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(Citation)
Journal of Mathematical Analysis and Applications,342(1):125-134

(Issue Date)
2008-06-01

(Resource Type)
journal article

(Version)
Accepted Manuscript

(URL)
https://hdl.handle.net/20.500.14094/90000822

Parameter identification problem for the equation of motion of membrane with strong viscosity

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Abstract The parameter identification problem of constant parameters in the equation of membrane with strong viscosity is studied. The problem is formulated by a minimization of quadratic cost functionals by distributive measurements. The existence of optimal parameters and necessary optimality conditions for the parameters are proved.

Key words: Identification problem; Equation of membrane with strong viscosity; Quadratic cost functionals; Necessary optimality conditions

1 Introduction

Let Ω be an open bounded set of $\mathbf{R}^n$ with the smooth boundary Γ. The inner product of $\mathbf{R}^n$ is denoted by $x \cdot y$ for $x, y \in \mathbf{R}^n$. We put $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$ for $T > 0$. We study an identification problem for diffusive constants α, β in the equation of motion of membrane with strong viscosity

$$
\begin{cases}
\dfrac{\partial^2 y}{\partial t^2} - \alpha \mathrm{div}\Big(\dfrac{\nabla y}{\sqrt{1+|\nabla y|^2}}\Big) - \beta \Delta \dfrac{\partial y}{\partial t} = f \quad \text{in} \ \ Q, \\
y = 0 \qquad \text{on } \Sigma, \\
y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \ \ \text{in } \Omega,
\end{cases}
\tag{1.1}
$$

where f is a forcing function, y_0 and y_1 are initial conditions. In previous paper Hwang and Nakagiri [5], we studied the quadratic optimal control problems for (1.1) and established the necessary conditions for the costs of distributive and

*Supported by Grant-in-Aid for Scientific Research (C)(2) 16540194.

terminal values observations based on the well posedness of weak solutions in [4].

The system (1.1) is proposed to be a model equation of the dynamics of longitudinal motion of vibrating membrane surrounding Ω by taking into accounts of strong (diffusive) viscosity effects (cf. Gilbarg and Trudinger [7]). It is an important problem that whether the physical constants α, β can be estimated or not by the possible observed measurements. In this paper we study a parameter identification problem of finding best parameters α, β for which the solution of (1.1) gives a minimum of the given functionals. For the approach of identification problems based on the optimal control theory due to Lions [6], we refer to Ahmed [1] for abstract evolution equations and Ha and Nakagiri [3] for damped Klein-Gordon equations. We take the same approach as in [3].

We explain our identification problem precisely as follows. First we replace the positive constants α, β in (1.1) by $\alpha^2 + \alpha_0, \beta^2 + \beta_0$ in order to take the parameter space $\mathcal{P} = \mathbf{R}^2$ as the set of parameters (α, β). Let $q = (\alpha, \beta) \in \mathcal{P}$ and let $y(q) = y(q; t, x)$ be the solution for a given $q \in \mathcal{P}$. Let $\mathcal{P}_{ad} \subset \mathcal{P}$ be an admissible parameter set. We consider the following two quadratic distributive functionals:

$$J_1(q) = \int_0^T \int_\Omega |y(q; t, x) - z_{d_1}(t, x)|^2 dx dt \quad \text{for} \quad q \in \mathcal{P}, \tag{1.2}$$

$$J_2(q) = \int_0^T \int_\Omega \left| \frac{\partial y(q; t, x)}{\partial t} - z_{d_2}(t, x) \right|^2 dx dt \quad \text{for} \quad q \in \mathcal{P}, \tag{1.3}$$

where $z_{d_i} \in L^2(Q),\ i = 1, 2$ are the desired values. We remark that the cost of velocity mesurements (1.3) has not treated in [3]. The parameter identification problem for (1.1) with the cost $J = J_1$ in (1.2) or $J = J_2$ in (1.3) is to find and characterize an optimal parameters $q^* = (\alpha^*, \beta^*) \in \mathcal{P}_{ad}$ satisfying

$$J(q^*) = \inf \{J(q) :\ q \in \mathcal{P}_{ad}\}. \tag{1.4}$$

We prove the existence of an optimal parameter q^* by using the continuity of solutions on parameters and establish the necessary optimality conditions by introducing appropriate adjoint systems. For this we prove the strong Gâteaux differentiability of the nonlinear mapping $q \to y(q)$. We also emphasize that in the velocity's measurements case (1.3), a first order Volterra integro-differential equation is utilized as a proper adjoint system as in [5].

