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Sharp conditions for the existence of sign-changing solutions to the equations involving one-dimensional p-Laplacian

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Abstract

We consider the boundary value problem involving one-dimensional p-Laplacian

$$(|u'|^{p-2}u')' + a(x)f(u) = 0, \quad 0 < x < 1, \qquad u(0) = u(1) = 0,$$

where $p > 1$. We establish sharp conditions for the existence of solutions with prescribed numbers of zeros in terms of the ratio $f(s)/s^{p-1}$ at infinity and zero. Our argument is based on the shooting method together with the qualitative theory for half-linear differential equations.

Keywords: Two-point boundary value problems, One-dimensional p-Laplacian, Half-linear differential equations, Shooting method

1. Introduction

In this paper we consider the existence of sign-changing solutions for the boundary value problem

$$\begin{cases} (|u'|^{p-2}u')' + a(x)f(u) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$. In (1.1) we assume that a satisfies

$$a \in C^1[0,1], \quad a(x) > 0 \quad \text{for } 0 \le x \le 1, \tag{1.2}$$

and that f is locally Lipschitz continuous on $\mathbf{R} \setminus \{0\}$ and satisfies

$$f \in C(\mathbf{R}), \quad sf(s) > 0 \quad \text{for } s \ne 0. \tag{1.3}$$

We note that (1.3) implies $f(0) = 0$.

By a solution u of (1.1) we mean a function $u \in C^1[0,1]$ with $|u'|^{p-2}u' \in C^1[0,1]$ which satisfies (1.1).

Problems of the form (1.1) describe some nonlinear phenomena in mathematical sciences and have been studied in recent years by many authors (see [1], [2], [6], [8],

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[11], [14], [15], [17]–[21], [24], [25], [26] and references therein). Let us consider the problem of the form

$$\begin{cases} (|u'|^{p-2}u')' + g(x,u) = 0, \quad 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.4}$$

where $g : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a continuous function with $g(x,0) \equiv 0$. Put

$$\mu_k = (p-1)(k\pi_p)^p \quad \text{for } k = 1, 2, \dots,$$

where $\pi_p = (2\pi)/(p\sin(\pi/p))$. It is well known by [2] that the problem (1.4) possesses a nontrivial solution if there exist positive integers j, k such that $(j-k)$ is an odd integer and

$$\mu_k \le \liminf_{|u|\to\infty} \frac{g(t,u)}{u^{p-1}} \le \limsup_{|u|\to\infty} \frac{g(t,u)}{u^{p-1}} \le \mu_{k+1} \tag{1.5}$$

and

$$\mu_j \le \liminf_{u\to 0} \frac{g(t,u)}{u^{p-1}} \le \limsup_{u\to 0} \frac{g(t,u)}{u^{p-1}} \le \mu_{j+1}, \tag{1.6}$$

where, in (1.5) and (1.6), the limits are supposed to hold uniformly on $[0,1]$ and the first and last inequalities being strict in some subsets of positive measure in $[0,1]$. Their method is based on the degree argument. The problem of the existence of positive solutions was studied by [1], [8], [11], [20], [24], [25], and it was shown in [20] that the problem (1.4) has at least one positive solution if

$$\liminf_{u\to 0^+} \frac{g(t,u)}{u^{p-1}} > -\infty$$

and

$$\limsup_{u\to 0^+} \frac{g(t,u)}{u^{p-1}} < \mu_1 < \liminf_{u\to\infty} \frac{g(t,u)}{u^{p-1}},$$

where the limits are supposed to hold uniformly for almost every $t \in [0,1]$.

In this paper we will show the precise conditions for the existence of solutions with prescribed nodal properties to the problem (1.1). Our approach is based on the shooting method together with the qualitative theory for half-linear differential equations.

Following the idea by [26], we consider the weighted eigenvalue problem

$$\begin{cases} (|\varphi'|^{p-2}\varphi')' + \lambda a(x)|\varphi|^{p-2}\varphi = 0, \quad 0 < x < 1, \\ \varphi(0) = \varphi(1) = 0. \end{cases} \tag{1.7}$$

Let λ_k be the k-th eigenvalue of (1.7), and let φ_k be an eigenfunction corresponding to λ_k. It is known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and that φ_k has exactly $k-1$ zeros in $(0,1)$. (See, e.g., [3], [4], [13], [26].) For convenience, we put $\lambda_0 = 0$.

For each $k \in \mathbf{N} = \{1, 2, \ldots\}$, we denote by S_k^+ (respectively S_k^-) the set of all solutions u for (1.1) which have exactly $k-1$ zeros in $(0,1)$ and satisfy $u'(0) > 0$ (respectively $u'(0) < 0$). Our main result is the following.

Theorem 1. *Assume that there exists an integer $k \in \mathbf{N}$ such that either*

$$\limsup_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} < \lambda_k < \liminf_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s} \tag{1.8}$$

or

$$\limsup_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s} < \lambda_k < \liminf_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s}. \tag{1.9}$$

Then $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$.

As a corollary, we obtain the multiplicity of the existence of solutions to (1.1).

Corollary 1. *Assume that there exist some integers $j, k \in \mathbf{N}$ with $j \geq k$ such that either*

$$\limsup_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} < \lambda_k \quad \textit{and} \quad \liminf_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s} > \lambda_j$$

or

$$\limsup_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s} < \lambda_k \quad \textit{and} \quad \liminf_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} > \lambda_j.$$

Then the problem (1.1) *has at least* $2(j-k+1)$ *nontrivial solutions.*

Remark 1. It is known by Lee and Sim [15] that (1.1) has no non-trivial solution when the range of $f(s)/|s|^{p-2}s$ contains no eigenvalue of the problem (1.7). Precisely, it was shown in [15, Theorem 2.6] that if there exists an integer $k \in \mathbf{N}$ such that

$$\lambda_k < \frac{f(s)}{|s|^{p-1}s} < \lambda_{k+1} \quad \text{for } s \neq 0,$$

then the problem (1.1) has no non-trivial solution.

Corollary 2. *Assume that either*

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} = 0 \quad \textit{and} \quad \lim_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s} = \infty$$

or

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} = \infty \quad \textit{and} \quad \lim_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s} = 0$$

holds. Then $S_k^+ \neq \emptyset$ *and* $S_k^- \neq \emptyset$ *for each* $k \in \mathbf{N}$*. In particular, the problem* (1.1) *has infinity many solutions.*

Remark 2. (i) Under the conditions in Corollary 2, Wang [24] has showed the existence of a positive solution of (1.1) subject to nonlinear boundary conditions by using a fixed point theorem in cones.

(ii) In the autonomous case $a(x) \equiv$ const, the existence of solutions to (1.1) has been studied by using a time-mapping method. See, e.g., [6], [21].

