2 = 0$. Π_i^D is still zero for all firms. This relocation increases Π^M and decreases Π_i^D for $i \geq 3$. We compare Π_1^D with Π_n^D. $\pi_1^D(y) = \pi_n^D(x_n - y)$ for all $y \leq x_n$. In other words, both firms obtain the same profits from markets lying to the left of firm n's location. $\pi_1^D(y) < \pi_n^D(y)$ for all $y > x_n$ because $T_1(d(y,0)) > T_n(d(y,x_n))$. In other words, firm 3 obtains a larger profit than firm 1 from markets to the right of firm n's location. Thus $\Pi_1^D \leq \Pi_n^D$, so $\delta_1^* \leq \delta_n^*$. Since the relocation decreases δ_n^* and $\delta_n^* > \delta_1^*$, the relocation decreases δ^*, which is a contradiction.

Next, we show that the location pattern $x_1 < x_2 =, ..., = x_n$ does not minimize δ^*. We prove it by contradiction. Suppose that the location $x_1 < x_2 =, ..., = x_n$ minimizes δ^*. Suppose that $\delta_1^* > \delta_n^*$. If firm 2 relocates (chooses) $x_2 = x_1$. It reduces Π_1^C but does not change Π^M or Π_i^D. Thus, it reduces δ_1^* without changing δ_n^*, so it reduces δ^*, which is a contradiction. Therefore, if the location $x_1 < x_2 =, ..., = x_n$ minimizes δ^*, then $\delta_1^* \leq \delta_n^*$.

We consider two cases: (i) $0 = x_1 < x_2 =, \ldots, = x_n$ and (ii) $0 < x_1 < x_2 =, \ldots, = x_n$. We now show that, in both cases, δ^* is not minimized. We prove them by contradiction.

Consider case (i). We assume that $0 = x_1 < x_2 =, \ldots, = x_n$ and $\delta_1^* \leq \delta_n^*$, and that the location pattern minimizes δ^*. We relocate firm 2 at $x_1 = 0$. This relocation decreases δ_1^* without affecting δ_n^*, and, then, $\delta_1^* < \delta_n^*$. We again relocate firms 1 and 2 from 0 to ε, where ε is positive and sufficiently small. The second-round relocation increases Π^M and reduces Π_n^D. Thus, it reduces δ_n^*. The second-round relocation increases δ_1^*. Since $\delta_1^* < \delta_n^*$ and these are continuous with respect to x_1 and x_2, there exists $\varepsilon(> 0)$, such that $\delta_1^* \leq \delta_n^*$. Therefore, the second relocation decreases δ^*, which is a contradiction.

Consider case (ii). We assume that $0 < x_1 < x_2 =, ..., = x_n$, $\delta_1^* \leq \delta_n^*$ and that the location pattern minimizes δ^*. We relocate firm 2 to 0. This relocation decreases δ_1^* and δ_n^* since it increases Π^M and Π_i^D for $i = 1, 3, ..., n$. After the relocation, $\Pi_2^C < \Pi_1^C$ and $\Pi_2^D < \Pi_1^D$, so $\delta_2^* < \delta_1^*$. Since the relocation decreases δ_1^* and δ_n^* and $\delta_2^* < \delta_1^*$, it decreases δ^*, which is a contradiction. **Q.E.D.**

Proof of Proposition 5(iii): The joint profit is maximized by

$$(p_1(x), p_2(x), p_3(x), p_4(x)) = \begin{cases} (\frac{1+td(x,x_1)}{2}, +\infty, +\infty, +\infty) & \text{if } x \in \left[0, \frac{x_1+x_2}{2}\right], \\ (+\infty, \frac{1+td(x,x_2)}{2}, +\infty, +\infty) & \text{if } x \in \left[\frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}\right], \\ (+\infty, +\infty, \frac{1+td(x,x_3)}{2}, +\infty) & \text{if } x \in \left[\frac{x_2+x_3}{2}, \frac{x_3+x_4}{2}\right], \\ (+\infty, +\infty, +\infty, \frac{1+td(x,x_4)}{2}) & \text{if } x \in \left[\frac{x_3+x_4}{2}, 1\right]. \end{cases} \quad (7)$$