2 Weak solutions and energy equality

For the problem (1.1) we suppose that $f \in L^2(0, T; H^{-1}(\Omega))$, $y_0 \in H_0^1(\Omega)$ and $y_1 \in L^2(\Omega)$. As in [5] the solution space $W(0, T)$ of (1.1) is defined by

$$W(0, T) = \{g | g \in L^2(0, T; H_0^1(\Omega)),\ g' \in L^2(0, T; H_0^1(\Omega)),\ g'' \in L^2(0, T; H^{-1}(\Omega))\}$$

endowed with the norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;H^1_0(\Omega))} + \|g'\|^2_{L^2(0,T;H^1_0(\Omega))} + \|g''\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)^{\frac{1}{2}}.$$

The space $W(0,T)$ is continuously imbedded in $C([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))$ (cf. Dautray and Lions [2, p.555]). The scalar products and norms on $L^2(\Omega)$ and $H^1_0(\Omega)$ are denoted by (ϕ,ψ), $|\phi|$ and $(\phi,\psi)_{H^1_0(\Omega)}$, $\|\phi\|$, respectively. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by (ϕ,ψ) and $|\phi|$. Then the scalar product $(\phi,\psi)_{H^1_0(\Omega)}$ and the norm $\|\phi\|$ of $H^1_0(\Omega)$ are given by $(\nabla\phi,\nabla\psi)$ and $\|\phi\| = |(\nabla\phi,\nabla\phi)|^{\frac{1}{2}}$, respectively. The duality pairing between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle\phi,\psi\rangle$.

The nonlinear operator $G(\nabla\cdot) : H^1_0(\Omega) \to [L^2(\Omega)]^n$ is defined by

$$G(\nabla\phi)(x) = \frac{\nabla\phi(x)}{\sqrt{1+|\nabla\phi(x)|^2}} \quad \text{a.e. } x\in\Omega, \quad \forall\phi\in H^1_0(\Omega). \tag{2.1}$$

By the definition of $G(\nabla\cdot)$ in (2.1), we have the following property on $G(\nabla\cdot)$:

$$|G(\nabla\phi)| \le |\nabla\phi|, \quad |G(\nabla\phi)-G(\nabla\psi)| \le 2|\nabla\phi-\nabla\psi|, \qquad \forall\phi,\psi\in H^1_0(\Omega). \tag{2.2}$$

As in Dautray and Lions [2, p.555], we give the variational formulation of weak solutions of (1.1). A function y is said to be a weak solution of (1.1) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \langle y''(\cdot),\phi\rangle + \alpha(G(\nabla y(\cdot)),\nabla\phi) + \beta(\nabla y'(\cdot),\nabla\phi) = \langle f(\cdot),\phi\rangle \\ \qquad\qquad\qquad \text{for all } \ \phi\in H^1_0(\Omega) \ \text{ in the sense of } \ \mathcal{D}'(0,T), \\ y(0)=y_0\in H^1_0(\Omega), \quad y'(0)=y_1\in L^2(\Omega). \end{cases} \tag{2.3}$$

The following two theorems are proved in Hwang and Nakagiri [4] by the Galerkin method.

Theorem 2.1 *Assume that $\alpha,\beta>0$, $y_0\in H^1_0(\Omega)$, $y_1\in L^2(\Omega)$ and $f\in L^2(0,T;H^{-1}(\Omega))$. Then the problem (1.1) has a unique weak solution y in $W(0,T)$. Furthermore, y has the following estimate*

$$\begin{aligned} & |y'(t)|^2 + |\nabla y(t)|^2 + \int_0^t |\nabla y'(s)|^2 ds \\ \le\ & C(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;H^{-1}(\Omega))}), \qquad \forall t \ \in \ [0,T], \end{aligned} \tag{2.4}$$

where $C=C(\alpha,\beta)$ is a constant depending only on $\alpha,\beta>0$.

We will omit writing the integral variables in the definite integral without any confusion. For example, in (2.4) we will write $\int_0^t |\nabla y'|^2 ds$ instead of $\int_0^t |\nabla y'(s)|^2 ds$.

Remark 2.1 The constant $C(\alpha,\beta)$ in Theorem 2.1 can be chosen uniformly on any bounded set of (α,β), $\alpha,\beta>0$ as shown in [4].

