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Existence of non radially symmetric viscosity solutions to semilinear degenerate elliptic equations with radially symmetric coefficients in the plane, Part I

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Abstract

We prove the existence of non-radially symmetric solutions for semilinear degenerate elliptic equations with radially symmetric coefficients in the plane. We adapt the viscosity solution for the weak solution.

The key arguments consist of the analysis of the structure of 2π-periodic solutions for the associated Laplace Beltrami operator and construction of super and subsolutions which have the prescribed asymptotic structures.

1 Introduction

We consider the following semilinear degenerate elliptic equation:

$$-g(|x|)\Delta u(x) + u(x)|u(x)|^{p-1} = f(|x|) \quad \text{in } \mathbb{R}^N \ (N \geqq 2) \tag{1}$$

where $p > 1$ is a constant and $g : [0, \infty) \to [0, \infty)$ is a differentiable, nonnegative function with $g(T_0) = 0$ for some $T_0 > 0$.

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We are concerned with the existence and structure of continuous viscosity solutions of (1) which may not be bounded. In order to study the structure of continuous viscosity solutions of (1), it is important to investigate whether solutions of (1) are radially symmetric or not. In this paper we prove that there exist non-radially symmetric solutions of (1) and study the structure of them.

We then consider the following ordinary differential equation associated with the equation (1);

$$-g(t)\left(\frac{d^2y}{dt^2}(t)+\frac{(N-1)}{t}\frac{dy}{dt}(t)\right)+y(t)|y(t)|^{p-1}=f(t)\quad \text{on } [0,\infty), \qquad (2)$$

$$\frac{dy}{dt}(0)=0.$$

By our previous results in [1] and [2], we already know that the solution of (1) exists uniquely and the solution is radially symmetric if there exists a unique solution of (2). Then, if the equation (1) has a non-radially symmetric solution, it is necessary that there exist many solutions of (2).

In [2] we have investigated that whether the solution of (2) is unique or not. Moreover, in [3] we have studied that the non-radially symmetric solution of (1) does not exist though there are many solutions of (1) under the certain assumptions of f and g. The purpose of this paper is to show that there exists a non-radially symmetric solution of (1) and to study the structure of solutions.

In order to show our assertion we have to understand the existence of the nontrivial solutions on the sphere S^{N-1} for the Laplace-Beltrami equations associated with the equation (1). However, to the author's best knowledge, it seems to be extremely difficult to analyze the structure of solutions of the nonlinear Laplace-Beltrami equations in case of a general dimension. Hence as the first step of the study of this problem we are concerned with the case where the space dimension is two. In this case our problem is solved in the affirmative way under some assumptions.

The outline of the present paper is as follows. In section 2, after reviewing quickly the results in [2] and [3], we state the assumptions and our main theorem. In section 3 we show the properties of nonlinear Laplace-Beltrami equations in one dimensional case. Section 4 is devoted to the study of the existence of non-radially symmetric solutions.

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2 Assumptions and main theorem

In the first part of this section, we will review out previous results [2] and [3] and describe the assumptions we need. For the sake of simplicity, first we let g and f be smooth functions and $g \geqq 0$.

In [2] we have shown that the solution of (1) is unique and radially symmetric if $g(t)$ has infinitely many zeros $\{t_n\}$ satisfying $\lim_{n\to\infty} t_n = \infty$. In the case that the zeros of g are bounded above, there is a possibility of the existence of non-radially symmetric solutions to (1). Thus, we make the following basic assumption:

(A-0) There exists $T_0 > 0$ such that $g(T_0) = 0$ and $g(t) > 0$ for all $t > T_0$.

Although we treat the solution of (1) in $\mathbb{R}^N$, it is enough to consider the equation (1) in $\mathbb{R}^N \backslash B_{T_0}$ where $B_{T_0} = \{x \in \mathbb{R}^N : |x| < T_0\}$. Indeed, if $g \in C^1([0,\infty))$, then we can get continuous viscosity solution in $\mathbb{R}^N$ by combining continuous viscosity solutions on $\mathbb{R}^N \backslash B_{T_0}$ and that in B_{T_0}.

On the other hand, we have shown that a continuous viscosity solution in B_{T_0} exists uniquely and radially symmetric (see [1]).

In the following of this paper we consider the equation (1) only on $\mathbb{R}^N \backslash B_{T_0}$.

We recall that the asymptotic behavior of $g(t)$ and $f(t)$ at $t \to \infty$ are essentially important to investigate whether (1) has many solutions or not. In [2] and [3], we assumed that $g(t)$ and $f(t)$ satisfy the following assumptions:

$$g(t) = d_0 t^\ell + O(t^{\ell-1}) \text{ as } t \to \infty$$

$$0 \leqq K \equiv \lim_{t\to\infty} \frac{f(t)}{t^{\alpha p}} \leqq \infty$$

where $\alpha = \dfrac{\max\{\ell-2, 0\}}{p-1}$ and $\ell \geqq 0$.

If $0 < \ell \leqq 2$ or $K = \infty$, then the solution of (1) exists uniquely and radially symmetric (see [2]). To investigate the results for the case $K < \infty$ and $\ell > 2$, we consider the following algebraic equation

$$X|X|^{p-1} - \kappa^p - \alpha^2 X = 0. \tag{3}$$

Here, $K = \kappa^p$ and, for the sake of simplicity, we write down the equation only for the case $d_0 = 1$ and $N = 2$. The real roots of (3) are classified to the following three cases:

(C1) The equation (3) has only a positive root ω_+ which has single multiplicity.

(C2) The equation (3) has a positive single root ω_+ and a negative double root ω_0.

(C3) The equation (3) has a positive single root ω_+ and two nonpositive single roots ω_0 and ω_-.

For each case, we already have proved the following results.

Proposition 2.1 ([2]) *In the case* (C1), *there exists a unique solution* $y(x)$ *of* (1). *The solution* $y(x)$ *is radially symmetric and satisfies the asymptotic behavior* $\displaystyle\lim_{|x|\to\infty} \frac{y(x)}{|x|^\alpha} = \omega_+$.