(iii) Recently, Lee and Sim [14], [15] considered the problem (1.1) in the case $a \in L^1(0,1)$, and gave global analyses for sign-changing solutions employing a bifurcation argument.

(iv) In the case $p = 2$, related results can be found in [17]–[19], [22].

In the proof of Theorem 1, we first consider the solution $u(x;\mu)$ of (1.1) satisfying the initial condition $u(0) = 0$ and $u'(0) = \mu$, where $\mu \in \mathbf{R}$ is a parameter, and then we investigate the behavior of the solution $u(x;\mu)$ as $\mu \to 0$ and $\mu \to \infty$ by making use of the properties of solutions for half-linear differential equations of the form

$$(|v'|^{p-2}v')' + c(x)|v|^{p-2}v = 0, \tag{1.10}$$

where c is continuous. In particular, we will employ the generalized Prüfer transformation, the Sturmian theorems, and the Picone type identity for (1.10) in our arguments.

This paper is organized as follows. In Section 2, we show a global existence and uniqueness of the solution for the initial value problem. In Section 3, we give some comparison arguments. We show the proof of Theorem 1 in Section 4. In Section 5, we prove the lemmas stated in Section 4. In the appendix, we give the proof of the lemma used in Section 2.

2. Initial value problem

We denote by $u(x;\mu)$ the solution of the initial value problem

$$\begin{cases} (|u'|^{p-2}u')' + a(x)f(u) = 0, \quad 0 < x < 1, \\ u(0) = 0 \quad \text{and} \quad u'(0) = \mu, \end{cases} \tag{2.1}$$

where $\mu \in \mathbf{R}$ is a parameter. We show the following proposition.

Proposition 2.1. *For each* $\mu \in \mathbf{R}$*, the solution* $u(x;\mu)$ *exists on* $[0,1]$ *and is unique. Furthermore,* $u(x;\mu)$ *and* $u'(x;\mu)$ *are continuous functions of* $(x;\mu) \in [0,1] \times \mathbf{R}$*, and the number of zeros of* $u(x;\mu)$ *in* $[0,1]$ *is finite for each* $\mu \in \mathbf{R} \setminus \{0\}$.

Remark 3. (i) In the case where $f(s) = |s|^{\beta-1}s$ with $\beta > 0$, the existence and uniqueness of the initial value problem (2.1) was studied by Kitano and Kusano [12].

(ii) For more general initial value problems, an extensive list of uniqueness conditions was given by Reichel and Walter [23], together with examples of non-uniqueness.

For a solution u of (2.1), we denote by $I[u] \subset [0,1]$ the maximal interval of the existence for u. We define the energy function $E[u]$ by

$$E[u](x) = |u'(x)|^p + a(x)F(u(x)) \quad \text{for } x \in I[u], \tag{2.2}$$

where

$$F(s) = \frac{p}{p-1}\int_0^s f(t)\,dt \quad \text{for } s \in \mathbf{R}.$$

It is easy to see that $F(s) > 0$ for all $s \in \mathbf{R} \setminus \{0\}$, $F(0) = 0$, and $F(s)$ is strictly decreasing in $s \in (-\infty, 0)$ and strictly increasing in $s \in (0, \infty)$. Thus $E[u](x) \geq 0$ on $I[u]$, and $E[u](x) \equiv 0$ on $I[u]$ if and only if $u(x) \equiv 0$ on $I[u]$. Furthermore, we obtain the following properties.

Lemma 2.1. *Let u be a solution of* (2.1), *and let $x_0, x \in I[u]$. Then*

$$E[u](x) \leq E[u](x_0) \exp\left(\int_{x_0}^{x} \frac{[a'(t)]_+}{a(t)}dt\right) \quad \textit{for } x_0 < x, \tag{2.3}$$

and

$$E[u](x) \leq E[u](x_0) \exp\left(\int_{x}^{x_0} \frac{[a'(t)]_-}{a(t)}dt\right) \quad \textit{for } x < x_0, \tag{2.4}$$

where $[s]_+ = \max\{s, 0\}$ *and* $[s]_- = \max\{-s, 0\}$.

Proof. Put $v = |u'|^{p-2}u'$. Then we have $|u'|^p = |v|^{\frac{p}{p-1}}$ and

$$\frac{d}{dx}|u'|^p = \frac{p}{p-1}(|v|^{\frac{p}{p-1}-2}v)v' = \frac{p}{p-1}u'(|u'|^{p-2}u')'.$$

Then $E[u]$ satisfies

$$\frac{d}{dx}E[u](x) = a'(x)F(u(x)) \quad \text{for } x \in I[u]. \tag{2.5}$$

In view of (2.2), we have

$$\frac{d}{dx}E[u](x) \leq \frac{[a'(x)]_+}{a(x)}E[u](x) \quad \text{for } x \in I[u].$$

Then, for x_0, $x \in I[u]$ with $x_0 < x$, we obtain

$$\frac{d}{dx}\left(E[u](x)\exp\left(-\int_{x_0}^{x}\frac{[a'(t)]_+}{a(t)}dt\right)\right) \leq 0.$$

An integration of the above over $[x_0, x]$ gives inequality (2.3). Inequality (2.4) can be obtained by a similar argument. $\square$

In the proof of Proposition 2.1 we need the following lemma, which will be proven in the appendix below.

Lemma 2.2. *Let $x_0 \in [0,1]$, and let α, $\beta \in \mathbf{R}$. Then the initial value problem*

$$(|u'|^{p-2}u')' + a(x)f(u) = 0, \qquad u(x_0) = \alpha, \quad u'(x_0) = \beta, \tag{2.6}$$

has a unique local solution.

Proof of Proposition 2.1. From Lemma 2.2 the initial value problem (2.1) has a unique local solution $u(x;\mu)$ for each $\mu \in \mathbf{R}$, and the solution $u(x;\mu)$ is unique as far as the solution exists. Let $I[u]$ be the maximal interval of existence for $u(x;\mu)$. Lemma 2.1 with $x_0 = 0$ implies that

$$E[u(\,\cdot\,;\mu)](x) \le \mu^p \exp\left(\int_0^1 \frac{[a'(t)]_+}{a(t)}dt\right) \quad \text{for } x \in I[u].$$

This means that both $u(x;\mu)$ and $u'(x;\mu)$ are bounded as far as the solution $u(x;\mu)$ exists. Thus, by a standard argument, we conclude that $u(x;\mu)$ exists on $[0,1]$ and is unique.

By a general theory on the continuous dependence of solutions on parameters and initial conditions (see, for example, [7, Chap. V, Theorem 2.1]), it follows that $u(x;\mu)$ and $u'(x;\mu)$ are continuous in $(x;\mu)$ on the set $[0,1] \times \mathbf{R}$.