The joint-profit maximization is achieved by the market division because it economizes the transport costs. Note that we assume the symmetric locations, i.e., $x_1 = 1 - x_4$ and $x_2 = 1 - x_3$.

$$\begin{aligned}
\Pi^M &= \Bigg[\int_0^{x_1} \frac{(1-t(x_1-x))^2}{4}dx + \int_{x_1}^{\frac{x_1+x_2}{2}} \frac{(1-t(x-x_1))^2}{4}dx + \int_{\frac{x_1+x_2}{2}}^{x_2} \frac{(1-t(x_2-x))^2}{4}dx \\
&+ \int_{x_2}^{\frac{x_2+x_3}{2}} \frac{(1-t(x-x_2))^2}{4}dx + \int_{\frac{x_2+x_3}{2}}^{x_3} \frac{(1-t(x_3-x))^2}{4}dx \\
&+ \int_{x_3}^{\frac{x_3+x_4}{2}} \frac{(1-t(x-x_3))^2}{4}dx + \int_{\frac{x_3+x_4}{2}}^{x_4} \frac{(1-t(x_4-x))^2}{4}dx + \int_{x_4}^{1} \frac{(1-t(x-x_4))^2}{4}dx\Bigg] \\
&= \frac{12 - 6(1-4(1+x_1)x_2 + 6(x_1^2+x_2^2))t + (1-6x_2+12x_2^2+6(x_1-x_2)(x_1+x_2)^2)t^2}{48}. \quad (8)
\end{aligned}$$

We consider the one-shot gain when firm 2 deviates from the collusive behavior. Consider the firm 2's deviation. Given $p_i(x)$ in (7), firm 2 chooses $p_2(x)$ so as to maximize its own profits. Let p_2^D denote the price of firm 2 when it deviates.

$$p_2^D = \begin{cases} \frac{1+td(x,x_1)}{2}(-\varepsilon) & \text{if } x \in \left[0, \frac{x_1+x_2}{2}\right], \\ \frac{1+td(x,x_2)}{2} & \text{if } x \in \left[\frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}\right], \\ \frac{1+td(x,x_3)}{2}(-\varepsilon) & \text{if } x \in \left[\frac{x_2+x_3}{2}, \frac{x_3+x_4}{2}\right], \\ \frac{1+td(x,x_4)}{2}(-\varepsilon) & \text{if } x \in \left[\frac{x_3+x_4}{2}, 1\right]. \end{cases} \quad (9)$$

Let Π_2^D denote the one-shot profit of the deviator.

$$\begin{aligned}
\Pi_2^D &= \int_0^{x_1} \left(\frac{1+t(x_1-x)}{2} - t(x_2-x)\right)\frac{1-t(x_1-x)}{2}dx \\
&+ \int_{x_1}^{\frac{x_1+x_2}{2}} \left(\frac{1+t(x-x_1)}{2} - t(x_2-x)\right)\frac{1-t(x-x_1)}{2}dx \\
&+ \int_{\frac{x_1+x_2}{2}}^{x_2} \frac{(1-t(x_2-x))^2}{4}dx + \int_{x_2}^{\frac{x_2+x_3}{2}} \frac{(1+t(x-x_2))^2}{4}dx \\
&+ \int_{\frac{x_2+x_3}{2}}^{x_3} \left(\frac{1+t(x_3-x)}{2} - t(x-x_2)\right)\frac{1-t(x_3-x)}{2}dx \\
&+ \int_{x_3}^{\frac{x_3+x_4}{2}} \left(\frac{1+t(x-x_3)}{2} - t(x-x_2)\right)\frac{1-t(x-x_3)}{2}dx \\
&+ \int_{\frac{x_3+x_4}{2}}^{x_4} \left(\frac{1+t(x_4-x)}{2} - t(x-x_2)\right)\frac{1-t(x_4-x)}{2}dx \\
&+ \int_{x_4}^{1} \left(\frac{1+t(x-x_4)}{2} - t(x-x_2)\right)\frac{1-t(x-x_4)}{2}dx \\
&= \frac{6-6(1-2x_2+2x_2^2)t}{24} \quad (10) \\
&+ \frac{((1-x_1)(1+x_1+10x_1^2) - 6(1+x_1)x_2 + 3(5+2x_1)x_2^2 - 12x_2^3)t^2}{24}
\end{aligned}$$