Proposition 2.2 ([2]) *In the case* (C2) *or* (C3), *there exist multiple solutions* $y(t)$ *to* (2). *The solutions have the following asymptotic behavior:*

(i) *In the case* (C2), *a solution* $y(t)$ *of* (1) *satisfies* $\displaystyle\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \omega_+$ *or* ω_0.

(ii) *In the case* (C3), *a solution* $y(t)$ *of* (1) *satisfies* $\displaystyle\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \omega_+$, ω_0 *or* ω_-.

In both cases, the solution satisfying $\displaystyle\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \omega_+$ *or* ω_- *is unique, respectively. However, the set of solutions satisfying* $\displaystyle\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \omega_0$ *is continuum.*

Hence, as the continuation, we shall work under the same assumption of Proposition 2.2.

Now, in addition to (A-0), we assume the following conditions (A-1)–(A-4).

(A-1) Let $N = 2$, that is we consider the equation (1) in $\mathbb{R}^2$.

(A-2) The real roots of (3) satisfy either (C2) or (C3).

(A-3) The functions $f(t)$ and $g(t)$ satisfy

$$\begin{aligned} & f(t),\ g(t) \in C^3([0,\infty)), \\ & \left| \frac{d^i}{dt^i}(f(t) - \kappa^p t^{\alpha p}) \right| \leqq C t^{\alpha p - 1 - i}, \\ & \left| \frac{d^i}{dt^i}(g(t)^{-1} - t^{-\ell}) \right| \leqq C t^{-\ell - 1 - i} \end{aligned}$$

for a positive constant C and sufficiently large $t > 0$, where $\ell > 2$ $\alpha = \dfrac{\ell - 2}{p - 1}$, and $i = 1, 2$.

Note that we put $K = \kappa^p$. Under these assumptions we have proved the following proposition.

Proposition 2.3 ([3]) *Let α satisfy the inequality such that $1 + p|\omega_0|^{p-1} > \alpha^2$ and $p \geqq 2$. Then all continuous viscosity solutions to* (1) *are radially symmetric.*

In the following, we always assume (A-0)–(A-3). Moreover, we assume

(A-4) $1 + p|\omega_0|^{p-1} < \alpha^2$ and $p \geqq 2$.

Under (A-4), the case (C2) does not occur. Indeed, when (C2) holds, ω_0 is calculated as $\omega_0 = -(\alpha^2/p)^{1/(p-1)}$. This contradicts (A-4). Therefore we consider (A-4) with (C3).

We translate the original equation (1) by polar coordinate. Let $u(x) = u(t, \theta)$ and normalize somehow by $v(t, \theta) = \dfrac{u(t,\theta)}{t^\alpha} - \omega_0$. Then we have

$$\begin{aligned} & \frac{\partial^2 v}{\partial t^2} + \frac{2\alpha + 1}{t}\frac{\partial v}{\partial t} + \frac{1}{t^2}\frac{\partial^2 v}{\partial \theta^2} \\ & = t^{l-2} g(t)^{-1} |v + \omega_0|^{p-1}(v + \omega_0) - \alpha^2 t^{-2}(v + \omega_0) - t^{-\alpha} f(t) g(t)^{-1} \\ & = -F_0(v) t^{-2} + h(t, \theta, v) t^{-3}, \end{aligned} \tag{4}$$

where $h(t, \theta, v)$ is a higher order term with respect to v.

Recall that we have shown in [3] that if we set (4) as

$$\frac{\partial^2 v}{\partial \theta^2} + F_0(v) = \varepsilon(t, \theta),$$

then $\lim_{t\to\infty}\varepsilon(t,\theta) = 0$ uniformly on θ. Hence we consider the following nonlinear Laplace Beltrami equation associated with the equation (1):

$$\frac{\partial^2 w}{\partial\theta^2} + F_0(w) = 0. \tag{5}$$

Introducing the new parameter $\lambda = \alpha^2 - p|\omega_0|^{p-1}$, we write F_0 as

$$F_0(w) = \lambda w - (\omega_0 + w)|\omega_0 + w|^{p-1} - |\omega_0|^p + p|\omega_0|^{p-1}w. \tag{6}$$

3 Nonlinear Laplace Beltrami Equation on $[0, 2\pi]$

In this section we consider the nonlinear Laplace Beltrami Equation (5) on the interval $[0, 2\pi]$ and investigate the existence and properties of 2π periodic solutions. We say that a function $w \in C^2([0, 2\pi])$ is a 2π-periodic solution of (5) if and only if w satisfies the equation (5), $w(0) = w(2\pi)$ and $\frac{dw}{d\theta}(0) = \frac{dw}{d\theta}(2\pi) = 0$.

To investigate the periodic solution of (5), we follow W. S. Loud ([4]). The existence of periodic solutions w is already known in [4].

Consider the function

$$G(x) = \int_0^x F_0(\xi)d\xi$$

for $x \in \mathbb{R}$. A non-trivial periodic solutions of (5) corresponds to a closed curve in the phase plane with coordinates w and $v = \frac{dw}{d\theta}$. This curve satisfies the equation

$$\frac{v^2}{2} + G(v) = E.$$

Here E is a constant which represents the energy associated with the solution of the equation (5). We denote the maximum and the minimum of the solution $w(\theta)$ of (5) by w^* and w_*, respectively. Then we have $E = G(w^*) = G(w_*)$. Without loss of generality, we can assume that $w_* = w(0) = w(2\pi)$ and $(dw/d\theta)(0) = (dw/d\theta)(2\pi) = 0$. Since w_* is a minimum, it holds $(dw^2/d\theta^2)(0) \geq 0$. If $(dw^2/d\theta^2)(0) = 0$, then $F_0(w_*) = 0$

by (5) and hence w_* becomes a stationary solution. This is a contradiction. Therefore $d^2w(0)/d\theta > 0$ and $F_0(w_*) < 0$. In the same argument, we see $F_0(w^*) > 0$. Note that $F_0(v)$ has three zeros, $\omega_- - \omega_0$, 0 and $\omega_+ - \omega_0$ and $F_0(v)$ is positive in $(-\infty, \omega_- - \omega_0)$ or $(0, \omega_+ - \omega_0)$ and negative in $(\omega_- - \omega_0, 0)$ or $(\omega_+ - \omega_0, \infty)$. From $F_0(w_*) < 0$, it follows that either $\omega_- - \omega_0 < w_* < 0$ or $\omega_+ - \omega_0 < w_*$. Similarly, we have either $w^* < \omega_- - \omega_0$ or $0 < w^* < \omega_+ - \omega_0$. Since $w_* < w^*$, we conclude that $\omega_- - \omega_0 < w_* < 0 < w^* < \omega_+ - \omega_0$.

