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Resonant interaction of internal waves in a stratified shear flow

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The resonant interactions among internal wave trains in a flow consisting of a middle layer of homogeneous fluid in linear shear flow and continguous upper and lower stratified layers of constant velocities are investigated. Three wave trains satisfying the dispersion relations and the so-called resonant conditions, $k_1 = k_2 + k_3$ and $\sigma_1 = \sigma_2 + \sigma_3$, are considered graphically, and the resonant triads of the same mode are found to exist. Some multiple resonances, i.e., the resonant interactions among waves which are the members of different triads, are also found to happen. The explosive and decay instabilities of multiple resonances are considered qualitatively, and energy exchange is expected to occur among the resonant triads.

I. INTRODUCTION

The interaction of weakly nonlinear wave trains has been investigated rather intensively in recent years. Benjamin[1] and Whitham[2] have reported on the sideband instability of the Stokes wave. In more general cases for three dispersive wave trains, if their wavenumbers k_1, k_2 and k_3 and their corresponding frequencies σ_1, σ_2, and σ_3 satisfy the so-called resonant conditions

$$k_1 = k_2 + k_3 \quad \text{and} \quad \sigma_1 = \sigma_2 + \sigma_3, \tag{1}$$

then there is a resonant interaction among these wave trains at the second order. Phillips,[3] however, has shown that there cannot be such a resonance among purely gravitational surface waves in deep water at this order.

On the other hand, Davis and Acrivos[4] demonstrated analytically and experimentally that three interal waves of different modes which satisfy (1) are resonant and the flow is unstable. Hasselman[5] has shown on this result that if one of these three waves has a finite amplitude, then the other two infinitesimal waves grow exponentially in time. Yih[6] has presented the instability due to the resonant interaction between the two progressive wave trains of the same mode in a stratified shear flow in his simplified flow model.

Recently, in work with three-layer systems, Cairns[7] and Craik and Adam[8] have shown that when three interfacial waves, one of which has energy of a sign different from the others, have the resonant interaction, an explosive instability occurs and the amplitudes of the three waves increase simultaneously.

We shall consider here the infinitesimal waves in Yih's flow model[6] and show by a graphical method the existence of multiple resonances among two or three resonant triads consisting of the waves of the same mode. Then we shall predict qualitatively that when some of, or even all of, the triads are the explosive type, the multiple resonance as a whole can be either the decay type or the explosive type and that some energy transfer will take place among the triads.

II. GOVERNING EQUATION

A two-dimensional stratified flow will be considered, where the mean flow is in the horizontal direction, i.e., x direction with the mean velocity and the mean density varying in the vertical direction, i.e., y direction but without any disturbances. The fluid is assumed to be incompressible and nondiffusive. We shall use the Boussinesq approximation and let u and v, the velocity components in x and y direction, respectively, be measured in units of a velocity scale U_0, the density ρ in units of a constant density ρ_0, x, and y in units of a length scale d, and the pressure p and time t in units of $\rho_0 U_0^2$ and d/U_0, respectively. Then the equations of motion in a dimensionless form are

$$\frac{\partial u'}{\partial t} + (\bar{u} + u')\frac{\partial u'}{\partial x} + v'\frac{\partial}{\partial y}(\bar{u} + u') = -\frac{\partial p'}{\partial x}, \tag{2}$$

$$\frac{\partial v'}{\partial t} + (\bar{u} + u')\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} = -\frac{\partial p'}{\partial y} - F^{-2}\rho', \tag{3}$$

in which

$$F = U_0/(gd)^{1/2} \tag{4}$$

is the Froude number, overbars denote the values for the mean flow, and primes denote those of small perturbations. For simplicity the primes will be omitted hereafter. The equations continuity and incompressibility are, respectively,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5}$$

and

$$\frac{\partial \rho}{\partial t} + (\bar{u} + u)\frac{\partial \rho}{\partial x} + v\frac{\partial}{\partial y}(\bar{\rho} + \rho) = 0. \tag{6}$$

Using a stream function ψ and eliminating p' from (2) and (3), we obtain

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \psi_y\nabla^2\psi_x - \psi_x\nabla^2\psi_y - \psi_x\,\bar{u}_{yy} = F^{-2}\rho_x, \tag{7}$$

in which ∇^2 is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{8}$$

From (6) we find

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\rho + \beta\psi_x + \psi_y\,\rho_x - \psi_x\,\rho_y = 0, \tag{9}$$

in which

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$$\beta = -\frac{d\bar{\rho}}{dy}. \quad (10)$$