If firm 2 deviates from the collusive behavior, it earns Π_2^D now and Π_2^C thereafter. Π_2^C is:

$$\Pi_2^C = \begin{cases} \int_{\frac{x_1+x_2}{2}}^{x_2} (t(x-x_1) - t(x_2-x))(1-t(x-x_1))dx \\ + \int_{x_2}^{\frac{x_1+x_3}{2}} (t(x-x_1) - t(x-x_2))(1-t(x-x_1))dx \\ + \int_{\frac{x_1+x_3}{2}}^{\frac{x_2+x_3}{2}} (t(x_3-x) - t(x-x_2))(1-t(x_3-x))dx \\ = \frac{t(x_2-x_1)(12(1-2x_2)+(-3+9x_1+2x_1^2+(3-22x_1)x_2+8x_2^2)t)}{24}, \quad \text{if } x_2 \le \frac{x_1+x_3}{2}, \\ \int_{\frac{x_1+x_2}{2}}^{\frac{x_1+x_3}{2}} (t(x-x_1) - t(x_2-x))(1-t(x-x_1))dx \\ + \int_{\frac{x_1+x_3}{2}}^{x_2} (t(x_3-x) - t(x_2-x))(1-t(x_3-x))dx \\ + \int_{x_2}^{\frac{x_2+x_3}{2}} (t(x_3-x) - t(x-x_2))(1-t(x_3-x))dx \\ = \frac{t(1-2x_2)(12(x_2-x_1)+(2+9x_1-3x_1^2-(17+12x_1)x_2+23x_2^2)t)}{24}, \quad \text{if } x_2 \ge \frac{x_1+x_3}{2}. \end{cases} \tag{11}$$

Substituting these results and $i = 2$ into (2), we have that, when $x_2 < (1+x_1)/3$,

$$\begin{aligned} \delta_2^* \quad = \quad & \frac{36 - (6(7-6x_1^2) - 24(3-x_1)x_2 + 60x_2^2)t}{8Y} \\ & + \frac{(7 + 72x_1^2 - 86x_1^3 - 6(1+x_1)(7+x_1)x_2 + 54(2+x_1)x_2^2 - 90x_2^3)t^2}{8Y},, \end{aligned} \tag{12}$$

where $Y \equiv 6 - 6(1 - 2x_1 + 4x_1x_2 - 2x_2^2)t + (1 - 3x_1 + 18x_1^2 - 8x_1^3 - 3(1 + 4x_1 + 8x_1^2)x_2 + 12(1 + 3x_1)x_2^2 - 20x_2^3)t^2$, and that when $x_2 \ge (1+x_1)/3$,

$$\begin{aligned} \delta_2^* \quad = \quad & \frac{36 - (6(7-6x_1^2) - 24(3-x_1)x_2 + 60x_2^2)t}{8Z} \\ & + \frac{(7 + 72x_1^2 - 86x_1^3 - 6(1+x_1)(7+x_1)x_2 + 54(2+x_1)x_2^2 - 90x_2^3)t^2}{8Z}, \end{aligned} \tag{13}$$

where $Z \equiv 6 - 6(1 - 2x_1 + 4x_1x_2 - 2x_2^2)t - (1 + 9x_1 - 12x_1^2 + 10x_1^3 - 3(5 + 8x_1 - 2x_1^2)x_2 + 6(7 + 3x_1)x_2^2 - 34x_2^3)t^2$.