On the other hand, it is shown in [4] that the period of this solution is a function of w_* (w^*) and is given by the following formula:

$$T(w_*) = T(w^*) = 2\int_{w_*}^{w^*} \frac{dw}{\sqrt{2(E - G(w))}}. \tag{7}$$

Considering the phase plane we know that $T(w_*)$ is well defined for any $w_* : (\omega_- - \omega_0) < w_* < 0$ where ω_- and ω_0 are real roots in (C-3).

Lemma 3.1 *Let w_* be $(\omega_- - \omega_0) < w_* < 0$. Then we have the following properties.*

(1) *$T(w_*)$ is a differentiable function.*

(2) $\displaystyle\lim_{w_* \to \omega_- - \omega_0} T(w_*) = \infty$,

(3) $\displaystyle\lim_{w_* \to 0} T(w_*) = \lim_{w^* \to 0} T(w^*) = \frac{2\pi}{\sqrt{\alpha^2 - p|\omega_0|^{p-1}}}$.

The proof is obvious from [3] and [4].

Since $\lambda = \alpha^2 - p|\omega_0|^{p-1} > 1$ and from (3) of Lemma 3.1, we have $\lim_{w_* \to 0} T(w_*) < 2\pi$. Since $\lim_{w_* \to \omega_- - \omega_0} T(w_*) = \infty$, the intermediate value theorem yields the number w_1 such that $\omega_- - \omega_0 < w_1 < 0$ and $T(w_1) = 2\pi$. Then there exists a 2π periodic solution of (5). Using the same method as above, we get the following Lemma.

Lemma 3.2 *If λ satisfies $n^2 < \lambda \leqq (n+1)^2$ for some $n = 1, 2, \cdots$, then there exist $\dfrac{2\pi}{k}$ periodic solutions of (5) where $k = 1, 2, \cdots, n$.*

Let ω_0 be fixed. Then the period T is a function depending on $1 < \lambda$ and $0 < w^* < (\omega_- - \omega_0)^*$ where $(\omega_- - \omega_0)^*$ satisfies $G(\omega_- - \omega_0) = G((\omega_- - \omega_0)^*)$.

For $n^2 < \lambda \leqq (n+1)^2$ we investigate that whether $\dfrac{2\pi}{k}$ periodic solutions of (5) is only one or not. Let $\lambda(w^*)$ be the implicit function such that $T(\lambda, w^*) = \dfrac{2\pi}{k}$. From (3) in Lemma 3.1 it follows that $\lim_{w^* \to 0} \lambda(w^*) = k^2$. Note that the positivity of $\dfrac{d\lambda(w^*)}{dw^*}$ yields that $\dfrac{2\pi}{k}$ periodic solution of (5) is unique. Since we have $\dfrac{\partial T}{\partial \lambda}\dfrac{d\lambda}{d\omega^*} + \dfrac{\partial T}{\partial \omega^*} = 0$ from the implicit function, we will study the sign of $\dfrac{\partial T(\lambda, w^*)}{\partial w^*}$ and $\dfrac{\partial T(\lambda, w^*)}{\partial \lambda}$.

First, we investigate the sign of $\dfrac{\partial T(\lambda, w^*)}{\partial \lambda}$.

We define $G_\lambda(x)$ and $g_\lambda(x)$ by

$$G_\lambda(x) = \frac{\lambda}{2}x^2 - [\frac{1}{p+1}\{|x+\omega_0|^{p+1} - |\omega_0|^{p+1}\} + |\omega_0|^p x - \frac{p|\omega_0|^{p-1}}{2}x^2],$$
$$g_\lambda(x) = \frac{dG_\lambda(x)}{dx} = \lambda x - \{(x+\omega_0)|x+\omega_0|^{p-1} + |\omega_0|^p - p|\omega_0|^{p-1}x\}.$$

Define $(\omega_- - \omega_0)^*$ by $G(\omega_- - \omega_0) = G((\omega_- - \omega_0)^*)$. We also denote by $a_* = a_*(a) < 0$ such that $G_\lambda(a_*) = G_\lambda(a)$, for $a \in (0, (\omega_- - \omega_0)^*)$. We recall that the period T satisfies

$$T(a, \lambda) = \sqrt{2}\int_{a_*}^{a} \frac{dx}{\sqrt{G_\lambda(a) - G_\lambda(x)}}.$$

Lemma 3.3 *We have $a^2 < (a_*)^2$ for $a \in (0, (\omega_- - \omega_0)^*)$.*

Proof. For $x \geqq 0$ we define $f_1(x) = G_\lambda(x) - G_\lambda(-x)$. From $x > 0$ we see $\dfrac{d^2}{dx^2}f_1(x) = -p|x+\omega_0|^{p-1} + p|-x+\omega_0|^{p-1} > 0$. Since $f_1(0) = 0$ and $\dfrac{d}{dx}f_1(0) = 0$, it follows $f_1(x) > 0$ for any $x > 0$. Then we have $G_\lambda(x_*) = G_\lambda(x) > G_\lambda(-x)$. If $\omega_- - \omega_0 < y < 0$, then $G_\lambda(y)$ is decreasing since $\dfrac{dG_\lambda}{dy} = g_\lambda(y) < 0$. Thus we obtain $|x_*| > |-x| = x$. The proof is complete. ∎

