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Three-wave resonant interactions and multiple resonances in two-layer and three-layer flows

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The nonlinear interactions of waves on the interfaces of two- and three-layer flows are studied. It is shown that in both flows there are resonant triads in which three waves have the resonant interactions, and that these triads have interactions through their common waves. This is called the multiple resonance. The total energy of the waves which take part in the multiple resonance is conserved; however, it is not conserved in each triad, which is not the case when the triad does not have interaction with other triads, and the energy of a triad is transferred to others. The solutions of the evolution equations of the amplitudes are studied analytically and numerically. It is shown that soliton solutions exist, and that in some cases the evolution of the amplitudes becomes irregular and chaotic. In the three-layer flow the interactions are either of the "explosive type" or of the "decay type." These types can be switched by time from one to the other. On the other hand, all of the interactions in the two-layer flow are of the "decay type."

I. INTRODUCTION

The nonlinear three-wave resonant interactions in fluids have been widely studied.[1,2] They take place in the second order of approximation. In higher approximations more waves have resonant interactions. For example, interactions among four waves in the third-order approximation are detected in surface gravity waves,[1] among which the three-wave resonance does not occur. The multiple resonances in which some resonant triads have interaction through the common waves have been found to take place in an experiment of internal waves in a stratified fluid[3] and in an analysis.[4] In the experiment by Martin, Simmons, and Wunsch,[3] the multiple resonance between the (1,3,4) triad and the (3,4,7) triad, where the integers denote mode numbers, with two common waves of modes 2 and 4 are found to occur in a linearly stratified fluid. Tsutahara[4] shows that the multiple resonances consisting of an infinite number of resonant triads exist in a stratified shear flow. With the multiple resonance, four or more waves have the resonant interactions of the second order.

On the other hand, there are two types of interactions, i.e., the "explosive type" and the "decay type." [5,6] In the explosive interaction, the amplitudes of the waves grow simultaneously, and this type of interaction is discussed in connection with the waves of negative energy.

Here we consider internal waves in a two-layer flow and a three-layer flow with stepwise velocity and density profiles. We shall show that the multiple resonances take place in these flows and show some features of the multiple resonances, for example, that the evolution of the amplitudes becomes chaotic or that the types of the interaction can be switched by time. We also will describe the difference between the two flows.

II. MULTIPLE RESONANCES IN TWO- AND THREE-LAYER FLOWS

A. Two-layer flow

A two-layer flow as shown in Fig. 1 will be considered. The upper side and the lower side are both unbounded and the interface between the two fluids without any disturbances is taken to coincide with the x axis. The fluids are assumed to be incompressible and nondiffusive. The velocity is denoted by U and the density by ρ, and the subscripts 1 and 2 represent the upper fluid and the lower fluid, respectively.

Denoting the surface tension on the interface by γ, we have the well-known dispersion relation

$$D(k,\omega) = \rho_1 k^{-1}(U_1 k - \omega)^2 + \rho_2 k^{-1}(U_2 k - \omega)^2 + (\rho_1 - \rho_2)g - k^2\gamma = 0, \quad (1)$$

where k is the wavenumber, ω is the frequency, and g is the gravity acceleration.

If three waves satisfy the so-called resonant conditions

$$k_1 = k_2 + k_3, \quad \omega_1 = \omega_2 + \omega_3, \quad (2)$$

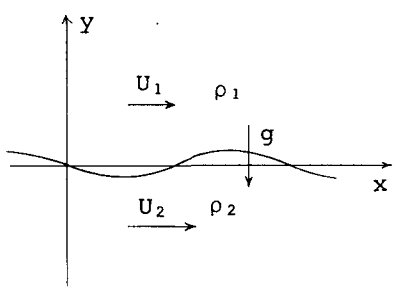

FIG. 1. The two-layer flow.

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these three waves which are called the resonant triad, have the resonant interactions with one another. We can easily find the resonant triad by using the graphs as presented by Craik and Adam,[5] and also by Tsutahara.[4] In their method, two graphs of the dispersion curves are used. The origin of one graph is moved along the curves of the other graph and if the intersections are found, there exist resonant triads. The dispersion curves for $\rho_1 = 1$, $\rho_2 = 1.02$, $U_1 = 0$, $U_2 = 5$, $\gamma = 4.13$, and $g = 9.81$ and the intersections of the curves are shown in Fig. 2. There are three intersections in this figure that represent the three resonant triads that have the common wave k_3. This constitutes the multiple resonance. If we put the origin of the dashed curve at the intersection A, then we obtain the multiple resonance in which k_1 of the one triad coincides with k_3 of other triads. Similarly we can form any multiple resonances with common k_1 and k_2 waves or k_1 and k_1 waves and so on. In this case the multiple resonance has an infinite number of member waves.