First, we will show that, given x_1, either $x_2 = x_1$ or $x_2 = 1/2$ minimizes δ_2^* under the assumptions that $x_2 \in [x_1, 1/2]$ and $t \in (0, 1/2]$. Since the derivation is quite technical, we show it at the last part of the proof and skip it for a moment.

Second, we show δ_2^* is minimized at $x_2 = x_1$ by comparing $\delta_2^*(x_1, x_1)$ and $\delta_2^*(x_1, 1/2)$.

$$\delta_2^*|_{x_2=x_1} \quad = \quad \frac{36 - 6(7 - 12x_1 + 8x_1^2)t + (7 - 42x_1 + 132x_1^2 - 128x_1^3)t^2}{8A_\alpha}, \tag{14}$$

$$\delta_2^*|_{x_2=1/2} = \frac{144-12(7+4x_1-12x_1^2)t+(7-42x_1+276x_1^2-344x_1^3)t^2}{8A_\beta},$$

$$\delta_2^*|_{x_2=1/2}-\delta_2^*|_{x_2=x_1} = \frac{3t(1-2x_1)^2(4-(1-2x_1)t)(3+6x_1(1-3x_1)t-x_1^2(3-8x_1)t^2)}{4A_\alpha A_\beta},$$

where $A_\alpha \equiv (6-6(1-2x_1+2x_1^2)t+(1-6x_1+18x_1^2-16x_1^3)t^2)$ and $A_\beta \equiv (24-12t+(1-6x_1+36x_1^2-40x_1^3)t^2)$. For any $t \in (0,1/2]$ and $x_1 \in [0,1/2]$, this is positive. Therefore, given x_1, δ_2^* is minimized at $x_2 = x_1$.

Third, we show that δ^* is minimized at $x_2 = x_1$. When $x_1 = x_2$, $\delta_1^* = \delta_2^*$. An increase of x_2 from $x_2 = x_1$ never decreases δ_2^*, so it never decreases $\delta^* = \max(\delta_1^*, \delta_2^*)$. Thus, $x_2 = x_1$ minimizes δ^*.

Fourth, we show that δ^* is minimized at $x_1 = x_2 = 0$. When $x_1 = x_2$, $\delta^* = \delta_1^* = \delta_2^*$. The value is $\delta_2^*|_{x_2=x_1}$ in (14). Differentiating it with respect to x_1, we have:

$$\frac{\partial \delta_2^*|_{x_2=x_1}}{\partial x_1} = \frac{3t[24x_1+3(1-4x_1-8x_1^2)t-(1-30x_1^2+64x_1^3-64x_1^4)t^2+x_1(1-4x_1)(1-3x_1+4x_1^2)t^3]}{2A_\alpha^2}.$$

For any $t \in (0,1/2]$ and $x_1 \in [0,1/2]$, this is positive. This implies that δ^* is minimized at $x_1 = x_2 = 0$.

Finally, we complete the skipped technical part of the proof. We show that δ_2^* is minimized at either $x_2 = x_1$ or $x_2 = 1/2$.

First, we show that for the range $x_2 \in [(x_1+1)/3, 1/2]$, δ_2^* is minimized when $x_2 = 1/2$.

Differentiating δ_2^* with respect to x_2, we have:

$$\frac{\partial \delta_2^*}{\partial x_2} = \frac{3k(x_2)}{8Y^2}, \tag{15}$$

where $k(x_2) \equiv 4t^2(80+9(1-2x_1)t)x_2^4-4t^2(8(51-62x_1)-(1-2x_1)(13-62x_1)t)x_2^3-4t(9(57-16x_1)-3(237-178x_1-138x_1^2)t+(49-255x_1+921x_1^2-920x_1^3)t^2)x_2^2-4(132-6(84+7x_1-12x_1^2)t+2(203-63x_1-48x_1^2-190x_1^3)t^2-(31-150x_1+639x_1^2-458x_1^3-348x_1^4)t^3)x_2+48(3+5x_1)-12(34+32x_1+3x_1^2-24x_1^3)t+2(135+56x_1-42x_1^2-36x_1^3-268x_1^4)t^2-(21-86x_1+368x_1^2+180x_1^3-968x_1^4+152x_1^5)t^3$ and Y is defined at (12).