Assume to the contrary that $u(x;\mu)$ has infinitely many zeros in the finite interval $[0,1]$ for some $\mu \neq 0$. Then, by the uniqueness we have $u(x;\mu) \equiv 0$ on $[0,1]$. This is a contradiction. Thus the number of zeros of $u(x;\mu)$ in $[0,1]$ is finite for each $\mu \in \mathbf{R}\backslash\{0\}$. This completes the proof of Proposition 2.1.

3. Comparison lemmas

In this section first we will recall the Sturmian comparison theorem and the Picone type identity for the half-linear differential equations. Let us consider a pair of half-linear differential equations

$$(|u'|^{p-2}u')' + c(x)|u|^{p-2}u = 0, \quad 0 \le x \le 1, \tag{3.1}$$

and

$$(|U'|^{p-2}U')' + C(x)|U|^{p-2}U = 0, \quad 0 \le x \le 1, \tag{3.2}$$

where c, $C \in C[0,1]$ satisfy $C(x) \geq c(x)$ for $x \in [0,1]$. The Sturm comparison theorem for the half-linear differential equation is formulated as follows:

Lemma 3.1. *Assume that a nontrivial solution u of* (3.1) *satisfies $u(x_1) = u(x_2) = 0$ with $x_1, x_2 \in [0,1]$. Then every nontrivial solution U of* (3.2) *has a zero in (x_1, x_2) or it is a multiple of the solution u. The last possibility is excluded if $C(x) \not\equiv c(x)$ for $x \in (x_1, x_2)$.*

For the proof, we refer to [4, Theorem 1.2.4]. (See also [3], [5] and [16].) The following Picone type identity for equations (3.1) and (3.2) is introduced by Jaroš and Kusano [9]. See also [3] and [4].

Lemma 3.2. *Define Φ and P, respectively, by*

$$\Phi(u) = |u|^{p-2}u \quad \textit{and} \quad P(u,v) = \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0,$$

where $q = p/(p-1)$. Let u and U be solutions of (3.1) *and* (3.2), *respectively. Then*

$$\begin{aligned} &\left[\frac{u}{\Phi(U)}\left(\Phi(u')\Phi(U) - \Phi(u)\Phi(U')\right)\right]' \\ &= [C(x) - c(x)]|u|^p + pP\left(u', \Phi\left(\frac{uU'}{U}\right)\right) \quad \textit{for } x \in [x_1, x_2]. \end{aligned}$$

In particular, we have

$$\left[\frac{u}{\Phi(U)}\Big(\Phi(u')\Phi(U) - \Phi(u)\Phi(U')\Big)\right]' \geq [C(x) - c(x)]|u|^p$$

for $x \in [x_1, x_2]$.

Next, let us consider the eigenvalue problem (1.7). Recall that λ_k is the k-th eigenvalue of (1.7).

Lemma 3.3. *Let $\{x_i\}_{i=0}^k$ be zeros of an eigenfunction φ_k corresponding to λ_k such that*

$$0 = x_0 < x_1 < x_2 < \cdots < x_{k-1} < x_k = 1. \tag{3.3}$$

(i) *Let $\lambda > \lambda_k$. Then, for each $i \in \{1, 2, \ldots, k\}$, there is a solution w_i of the equation*

$$(|w'|^{p-2}w')' + \lambda a(x)|w|^{p-2}w = 0, \tag{3.4}$$

which has at least two zeros in (x_{i-1}, x_i).

(ii) *Let $\lambda < \lambda_k$. Then, for each $i \in \{1, 2, \ldots, k\}$, there is a solution w_i of* (3.4) *such that $w_i(x) > 0$ on $[x_{i-1}, x_i]$.*

Proof. (i) Let $i \in \{1, 2, \ldots, k\}$ be fixed. Consider the initial condition

$$w(x_{i-1} + \varepsilon) = 0 \quad \text{and} \quad w'(x_{i-1} + \varepsilon) = 1 \tag{3.5}$$

with $\varepsilon \geq 0$. Note that $\lambda a(x) > \lambda_k a(x)$ on $[x_{i-1}, x_i]$ and $\varphi_k(x_{i-1}) = \varphi_k(x_i) = 0$. By Lemma 3.1 the solution of (3.4) satisfying (3.5) with $\varepsilon = 0$ has a zero z_0 in (x_{i-1}, x_i). By the continuous dependence of solutions on initial conditions, the solution of (3.4) and (3.5) with small $\varepsilon > 0$ has a zero z_ε near z_0. Hence, if $\varepsilon > 0$ is sufficiently small, the solution of (3.4) and (3.5) has two zeros $x_{i-1} + \varepsilon$ and z_ε in (x_{i-1}, x_i).

(ii) Fix $i \in \{1, 2, \ldots, k\}$, and consider the initial condition

$$w(x_{i-1}) = \varepsilon \quad \text{and} \quad w'(x_{i-1}) = 1 \tag{3.6}$$

with $\varepsilon \geq 0$. Since $\lambda a(x) < \lambda_k a(x)$ on $[x_{i-1}, x_i]$ and φ_k has no zero in (x_{i-1}, x_i), Lemma 3.1 shows that the solution of (3.4) and (3.6) with $\varepsilon = 0$ satisfies $w(x) > 0$ on $(x_{i-1}, x_i]$. By the continuous dependence of solutions on initial conditions, if $\varepsilon > 0$ is sufficiently small, the solution of (3.4) and (3.6) satisfies $w(x) > 0$ on $[x_{i-1}, x_i]$. □

4. Proof of Theorem 1

In order to show Theorem 1, we introduce the generalized trigonometric functions $\sin_p$, $\cos_p$ and $\tan_p$, which are generalizations of the classical trigonometric functions sin, cos and tan, respectively. (See [3], [4], [5] and [13].) The *generalized sine function* $\sin_p$ is defined as the solution of the specific half-linear differential equation

$$(|S'|^{p-2}S')' + (p-1)|S|^{p-2}S = 0$$

satisfying the initial condition $S(0) = 0$ and $S'(0) = 1$. The function $\sin_p$ is defined on $\mathbf{R}$ and is periodic with period $2\pi_p$, where $\pi_p = (2\pi)/(p\sin(\pi/p))$. Further, $\sin_p x$ is an odd function having zeros at $x = j\pi_p$, $j \in \mathbf{Z}$; it is positive for $2j\pi_p < x < (2j+1)\pi_p$, $j \in \mathbf{Z}$, and negative for $(2j+1)\pi_p < x < 2(j+1)\pi_p$, $j \in \mathbf{Z}$. The *generalized cosine function* $\cos_p$ is defined by $\cos_p x = (\sin_p x)'$, and the *generalized tangent function* $\tan_p x$ is defined by

$$\tan_p x = \frac{\sin_p x}{\cos_p x} \quad \text{for } x \neq \left(j + \frac{1}{2}\right)\pi_p, \ j \in \mathbf{Z}.$$

We see that the generalized Pythagorean identity holds:

$$|\sin_p x|^p + |\cos_p x|^p = 1, \quad x \in \mathbf{R},$$

and that $\tan_p x$ is strictly increasing for $-\pi_p/2 < x < \pi_p/2$ and there exists the inverse function $\operatorname{arctan}_p x$ of $\tan_p x$ which is multivalued and defined on $\mathbf{R}$.