Lemma 3.4 *Let $E = G_\lambda(a_*)$. Then the following equality holds.*

$$\int_{a_*}^{0} \frac{g_\lambda(a_*) - g_\lambda(x)}{(E - G_\lambda(x))^{3/2}} dx =$$
$$- \frac{2}{\sqrt{E}} + \frac{2g_\lambda(a_*)}{E} \int_{a_*}^{0} \left(\frac{G_\lambda(x)(dg_\lambda(x)/dx)}{g_\lambda(x)^2} - \frac{1}{2} \right) \frac{dx}{\sqrt{E - G_\lambda(x)}}.$$

Proof. We use the method of W. S. Loud ([4]). Let μ be $a_* < \mu < 0$. From

$$\frac{1}{(E - G_\lambda(x))^{3/2}} = \frac{1}{E} \frac{G_\lambda(x)}{(E - G_\lambda(x))^{3/2}} + \frac{1}{E} \frac{1}{(E - G_\lambda(x))^{1/2}},$$

we see that

$$\begin{aligned} &\int_{\mu}^{0} \frac{g_\lambda(\mu)}{(E - G_\lambda(x))^{3/2}} dx \\ &= \frac{1}{E} \int_{\mu}^{0} \frac{g_\lambda(\mu)}{(E - G_\lambda(x))^{1/2}} dx + \frac{1}{E} \int_{\mu}^{0} \frac{g_\lambda(\mu) G_\lambda(x)}{(E - G_\lambda(x))^{3/2}} dx \\ &= I_1 + I_2. \end{aligned}$$

Noting that

$$\frac{d}{dx} \frac{2}{\sqrt{E - G_\lambda(x)}} = \frac{g_\lambda(x)}{(E - G_\lambda(x))^{3/2}},$$
$$\frac{d}{dx} \frac{g_\lambda(\mu) G_\lambda(x)}{g_\lambda(x)} = \frac{g_\lambda(\mu)\{g_\lambda(x)^2 - G_\lambda(x)(dg_\lambda(x)/dx)\}}{g_\lambda(x)^2},$$

and applying the integration by parts to I_2 by rewriting as

$$I_2 = \frac{1}{E} \int_{\mu}^{0} \frac{g_\lambda(x)}{(E - G_\lambda(x))^{3/2}} \frac{g_\lambda(\mu) G_\lambda(x)}{g_\lambda(x)} dx,$$

we get

$$\begin{aligned} I_2 = &\frac{-G_\lambda(\mu)}{E} \frac{2}{\sqrt{E - G_\lambda(\mu)}} \\ &- \frac{2g_\lambda(\mu)}{E} \int_{\mu}^{0} \frac{\{g_\lambda(x)^2 - G_\lambda(x)(dg_\lambda(x)/dx)\}}{g_\lambda(x)^2} \frac{dx}{\sqrt{E - G_\lambda(x)}}. \end{aligned}$$

Since it holds

$$\int_\mu^0 \frac{g_\lambda(x)}{(E-G_\lambda(x))^{3/2}}dx = \frac{2}{\sqrt{E}} - \frac{2}{\sqrt{E-G_\lambda(\mu)}},$$

we have

$$\int_\mu^0 \frac{g_\lambda(\mu)-g_\lambda(x)}{(E-G_\lambda(x))^{3/2}}dx = \frac{-2}{\sqrt{E}} + \frac{2}{\sqrt{E-G_\lambda(\mu)}} + I_1 + I_2 = I_3.$$

From the calculation

$$\begin{aligned}&\frac{1}{E}\frac{g_\lambda(\mu)}{\sqrt{E-G_\lambda(x)}} - \frac{2g_\lambda(\mu)}{E}\left(1-\frac{G_\lambda(x)g'_\lambda(x)}{g_\lambda(x)^2}\right)\frac{1}{\sqrt{E-G_\lambda(x)}}\\ &= \frac{g_\lambda(\mu)^2}{E}\left(-1+\frac{2G_\lambda(x)g'_\lambda(x)}{g_\lambda(x)}\right)\frac{1}{\sqrt{E-G_\lambda(x)}},\end{aligned}$$

we get

$$\begin{aligned}I_3 =& -\frac{2}{\sqrt{E}} + \frac{2}{E}\frac{E-G_\lambda(\mu)}{\sqrt{E-G_\lambda(\mu)}}\\ &+\frac{g_\lambda(\mu)}{E}\int_\mu^0 \frac{\{2G_\lambda(x)(dg_\lambda(x)/dx)-g_\lambda(x)^2\}}{g_\lambda(x)^2}\frac{dx}{\sqrt{E-G_\lambda(x)}}.\end{aligned}$$

Letting μ to a_*, we have the desired equality. ∎

Proposition 3.1 *We have*

$$\frac{d}{d\lambda}T(a,\lambda) < 0 \quad \text{for } \lambda > 0.$$

Proof. Noting $G_\lambda(a) = G_\lambda(a_*)$, we divide the interval of integration as follows:

$$\begin{aligned}&\int_{a*}^a \frac{dx}{\sqrt{G_\lambda(a)-G_\lambda(x)}}\\ &= \int_0^a \frac{dx}{\sqrt{G_\lambda(a)-G_\lambda(x)}} + \int_{a*}^0 \frac{dx}{\sqrt{G_\lambda(a_*)-G_\lambda(x)}} = I_1 + I_2.\end{aligned}$$

Applying the change of variables in I_2, we have

$$I_2 = \int_0^{-a_*} \frac{dz}{\sqrt{zH(z,\lambda)}},$$

where $z = x - a_*$ and $H(z,\lambda) = \dfrac{G_\lambda(a_*) - G_\lambda(x)}{x - a_*}$.