Differentiating $k(x_2)$ four times, we have

$$k'(x_2) = 16t^2(80+9(1-2x_1)t)x_2^3-12t^2(8(51-62x_1)-(1-2x_1)(13-62x_1)t)x_2^2$$
$$-8t(9(57-16x_1)-3(237-178x_1-138x_1^2)t+(49-255x_1+921x_1^2-920x_1^3)t^2)x_2$$

$$
\begin{aligned}
& -4(132-6(84+7x_1-12x_1^2)t+2(203-63x_1-48x_1^2-190x_1^3)t^2) \\
& +4(31-150x_1+639x_1^2-458x_1^3-348x_1^4)t^3 \\
k''(x_2) &= 48t^2(80+9(1-2x_1)t)x_2^2-24t^2(8(51-62x_1)-(1-2x_1)(13-62x_1)t)x_2 \\
& -8t(9(57-16x_1)-3(237-178x_1-138x_1^2)t+(49-255x_1+921x_1^2-920x_1^3)t^2) \\
k'''(x_2) &= 96t^2(80+9(1-2x_1)t)x_2-24t^2(8(51-62x_1)-(1-2x_1)(13-62x_1)t) \\
k''''(x_2) &= 96t^2(80+9(1-2x_1)t).
\end{aligned}
$$

$k''''(x_2)$ is positive for any $x_1 \in [0, 1/2]$ and $t \leq 1/2$. $k'''(1/2) = -744t^2(1-2x_1)(8-(1-2x_1)t) < 0$. $k'''(x_2)$ is negative for any $x_1 \in [0, 1/2]$ and $t \leq 1/2$. Since $k'''(x_2)$ is negative, $k''((1+x_1)/3) < 0$ implies $k''(x_2) < 0$ for all $x_2 \in [(x_1+1)/3, 1/2]$.

$$
\begin{aligned}
k''((1+x_1)/3) &= -\frac{8t(1539-432x_1-(1069-1018x_1+406x_1^2)t)}{3} \\
& -\frac{8t(90-540x_1+2709x_1^2-3096x_1^3)t^2}{3} < 0.
\end{aligned}
$$

Since $k''(x_2) < 0$, $k'((1+x_1)/3) < 0$ implies $k'(x_2) < 0$ for all $x_2 \in [(x_1+1)/3, 1/2]$.

$$
\begin{aligned}
k'((1+x_1)/3) &= -\frac{8(1782-(2187-2754x_1+324x_1^2)t+(758-2334x_1+4128x_1^2-3796x_1^3)t^2)}{27} \\
& +\frac{8(54-432x_1+2403x_1^2-5562x_1^3+4104x_1^4)t^3}{27} < 0.
\end{aligned}
$$

Since $k'(x_2) < 0$, $k((1+x_1)/3) < 0$ implies $k(x_2) < 0$ for all $x_2 \in [(x_1+1)/3, 1/2]$.

$$
\begin{aligned}
k((1+x_1)/3) &= \frac{t(1-2x_1)(-2592+(2916+1944x_1-10368x_1^2)t)}{27} \\
& -\frac{t(1-2x_1)(958-1428x_1+2640x_1^2-5504x_1^3)t^2)}{27} \\
& +\frac{t(1-2x_1)(75-600x_1+3960x_1^2-9960x_1^3+7680x_1^4)t^3}{27} < 0.
\end{aligned}
$$

Since the numerator in (15) is negative for any $x_2 \geq (1+x_1)/3$, δ_2^* is minimized at $x_2 = 1/2$ among $x_2 \in [(x_1+1)/3, 1/2]$.

Next we consider the range $x_2 \in [x_1, (x_1+1)/3]$ and show that δ_2^* is minimized at either $x_2 = x_1$ or $x_2 = (x_1+1)/3$.