For the solution $u(x;\mu)$ of (2.1) with $\mu > 0$, we define the functions $r(x;\mu)$ and $\theta(x;\mu)$ by

$$\begin{cases} u(x;\mu) = r(x;\mu)\sin_p\theta(x;\mu), \\ u'(x;\mu) = r(x;\mu)\cos_p\theta(x;\mu), \end{cases}$$

where $' = d/dx$. Since $u(x;\mu)$ and $u'(x;\mu)$ cannot vanish simultaneously, $r(x;\mu)$ and $\theta(x;\mu)$ can be written in the forms

$$r(x;\mu) = \left(|u(x;\mu)|^p + |u'(x;\mu)|^p\right)^{1/p} > 0$$

and

$$\theta(x;\mu) = \arctan_p \frac{u(x;\mu)}{u'(x;\mu)},$$

respectively. It can be shown that

$$\theta'(x;\mu) = |\cos_p\theta(x;\mu)|^p + \frac{a(x)f(r(x;\mu)\sin_p\theta(x;\mu))\sin_p\theta(x;\mu)}{(p-1)[r(x;\mu)]^{p-1}} > 0$$

for $x \in [0,1]$, which implies that $\theta(x;\mu)$ is strictly increasing in $x \in [0,1]$ for each fixed $\mu > 0$. (See, for example, [3] or [4].) From the initial condition in (2.1) it follows that $\theta(0;\mu) \equiv 0 \pmod{2\pi_p}$. For simplicity we take $\theta(0;\mu) = 0$. Proposition 2.1 implies that $\theta(x;\mu)$ is continuous in $(x;\mu) \in [0,1] \times (0,\infty)$. We easily see that $u(x;\mu)$ has exactly $k-1$ zeros in $(0,1)$ if and only if

$$(k-1)\pi_p < \theta(1;\mu) \le k\pi_p.$$

To show Theorem 1, we need Lemmas 4.1–4.4 below.

Lemma 4.1. *Assume that*

$$\limsup_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} < \lambda_k \tag{4.1}$$

for some $k \in \mathbf{N}$. Then there exists $\mu_* > 0$ such that, for each $\mu \in (0,\mu_*]$, the solution $u(x;\mu)$ has at most $k-1$ zeros in $(0,1)$.

Lemma 4.2. *Assume that*

$$\liminf_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} > \lambda_k \tag{4.2}$$

for some $k \in \mathbf{N}$. Then there exists $\mu_* > 0$ such that, for each $\mu \in (0,\mu_*]$, the solution $u(x;\mu)$ has at least k zeros in $(0,1)$.

Lemma 4.3. *Assume that*

$$\liminf_{|s|\to \infty} \frac{f(s)}{|s|^{p-2}s} > \lambda_k \tag{4.3}$$

for some $k \in \mathbf{N}$. Then there exists $\mu^* > 0$ such that, for each $\mu \geq \mu^*$, the solution $u(x;\mu)$ has at least k zeros in $(0,1)$.

Lemma 4.4. *Assume that*

$$\limsup_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s} < \lambda_k \tag{4.4}$$

for some $k \in \mathbf{N}$. Then there exists $\mu^* > 0$ such that, for each $\mu \geq \mu^*$, the solution $u(x;\mu)$ has at most $k-1$ zeros in $(0,1)$.

Proofs of Lemmas 4.1–4.4 will be given in the next section.

Proof of Theorem 1. First assume that (1.8) holds. Then, from Lemma 4.1, there exists $\mu_* > 0$ such that $u(x;\mu)$ has at most $k-1$ zeros in $(0,1)$ for $\mu \in (0,\mu_*]$, that is, $\theta(1;\mu) \leq k\pi_p$ for $\mu \in (0,\mu_*]$. Lemma 4.3 implies that there exists $\mu^* > 0$ such that $\theta(1;\mu) > k\pi_p$ for $\mu \geq \mu^*$. Then $\mu_* < \mu^*$. Since $\theta(1;\mu)$ is continuous in $\mu \in (0,\infty)$, there exists a number $\mu_k \in [\mu_*, \mu^*)$ such that $\theta(1;\mu_k) = k\pi_p$. This implies that $u(x;\mu_k) \in S_k^+$.

Put $g(s) = -f(-s)$, and let us consider the boundary value problem

$$\begin{cases} (|v'|^{p-2}v')' + \lambda a(x)g(v) = 0, \quad 0 < x < 1, \\ v(0) = v(1) = 0, \end{cases} \tag{4.5}$$

We note here that (1.8) implies that

$$\limsup_{|s|\to 0} \frac{g(s)}{|s|^{p-2}s} < \lambda_k < \liminf_{|s|\to\infty} \frac{g(s)}{|s|^{p-2}s}$$

Then, by the argument above, the problem (4.5) has a solution v_k which has exactly $k-1$ zeros in $(0,1)$ and $v_k'(0) > 0$. Put $u_k = -v_k$. Then we easily see that $u_k \in S_k^-$.

In the case where (1.9) holds, we can show $S_k^+ \neq \emptyset$ and $S_k^- \neq \emptyset$ by employing Lemmas 4.2 and 4.4 with a similar argument. □

5. Proofs of Lemmas 4.1–4.4

In the proofs of Lemmas 4.1–4.4, the following notation will be used:

$$a_* = \min\{a(x) : 0 \leq x \leq 1\}, \quad a^* = \max\{a(x) : 0 \leq x \leq 1\},$$

$$A_* = \exp\left(-\int_0^1 \frac{[a'(t)]_-}{a(t)}dt\right), \quad A^* = \exp\left(\int_0^1 \frac{[a'(t)]_+}{a(t)}dt\right),$$

$$F^*(u) = \max\{F(u), F(-u)\}, \quad F_*(u) = \min\{F(u), F(-u)\}.$$

Since $F(u)$ and $F(-u)$ are strictly increasing in $u \in (0, \infty)$, we easily see that $F^*(u)$ and $F_*(u)$ are strictly increasing in $u \in (0, \infty)$.

Lemma 5.1. (i) *Let* $\mu > 0$. *Then*

$$A_* \mu^p \le E[u(\,\cdot\,;\mu)](x) \le A^* \mu^p \quad \textit{for } 0 \le x \le 1, \tag{5.1}$$

where $E[u]$ *is the function defined by* (2.2).

(ii) *Let* $M > 0$. *If*

$$0 < \mu \le \left[\frac{a_* F_*(M)}{A^*} \right]^{1/p}, \tag{5.2}$$

then $|u(x;\mu)| \le M$ *for* $0 \le x \le 1$.