3 Identification problems

Let $\mathcal{P} = \mathbf{R}^2$ be the space of parameters $q = (\alpha, \beta)$ with the Euclidian norm. In this section we study the identification problem for the unknown parameters $q = (\alpha, \beta) \in \mathcal{P}$ in the problem

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - (\alpha_0 + \alpha^2)\mathrm{div}\Big(\dfrac{\nabla y}{\sqrt{1 + |\nabla y|^2}}\Big) - (\beta_0 + \beta^2)\Delta \dfrac{\partial y}{\partial t} = f \quad \text{in} \quad Q, \\ y = 0 \qquad \text{on} \quad \Sigma, \\ y(0, x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0, x) = y_1(x) \qquad \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\alpha_0,\ \beta_0 > 0$, $y_0 \in H_0^1(\Omega),\ y_1 \in L^2(\Omega)$, and $f \in L^2(0, T; H^{-1}(\Omega))$ are fixed. The pair of diffusive and viscosity coefficients $q = (\alpha, \beta)$ in (3.1) is an unknown parameter should be identified. In this setting we take the two dimensional Euclidean space $\mathcal{P} = \mathbf{R}^2$ as the set of parameters (α, β). By Theorem 2.1 we have that for each $q \in \mathcal{P}$ there exists a unique weak solution $y = y(q) \in W(0, T)$ of (3.1).

First we shall show the continuous dependence of solutions on parameters $q = (\alpha, \beta)$.

Theorem 3.1 *The solution map $q \to y(q)$ from $\mathcal{P} = \mathbf{R}^2$ into $W(0, T)$ is continuous.*

Proof. Let $q = (\alpha, \beta)$ be arbitrarily fixed. Suppose $q_m = (\alpha_m, \beta_m) \to q = (\alpha, \beta)$ in $\mathcal{P}$. Let $y_m = y(q_m)$ and $y = y(q)$ be the solutions of (3.1) for $q = q_m$ and for q, respectively. Since $\{q_m\}$ is bounded in $\mathcal{P}$, by Theorem 2.1 and Remark 2.1, we see that

$$|y_m'(t)|^2 + |\nabla y_m(t)|^2 + \int_0^t |\nabla y_m'|^2 ds \le C_0 < \infty, \qquad \forall t \ \in \ [0, T], \tag{3.2}$$

where $C_0 > 0$ is a constant depending only on $\alpha_0, \beta_0, y_0, y_1$ and f. We set $z_m = y_m - y$. Then z_m satisfies

$$\begin{cases} \dfrac{\partial^2 z_m}{\partial t^2} - (\alpha_0 + \alpha_m^2)\mathrm{div}\Big(G(\nabla y_m) - G(\nabla y)\Big) - (\beta_0 + \beta_m^2)\Delta \dfrac{\partial z_m}{\partial t} \\ \qquad = -(\alpha^2 - \alpha_m^2)\mathrm{div}\, G(\nabla y) - (\beta^2 - \beta_m^2)\Delta \dfrac{\partial y}{\partial t} \qquad \text{in} \quad Q, \\ z_m = 0 \qquad \text{on} \quad \Sigma, \\ z_m(0, x) = 0, \quad \dfrac{\partial z_m}{\partial t}(0, x) = 0 \qquad \text{in } \Omega \end{cases} \tag{3.3}$$

in the weak sense. Multiply the weak form of (3.3) by z_m' and z_m, integrate them over $[0, t]$ and add the integrals, then we have

$$|z_m'(t)|^2 + (\beta_0 + \beta_m^2)|\nabla z_m(t)|^2 + 2(\beta_0 + \beta_m^2)\int_0^t |\nabla z_m'|^2 ds$$

$$
\begin{aligned}
= \quad & \int_0^t (2(\alpha^2-\alpha_m^2)G(\nabla y)+2(\beta^2-\beta_m^2)\nabla y', \nabla z_m + \nabla z_m')ds \\
& - 2(\alpha_0+\alpha_m^2)\int_0^t (G(\nabla y_m)-G(\nabla y), \nabla z_m+\nabla z_m')ds \\
& - 2(z_m'(t), z_m(t)) + 2\int_0^t |z_m'|^2 ds. \qquad (3.4)
\end{aligned}
$$

We estimate the right hand side of (3.4). We put

$$
\Phi_m(t) = \int_0^t (2(\alpha^2-\alpha_m^2)G(\nabla y)+2(\beta^2-\beta_m^2)\nabla y', \nabla z_m+\nabla z_m')ds. \qquad (3.5)
$$