For simplicity, a multiple resonance consisting of five waves with one common wave k_3 which consists of two triads that will be considered here, and in Sec. IV, a seven-wave multiple resonance will be considered. Then the resonant conditions are

$$k_1 = k_2 + k_3, \quad \omega_1 = \omega_2 + \omega_3, \qquad k_4 = k_5 + k_3, \quad \omega_4 = \omega_5 + \omega_3. \tag{3}$$

If we let A_j $(j = 1,2,...,5)$ be the amplitudes of the five waves, then we obtain the interaction equations,

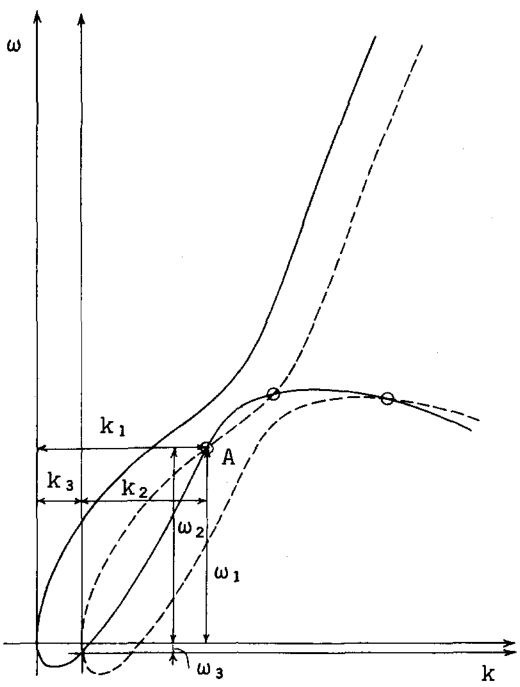

FIG. 2. The dispersion curves and the intersections.

$$\begin{aligned}
&\frac{\partial A_1}{\partial t_1} + Cg_1 \frac{\partial A_1}{\partial x_1} = iQ_1 A_2 A_3,\\
&\frac{\partial A_2}{\partial t_1} + Cg_2 \frac{\partial A_2}{\partial x_1} = iQ_2 A_1 A_3^*,\\
&\frac{\partial A_3}{\partial t_1} + Cg_3 \frac{\partial A_3}{\partial x_1} = iQ_3 A_1 A_2^* + iQ_6 A_4 A_5^*,\\
&\frac{\partial A_4}{\partial t_1} + Cg_4 \frac{\partial A_4}{\partial x_1} = iQ_4 A_5 A_3,\\
&\frac{\partial A_5}{\partial t_1} + Cg_5 \frac{\partial A_5}{\partial x_1} = iQ_5 A_4 A_3^*,
\end{aligned} \tag{4}$$

where the asterisk denotes the complex conjugate, and t_1 and x_1 are large scales of time and space. Letting ϵ be the order of the product of the wavenumber and the amplitude, they are expressed by

$$t_1 = \epsilon t, \quad x_1 = \epsilon x, \tag{5}$$

and A_j is their function. Here C_{gj} is the group velocity of k_j waves and is expressed as

$$C_{gj} = \frac{2[\rho_1 U_1 (U_1 k_j - \omega_j) + \rho_2 U_2 (U_2 k_j - \omega_j) - 2k_j \gamma]}{k_j [\partial D(k_j,\omega_j)/\partial \omega_j]}, \quad (j = 1,2,3,4,5). \tag{6}$$