Since we have

$$\lim_{z\to 0} H(z,\lambda) = \lim_{x\to a_*} \frac{-g_\lambda(x)}{1} > 0,$$

it follows that

$$\frac{d}{d\lambda} I_2 = \frac{-1}{\sqrt{G_\lambda(a_*)}} \frac{da_*}{d\lambda} + \int_0^{-a_*} \left(\frac{d}{d\lambda} \frac{1}{\sqrt{zH(z,\lambda)}} \right) dz. \tag{8}$$

From the equality

$$\frac{d}{d\lambda} \frac{1}{\sqrt{zH(z,\lambda)}} = -\frac{d(G_\lambda(a_*) - G_\lambda(a_* + z))/d\lambda}{2((G_\lambda(a_*) - G_\lambda(a_* + z)))^{3/2}}$$

and

$$\frac{d}{d\lambda}(G_\lambda(a_*) - G_\lambda(a_* + z)) = \frac{1}{2}((a_*)^2 - (z + a_*)^2) + (g_\lambda(a_*) - g_\lambda(a_* + z))\frac{da_*}{d\lambda},$$

we have

$$\begin{aligned} &\int_0^{-a_*} \frac{d}{d\lambda} \frac{dz}{\sqrt{zH(z,\lambda)}} \\ &= -\frac{1}{4}\int_{a_*}^0 \frac{((a_*)^2 - (x)^2)}{(G_\lambda(a_*) - G_\lambda(x))^{3/2}}\, dx - \frac{1}{2}\int_{a_*}^0 \frac{(g_\lambda(a_*) - g_\lambda(x))}{(G_\lambda(a_*) - G_\lambda(x))^{3/2}} \frac{da_*}{d\lambda}\, dx \\ &= I_3 + I_4, \end{aligned} \tag{9}$$

where we used the change of variables $z + a_* = x$.

Note that by differentiating $G_\lambda(a_*) - G_\lambda(a) = 0$ at λ, we have

$$\frac{(a_*)^2 - a^2}{2} + g_\lambda(a_*)\frac{da_*}{d\lambda} = 0.$$

Then it holds $\dfrac{da_*}{d\lambda} = \dfrac{(a_*)^2 - a^2}{-2g_\lambda(a_*)}$. Therefore we get

$$I_4 = \frac{(a_*)^2 - a^2}{4g_\lambda(a_*)} \int_{a_*}^0 \frac{(g_\lambda(a_*) - g_\lambda(x))}{(G_\lambda(a_*) - G_\lambda(x))^{3/2}} dx.$$

Applying Lemma 3.4 to I_4 we see

$$I_4 = -\frac{(a_*)^2 - a^2}{2g_\lambda(a_*)\sqrt{G_\lambda(a_*)}} + \frac{(a_*)^2 - a^2}{2G_\lambda(a_*)} \int_{a_*}^0 \left(\frac{G_\lambda(x)(dg_\lambda(x)/dx)}{g_\lambda(x)^2} - \frac{1}{2} \right) \frac{dx}{\sqrt{G_\lambda(a_*) - G_\lambda(x)}}. \quad (10)$$

Combining these relations (8), (9) and (10), we have

$$\frac{d}{d\lambda} I_2 = -\frac{1}{4} \int_{a_*}^0 \frac{((a_*)^2 - (x)^2)}{(G_\lambda(a_*) - G_\lambda(x))^{3/2}} dx + \frac{(a_*)^2 - a^2}{2G_\lambda(a_*)} \int_{a_*}^0 \left(\frac{G_\lambda(x)(dg_\lambda(x)/dx)}{g_\lambda(x)^2} - \frac{1}{2} \right) \frac{dx}{\sqrt{G_\lambda(a_*) - G_\lambda(x)}}.$$

For $x < 0$, we denote $f_2(x) = 2G_\lambda(x)\dfrac{dg_\lambda(x)}{dx} - g_\lambda(x)^2$. Since $\dfrac{df_2}{dx}(x) = 2G_\lambda(x)\dfrac{d^2 g_\lambda(x)}{dx^2} > 0$, the fact $f_2(0) = 0$ yields $f_2(x) < 0$ for any $x < 0$. Then, noting $(a_*)^2 - a^2 > 0$ form Lemma 3.3, we get

$$\frac{d}{d\lambda} I_2 < 0.$$

Next we consider I_1. Since the equality

$$\frac{d}{d\lambda} \frac{1}{\sqrt{G_\lambda(a) - G_\lambda(x)}} = -\frac{1}{4} \frac{a^2 - x^2}{(G_\lambda(a) - G_\lambda(x))^{3/2}}$$

yields the integrability of the right hand side, it follows that

$$\frac{d}{d\lambda} \frac{1}{\sqrt{G_\lambda(a) - G_\lambda(x)}} < 0.$$

Hence the proof is complete. ∎

We next study the sign of $\dfrac{\partial T(\lambda, a)}{\partial a}$. We recall the assumption $\lambda > 1$ and $p \geqq 2$. For a while we fix λ. Then we regard the function $T(a, \lambda)$ as the function of one variable a and write $T(a) = T(a, \lambda)$. We also write $G_\lambda(x)$ and $g_\lambda(x)$ by $G(x)$ and $g(x)$, respectively.

First we prepare several lemmas.