Proof. (i) From Lemma 2.1 we have

$$E[u(\,\cdot\,;\mu)](x) \le E[u(\,\cdot\,;\mu)](0) \exp\left(\int_0^x \frac{[a'(t)]_+}{a(t)} dt \right) \quad \text{for } x > 0$$

and

$$E[u(\,\cdot\,;\mu)](0) \le E[u(\,\cdot\,;\mu)](x) \exp\left(\int_0^x \frac{[a'(t)]_-}{a(t)} dt \right) \quad \text{for } x > 0.$$

Hence, noting that $E[u(\,\cdot\,;\mu)](0) = \mu^p$, we obtain (5.1).

(ii) From (5.1) and (5.2) we have

$$E[u(\,\cdot\,;\mu)](x) \le A^* \mu^p \le a_* F_*(M) \quad \text{for } 0 \le x \le 1.$$

From $a_* F_*(u(x;\mu)) \le E[u(\,\cdot\,;\mu)](x)$, it follows that

$$a_* F_*(u(x;\mu)) \le a_* F_*(M).$$

This implies that $|u(x;\mu)| \le M$ for $0 \le x \le 1$. □

We observe that $u = u(x;\mu)$ satisfies the equation

$$(|u'|^{p-2}u')' + b(x)|u|^{p-2}u = 0, \tag{5.3}$$

where

$$b(x) = a(x) \frac{f(u(x;\mu))}{|u(x;\mu)|^{p-2} u(x;\mu)}. \tag{5.4}$$

Proof of Lemma 4.1. From (4.1) there exist $\lambda < \lambda_k$ and $M > 0$ such that

$$\frac{f(s)}{|s|^{p-2}s} < \lambda \quad \text{for } 0 < |s| \le M.$$

Put $\mu_* > 0$ so that (5.2) holds for all $\mu \in (0, \mu_*]$. Let $\mu \in (0, \mu_*]$. From Lemma 5.1 (ii), we obtain $|u(x;\mu)| \le M$ for $x \in [0,1]$. Then, the function b given by (5.4) satisfies

$$b(x) < \lambda a(x) \quad \text{for a.e. } 0 \le x \le 1. \tag{5.5}$$

Let $\{x_i\}_{i=0}^k$ be zeros of an eigenfunction φ_k corresponding to λ_k satisfying (3.3). We will show that $u(x;\mu)$ with $\mu \in (0, \mu^*]$ has at most one zero in $[x_{i-1}, x_i)$ for each $i = 1, 2, \ldots, k$. Assume to the contrary that $u(x;\mu)$ has at least two zeros in $[x_{i-1}, x_i)$ for some $i \in \{1, 2, \ldots, k\}$. Let t_1 and t_2 be zeros of $u(x;\mu)$ such that $x_{i-1} \le t_1 < t_2 < x_i$ and $|u(x;\mu)| > 0$ for $t_1 < x < t_2$. Then $u = u(x;\mu)$ satisfies (5.3) for $t_1 < x < t_2$ with $b \in C(t_1, t_2)$. By Lemma 3.3 (ii), there exists a solution w_i of (3.4) such that $w_i(x) > 0$ on $[x_{i-1}, x_i]$. By applying Lemma 3.2 with $u = u(x;\mu)$ and $U = w_i$, we have

$$\frac{d}{dx}\left[\frac{u}{\Phi(w_i)}\Big(\Phi(u')\Phi(w_i) - \Phi(u)\Phi(w_i')\Big)\right] \ge (\lambda a(x) - b(x))|u|^p$$

for $t_1 < x < t_2$. Integrating this inequality on $[s, t_2]$ with $t_1 < s < t_2$, and letting $s \to t_1$, we obtain

$$0 \ge \int_{t_1}^{t_2} [\lambda a(x) - b(x)]|u|^p dx.$$

This contradicts (5.5). Thus u has at most one zero in $[x_{i-1}, x_i)$ for each $i = 1, 2, \ldots, k$. Note that u has no zero in (x_0, x_1), since $x_0 = 0$ is a zero of u in $[x_0, x_1)$. Hence u has at most $k-1$ zeros in $(0,1)$. □

Proof of Lemma 4.2. From (4.2) there exist $\lambda > \lambda_k$ and $M > 0$ such that

$$\lambda < \frac{f(s)}{|s|^{p-2}s} \quad \text{for } 0 < |s| \le M.$$

Take $\mu_* > 0$ so that (5.2) holds for all $\mu \in (0, \mu_*]$. Let $\mu \in (0, \mu_*]$. Lemma 5.1 (ii) implies that $|u(x;\mu)| \le M$ for $x \in [0,1]$. Let $\{x_i\}_{i=0}^k$ be zeros of an eigenfunction φ_k corresponding to λ_k satisfying (3.3). We will show that $u(x;\mu)$ has at least one zero in each interval (x_{i-1}, x_i) for $i = 1, 2, \ldots, k$. Assume to the contrary that $u(x;\mu)$ has no zero in (x_{i-1}, x_i) with some $i \in \{1, 2, \ldots, k\}$. By Lemma 3.3 (i), there exists a solution w_i of (3.4) such that $w_i(t_1) = w_i(t_2) = 0$ for some t_1, $t_2 \in (x_{i-1}, x_i)$. We see that the function b given by (5.4) is continuous in $[t_1, t_2]$, and satisfies

$$b(x) > \lambda a(x), \quad t_1 \le x \le t_2.$$

Then Lemma 3.1 implies that $u(x;\mu)$ has at least one zero in $(t_1, t_2) \subset (x_{i-1}, x_i)$. This is a contradiction. Thus $u(x;\mu)$ has at least one zero in each interval (x_{i-1}, x_i) for $i = 1, 2, \ldots, k$, which implies that $u(x;\mu)$ has at least k zeros in $(0,1)$. □

Lemma 5.2. *Let $M > 0$. If $\mu > 0$ satisfies*

$$\mu > \left[\frac{a^* F^*(M)}{A_*}\right]^{1/p}, \tag{5.6}$$

then the solution $u(x;\mu)$ has the following properties (i)–(iii):

(i) *If $u'(x_0;\mu) = 0$ for some $x_0 \in (0,1]$, then $|u(x_0;\mu)| > M$;*

(ii) *Assume that $u(x;\mu)$ has no zero in (x_1, x_2) and satisfies $|u(x;\mu)| \le M$ on $[x_1, x_2]$ for some $x_1, x_2 \in [0,1]$. Then we have $x_2 - x_1 \le \delta$, where*

$$\delta = M\Big(A_*\mu^p - a^* F^*(M)\Big)^{-1/p} > 0. \tag{5.7}$$

(iii) *Define δ by (5.7). Assume that $u(x;\mu)$ has no zero in (α, β) for some $\alpha, \beta \in [0,1]$ satisfying $\beta - \alpha > 2\delta$. Then $|u(x;\mu)| > M$ for $x \in (\alpha + \delta, \beta - \delta)$.*

Proof. (i) From $u'(x_0;\mu) = 0$, we have

$$E[u(\,\cdot\,;\mu)](x_0) = a(x_0)F(u(x_0;\mu)) \le a^* F^*(u(x_0;\mu)).$$

From (5.1) and (5.6) we obtain

$$E[u(\,\cdot\,;\mu)](x_0) \ge A_*\mu^p > a^* F^*(M).$$

Then it follows that $a^*F^*(M) < a^*F^*(u(x_0;\mu))$. Thus we obtain $M < |u(x_0;\mu)|$.