On the other hand, Q_j is

$$\begin{aligned}
Q_1 &= -\frac{\lambda_1}{[\partial D(k_1,\omega_1)/\partial\omega_1]}, & Q_2 &= -\frac{\lambda_1}{[\partial D(k_2,\omega_2)/\partial\omega_2]},\\
Q_3 &= -\frac{\lambda_1}{[\partial D(k_3,\omega_3)/\partial\omega_3]}, & Q_4 &= -\frac{\lambda_2}{[\partial D(k_4,\omega_4)/\partial\omega_4]},\\
Q_5 &= -\frac{\lambda_2}{[\partial D(k_5,\omega_5)/\partial\omega_5]}, & Q_6 &= -\frac{\lambda_2}{[\partial D(k_3,\omega_3)/\partial\omega_3]},
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
\lambda_1 &= 2[\rho_2 (U_2 k_2 - \omega_2)(U_2 k_3 - \omega_3) - \rho_1 (U_1 k_2 - \omega_2)(U_1 k_3 - \omega_3)],\\
\lambda_2 &= 2[\rho_2 (U_2 k_5 - \omega_5)(U_2 k_3 - \omega_3) - \rho_1 (U_1 k_5 - \omega_5)(U_1 k_3 - \omega_3)].
\end{aligned} \tag{8}$$

Comparing the interaction equations in (4) with those for three-wave resonances, the evolution equation for A_3 is coupled because the k_3 wave is common to the two triads. But it is linearly coupled, so that in order to obtain the interaction equations for more than five waves, we simply couple some sets of the evolution equations for three waves only in the equations for the common waves.

B. Three-layer flow

We will also consider a three-layer flow as shown in Fig. 3, consisting of the upper layer, the lower layer, and the middle layer of width d between them. This flow has been studied by Craik and Adam,[5] and their results will be used. The group velocity C_{gj} in this case will be obtained by

$$C_{gj} = \frac{\partial \omega_j}{\partial k_j} \quad (j = 1,2,3,4,5)$$

and Q_j are given in their paper as

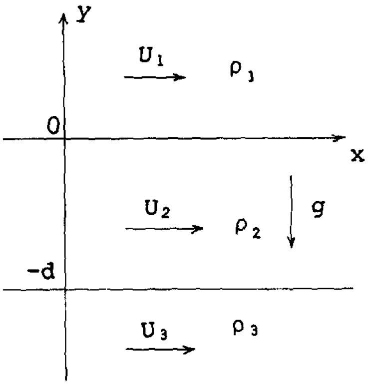

FIG. 3. The three-layer flow.

$$Q_j = \frac{i}{2} \frac{\lambda}{(\partial D(\omega_j,k_j)/\partial \omega_j)} . \tag{9}$$

The exact form of λ, which is very complicated, is also presented in the Craik and Adam paper.

III. ENERGY CONSERVATION

We shall show that the total energy is conserved within the multiple resonance by assuming that A_j is a function of time only. The interaction equations become

$$\frac{dA_1}{dt} = iQ_1A_2A_3 , \quad \frac{dA_2}{dt} = iQ_2A_1A_3^* ,$$
$$\frac{dA_3}{dt} = iQ_3A_1A_2^* + iQ_6A_4A_5^* , \tag{10}$$
$$\frac{dA_4}{dt} = iQ_4A_5A_3 , \quad \frac{dA_5}{dt} = iQ_5A_4A_3^* ,$$

where t_1 is replaced by t for simplicity. Equations in (10) are reduced to the following three equations:

$$\frac{d|A_1|^2}{dt} = K_1 \frac{d|A_2|^2}{dt} , \tag{11a}$$
$$\frac{d|A_3|^2}{dt} = K_2 \frac{d|A_1|^2}{dt} + K_3 \frac{d|A_4|^2}{dt} , \tag{11b}$$
$$\frac{d|A_4|^2}{dt} = K_4 \frac{d|A_5|^2}{dt} , \tag{11c}$$

where

$$K_1 = -Q_1/Q_2 , \quad K_2 = -Q_3/Q_1 ,$$
$$K_3 = -Q_6/Q_4 , \quad K_4 = -Q_4/Q_5 . \tag{12}$$

The energies of k_j ($j = 1,2,...,5$) waves are written as[6]

$$E_j = \frac{1}{4} \omega_j \frac{\partial D(k_j,\omega_j)}{\partial \omega_j} |A_j|^2 , \tag{13}$$

so that we have an energy conservation equation

$$\frac{d}{dt} \sum_{j=1}^{5} E_j = \frac{i}{8} [\lambda_1(\omega_2 + \omega_3 - \omega_1)x \operatorname{Re}(A_1^*A_2A_3)$$
$$+ \lambda_2(\omega_5 + \omega_3 - \omega_4)x \operatorname{Re}(A_4^*A_5A_3)] = 0 . \tag{14}$$

This means that the total energy of the multiple resonance is conserved but not in each triad, and, in general, the energy of one triad is transferred to other triads through the common wave. If the multiple resonance includes more triads, the same is true.