Combining (7) and (9) gives

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 \nabla^2\psi + \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)(\psi_y \nabla^2\psi_x - \psi_x \nabla^2\psi_y - \psi_x \bar{u}_{yy}) = F^{-2}\left(-\beta\psi_{xx} + \frac{\partial}{\partial x}(\psi_x \rho_y - \psi_y \rho_x)\right). \quad (11)$$

III. THE METHOD OF MULTIPLE SCALES

To find an approximate solution, we shall use the method of multiple scales by introducing the temporal scales

$$t_0 \equiv t, \quad t_1 \equiv \epsilon t, \quad t_2 \equiv \epsilon^2 t, \cdots, \quad (12)$$

and the spacial scales

$$x_0 \equiv x, \quad x_1 \equiv \epsilon x, \quad x_2 \equiv \epsilon^2 x, \cdots, \quad (13)$$

in which ϵ is the maximum steepness ratio and is assumed to be small but finite. Then the differentiations with respect to t and x become

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \epsilon^2\frac{\partial}{\partial t_2} + \cdots, \quad (14)$$

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon\frac{\partial}{\partial x_1} + \epsilon^2\frac{\partial}{\partial x_2} + \cdots, \quad (15)$$

respectively.

We assume that ψ, ρ, and p possess uniformly valid expansions of the form

$$\psi = \epsilon\psi_0(x_0, x_1, \cdots, t_0, t_1, \cdots) + \epsilon^2\psi_1 + \epsilon^3\psi_2 + \cdots, \quad (16)$$

$$\rho = \epsilon\rho_0 + \epsilon^2\rho_1 + \cdots, \quad (17)$$

$$p = \epsilon p_0 + \epsilon^2 p_1 + \cdots. \quad (18)$$

Substituting (14)–(18) into (10) and equating coefficients of powers of ϵ, we find for terms of order ϵ,

$$\left(\frac{\partial}{\partial t_0} + \bar{u}\frac{\partial}{\partial x_0}\right)^2 \nabla_0^2\,\psi_0 - \bar{u}_{yy}\left(\frac{\partial}{\partial t_0} + \bar{u}\frac{\partial}{\partial x_0}\right) \times \psi_{0x_0} + F_i^{-2}\psi_{0x_0x_0} = 0, \quad (19)$$

in which F_i is an internal Froude number

$$F_i^{-2} = F^{-2}\beta, \quad (20)$$

and the operator ∇_0^2 is

$$\nabla_0^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}. \quad (21)$$

For terms of order ϵ^2 we have

$$\left(\frac{\partial}{\partial t_0} + \bar{u}\frac{\partial}{\partial x_0}\right)^2 \nabla_0^2\,\psi_1 + F_i^{-2}\psi_{1x_0x_0} = -\left(\frac{\partial}{\partial t_1} + \bar{u}\frac{\partial}{\partial x_1}\right)\left(\frac{\partial}{\partial t_0} + \bar{u}\frac{\partial}{\partial x_0}\right)\nabla_0^2\,\psi_0 - \left(\frac{\partial}{\partial t_0} + \bar{u}\frac{\partial}{\partial x_0}\right)^2\left(\frac{\partial^2}{\partial x_0\,\partial x_1}\psi_0\right) - \left(\frac{\partial}{\partial t_0} + \bar{u}\frac{\partial}{\partial x_0}\right)(\psi_{0y}\nabla_0^2\,\psi_{0x_0} - \psi_{0x_0}\nabla_0^2\,\psi_{0y}) + F^{-2}\left(-\beta\psi_{0_{x_0x_1}} + \frac{\partial}{\partial x_0}(\psi_{0_{x_0}}\rho_{0_y} - \psi_{0_y}\rho_{0_{x_0}})\right). \quad (22)$$

Similarly, for the equation of incompressibility, we have to the order ϵ,

$$\frac{\partial\rho_0}{\partial t_0} + \bar{u}\frac{\partial\rho_0}{\partial x_0} - \psi_{0x_0}\frac{\partial\bar{\rho}}{\partial y} = 0. \quad (23)$$

On the other hand, from (2) and (18) we obtain to the order ϵ,

$$-\frac{\partial p_0}{\partial x_0} = \frac{\partial\Psi_{0y}}{\partial t_0} + \bar{u}\frac{\partial\Psi_{0y}}{\partial x_0} - \psi_{0_{x_0}}\frac{\partial\bar{u}}{\partial y}. \quad (24)$$