Since $|z_m| \leq |y| + |y_m|$, we can verify by (2.2), (3.2) and Schwartz inequality that

$$
|\Phi_m(t)| \leq K_0(|\alpha^2-\alpha_m^2| + |\beta^2-\beta_m^2|), \quad \forall t \in [0,T] \qquad (3.6)
$$

for some $K_0 > 0$. By (2.2), we see

$$
|G(\nabla y_m) - G(\nabla y)| \leq 2|\nabla z_m|. \qquad (3.7)
$$

We notice here that $\alpha_0 + \alpha_m^2$ is uniformly bounded in m. Let $\epsilon > 0$ be an arbitrary number. Then, by (3.7) and Schwartz inequality, we can deduce

$$
\begin{aligned}
& 2\Big|(\alpha_0+\alpha_m^2)\int_0^t (G(\nabla y_m)-G(\nabla y), \nabla z_m+\nabla z_m')ds\Big| \\
\leq \quad & K_1(\epsilon)\int_0^t |\nabla z_m|^2 ds + \epsilon\int_0^t |\nabla z_m'|^2 ds, \qquad (3.8)
\end{aligned}
$$

where $K_1(\epsilon)$ is a positive constant depending on ϵ. We choose $\epsilon = \beta_0 > 0$. Then by (3.4), (3.8) and the positivity of β_0, we can obtain

$$
\begin{aligned}
& |z_m'(t)|^2 + |\nabla z_m(t)|^2 + \int_0^t |\nabla z_m'|^2 ds \\
\leq \quad & C_1\Phi_m(t) + C_2\int_0^t (|z_m'|^2 + |\nabla z_m|^2)ds, \qquad (3.9)
\end{aligned}
$$

where $C_1, C_2 > 0$. Hence applying Gronwall's inequality to (3.9), we have

$$
\begin{aligned}
& |z_m'(t)|^2 + |\nabla z_m(t)|^2 + \int_0^t |\nabla z_m'|^2 ds \\
\leq \ & C_1\Phi_m(t) + C_1C_2\exp(C_2T)\int_0^t \Phi_m ds. \qquad (3.10)
\end{aligned}
$$

Since $\Phi_m(t) \to 0$ as $m \to \infty$ for all $t \in [0,T]$ by (3.6), we obtain from (3.10) that

$$
\begin{aligned}
z_m(\cdot) \quad &\to \quad 0 \quad \text{in } \ C([0,T]; H_0^1(\Omega)), \\
z_m'(\cdot) \quad &\to \quad 0 \quad \text{in } \ C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)),
\end{aligned}
$$

so that

$$y_m(\cdot) \;\to\; y(\cdot) \quad \text{strongly in} \;\; W(0,T).$$

This proves Theorem 3.1.

The cost functionals given in Introduction are represented by

$$J_1(q) = \|y(q) - z_{d_1}\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{for} \;\; q \in \mathcal{P}, \tag{3.11}$$

$$J_2(q) = \|y'(q) - z_{d_2}\|^2_{L^2(0,T;L^2(\Omega))} \quad \text{for} \;\; q \in \mathcal{P}, \tag{3.12}$$

respectively, where $z_{d_i} \in L^2(0,T;L^2(\Omega))$, $i = 1,2$. We choose the cost $J = J_1$ or $J = J_2$ for the identification of $q = (\alpha, \beta)$.

Assume that an admissible subset $\mathcal{P}_{ad}$ of $\mathcal{P}$ is convex and closed. If $\mathcal{P}_{ad}$ is compact, then for the minimizing sequence $\{q_m\}$ such as $J(q_m) \to J^* = \inf\{J(q) : q \in \mathcal{P}_{ad}\}$ we can choose a subsequence $\{q_{mj}\}$ of $\{q_m\}$ such that $q_{mj} \to q^* \in \mathcal{P}_{ad}$ and $y(q_{mj}) \to y(q^*)$ strongly in $W(0,T)$ by Theorem 3.1. Due to the continuous imbedding $W(0,T) \hookrightarrow C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ we have $J^* = J(q^*)$ for the costs (3.11) and (3.12). Then we have the following corollary.

Corollary 3.1 *If $\mathcal{P}_{ad}$ is compact, then there exists at least one optimal parameter $q^* \in \mathcal{P}_{ad}$ for the cost J_1 in* (3.11) *or J_2 in* (3.12).

Let the cost $J = J(q)$ be given over $\mathcal{P}$. Let the admissible set $\mathcal{P}_{ad}$ be closed and convex in $\mathcal{P}$ and let $q^* = (\alpha^*, \beta^*)$ be an optimal parameter on $\mathcal{P}_{ad}$ for the cost J. As is well known the necessary optimality condition for the cost J is given by

$$DJ(q^*)(q - q^*) \ge 0 \;\; \text{for all} \;\; q \in \mathcal{P}_{ad}, \tag{3.13}$$

where $DJ(q^*)$ denotes the Gâteaux derivative of $J(q)$ at $q = q^*$.

If the cost $J(q)$ is given by a quadratic functional $J(q) = J(y(q))$ over the solution space $W(0,T)$, then the Gâteaux differentiability of $J(q)$ follows from that of $y(q)$ in q. We can prove that the solution map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is Gâteaux differentiable. The following theorem gives the characterization of the Gâteaux derivatives as in [5].