IV. CHAOTIC BEHAVIOR OF THE EVOLUTION OF AMPLITUDES

Now we will show that the time evolution of the amplitudes exhibit a chaotic behavior in some multiple resonances. Here we will consider the multiple resonance in a two-layer flow for $U_1 = 2.9$, $U_2 = 0$, $\rho_1 = 1.0$, $\rho_2 = 1.2$, $\gamma = 4.13$, and $g = 9.81$ and multiple resonance including the triad $(k_1,k_2,k_3) = (1.19492, 0.49492, 0.7)$ and $(\omega_1,\omega_2,\omega 3) = (2.02622, 0.93685, 1.08937)$. In this example, if the multiple resonance consists of only one resonant triad, that is, the resonant triad itself, the trajectory of the solution for the evolution equations projected onto the $A_1 - A_3$ plane with initial values $(A_1,A_2,A_3) = (0.2, 0.1, -0.15)$ is a simple ellipse as shown in Fig. 4, so that the solution is simple and periodic.

If the multiple resonance includes one more triad

$$(k_4,k_5,k_3) = (0.94987, 0.24987, 0.7)$$

and

$$(\omega_4,\omega_5,\omega_3) = (1.08409, -0.00528, 1.08937),$$

that is, if it consists of two triads, the trajectory with initial values

$$(A_1,A_2,A_3,A_4,A_5) = (0.2, 0.1, -0.15, 0.1, 0.1)$$

becomes a little complex but still not as irregular as that of Fig. 5. But when one more triad

$$(k_6,k_7,k_3) = (0.81819, 0.11819, 0.7)$$

and

$$(\omega_6,\omega_7,\omega_3) = (0.96530, -0.12407, 1.08937)$$

is included, then the trajectory becomes irregular and chaotic as shown in Fig. 6, where the initial values are

$$(A_1,A_2,A_3,A_4,A_5,A_6,A_7) = (0.2, 0.1, -0.15, 0.1, 0.1, 0.15, 0.1) .$$

In the last case, the interaction equation can easily be obtained by the procedure described in Sec. II A.

Errors in the above numerical calculations are estimated by

$$e = (E_0 - E)/E_0 ,$$

where E_0 is the initial value of the total energy of the member

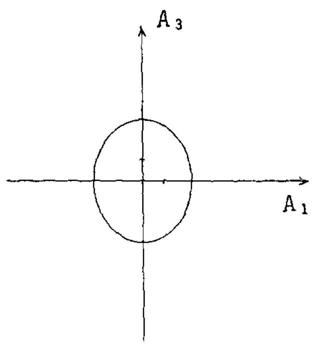

FIG. 4. The trajectory of the evolution of the amplitudes for one triad.

waves, each of which is expressed by (13), and E represents the total energy at each time step. The maximum error is less than 0.001% for the simple triad represented in Fig. 4, less than 1% for the two triads represented in Fig. 5, and less than 3% for the three triads represented in Fig. 6.

The trajectories in the three figures may depend on the initial values of A_j. But the tendency that the more triads are included in a multiple resonance, the more complicated it becomes, is the same for other initial values in this system of differential equations. Of course, if the system is changed, for instance, k_j and ω_j are changed, the trajectory may become more complicated even in a simple triad or in two triads.

The point stated here is to show that there are some cases in which the evolution of the amplitudes keeping regular cycles in a simple triad can become very irregular when the triad is coupled with other triads. Therefore, this is not always the case.

On the other hand, the spatial dependence of the amplitudes will be estimated in coordinates moving with a constant speed, which will be introduced in the next section. The evolution of the amplitudes for this case will be discussed in the same manner as presented in this section.

V. SOLITON SOLUTION

In some cases the evolution in (4) has a solution expressed by elliptic functions and a soliton solution. For three-wave interactions, Taniuti and Nozaki[7] have shown that the evolution equations have soliton solutions. The analysis described here is an extension of their results.