Differentiating δ_2^* with respect to x_2, we have:

$$
\frac{\partial \delta_2^*}{\partial x_2} = \frac{l(x_2)}{8Y^2}, \tag{16}
$$

where $l(x_2) \equiv -40t^2(19-9(1-2x_1)t)x_2^4 + 20t^2(8(6+7x_1)-(1-2x_1)(19+34x_1)t)x_2^3 - 2t(18(3-16x_1)+60(3+19x_1-9x_1^2)t-5(11+48x_1-66x_1^2-136x_1^3)t^2)x_2^2 - 4(132-6(30-47x_1-12x_1^2)t+12(4-37x_1+24x_1^2-3x_1^3)t^2-(4-87x_1+153x_1^2-242x_1^3+444x_1^4)t^3)x_2 + 48(3+5x_1)-12(16-4x_1-15x_1^2-24x_1^3)t+6(11-44x_1-4x_1^2+60x_1^3-56x_1^4)t^2-(7-54x_1+78x_1^2-32x_1^3-324x_1^4+672x_1^5)t^3$ and Y is defined at (12).

Differentiating $l(x_2)$, we have:

$$
\begin{aligned}
l'(x_2) \quad = \quad & -160t^2(19-9(1-2x_1)t)x_2^3 + 60t^2(8(6+7x_1)-(1-2x_1)(19+34x_1)t)x_2^2 \\
& -4t(18(3-16x_1)+60(3+19x_1-9x_1^2)t-5(11+48x_1-66x_1^2-136x_1^3)t^2)x_2 \\
& -4(132-6(30-47x_1-12x_1^2)t+12(4-37x_1+24x_1^2-3x_1^3)t^2) \\
& -4(4-87x_1+153x_1^2-242x_1^3+444x_1^4)t^3.
\end{aligned}
$$

We define $g(t) \equiv l'(x_2)/(12t)$. We show that $g(t)$ is negative for any t. When $t = 1/2$, $g(t)$ is:

$$
\begin{aligned}
g(1/2) \quad = \quad & \frac{-428-327x_1-711x_1^2-170x_1^3+444x_1^4-(521+888x_1-750x_1^2+680x_1^3)x_2}{8} && (17) \\
& + \frac{15(77+116x_1+68x_1^2)x_2^2-40(29+18x_1)x_2^3}{8} < 0. && (18)
\end{aligned}
$$

Differentiating $g(t)$ twice, we have:

$$
\begin{aligned}
g'(t) \quad = \quad & 6(30-47x_1-12x_1^2)-18(3-16x_1)x_2-24(4-37x_1+24x_1^2-3x_1^3)t \\
& +(120(3+19x_1-9x_1^2)x_2-240(6+7x_1)x_2^2+1520x_2^3)t \\
& +(3(4-87x_1+153x_1^2-242x_1^3+444x_1^4)+15(11+48x_1-66x_1^2-136x_1^3)x_2)t^2 \\
& -(45(1-2x_1)(19+34x_1)x_2^2-1080(1-2x_1)x_2^3)t^2 \\
g''(t) \quad = \quad & -24(4-37x_1+24x_1^2-3x_1^3)+120(3+19x_1-9x_1^2)x_2-240(6+7x_1)x_2^2+1520x_2^3 \\
& +6((4-87x_1+153x_1^2-242x_1^3+444x_1^4)+5(11+48x_1-66x_1^2-136x_1^3)x_2)t \\
& -90((1-2x_1)(19+34x_1)x_2^2-24(1-2x_1)x_2^3)t.
\end{aligned}
$$

$g''(t)$ is a linear function with respect to t. Thus $g''(0) < 0$ and $g''(1/2) < 0$ implies $g''(t) < 0$ for any t.

$$
g''(0) \quad = \quad -8(12-111x_1+72x_1^2-9x_1^3+(45+285x_1-135x_1^2)x_2-(180+210x_1)x_2^2+190x_2^3) < 0
$$

$$g''(1/2) = -84 + 627x_1 - 117x_1^2 - 654x_1^3 + 1332x_1^4 - (195 + 1560x_1 - 90x_1^2 + 2046x_1^3)x_2 + (586 + 1860x_1 + 3060x_1^2)x_2^2 - (440 + 2160x_1)x_2^3 < 0.$$