Theorem 3.2 *The map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is Gâteaux differentiable at $q = q^*$ and such the Gâteaux derivative of $y(q)$ at $q = q^*$ in the direction $q - q^* \in \mathcal{P}$, say $z = Dy(q^*)(q - q^*)$, is a unique weak solution of the following linear problem*

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2} - (\alpha_0 + {\alpha^*}^2)\mathrm{div}\Big(\dfrac{\nabla z}{\sqrt{1 + |\nabla y^*|^2}} - \dfrac{\nabla y^* \cdot \nabla z}{\big(1 + |\nabla y^*|^2\big)^{\frac{3}{2}}}\nabla y^*\Big) \\ \qquad -(\beta_0 + {\beta^*}^2)\Delta \dfrac{\partial z}{\partial t} = 2\alpha^*(\alpha - \alpha^*)\mathrm{div}\, G(\nabla y^*) + 2\beta^*(\beta - \beta^*)\Delta \dfrac{\partial y^*}{\partial t} \;\; \text{in} \;\; Q, \\ z = 0 \quad \text{on} \;\; \Sigma, \\ z(0,x) = 0, \quad \dfrac{\partial z}{\partial t}(0,x) = 0 \quad \text{in} \;\; \Omega, \end{cases} \tag{3.14}$$

where $y^* = y(q^*)$.

Proof. Let $\lambda \in [0,1]$, and let y_λ and y^* be the weak solutions of (3.1) corresponding to $q^* + \lambda(q - q^*)$ and q^*, respectively. We set $z_\lambda = \lambda^{-1}(y_\lambda - y^*)$, $\lambda \neq 0$. Then z_λ is a solution of the following problem in the weak sense:

$$\begin{cases} \dfrac{\partial^2 z_\lambda}{\partial t^2} - (\alpha_0 + {\alpha^*}^2)\mathrm{div}\, \dfrac{1}{\lambda}\Big(G(\nabla y_\lambda) - G(\nabla y^*)\Big) - (\beta_0 + {\beta^*}^2)\Delta \dfrac{\partial z_\lambda}{\partial t} \\ \qquad = F_\lambda \quad \text{in } \ Q, \\ z_\lambda = 0 \quad \text{on } \ \Sigma, \\ z_\lambda(0,x) = 0, \quad \dfrac{\partial z_\lambda}{\partial t}(0,x) = 0 \quad \text{in } \ \Omega, \end{cases} \tag{3.15}$$

where

$$\begin{aligned} F_\lambda \ &= \ (2\alpha^*(\alpha - \alpha^*) + \lambda(\alpha - \alpha^*)^2)\mathrm{div}\, G(\nabla y_\lambda) \\ &\quad + (2\beta^*(\beta - \beta^*) + \lambda(\beta - \beta^*)^2)\Delta \frac{\partial y_\lambda}{\partial t}. \end{aligned} \tag{3.16}$$

It is verified by (2.4) and Remark 2.1 that $F_\lambda \in L^2(0,T;H^{-1}(\Omega))$ and

$$\|F_\lambda\|_{L^2(0,T;H^{-1}(\Omega))} \le C(\lambda)\|y_\lambda\|_{W(0,T)} \le C_3 < \infty, \tag{3.17}$$

where C_3 is a constant independent of λ. We set

$$\mathcal{G}_\lambda = \frac{1}{\lambda}\Big(G(\nabla y_\lambda) - G(\nabla y^*)\Big). \tag{3.18}$$

Then we see easily by (2.2) that

$$|\mathcal{G}_\lambda| \le 2|\nabla z_\lambda|. \tag{3.19}$$

Hence, by repeating similar calculations using Gronwall's inequality as in the proof of Theorem 3.1, we can deduce

$$|\nabla z_\lambda(t)|^2 + |z'_\lambda(t)|^2 + \int_0^t |\nabla z'_\lambda|^2 ds \le K_2\|F_\lambda\|^2_{L^2(0,T;H^{-1}(\Omega))} \le K_2 C_3 \tag{3.20}$$

for some $K_1 > 0$. Therefore there exists a $z \in W(0,T)$ and a sequence $\{\lambda_k\} \subset [0,1]$ tending to 0 such that

$$\begin{cases} z_{\lambda_k} \to z \ \text{ weakly star in } \ L^\infty(0,T;H_0^1(\Omega)) \\ \qquad\qquad \text{and weakly in } \ L^2(0,T;H_0^1(\Omega)) \ \text{ as } \ k \to \infty, \\ z'_{\lambda_k} \to z' \ \text{ weakly star in } \ L^\infty(0,T;L^2(\Omega)) \\ \qquad\qquad \text{and weakly in } \ L^2(0,T;H_0^1(\Omega)) \ \text{ as } \ k \to \infty, \\ z(0) = 0, \quad z'(0) = 0. \end{cases} \tag{3.21}$$