In Eq. (4), A_j is assumed to be a function of $\xi = x_1 - ct_1$, where c is the velocity of the soliton, and then we have

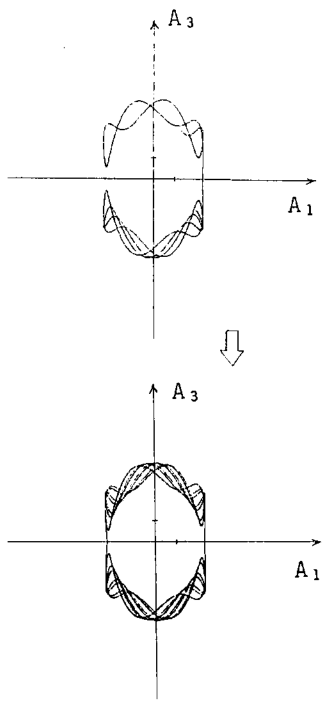

FIG. 5. The trajectory for two triads.

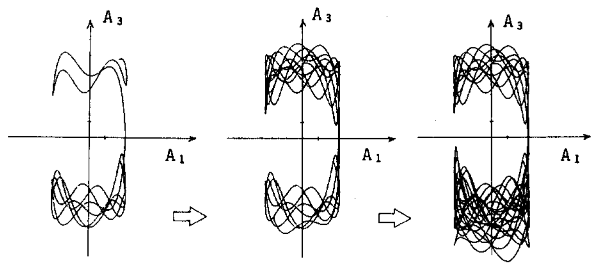

FIG. 6. The trajectory for three triads.

$$\frac{\partial}{\partial t_1} = -c\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}.$$

We also put

$$\widetilde{Q}_j = \frac{Q_j}{C_{gj} - c} \quad (j = 1,2,3,4,5), \quad \widetilde{Q}_6 = \frac{Q_6}{C_{g_3} - c}. \tag{15}$$

Then we obtain the equations that have the same form as that of (10) replacing t by ξ, and Q_j by $\widetilde{Q}_j$. Accordingly we also obtain the equations of the form used in (11). Here we use (10) and (11), with the above replacements made.

From (11a) and (11b) we obtain

$$\frac{d\,|A_1|^2}{d\xi} = \widetilde{Q}_1^2 |A_2|^2 |A_3|^2. \tag{16}$$

The solution for $|A_1|$ with the initial value, that is the value at $\xi = 0$, being zero may be taken as

$$|A_1|^2 = a_1 \,\mathrm{sn}^2(b_1^{1/2}\xi, m) \quad (a_1, b_1 > 0). \tag{17}$$

A solution for $|A_2|^2$ in (11a) may be

$$|A_2|^2 = -(a_1/K_1)\mathrm{cn}^2(b_1^{1/2}\xi, m). \tag{18}$$

Substituting (17) and (18) into (16), we obtain

$$|A_3|^2 = -(b_1K_1/\widetilde{Q}_1^2)\mathrm{dn}^2(b_1^{1/2}\xi, m). \tag{19}$$

On the other hand, if we let

$$|A_4|^2 = a_2 \,\mathrm{sn}^2(b_2^{1/2}\xi, m) \quad (a_2, b_2 > 0), \tag{20}$$

then we have from (11c)

$$|A_5|^2 = -(a_2/K_4)\mathrm{cn}^2(b_2^{1/2}\xi, m) \tag{21}$$

and from (11b)

$$\begin{aligned}(b_1^{1/2}K_1/\widetilde{Q}_1^2)m^2\,\mathrm{dn}(b_1^{1/2}\xi,m)\mathrm{sn}(b_1^{1/2}\xi,m)\mathrm{cn}(b_1^{1/2}\xi,m) \\ = K_2a_1b_1^{1/2}\,\mathrm{dn}(b_1^{1/2}\xi,m)\mathrm{sn}(b_1^{1/2}\xi,m)\mathrm{cn}(b_1^{1/2}\xi,m) \\ + K_3a_2b_2^{1/2}\,\mathrm{dn}(b_2^{1/2}\xi,m)\mathrm{sn}(b_2^{1/2}\xi,m)\mathrm{cn}(b_2^{1/2}\xi,m).\end{aligned} \tag{22}$$

It must hold from (22) that

$$b_1 = b_2, \quad m^2 = \widetilde{Q}_1^2(K_2a_1 + K_3a_2)/b_1K_1. \tag{23}$$

If we start with (20), we obtain

$$m^2 = \widetilde{Q}_4^2(K_2a_1 + K_3a_2)/b_1K_4. \tag{24}$$

In the limiting case of $m = 1$, (23) and (24) become

$$\widetilde{Q}_1\widetilde{Q}_2(K_2a_1 + K_3a_2) = \widetilde{Q}_4\widetilde{Q}_5(K_2a_1 + K_3a_2) = b_1^2 \tag{25}$$

and we have

$$|A_1|^2 = a_1 \tanh^2(b\xi)\,, \tag{26a}$$
$$|A_2|^2 = -(a_1/K_1)\operatorname{sech}^2(b\xi)\,, \tag{26b}$$
$$|A_3|^2 = -(b^2K_1/\tilde{Q}_1^2)\operatorname{sech}^2(b\xi)\,, \tag{26c}$$
$$|A_4|^2 = a_2 \tanh^2(b\xi)\,, \tag{26d}$$
$$|A_5|^2 = -(a_2/K_4)\operatorname{sech}^2(b\xi)\,, \tag{26e}$$

where $b = b_1^{1/2}$.