Lemma 3.5 *Let y and x satisfy $\omega_- - \omega_0 < y < 0 < x < 2|\omega_0|$ and $G(x) = G(y)$. Then we have $|y|g''(y) \geqq x|g''(x)|$.*

Proof. Recall $p \geqq 2$. For any $z < |\omega_0|$ it follows that $g''(z) = p(p-1)|z+\omega_0|^{p-2}$. Then $g''(z)$ is not increasing. Hence we have $g''(y) \geqq g''(x) > 0$ for $y < 0 < x \leqq |\omega_0|$. Therefore, Lemma 3.3 yields that $|y|g''(y) \geqq x|g''(x)|$. Next, for $x > |\omega_0|$ we see $g''(-x) - |g''(x)| = p(p-1)(|-x+\omega_0|^{p-2} - |x+\omega_0|^{p-2}) \geqq 0$. Since $g''(z)$ $(z < |\omega_0|)$ is not increasing and $|y| > x$ from Lemma 3.3, it follows that $g''(y) \geqq g''(-x)$. Then, we have $|y|g''(y) \geqq x|g''(x)|$. ∎

Lemma 3.6 *Under the same assumptions in Lemma* 3.5 *we have $|y||g(y)| < xg(x)$.*

Proof. Let $f_3(x)$ be $f_3(x) = 2G(x) - xg(x)$ for $x < 2|\omega_0|$. Then we see that $f_3(0) = 0$ and $f_3(2|\omega_0|) = 0$. From $f_3'(x) = g(x) - xg'(x)$ and $f_3''(x) = xp(p-1)(x+\omega_0)|x+\omega_0|^{p-3}$, it follows that $f_3'(0) = 0$, $f_3'(2|\omega_0|) = 2(p-1)|\omega_0|^p > 0$ and $f_3''(x) \leqq 0$ for $0 < x < |\omega_0|$, $f_3''(x) \geqq 0$ for $|\omega_0| < x < 2|\omega_0|$. Thus we have $f_3(x) \leqq 0$ for $0 < x < 2|\omega_0|$. On the other hand, since $f_3''(y)$ is nonnegative for $y < 0$, we see $f_3(y) \geqq 0$ for $y < 0$. Therefore, $G(y) = G(x)$ yields the proof of this lemma. ∎

Lemma 3.7 *Let x satisfy $0 < x < 2|\omega_0|$, $\omega_- - \omega_0 < y < 0$ and $G(y) = G(x)$. Then we have the following inequality*

$$\frac{g''(y)}{|g(y)|}g(x) > |g''(x)|.$$

Proof. The proof is easy from Lemma 3.5 and Lemma 3.6. ∎

Lemma 3.8 *Let $E = G(a_*) = G(a)$. Then we have*

$$\frac{dT(a)}{da} = -\frac{\sqrt{2}g(a)}{E}\int_{a_*}^{a}\left(\frac{G(x)g'(x)}{g^2(x)} - \frac{1}{2}\right)\frac{dx}{\sqrt{E-G(x)}}.$$

Proof. See [4]. ∎

Remark 3.1 *We recall that*

$$g'(x) = \lambda - p(|\omega_0 + x|^{p-1} - |\omega_0|^{p-1})$$
$$g''(x) = -p(p-1)(x+\omega_0)|x+\omega_0|^{p-3}.$$

Thus, it follows that $g''(x) > 0$ *if* $x < |\omega_0|$ *and* $g''(x) < 0$ *if* $x > |\omega_0|$.

Lemma 3.9 *Let* $f_4(z) = 2G(z)g'(z) - g^2(z)$ *for* $\omega_- - \omega_0 < z < (\omega_- - \omega_0)^*$, *where* $G((\omega_- - \omega_0)^*) = G(\omega_- - \omega_0)$. *Then there exists a number* a_1 *such that*

$$\begin{aligned} &f_4(a_1) = 0, \ (|\omega_0| < a_1 < 2|\omega_0|), \\ &f_4(z) < 0 \quad \text{if } z < 0, \\ &f_4(z) > 0 \quad \text{if } 0 < z < a_1, \\ &f_4(z) < 0 \quad \text{if } a_1 < z. \end{aligned}$$

Proof. If $z < 0$, then Remark 3.1 implies that $f_4'(z) > 0$. Since $f_4(0) = 0$, it follows that $f_4(z) < 0$. Noting again that $f_4(0) = 0$ and $f_4'(z) > 0$ for $z: \ 0 < z < |\omega_0|$, we see that $f_4(z) > 0$. Since $f_4(|\omega_0|) > 0$ and

$$f_4(2|\omega_0|) = 4\{-\lambda(p-1)|\omega_0|^{p+1} - (p-1)^2|\omega_0|^{2p}\} < 0,$$

the intermediate value theorem yields the existence of a_1. Moreover, from $f_4'(z) < 0$ for $z > |\omega_0|$, we have that a_1 is unique. Then, our assertion is proved. ∎

Lemma 3.10 *For any* x *and* y *satisfying* $y < 0 < x < a_1$ *and* $G(x) = G(y)$, *we put*

$$\Lambda(x) = (2G(x)g'(x) - g^2(x))g^3(y) - (2G(y)g'(y) - g^2(y))g^3(x).$$

Then we have $\Lambda(x) > 0$.

Proof. From $(2G(x)g'(x) - g^2(x))' = 2G(x)g''(x)$, we see that

$$\Lambda(x) = g^3(y)\int_0^x 2G(\eta)g''(\eta)d\eta - g^3(x)\int_0^y 2G(\xi)g''(\xi)d\xi.$$

Change the variables from ξ to η by the relation $G(\xi) = G(\eta)$ with $y < \xi < 0$ and $0 < \eta < x$. Since $g(\xi)d\xi = g(\eta)d\eta$, we have

$$-\int_0^y 2G(\xi)g''(\xi)d\xi = -\int_0^x 2G(\eta)g''(\xi)\frac{g(\eta)}{g(\xi)}d\eta = I_1.$$

From Lemma 3.7, it follows that $I_1 > \int_0^x 2G(\eta)|g''(\eta)|d\eta$.
Then we have

$$\begin{aligned}\Lambda &\geqq g^3(y)\int_0^x 2G(\eta)g''(\eta)d\eta + g(x)^3\int_0^x 2G(\eta)|g''(\eta)|\,d\eta \\ &\geqq (g^3(y)+g^3(x))\int_0^x 2G(\eta)|g''(\eta)|\,d\eta.\end{aligned}$$

Combining Lemma 3.3 and Lemma 3.6, we see that $1 < \dfrac{|y|}{x} < \dfrac{g(x)}{|g(y)|}$. Thus, we get $g^3(y) + g^3(x) > 0$. Therefore, we obtain that $\Lambda > 0$. The proof is complete. ∎

Lemma 3.11 *Let β satisfy $0 < \beta < \min\{a_1,\ a\}$. We denote by β_* a number satisfying $G(\beta_*) = G(\beta)$ and $\beta_* < 0 < \beta$. Then we have*

$$\int_{\beta_*}^{\beta} \frac{2G(x)g'(x) - g^2(x)}{2g^2(x)} \frac{dx}{\sqrt{E - G(x)}} < 0$$

where $E = G(a)$.