By the strong convergence $y_\lambda \to y^*$ in $W(0,T)$ as $\lambda \to 0$ and (3.21), we can prove as in [5] that

$$\begin{aligned} \mathrm{div}\, \mathcal{G}_{\lambda_k} \to \mathrm{div}\Big(\frac{\nabla z}{\sqrt{1 + |\nabla y^*|^2}} - \frac{\nabla y^* \cdot \nabla z}{\big(1 + |\nabla y^*|^2\big)^{\frac{3}{2}}}\nabla y^*\Big) \\ \text{weakly in } \ L^2(0,T;H^{-1}(\Omega)) \end{aligned} \tag{3.22}$$

as $\lambda_k \to 0$. At the same time we can prove that

$$F_\lambda \to F \equiv 2\alpha^*(\alpha-\alpha^*)\mathrm{div}\, G(\nabla y^*) + 2\beta^*(\beta-\beta^*)\Delta\frac{\partial y^*}{\partial t}$$
$$\text{strongly in } \ L^2(0,T;H^{-1}(\Omega)) \qquad (3.23)$$

Consequently from (3.21), (3.22) and (3.23), z is a unique weak solution satisfying (3.14). Hence, by the uniqueness of solutions of (3.14), $z_\lambda \to z$ weakly in $W(0,T)$. This proves that the map $q \to y(q)$ of $\mathcal{P}$ into $W(0,T)$ is weakly Gâteaux differentiable and the Gâteaux derivative of $y(q)$ at $q = q^*$ in the direction $q - q^* \in \mathcal{P}$ is given by the unique solution z of (3.14). Further, we can prove the strong convergence of $\{z_\lambda\}$ to z in $W(0,T)$. The proof is quite similar, by adding the forcing term $F_\lambda - F$ in the equations for $z_\lambda - z$, to that of [5, pp.336-337]. This completes the proof.

Now we put

$$\mathcal{F}(q-q^*;y^*) = \left(2\alpha^*(\alpha-\alpha^*)G(\nabla y^*) + 2\beta^*(\beta-\beta^*)\nabla\,\frac{\partial y^*}{\partial t}\right) \qquad (3.24)$$

and

$$\mathcal{G}(\phi,\psi) = \Big(\frac{\nabla\phi}{\sqrt{1+|\nabla\psi|^2}} - \frac{\nabla\psi\cdot\nabla\phi}{\big(1+|\nabla\psi|^2\big)^{\frac{3}{2}}}\nabla\psi\Big), \qquad \forall\phi,\ \psi \in H_0^1(\Omega). \qquad (3.25)$$

Note that $F = \mathrm{div}\,\mathcal{F}(q-q^*;y^*)$ in (3.23) and $\mathcal{G}(\phi,\psi)$ is linear in ϕ for fixed ψ.

3.1 Case of distributive observations

The cost functional J_1 in (3.11) is represented by

$$J_1(q) = \int_0^T |y(q;t) - z_{d_1}(t)|^2 dt, \quad q \in \mathcal{P}. \qquad (3.26)$$

Then it is easily verified that the optimality condition (3.13) is written as

$$\int_0^T (y(q^*;t) - z_{d_1}(t), Dy(q^*)(q-q^*)(t))dt \geq 0, \quad \forall q \in \mathcal{P}_{ad}, \qquad (3.27)$$

where $q^* = (\alpha^*, \beta^*)$ is the optimal parameter for (3.26) and $z = Dy(q^*)(q-q^*)$ is a weak solution of (3.14).

Theorem 3.3 *The optimal parameter $q^* = (\alpha^*, \beta^*)$ for (3.26) is characterized by the following system of equations and inequality.*

$$\begin{cases} \dfrac{\partial^2 y^*}{\partial t^2} - (\alpha_0 + {\alpha^*}^2)\mathrm{div}\Big(\dfrac{\nabla y^*}{\sqrt{1+|\nabla y^*|^2}}\Big) - (\beta_0 + {\beta^*}^2)\Delta\dfrac{\partial y^*}{\partial t} = f \quad \text{ in } \ Q, \\ y^* = 0 \quad \text{ on } \ \Sigma, \\ y^*(0,x) = y_0(x), \quad \dfrac{\partial y^*}{\partial t}(0,x) = y_1(x) \quad \text{ in } \ \Omega. \end{cases}$$