In the soliton solutions, only two variables can be chosen arbitrarily among the four variables, that is, the phase velocity of the soliton c, the width of the pulse b, and the a_1 and a_2 in the amplitudes, because the two equations in (25) hold for the four variables.

In the above analysis, if we choose

$$|A_2|^2 = (a_1/K_1)\operatorname{sn}^2(b_2^{1/2}\xi,m) \tag{27}$$

as a solution for $|A_2|^2$ in (11a) instead of (18), then $A_3 \to \infty$ when $\xi \to 0$. Accordingly, (27) must be omitted for the soliton solutions when $|A_1|$ is expressed by (17). Of course, $|A_1|^2$ and $|A_2|^2$ can be chosen as

$$|A_1|^2 = a_1 \operatorname{cn}^2(b_2^{1/2}\xi,m)\,,$$
$$|A_2|^2 = -(a_1/K_1)\operatorname{sn}^2(b_2^{1/2}\xi,m), \tag{28}$$

and $|A_3|^2$ has the same form as that in (19). In this case a_1 in (23) or (25) should be replaced by $-a_1$. Similarly, if we choose

$$|A_4|^2 = a_2 \operatorname{cn}^2(b_2^{1/2}\xi,m) \tag{29}$$

instead of (20), a_2 in (24) or (25) should be replaced by $-a_2$.

Consequently, the conditions for existence of the soliton solutions are

$$K_1 < 0\,, \quad K_4 < 0\,, \tag{30}$$

from (18) and (21), and also (25) must hold. From (25), it should be noted that

$$\pm K_2 a_1 \pm K_3 a_2 < 0 \tag{31}$$

where the sign $+$ or $-$ corresponds to the sn function or the cn function of $|A_1|^2$ and $|A_4|^2$.

For the multiple resonances consisting of more than five waves, the soliton solutions can be proved to exist in the same manner.

A soliton solution represented by (26) is illustrated in Fig. 7, where the amplitudes are expressed as functions of ξ. We check the energy exchange among the triads, which was described in Sec. IV for the soliton solution. When $\xi < \xi_1$, the energy of the (k_1,k_2,k_3) triad increases and that of the (k_4,k_5,k_3) triad decreases as ξ increases, and vice versa when $\xi > \xi_1$. Therefore the energy of the (k_4,k_5,k_3) triad transfers to the (k_1,k_2,k_3) triad when $\xi < \xi_1$, and it completely returns to the (k_4,k_5,k_3) triad when $\xi > \xi_1$.

VI. TYPE OF RESONANCE

In the wave interactions, there are two types, which are the "decay" type and the "explosive" type. In the decay type interactions, some waves grow in amplitude but the others decrease. On the other hand, in the explosive interactions, all the waves grow simultaneously.

Here we consider the type of two resonant triads and that of the multiple resonance composed by them. The amplitudes are assumed to be functions of time only as those in Sec. III, so that (10)–(12) are used in the following analysis.

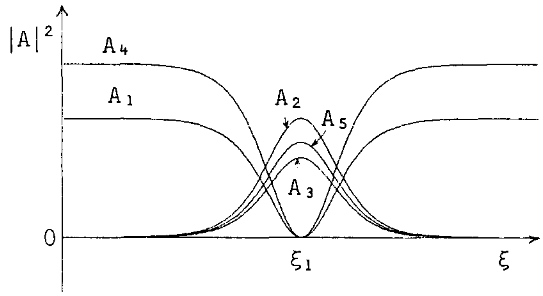

FIG. 7. Five solitons.