Proof. Let x and y be such that $G(x) = G(y)$ and $y < 0 < x$. Then we see $g(x) = g(y)\dfrac{dy}{dx}$. Using the change of variables from y to x we have

$$\begin{aligned}&\int_{\beta_*}^{0} \frac{2G(y)g'(y) - g^2(y)}{2g^2(y)} \frac{dy}{\sqrt{E - G(y)}} \\ &= \int_{\beta}^{0} \frac{2G(y(x))g'(y(x)) - g^2(y(x))}{2g^2(y(x))} \frac{g(x)}{g(y(x))\sqrt{E - G(x)}}\,dx.\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\int_{\beta_*}^{\beta} \frac{2G(x)g'(x) - g^2(x)}{2g^2(x)} \frac{dx}{\sqrt{E - G(x)}} \\ &= \int_0^{\beta} g(x)\frac{(2G(x)g'(x) - g^2(x))g(y)^3 - (2G(y)g'(y) - g^2(y))g^3(x)}{2\sqrt{E - G(x)}g^3(x)g^3(y)}\,dx.\end{aligned}$$

Since $\Lambda(x) > 0$ from Lemma 3.10, $g(y) < 0$ and $g(x) > 0$, the above equality yields the conclusion of the lemma. ∎

Now we can calculate the sign of $\dfrac{\partial T(a,\lambda)}{\partial a}$.

Proposition 3.2 *Let $p \geqq 2$ and $\lambda > 1$. Then we have* $\dfrac{\partial T(a, \lambda)}{\partial a} > 0$.

Proof. If $a \leqq a_1$, then the proof is complete from Lemma 3.11.
Let a be $a > a_1$. Then Lemma 3.9 implies that

$$\int_{a_1}^{a} \frac{2G(x)g'(x) - g^2(x)}{2g^2(x)} \frac{dx}{\sqrt{E - G(x)}} < 0$$

and

$$\int_{a_*}^{(a_1)_*} \frac{2G(x)g'(x) - g^2(x)}{2g^2(x)} \frac{dx}{\sqrt{E - G(x)}} < 0.$$

Combining Lemma 3.11 and the above inequalities, we get our assertion. ∎

Combining Propositions 3.1 and 3.2, we have that $\dfrac{d\lambda(a)}{da}$ is positive. Thus, we get the following proposition which states the existence and the number of periodic solutions.

Proposition 3.3 *Let n be a natural number. Under the assumptions $(n+1)^2 \geqq \lambda > n^2$ and $p \geqq 2$, there exists the non trivial $\dfrac{2\pi}{k}$ periodic solution of (5) and this solution is unique where $k = 1, 2, \cdots, n$. Hence there exist just n periodic solutions.*

Here we show some results of numerical computation. In this computation we choose $p = 3$ and $\omega_0 = -1$.

Figure 1 shows the bifurcation diagram of periodic solutions with parameter λ and negative amplitude of solutions. The thin line starting from the origin indicates the value w_*.

4 Construction of super- and sub-solution

In this section we construct a super-solution u_+ and a sub-solution u_- of (1) satisfying

$$u_\pm(T_0) = f^{1/p}(T_0),$$
$$\lim_{t \to \infty} \frac{u_\pm(t, \theta)}{t^\alpha} = \omega_0 + \phi(\theta),$$
$$u_+(t, \theta) \geqq u_-(t, \theta)$$

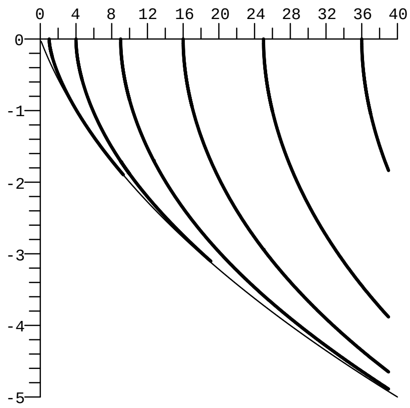

Figure 1: Bifurcation diagram

for any $(t, \theta) \in [t_0, \infty) \times [0, 2\pi)$, where ϕ is a nontrivial periodic solution of (5).

Let L be an operator defined by

$$L(u) = -g(|x|)\Delta u(x) + u(x)|u(x)|^{p-1} - f(|x|)$$

for $|x| > T_0$. We call that $u \in C^2$ is a super-solution or sub-solution if u satisfies $L(u) \geqq 0$ or $L(u) \leqq 0$, respectively.

If we construct such super- and sub-solution, then we can prove the existence of the non radial symmetric solution of (1) using the following proposition.

Proposition 4.1 (E. S. Noussair and C. A. Swanson [5]) *Let u_+, u_- be super and sub C^2-solution of* (1) *such that $u_+(t, \theta) \geqq u_-(t, \theta)$. Then, there exists a C^2-solution $u(t, \theta)$ of* (1) *such that*

$$u_+(t, \theta) \geqq u(t, \theta) \geqq u_-(t, \theta).$$

Since it is difficult to construct super- and sub-solution of (1) under the assumption (A-3), we need a slightly stronger assumption. In the following we assume that

(A-5) The functions $f(t)$ and $g(t)$ satisfy

$$\begin{aligned} f(t) &= \kappa^p t^{\alpha p}(1 + O((t^{-\alpha}\log t))^p) \\ g(t) &= t^\ell(1 + O((t^{-\alpha}\log t)^p)). \end{aligned}$$

Recall that y_+ and y_- are solutions of (2) satisfying $\lim_{t\to\infty} \frac{y(t)}{t^\alpha} = \omega_+$ and ω_-, respectively.