We show that $g'(1/2)$ is positive.

$$4g'(1/2) = 540 + 387x_1 - 981x_1^2 - 582x_1^3 + 1332x_1^4 - (771 + 2668x_1 - 1170x_1^2 + 2040x_1^3)x_2 + (2025 + 3540x_1 + 3060x_1^2)x_2^2 - (1960 + 2160x_1)x_2^3 > 0.$$

Since $g''(t) < 0$, $g'(1/2) > 0$ implies that $g'(t) > 0$ for any $t \leq 1/2$. As mentioned earlier, $g(1/2)$ is negative. Since $g'(t) > 0$, we have $g(t) < 0$ for any t. As defined earlier, $g(t) = l'(x_2)/(12t)$ and, then, $l'(x_2)$ is negative for any x_2. After tedious calculation, we obtain that $l(x_1)$ is positive and $l(1/2)$ is negative. Thus, among $x_2 \in [x_1, (x_1 + 1)/3]$, either $x_2 = x_1$ or $x_2 = (x_1 + 1)/3$ minimizes δ_2^*.

From these facts, we have that δ_2^* is minimized at $x_2 = x_1$, $x_2 = (x_1 + 1)/3$, or $x_2 = 1/2$. As shown above, among $x_2 \in [(x_1 + 1)/3, 1/2]$, δ_2^* is minimized when $x_2 = 1/2$. Thus, δ_2^* is never minimized when $x_2 = (x_1 + 1)/3$. **Q.E.D.**

Proof of Proposition 2': First, we assume that $x_1 = x_2 =, ..., = x_n$ and show that it does not minimize δ^{**}. We prove it by contradiction. Suppose that the location $x_1 = x_2 =, ..., = x_n$ minimizes δ^{**}. Without loss of generality, we assume that $x_1 = 0$. Suppose that firm $n-1$ and firm n relocate and choose $x_{n-1} = x_n = 1/2$. Π_i^C is still zero for all firms. This relocation increases Π^M, and it decreases Π_i^D for $i \leq n-2$. By the symmetry of the circular city, after the relocation, $\Pi_1^D = \Pi_n^D$.

Since the situations are symmetric both before and after the relocation, all firms obtain the equal share in the monopoly profit. Since the above relocation increases Π_i^M, reduces Π_i^D and does not affect Π_i^C, it reduces δ^{**}, which is a contradiction.

Next, we assume that $0 = x_1 < x_2 =, ..., = x_n$ and show that it does not minimize δ^*. We prove it by contradiction. Suppose that the location $x_1 < x_2 =, ..., = x_n$ minimizes δ^{**}. Under the locations, $\Pi_1^C > 0 = \Pi_2^C =, ..., = \Pi_n^D$. By the symmetry of the circular city, $\Pi_1^D = \Pi_2^D =, ..., = \Pi_n^D$. Thus, $\delta^* = \delta_1^* > \delta_2^* =, ..., = \delta_n^*$. This implies that $\Pi_1^M > \Pi_i^M/n > \Pi_n^M$. Suppose that firm 2 relocates to 0. After the relocation, $\Pi_1^C = 0$. This relocation does not affect Π^M and Π_i^D for all $i \in N$. By the symmetry, $\Pi_1^M = \Pi_n^M = \Pi_i^M/n$. Since the relocation increases Π_n^M and does not affect neither Π_n^C or Π_n^D, it reduces δ^{**}, which is a contradiction. **Q.E.D.**

Proof of Proposition 4': (i) We prove it by contradiction. Suppose that the location $x_1 = x_2 = , ..., = x_n$ minimizes δ^{**}. Without loss of generality, we assume that $x_n \geq 1/2$. Suppose that firm 1 and firm 2 relocate and choose $x_1 = x_2 = 0$. Π_i^C is still zero for all firms. As is shown in the Proof of Proposition 2, this relocation increases Π^M and decreases Π_i^D. Thus, the relocation decreases δ^{**}, which is a contradiction.