In the equations in (11):

(a) when $K_1,K_2,K_3,K_4 > 0$, both resonant triads are of the explosive type if they are considered to be uncoupled. This happens if

$$\operatorname{sign}(Q_1) \neq \operatorname{sign}(Q_2) = \operatorname{sign}(Q_3)\,,$$
$$\operatorname{sign}(Q_4) \neq \operatorname{sign}(Q_5) = \operatorname{sign}(Q_3)\,. \tag{32}$$

But when they are coupled, that is, they are multiply resonant and if

$$\frac{d|A_3|^2}{dt} < K_2 \frac{d|A_1|^2}{dt} \quad \text{or} \quad \frac{d|A_3|^2}{dt} < K_3 \frac{d|A_4|^2}{dt} \tag{33}$$

then

$$\operatorname{sign}\frac{d|A_3|^2}{dt} = \operatorname{sign}\frac{d|A_1|^2}{dt} \neq \operatorname{sign}\frac{d|A_4|^2}{dt}$$

or

$$\operatorname{sign}\frac{d|A_3|^2}{dt} = \operatorname{sign}\frac{d|A_4|^2}{dt} \neq \operatorname{sign}\frac{d|A_1|^2}{dt}\,.$$

Therefore the multiple resonance is of the decay type as a whole. However, if the above inequalities do not hold, that is, if

$$\frac{d|A_3|^2}{dt} > K_2 \frac{d|A_1|^2}{dt}\,, \quad \frac{d|A_3|^2}{dt} > K_3 \frac{d|A_4|^2}{dt} \tag{34}$$

the multiple resonance becomes the explosive type and all five amplitudes grow simultaneously.

(b) When $K_3 < 0$ and $K_1,K_2,K_4 > 0$, the (k_1,k_2,k_3) triad is of the explosive type and the (k_4,k_5,k_3) triad is of the decay type, if they are uncoupled. When they are coupled and if

$$\left| K_2 \frac{d|A_1|^2}{dt} \right| > \left| K_3 \frac{d|A_4|^2}{dt} \right| \tag{35}$$

all the amplitudes of the five waves grow simultaneously. Then the multiple resonance is explosive. If the above inequality does not hold, the multiple resonance is of the decay type.

When $K_2 < 0$ and $K_1, K_3, K_4 > 0$, the same discussion can be employed and the multiple resonance can be either of the decay type or of the explosive type.

(c) In all the other cases, the multiple resonance are all of the decay type.

We should note here that the above inequalities in case (a) and (b) depend on time, so that the type of the multiple resonance can switch from one to the other.

All the resonant triads in the two-layer flow are of the decay type, therefore all the multiple resonances are also of the decay type. But some of them in the three-layer flow are of the explosive type. One of the explosive-type triads is represented by the intersection shown in Fig. 8. In this figure, the intersection marked by the triangle represents the explosive interaction, but the other represents the decay interaction.

We shall show an example here in which a multiple resonance can be either of the decay type or the explosive type, depending on the initial conditions of the amplitudes. The multiple resonance consisting of the five waves $(k_j,\omega_j) = (0.91808, 0.4555)$, $(0.76808, 0.27807)$, $(0.15, 2.674)$, $(0.61206, 0.21712)$, and $(0.46206, 0.03969)$ will be considered.

In this example, the (k_1,k_2,k_3) triad is of the explosive type and the (k_4,k_5,k_3) is of the decay type. If the initial conditions are chosen as $A_1 = 0.02$, $A_2 = 0.05$, $A_3 = 0.002$, $A_4 = 0.0002$, and $A_5 = 0.0003$, then the multiple resonance is of the decay type as shown in Fig. 9, because A_1, A_2, and A_3 decrease until t_1 and after that A_4 and A_5 decrease. But if the initial conditions A_2 and A_3 are changed to $A_2 = 0.04$ and $A_3 = 0.004$ and the others remain the same, the multiple resonance becomes explosive. In Fig. 10, all the amplitudes decrease until t_1, which is included in the explosive interactions, and then they increase simultaneously; especially A_4 and A_5 become infinite at finite times. In this case, it is shown that a decay-type triad can be explosive when it is coupled with a explosive-type triad. In Figs. 9 and 10, the scale of $|A|^2$ is suitably changed for each A_j so that the behavior of the curves can be clearly shown.