We construct the super- and sub-solutions of the following form.

$$u_\pm(t,\theta) = c_2(t)(\phi(\theta) + \omega_0)t^\alpha + c_1(t)y_\pm(t) \pm A\log\frac{t}{T_0}$$

where c_1, $c_2 \in C^2([T_0,\infty))$ are functions satisfying

$$c_1(t) = \begin{cases} 1 & \text{for } T_0 < t < T_1, \\ 0 & \text{for } T_1 + 1 < t, \end{cases} \qquad c_2(t) = \begin{cases} 0 & \text{for } T_0 < t < T_2, \\ 1 & \text{for } T_2 + 1 < t, \end{cases}$$
$$0 \leqq c_1(t),\ c_2(t) \leqq 1.$$

Here, A, T_1 and T_2 are positive constants satisfying $T_1 < T_2$ and suitably chosen in the following.

We first show that u_+ is a super-solution. Define the operator $L_1(u)$ from $C^2([T_2,\infty))$ to $C([T_2,\infty))$ by

$$L_1(u) = -|x|^\ell \Delta u(x) + u(x)|u(x)|^{p-1} - \kappa^p t^{\alpha p}.$$

Then we have $L_1((\phi(\theta) + \omega_0)t^\alpha) = 0$.

Lemma 4.1 *For any $Y > 0$ and any $X \in \mathbb{R}^1$, it holds that*

$$\begin{aligned} (X+Y)|X+Y|^{p-1} &\geqq X|X|^{p-1} + 2^{1-p}Y^p \\ (X-Y)|X-Y|^{p-1} &\leqq X|X|^{p-1} - 2^{1-p}Y^p. \end{aligned}$$

We omit the proof.

Lemma 4.2 *For any sufficiently large positive number A, it holds*

$$L((\phi(\theta) + \omega_0)t^\alpha + A\log\frac{t}{T_0})) > 0$$

for any $t > T_2$.

Proof. From the assumption (A-5), we see that $|g(t)-t^\ell| < Ct^{2-\alpha}(\log t)^p$ and $|f(t)-\kappa^p t^{\alpha p}| < C(\log t)^p$ for some constant C. Let $v_1 = (\phi(\theta)+\omega_0)t^\alpha + A\log t$. Then, we have the following inequality:

$$|L(v_1) - L_1(v_1)| < Ct^{2-\alpha}(\log t)^p|\Delta((\phi(\theta)+\omega_0)t^\alpha)| + C(\log t)^p.$$

Applying Lemma 4.1 as $X = (\phi(\theta)+\omega_0)t^\alpha$ and $Y = A\log t$, we get

$$L_1(v_1) = |v_1|^{p-1}v_1 - |v_1 - A\log t|^{p-1}(v_1 - A\log t) > \varepsilon_0(A\log t)^p.$$

Then we obtain $L(v_1) > \varepsilon_0(A\log t)^p - C(\log t)^p$. Since A is large enough, the assertion is proved. ∎

We next prove that u_+ is a super-solution. For $T_0 < t < T_1$, it follows $L(u_+) = (y_+(t) + A\log\frac{t}{T_0})^p - y_+(t)^p > 0$. Let $t \geqq T_1$ and $t < T_2+1$. If $C(T_2)$ is a constant independent of A satisfying

$$\max_{T_1 \leqq t \leqq T_2+1}\{|g(t)\Delta(c_2(t)(\phi(\theta)+\omega_0)t^\alpha + c_1(t)y_+(t))| \\ + |f(t)| + |c_2(t)(\phi(\theta)+\omega_0)t^\alpha + c_1(t)y_+(t)^p\} < C(T_2),$$

we have $L(u_+(t)) > \varepsilon_0(A\log t)^p - C(T_2)$ from Lemma 4.1 and $\Delta\log t/T_0 = 0$. Choose A sufficiently large to have $L(u_+(t)) > 0$.

Next, let $t \geqq T_2+1$. From Lemma 4.2, we see $L(u_+(t)) > 0$. Therefore, $u_+(t)$ is a super-solution.

Using the similar method, we can show the existence of a sub-solution.

Combining Proposition 4.1 and the above arguments, we can get the main theorem.

Theorem 4.1 *Let the assumptions* (A-0)-(A-2), (A-4) *and* (A-5) *hold. Assume $m^2 < \alpha^2 - p|\omega_0|^{p-1} \leqq (m+1)^2$. Then, we have at least m non radially symmetric solutions $\{u\}$ of* (1) *such that*

$$\lim_{|x|\to\infty}\frac{u(x)}{|x|^\alpha} = \omega_0 + \phi(\theta)$$

where $\phi(\theta)$ are the nontrivial periodic solutions of (5).

Proof. Let u_+ and u_- are super- and sub-solution, respectively, which we constructed in above. For sufficiently large number t, it holds that

$$u_+(t,\theta) = t^\alpha(\omega_0 + \phi(\theta)) + A\log\frac{t}{T_0},$$
$$u_-(t,\theta) = t^\alpha(\omega_0 + \phi(\theta)) - A_1\log\frac{t}{T_0}.$$

We have

$$\liminf_{t\to\infty} \frac{u_-(t,\theta)}{t^\alpha} \leqq \liminf_{t\to\infty} \frac{u(t,\theta)}{t^\alpha} \leqq \limsup_{t\to\infty} \frac{u(t,\theta)}{t^\alpha} \leqq \limsup_{t\to\infty} \frac{u_+(t,\theta)}{t^\alpha}.$$

Since

$$\liminf_{t\to\infty} \frac{u_-(t,\theta)}{t^\alpha} = \limsup_{t\to\infty} \frac{u_+(t,\theta)}{t^\alpha} = \omega_0 + \phi(\theta),$$

we obtain

$$\lim_{t\to\infty} \frac{u(t,\theta)}{t^\alpha} = \omega_0 + \phi(\theta)$$

which completes the proof. ∎

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