$$
\begin{cases}
\dfrac{\partial^2 p}{\partial t^2} - (\alpha_0 + {\alpha^*}^2)\mathrm{div}\Big(\dfrac{\nabla p}{\sqrt{1+|\nabla y^*|^2}} - \dfrac{\nabla y^* \cdot \nabla p}{(1+|\nabla y^*|^2)^{\frac{3}{2}}}\nabla y^*\Big) \\
\qquad\qquad\qquad\qquad + (\beta_0 + {\beta^*}^2)\Delta \dfrac{\partial p}{\partial t} = y^* - z_{d_1} \quad \text{in} \ \ Q, \\
p = 0 \quad \text{on} \ \ \Sigma, \\
p(T,x) = 0, \quad \dfrac{\partial p}{\partial t}(T,x) = 0 \quad \text{in} \ \ \Omega.
\end{cases} \tag{3.28}
$$

$$
\int_Q \nabla p \cdot \Big(2(\alpha - \alpha^*)\alpha^* G(\nabla y^*) + 2(\beta - \beta^*)\beta^* \nabla \frac{\partial y^*}{\partial t}\Big) dx dt \leq 0, \qquad \forall q = (\alpha, \beta) \in \mathcal{P}_{ad}. \tag{3.29}
$$

Proof. Since $y^* - z_{d_1} \in L^2(0,T;L^2(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))$, it is vertified by the time reversion $t \to T - t$ via [2, p.558] there is an unique weak solution $p \in W(0,T)$ of (3.28). We shall show (3.29). We proceed the calculations in the Gelfand triple space $(H_0^1(\Omega), L^2(\Omega), H^{-1}(\Omega))$ as in [5]. Multiplying both sides of the weak form of (3.28) by $z = Dy(q^*)(q - q^*)$ and integrating it by parts on $[0,T]$, we have that

$$
\begin{aligned}
\int_0^T (y^* - z_{d_1}, z) dt &= \int_0^T \langle p'' - A^* \mathrm{div}\, \mathcal{G}(p, y^*) + B^* \Delta p', z \rangle dt \\
&= \int_0^T \langle p, z'' - A^* \mathrm{div}\, \mathcal{G}(z, y^*) - B^* \Delta z' \rangle dt \\
&= \int_0^T \langle p, \mathrm{div}\, \mathcal{F}(q - q^*; y^*) \rangle dt \\
&= -\int_0^T (\nabla p, \mathcal{F}(q - q^*; y^*)) dt,
\end{aligned} \tag{3.30}
$$

where $A^* = \alpha_0 + {\alpha^*}^2$ and $B^* = \beta_0 + {\beta^*}^2$. Therefore, (3.30) and (3.27) imply that the required optimality condition is given by (3.29). This proves Theorem 3.3.

3.2 Case of velocity observations

The cost functional J_2 in (3.12) is represented by

$$
J_2(q) = \int_0^T |y'(q;t) - z_{d_2}(t)|^2 dt, \quad q \in \mathcal{P}. \tag{3.31}
$$

The optimality condition (3.13) for (3.31) is given by

$$
\int_0^T (y'(q^*;t) - z_{d_2}(t), Dy(q^*)(q - q^*)'(t)) dt \geq 0, \quad \forall q \in \mathcal{P}_{ad}, \tag{3.32}
$$

where $z = Dy(q^*)(q - q^*)$ is a weak solution of (3.14). As indicated in [5], we introduce an adjoint system represented by the following first order integro-differential equation

$$\begin{cases} \dfrac{\partial p}{\partial t} + (\alpha_0 + {\alpha^*}^2) \displaystyle\int_t^T \operatorname{div} \mathcal{G}(p(s), y^*(s)) ds + (\beta_0 + {\beta^*}^2) \Delta p \\ \qquad\qquad\qquad\qquad = \frac{\partial y^*}{\partial t} - z_{d_2} \quad \text{in} \ \ Q, \\ p = 0 \quad \text{on} \ \ \Sigma, \\ p(T, x) = 0 \quad \text{in} \ \ \Omega. \end{cases} \tag{3.33}$$

Since $\frac{\partial y^*}{\partial t} - z_{d_2} \in L^2(Q) = L^2(0, T; L^2(\Omega))$, by reversing the direction of time $t \to T - t$ and applying the result of [2, pp. 656-662] to the problem (3.33), we have a unique weak solution p of (3.33) satisfying

$$p \ \in \ W(H_0^1(\Omega), L^2(\Omega)) \cap C([0, T]; H_0^1(\Omega)). \tag{3.34}$$