(ii) For simplicity, we denote $u(x;\mu)$ by $u(x)$. We may assume that $u(x) > 0$ on (x_1, x_2). Then we have

$$0 \le u(x) \le M \quad \text{for } x_1 \le x \le x_2 \tag{5.8}$$

and

$$E[u(\,\cdot\,;\mu)](x) \le |u'(x)|^p + a^*F^*(M) \quad \text{for } x_1 \le x \le x_2.$$

From (5.1), we have

$$A_*\mu^p \le |u'(x)|^p + a^*F^*(M) \quad \text{for } x_1 \le x \le x_2.$$

This implies that

$$|u'(x)| \ge \Big(A_*\mu^p - a^*F^*(M)\Big)^{1/p} = \frac{M}{\delta} \quad \text{for } x_1 \le x \le x_2.$$

Therefore we have either $u'(x) \ge M/\delta$ on $[x_1, x_2]$ or $-u'(x) \ge M/\delta$ on $[x_1, x_2]$.

If $u'(x) \ge M/\delta$ on $[x_1, x_2]$, then (5.8) implies that

$$M \ge u(x_2) = u(x_1) + \int_{x_1}^{x_2} u'(x)dx \ge \frac{M}{\delta}(x_2 - x_1),$$

and hence $x_2 - x_1 \le \delta$. If $-u'(x) \ge M/\delta$ on $[x_1, x_2]$, then

$$M \ge u(x_1) = u(x_2) - \int_{x_1}^{x_2} u'(x)dx \ge \frac{M}{\delta}(x_2 - x_1),$$

which means that $x_2 - x_1 \le \delta$.

(iii) We may assume that $u(x;\mu) > 0$ on (α, β). For simplicity, we denote $u(x;\mu)$ by $u(x)$. In view of (2.1) we see that $(|u'|^{p-2}u')' \le 0$ on $[\alpha, \beta]$, so that u' is nonincreasing on $[\alpha, \beta]$. Assume to the contrary that there is a number $\gamma \in (\alpha + \delta, \beta - \delta)$ such that $u(\gamma) \le M$.

First suppose that $u'(\gamma) \ge 0$. Since u' is nonincreasing on $[\alpha, \beta]$, we find that $u'(x) \ge u'(\gamma) \ge 0$ for $x \in [\alpha, \gamma]$, so that u is nondecreasing on $[\alpha, \gamma]$. Then we have $0 \le u(x) \le u(\gamma) \le M$ on $[\alpha, \gamma]$. From Lemma 5.2 (ii), we have $\gamma - \alpha \le \delta$. This contradicts $\gamma \in (\alpha + \delta, \beta - \delta)$.

Next we assume that $u'(\gamma) < 0$. Since u' is nonincreasing on $[\alpha, \beta]$, we have $u'(x) \le u'(\gamma) < 0$ for $x \in [\gamma, \beta]$. Then $0 \le u(x) \le u(\gamma) \le M$ for $x \in [\gamma, \beta]$. From Lemma 5.2 (ii), we obtain $\beta - \gamma \le \delta$. This is a contradiction. Therefore, $u(x;\mu) > M$ for $x \in (\alpha + \delta, \beta - \delta)$. □

Proof of Lemma 4.3. From (4.3) there exists $\lambda > \lambda_k$ and $M > 0$ such that

$$\frac{f(s)}{|s|^{p-2}s} > \lambda \quad \text{for } |s| \ge M. \tag{5.9}$$

Let $\{x_i\}_{i=0}^k$ be zeros of an eigenfunction φ_k corresponding to λ_k satisfying (3.3). By Lemma 3.3 (i), for each $i \in \{1, 2, \dots, k\}$, there exists a solution w_i of (3.4) having at least two zeros in (x_{i-1}, x_i).

Now fix $i \in \{1, 2, \dots, k\}$. Let t_1 and t_2 be zeros of w_i such that $x_{i-1} < t_1 < t_2 < x_i$. Define $\delta > 0$ by (5.7). We remark that $\delta \to 0$ as $\mu \to \infty$. Then we can take $\mu_i > 0$ so large that, for all $\mu \ge \mu_i$, we have (5.6), $x_i - x_{i-1} > 2\delta$, and $[t_1, t_2] \subset (x_{i-1} + \delta, x_i - \delta)$. Let $\mu \ge \mu_i$. We will show that $u(x;\mu)$ has at least one zero in (x_{i-1}, x_i).

Assume to the contrary that $u(x;\mu)$ has no zero in (x_{i-1}, x_i). From Lemma 5.2 (iii) we obtain $|u(x;\mu)| > M$ for $x \in (x_{i-1} + \delta, x_i - \delta)$. From (5.9), the function b given by (5.4) satisfies

$$\lambda a(x) < b(x) \quad \text{for} \quad x \in [t_1, t_2] \subset (x_{i-1} + \delta, x_i - \delta).$$

Then Lemma 3.1 implies that $u(x;\mu)$ has at least one zero in (t_1, t_2). This is a contradiction. Thus $u(x;\mu)$ with $\mu \ge \mu_i$ has at least one zero in (x_{i-1}, x_i).