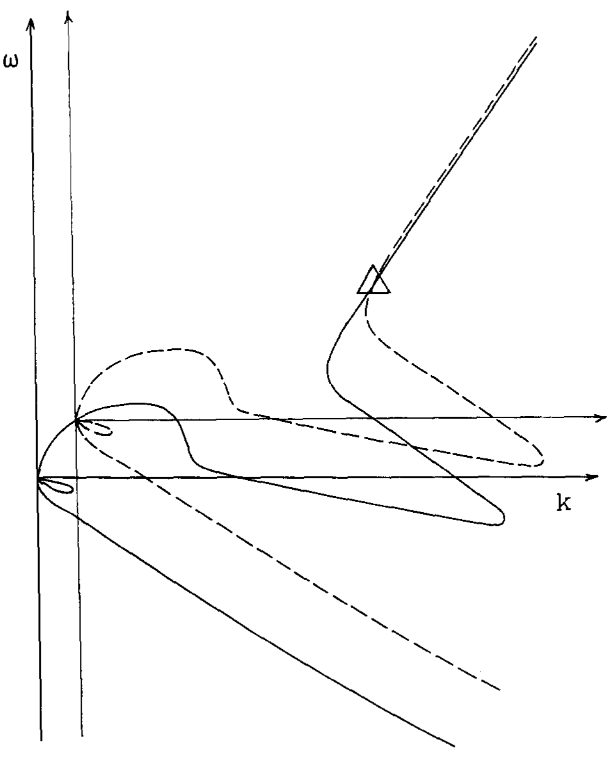

FIG. 8. The dispersion curves of the three-layer flow and an intersection representing an explosive interaction.

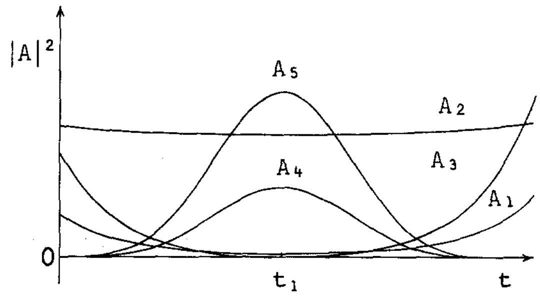

FIG. 9. The evolution of the amplitude of waves in a decay-type multiple resonance.

VII. DISCUSSION

As shown in the two-layer flow and the three-layer flow, the multiple resonances are generally seen as phenomena in internal waves. The consideration of a resonant triad independently of other triads is sometimes incorrect when it is coupled. For instance, the total energy is not conserved but is transferred and the type of interaction may be changed and the evolution of the amplitudes becomes more complex.

In most cases, infinitely many waves constitute a multiple resonance, which may include both the explosive- and the decay-type triads in three-layer flows, so that we seem to have to consider all those waves. However, because of the surface tension or the damping through the viscosity, the amplitudes of the waves of large wavenumbers are small and negligible or even damped out. Therefore, it may be enough to consider some finite number of waves, but they are still likely to be many. If we consider an initial stage where some small number of waves are excited, an analysis similar to the one presented here may be useful.

VIII. CONCLUSIONS

Resonant interactions of the interfacial waves in a two-layer flow and a three-layer flow are studied and the following conclusions are obtained.

(1) There are multiple resonances consisting of many resonant triads in two-layer and three-layer flows.

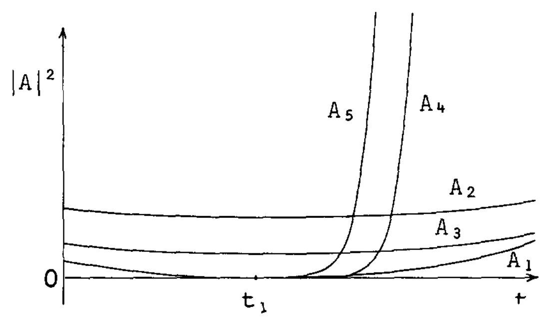

FIG. 10. The evolution of the amplitude of waves in an explosive-type multiple resonance.

(2) The nonlinear evolution equations for amplitudes of the waves constituting the multiple resonance have soliton solutions.

(3) The more waves the multiple resonance consists of, the more complicated the evolution of the amplitudes becomes, and sometimes it is chaotic.

(4) All the resonant triads in the two-layer flow dealt with in this study are of the decay type, but some of those in the three-layer flow are of the explosive type. Therefore the multiple resonances in the former are all of the decay type, but those in the latter can be either of the decay type or of the explosive type, and the type of the multiple resonance can switch from one to the other with time.

(5) Whether the multiple resonances are of the explosive type or of the decay type, the total energy is conserved within them, but the energy of each resonant triad is not conserved and is transferred to others.

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