Theorem 3.4 *The optimal parameter $q^* = (\alpha^*, \beta^*)$ for (3.31) is characterized by the following system of equations and inequality.*

$$\begin{cases} \dfrac{\partial^2 y^*}{\partial t^2} - (\alpha_0 + {\alpha^*}^2) \operatorname{div}\Big(\dfrac{\nabla y^*}{\sqrt{1 + |\nabla y^*|^2}} \Big) - (\beta_0 + {\beta^*}^2) \Delta \dfrac{\partial y^*}{\partial t} = f \quad \text{in} \ \ Q, \\ y^* = 0 \quad \text{on} \ \ \Sigma, \\ y^*(0, x) = y_0(x), \quad \dfrac{\partial y^*}{\partial t}(0, x) = y_1(x) \quad \text{in} \ \ \Omega. \end{cases}$$

$$\begin{cases} \dfrac{\partial p}{\partial t} + (\alpha_0 + {\alpha^*}^2) \displaystyle\int_t^T \operatorname{div}\Big(\frac{\nabla p}{\sqrt{1 + |\nabla y^*|^2}} - \frac{\nabla y^* \cdot \nabla p}{(1 + |\nabla y^*|^2)^{\frac{3}{2}}} \nabla y^* \Big) ds \\ \qquad\qquad\qquad + (\beta_0 + {\beta^*}^2) \Delta p = \dfrac{\partial y^*}{\partial t} - z_{d_2} \quad \text{in} \ \ Q, \\ p = 0 \quad \text{on} \ \ \Sigma, \\ p(T, x) = 0 \quad \text{in} \ \ \Omega. \end{cases} \tag{3.35}$$

$$\int_Q \nabla p \cdot \Big(2(\alpha - \alpha^*) \alpha^* G(\nabla y^*) + 2(\beta - \beta^*) \beta^* \nabla \frac{\partial y^*}{\partial t} \Big) dx dt \geq 0, \\ \forall q = (\alpha, \beta) \in \mathcal{P}_{ad}. \tag{3.36}$$

Proof. Multiplying both sides of the weak form of (3.35) by $z' = Dy(q^*)(q - q^*)'$, taking dual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and integrating it by parts on $[0, T]$, we have that

$$\begin{aligned} & \int_0^T ({y^*}' - z_{d_2}, z') dt \\ = \ & \int_0^T \langle p' + A^* \int_t^T \operatorname{div} \mathcal{G}(p(s), y^*(s)) ds + B^* \Delta p, z' \rangle dt \\ = \ & -\int_0^T \langle p, z'' - A^* \operatorname{div} \mathcal{G}(p(t), y^*(t)) - B^* \Delta z' \rangle dt \\ = \ & -\int_0^T \langle p, \operatorname{div} \mathcal{F}(q - q^*; y^*) \rangle dt \ = \ \int_0^T (\nabla p, \mathcal{F}(q - q^*; y^*)) dt, \end{aligned} \tag{3.37}$$

where $A^* = \alpha_0 + {\alpha^*}^2$ and $B^* = \beta_0 + {\beta^*}^2$. Hence, (3.37) and (3.32) imply that the required optimality condition is given by (3.36).

Let us deduce the bang-bang principle from (3.36) for the case where $\mathcal{P}_{ad}$ is given by $\mathcal{P}_{ad} = [0, \alpha_1] \times [0, \beta_1]$. In this case the necessary condition (3.36) is equivalent to

$$\alpha^*(\alpha - \alpha^*) \int_Q \nabla p \cdot G(\nabla y^*) dxdt \geq 0, \quad \forall \alpha \in [0, \alpha_1], \tag{3.38}$$

$$\beta^*(\beta - \beta^*) \int_Q \nabla p \cdot \nabla \frac{\partial y^*}{\partial t} dxdt \geq 0, \quad \forall \beta \in [0, \beta_1]. \tag{3.39}$$

First we consider (3.38). Put $a = \int_Q \nabla p \cdot G(\nabla y^*) dxdt$ and assume that $a \neq 0$. Then (3.38) is rewritten simply by

$$\alpha^*(\alpha - \alpha^*) a \geq 0, \quad \forall \alpha \in [0, \alpha_1].$$

It is easily verified from the above inequality that α^* is given by

$$\alpha^* = -\frac{1}{2}\{\text{sign}(a) - 1\}\alpha_1 \quad \text{or} \quad \alpha^* = 0.$$

Similarly, if $\int_Q \nabla p \cdot \nabla \frac{\partial y^*}{\partial t} dxdt = b (\neq 0)$, then

$$\beta^* = -\frac{1}{2}\{\text{sign}(b) - 1\}\beta_1 \quad \text{or} \quad \beta^* = 0.$$

These are the so called bang-bang principle for the optimal parameter $q^* = (\alpha^*, \beta^*)$.

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