Put $\mu^* = \max\{\mu_i : i = 1, 2, \dots, k\}$. If $\mu \ge \mu^*$, then $u(x;\mu)$ has at least one zero in (x_{i-1}, x_i) for each $i = 1, 2, \dots, k$, which means that $u(x;\mu)$ has at least k zeros in $(0, 1)$. □

Proof of Lemma 4.4. From (4.4), there exists $\lambda < \lambda_k$ and $M > 0$ such that

$$\frac{f(s)}{|s|^{p-2}s} < \lambda \quad \text{for } |s| \geq M. \tag{5.10}$$

Let $\{x_i\}_{i=0}^k$ be zeros of an eigenfunction φ_k satisfying (3.3). By Lemma 3.3 (ii), for every $i \in \{1, 2, \dots, k\}$, there exists a solution w_i of (3.4) such that $w_i(x) > 0$ on $[x_{i-1}, x_i]$. For each $i \in \{1, 2, \dots, k\}$, we define W_i by

$$W_i = \max\left\{\frac{|w_i'(x)|}{w_i(x)} : x \in [x_{i-1}, x_i]\right\}.$$

There exists a number $\mu^* > 0$ so large that, if $\mu \geq \mu^*$, then (5.6) and

$$\Big(A_*\mu^p - a^*F^*(M)\Big)^{1/p} M^{-1} \geq \max\{W_i : i = 1, 2, \dots, k\} \tag{5.11}$$

hold. Let $\mu \geq \mu^*$. We will show that $u(x;\mu)$ has at most one zero in $[x_{i-1}, x_i)$ for each $i \in \{1, 2, \dots, k\}$. For simplicity, we denote $u(x;\mu)$ by $u(x)$. Suppose that $u(x)$ has at least two zeros in $[x_{i-1}, x_i)$ for some $i \in \{1, 2, \dots, k\}$. Then there exist α, $\beta \in [x_{i-1}, x_i)$ with $\alpha < \beta$ such that $u(\alpha) = u(\beta) = 0$. We may assume that $u(x) > 0$ on (α, β). Take $\gamma \in (\alpha, \beta)$ such that $u(\gamma) = \max_{x \in [\alpha,\beta]} u(x)$. Then $u'(\gamma) = 0$ and $u(\gamma) > M$ from Lemma 5.2 (i). Thus there exists t_1 and t_2 such that

$$\alpha < t_1 < t_2 < \beta, \quad u(t_1) = u(t_2) = M \quad \text{and} \quad u(x) > M \quad \text{for } x \in (t_1, t_2).$$

We note that $u'(t_1) > 0$ and $u'(t_2) < 0$. From (5.1) we have

$$A_*\mu^p \leq E[u(\,\cdot\,;\mu)](t_j) \leq |u'(t_j)|^p + a^*F^*(M) \quad \text{for } j = 1, 2.$$

This implies that

$$|u'(t_j)|^p \geq A_*\mu^p - a^*F^*(M) \quad \text{for } j = 1, 2.$$

From $u(t_1) = u(t_2) = M$ and (5.11), we obtain

$$\frac{|u'(t_j)|}{u(t_j)} \geq \Big(A_*\mu^p - a^*F(M)\Big)^{1/p} M^{-1} \geq W_i \quad \text{for } j = 1, 2. \tag{5.12}$$

In view of $u'(t_1) > 0$ we have

$$\frac{u'(t_1)}{u(t_1)} \geq W_i \geq \frac{w_i'(t_1)}{w_i(t_1)}.$$

Since $\Phi(u) = |u|^{p-2}u$ is increasing for $u \in \mathbf{R}$, we obtain

$$\Phi\left(\frac{u'(t_1)}{u(t_1)}\right) \geq \Phi\left(\frac{w_i'(t_1)}{w_i(t_1)}\right), \quad \text{that is} \quad \frac{\Phi(u'(t_1))}{\Phi(u(t_1))} \geq \frac{\Phi(w_i'(t_1))}{\Phi(w_i(t_1))}.$$

Hence,

$$\Phi(u'(t_1))\Phi(w_i(t_1)) - \Phi(u(t_1))\Phi(w_i'(t_1)) \geq 0. \tag{5.13}$$

Recall that u satisfies (5.3) on $[t_1, t_2]$. By applying Lemma 3.2, we have

$$\frac{d}{dx}\left[\frac{u}{\Phi(w_i)}\Big(\Phi(u')\Phi(w_i) - \Phi(u)\Phi(w_i')\Big)\right] \geq (\lambda a(x) - b(x))|u|^p \tag{5.14}$$

for $t_1 \leq x \leq t_2$. From (5.10) and the fact that $u(x) \geq M$ for $t_1 \leq x \leq t_2$, we have $\lambda a(x) - b(x) > 0$ on $[t_1, t_2]$. Then, by integrating (5.14) on $[t_1, t_2]$, we obtain

$$\left[\frac{u}{\Phi(w_i)}\Big(\Phi(u')\Phi(w_i) - \Phi(u)\Phi(w_i')\Big)\right]_{t_1}^{t_2} \geq \int_{t_1}^{t_2} [\lambda a(x) - b(x)]|u|^p dx > 0.$$

Therefore from (5.13) we have

$$\Phi(u'(t_2))\Phi(w_i(t_2)) - \Phi(u(t_2))\Phi(w_i'(t_2)) > 0.$$

Then it follows that

$$\frac{\Phi(u'(t_2))}{\Phi(u(t_2))} > \frac{\Phi(w_i'(t_2))}{\Phi(w_i(t_2))}, \quad \text{and hence} \quad \frac{u'(t_2)}{u(t_2)} > \frac{w_i'(t_2)}{w_i(t_2)}.$$

This implies that

$$\frac{|u'(t_2)|}{u(t_2)} = \frac{-u'(t_2)}{u(t_2)} < \frac{-w_i'(t_2)}{w_i(t_2)} \leq \frac{|w_i'(t_2)|}{w_i(t_2)} \leq W_i.$$

This contradicts (5.12). Consequently u has at most one zero in $[x_{i-1}, x_i)$ for each $i = 1, 2, \ldots, k$. Note that u has no zero in (x_0, x_1), since $x_0 = 0$ is a zero of u in $[x_0, x_1)$. Hence u has at most $k-1$ zeros in $(0,1)$. □

Appendix: Proof of Lemma 2.2

Note that the initial value problem (2.6) can be written by

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} |v|^{\frac{1}{p-1}-1}v \\ -a(x)f(u) \end{pmatrix}, \qquad u(x_0) = \alpha, \quad v(x_0) = |\beta|^{p-2}\beta. \tag{A.1}$$

Then the existence of a local solution of (2.6) is guaranteed by the Peano existence theorem. Since $|v|^{\frac{1}{p-1}-1}v$ and $f(u)$ are locally Lipschitz continuous in $u, v \in \mathbf{R} \setminus \{0\}$, the local solution of (A.1) is unique for any $\alpha \neq 0$ and $\beta \neq 0$. Thus it suffices to show the uniqueness of a local solution of the problem in the case $\alpha\beta = 0$. We divide the proof into the following three cases: (i) $\alpha = 0$, $\beta = 0$; (ii) $\alpha = 0$, $\beta \neq 0$; (iii) $\alpha \neq 0$, $\beta = 0$. We will show the uniqueness in a right-neighborhood of x_0 only. In the same way we can prove the uniqueness in a left-neighborhood of x_0.

(i) Let $\alpha = 0$, $\beta = 0$. In this case we will show $u \equiv 0$ by using the function $E[u]$ defined by (2.2). Let u be a local solution of (2.6) with $\alpha = 0$ and $\beta = 0$. Since $E[u](x) \geq 0$ and $E[u](x_0) = 0$, Lemma 2.1 implies that $E[u](x) \equiv 0$ for $x \in I[u]$. Thus $u(x) \equiv 0$ for $x \in I[u]$.