(ii) We assume that $\delta^{**} \geq 1/2$. We prove Proposition 4'(ii) by contradiction. Suppose that the location $x_1 < x_2 =, ..., = x_n$ minimizes δ^{**}.

Suppose that $\Pi_1^M \leq \Pi_n^M$. Suppose that firm 2 relocates and chooses $x_2 = x_1$. It reduces Π_1^C and Π^M, Π_1^D, and Π_n^D remain unchanged. Thus, it reduces Π_2^M, resulting in an increase of Π_n^M and a decrease of δ^{**}, which is a contradiction.

Suppose that $\Pi_1^M \geq \Pi_n^M$. Suppose that firm 2 relocates and chooses $x_2 = x_1$. Define $\Pi_1^{M'}$ ($i = 1, 2$) as follows:

$$\frac{\Pi_1^D - \Pi_1^{M'}}{\Pi_1^D} = \delta^{**}.$$

$\Pi_1^{M'}$ is the required firm 1's profit in collusive phase after the relocation under the condition that the relocation does not affect firm 1's critical discount factor. By the symmetry between firms 1 and 2 after the relocation, firm 2 also obtains $\Pi_1^{M'}$ after the relocation so as to maintain the critical discount factor.

Since the relocation reduces firm 1's profit in competitive phase and does not affect Π_1^D, the relocation saves firm 1's share of monopoly profit (i.e., $\Pi_i^{M''} - \Pi_1^{M'}$ must be positive, where $\Pi_i^{M''}$ is firm i's profit in the collusive phase before the relocation). On the other hand, the relocation may increase Π_2^D, so it may increase firm 2's share of monopoly profit (i.e., $\Pi_1^{M'} - \Pi_n^{M''}$ can be either positive or negative). If $\Pi_i^{M''} - \Pi_1^{M'} > \Pi_1^{M'} - \Pi_n^{M''}$, firms can maintain δ^{**} and can re-distribute $\Pi_i^M - \Pi_1^{M'} - (\Pi_1^{M'} - \Pi_n^M)$ to all firms, resulting in the increase of δ^{**}, which is a contradiction. We then show that it is in fact positive.

We have that $\Pi_1^{M''} - \Pi_1^{M'} = \delta^{**}\Pi_1^{C''}$, where $\Pi_1^{C''}$ is firm 1's profit in the competitive phase before the relocation, and it is

$$\Pi_1^{C''} = \int_0^{(x_1+x_n)/2} q\Big(T(d(x_n,x))\Big)\Big(T(d(x_n,x)) - Td((x_1,x))\Big)dx.$$

We also have $\Pi_1^{M'} - \Pi_n^{M''} = (1-\delta^{**})(\Pi_1^{D''} - \Pi_n^{D''})$, where $\Pi_i^{D''}$ is firms i's one-shot deviation profit before the relocation. If $x_1 \leq 1 - x_n$, we have $\Pi_1^{D''} - \Pi_n^{D''} \leq 0$ so $\Pi_i^{M''} - \Pi_1^{M'} > \Pi_1^{M'} - \Pi_n^{M''}$ holds. If $x_1 > 1 - x_n$, we have

$$\Pi_1^{D''} - \Pi_n^{D''} = \int_0^{x_1-(1-x_n)} q\Big(p^M(T(d(x_1,x))\Big)\Big(T(d(x_n,x)) - T(d(x_1,x))\Big)dx.$$

Since the monopoly price is higher than the competitive price (i.e., $q\Big(p^M(T(d(x_1,x)))\Big) < q\Big(T(d(x_n,x))\Big)$) and $(x_1+x_n)/2 > x_1 - (1-x_n)$, we have $\Pi_1^{C''} > \Pi_1^{D''} - \Pi_2^{D''}$. Under these conditions $\Pi_i^{M''} - \Pi_1^{M'} > \Pi_1^{M'} - \Pi_n^{M''}$ if $\delta^{**} \geq 1/2$, which is a contradiction. **Q.E.D.**

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