(ii) Let $\alpha = 0$, $\beta \neq 0$. In this case, we show the uniquenss of solutions following the idea by Kajikiya [10]. We may suppose that $\beta > 0$ without loss of generality. Let u_1 and u_2 be local solutions of (2.6) with $\alpha = 0$ and $\beta > 0$. Then there exists an $x_1 \in (x_0, 1]$ such that

$$u_j'(x) \geq \frac{\beta}{2} \quad \text{for } x \in [x_0, x_1] \text{ and } j = 1, 2. \tag{A.2}$$

In view of (2.5), we obtain

$$E[u_j](x) - E[u_j](x_0) = \int_{x_0}^{x} a'(t)F(u_j(t))dt$$

for $x \in (x_0, x_1]$ and $j = 1$, 2. Noting that $E[u_1](x_0) = E[u_2](x_0)$, we have

$$E[u_1](x) - E[u_2](x) = \int_{x_0}^{x} a'(t)\left[F(u_1(t)) - F(u_2(t))\right]dt$$

for $x \in [x_0, x_1]$. Using (2.2), it follows that

$$\begin{aligned} |u_1'(x)|^p - |u_2'(x)|^p = & -a(x)[F(u_1(x)) - F(u_2(x))] \\ & + \int_{x_0}^{x} a'(t)\left[F(u_1(t)) - F(u_2(t))\right]dt \end{aligned} \tag{A.3}$$

for $x \in [x_0, x_1]$. From the mean value theorem and (A.2), we have

$$\left||u_1'|^p - |u_2'|^p\right| \geq p\left(\frac{\beta}{2}\right)^{p-1}|u_1' - u_2'| \quad \text{for } x \in [x_0, x_1]. \tag{A.4}$$

Set

$$\delta = \max_{j=1,2}\max_{x\in[x_0,x_1]}|u_j(x)| \quad \text{and} \quad M = \max_{|s|\leq\delta}|f(s)|.$$

Then we have

$$|F(u_1(x)) - F(u_2(x))| = \left|\frac{p}{p-1}\int_{u_2(x)}^{u_1(x)} f(s)ds\right| \leq \frac{pM}{p-1}|u_1(x) - u_2(x)| \tag{A.5}$$

for $x \in [x_0, x_1]$. Put $U(x) = u_1(x) - u_2(x)$. Using (A.3)–(A.5), we obtain

$$|U'(x)| \leq C|U(x)| + C\int_{x_0}^{x}|U(t)|dt \quad \text{for } x \in [x_0, x_1] \tag{A.6}$$

with some constant $C > 0$. Note that $|x_1 - x_0| < 1$. Integrating (A.6) over $[x_0, x]$, we have

$$|U(x)| \leq C\int_{x_0}^{x}|U(t)|dt + C\int_{x_0}^{x}(x-t)|U(t)|dt \leq 2C\int_{x_0}^{x}|U(t)|dt$$

for $x \in [x_0, x_1]$. Then by Gronwall's inequality we find that $U(x) = u_1(x) - u_2(x) \equiv 0$ for $x \in [x_0, x_1]$.

(iii) Let $\alpha \neq 0$, $\beta = 0$. If $1 < p \leq 2$, then $|v|^{\frac{1}{p-1}-1}v$ is locally Lipschitz continuous on $v \in \mathbf{R}$. By recalling (A.1), the Picard theorem ensures that the local solution of (2.6) with $\alpha \neq 0$ and $\beta = 0$ is unique.

Now assume that $p > 2$. We may suppose that $\alpha < 0$ without loss of generality. Let u_1 and u_2 be local solutions of (2.6) with $\alpha < 0$ and $\beta = 0$. Then there exists a number $x_1 \in (x_0, 1)$ such that

$$-a(x)f(u_j(x)) \geq \gamma > 0 \quad \text{for } x \in [x_0, x_1],\ j = 1, 2, \tag{A.7}$$

where $\gamma = -a(x_0)f(\alpha)/2$. Integrating the equation over $[x_0, x]$ and using (A.7), we have

$$u_j'(x) = \left(-\int_{x_0}^{x} a(t)f(u_j(t))dt\right)^{\frac{1}{p-1}} \quad \text{and} \quad -\int_{x_0}^{x} a(t)f(u_j(t))dt \geq \gamma(x - x_0)$$

for $x \in [x_0, x_1]$, $j = 1, 2$. By the mean value theorem, it follows that

$$u_1'(x) - u_2'(x) \leq \frac{1}{p-1}\left(\gamma(x - x_0)\right)^{-\frac{p-2}{p-1}} \int_{x_0}^{x} a(t)|f(u_1(t)) - f(u_2(t))|dt \tag{A.8}$$

for $x \in (x_0, x_1]$. Since f is locally Lipschitz continuous on $\mathbf{R} \setminus \{0\}$, there exist $\delta > 0$ and $L > 0$ such that $\alpha + \delta < 0$ and

$$|f(s) - f(t)| \leq L|s - t| \quad \text{for } s, t \in [\alpha - \delta, \alpha + \delta]. \tag{A.9}$$

There exists $x_2 \in (x_0, x_1]$ such that

$$\alpha - \delta \leq u_j(x) \leq \alpha + \delta \quad \text{for } x \in [x_0, x_2],\ j = 1, 2. \tag{A.10}$$

Combining (A.8)–(A.10), we obtain

$$u_1'(x) - u_2'(x) \leq C(x - x_0)^{-\frac{p-2}{p-1}} \int_{x_0}^{x} |u_1(t) - u_2(t)|dt \quad \text{for } x \in (x_0, x_2]$$

with some constant $C > 0$. Put $U(x) = \max_{t \in [x_0, x]} |u_1(t) - u_2(t)|$. Then we have

$$u_1'(x) - u_2'(x) \leq C(x - x_0)^{-\frac{p-2}{p-1}}(x - x_0)U(x) = C(x - x_0)^{\frac{1}{p-1}}U(x) \tag{A.11}$$

for $x \in (x_0, x_2]$. Therefore (A.11) holds even for $x = x_0$. Integrating (A.11) over $[x_0, x_2]$, we find that

$$u_1(x) - u_2(x) \leq \int_{x_0}^{x} C(t - x_0)^{\frac{1}{p-1}}U(t)dt \quad \text{for } x \in [x_0, x_2],$$

which implies that

$$U(x) \leq \int_{x_0}^{x} C(t-x_0)^{\frac{1}{p-1}} U(t)dt \quad \text{for } x \in [x_0, x_2].$$

Then by Gronwall's inequality we obtain $U(x) \equiv 0$ for $x \in [x_0, x_2]$, and hence $u_1(x) = u_2(x)$ for $x \in [x_0, x_2]$.

This completes the proof of Lemma 2.2.

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