IV. THE MEAN FLOW AND THE INTERNAL WAVES

We shall consider a two-dimensional flow model presented by Yih[6] which consists of three layers. The indices I, II, and III are assigned to the lower, middle, and upper layers, respectively. The depth of each layer is $2d$, with the origin at the midpoint of the middle layer. Then, the velocity in the middle layer is $U_0 y/d$, the velocity in the upper layer is U_0, and the velocity in the lower layer is $-U_0$. The flow is supposed to be confined by two rigid boundaries at the upper and lower ends. The density in the middle layer is constant and is denoted by ρ_c. The mean density $\bar{\rho}$ in the upper and the lower layer is such that $d\bar{\rho}/dy = -\beta\rho_c/d < 0$ when β is a constant. Then in the nondimensional notation, we have

$$\bar{u} = -1 \quad \text{and } \bar{\rho}_y = -\beta, \quad \text{for } -3 \leqslant y \leqslant -1, \quad (25a)$$

$$\bar{u} = y \quad \text{and } \bar{\rho} = 1, \quad \text{for } -1 \leqslant y \leqslant 1, \quad (25b)$$

$$\bar{u} = 1 \quad \text{and } \bar{\rho}_y = -\beta, \quad \text{for } 1 \leqslant y \leqslant 3. \quad (25c)$$

By introducing this model, singularities of (19) and (22) can be avoided and the effects of the shear flow on the resonant interaction among the internal waves can be considered without having to deal with any singularities.

We assume that ψ_0 has a form

$$\psi_0 = f(y)\exp[i(\alpha x_0 - \sigma t_0)], \quad (26)$$

in which α is the wavenumber and σ is the frequency, and both are real. Substituting (26) into (23) and (24), we have

$$\rho_0 = -\beta[\alpha f/(\bar{u}\alpha - \sigma)]\exp[i(\alpha x_0 - \sigma t_0)] \quad (27)$$

for the density, and

$$p_0 = \bar{\rho}(\bar{u}_y f - \bar{u}f' + \sigma f'/\alpha)\exp[i(\alpha x_0 - \sigma t_0)] \quad (28)$$

for the pressure.

Substituting (26) into (19) and using (25), we obtain the ordinary differential equations

$$f_{\rm I}'' - \alpha^2[1 - N/(\sigma + \alpha)^2]f_{\rm I} = 0, \quad (29a)$$

$$f''_{\rm II} - \alpha^2 f_{\rm II} = 0, \tag{29b}$$

$$f''_{\rm II} - \alpha^2[1 - N/(\sigma - \alpha)^2] f_{\rm III} = 0, \tag{29c}$$

in which the primes indicate differentiation with respect to y, and the indices I, II, and III are assigned to the lower, middle, and upper layers, respectively, and $N = F_i^{-2}$.

The boundary conditions at the lower and the upper rigid boundaries are

$$f_{\rm I}(-3) = 0 = f_{\rm III}(3), \tag{30}$$

and the interfacial conditions are written as

$$f_{\rm I}(-1) = f_{\rm II}(-1), \tag{31a}$$

$$f'_{\rm I}(-1) = f'_{\rm II}(-1) + [\alpha/(\sigma + \alpha)] f_{\rm II}(-1), \tag{31b}$$

$$f_{\rm II}(1) = f_{\rm III}(1), \tag{31c}$$

$$f'_{\rm III}(1) = f'_{\rm II}(1) + [\alpha/(\sigma - \alpha)] f_{\rm II}(1), \tag{31d}$$

in which (31a) and (31c) represent the continuity of the stream function, and (31b) and (31d) represent the continuity of the pressure at the interfaces. The differential system consisting of (29), (30), and (31) defines an eigenvalue problem.

As described by Yih,[6] the internal conditions in (31) are natural conditions and the characteristics of the eigenvalues are considered by the Sturm–Liouville theory without concern about (31). A consideration of the infinitesimal waves in this flow model by the Sturm–Liouville theory is presented by Tsutahara.[9] Here we shall calculate actually the relationship between α and σ for given N, which will be the dispersive relation of the wave trains, and show the existence of the resonant triads which satisfy the resonant conditions in (1).

V. THE DISPERSION RELATION

The solution to (29) has a form

$$f_{\rm I} = A \sinh[\sqrt{G_1}(y + 3)], \tag{32a}$$

$$f_{\rm II} = Be^{\alpha y} + Ce^{-\alpha y}, \tag{32b}$$

$$f_{\rm III} = D \sinh[\sqrt{G_3}(y - 3)], \tag{32c}$$

where

$$G_1 = \alpha^2[1 - N/(\sigma + \alpha)^2]$$

and

$$G_3 = \alpha^2[1 - N/(\sigma - \alpha)^2],$$

and A, B, C, and D are constants.

Substituting (32) into (31), we obtain the secular equation

$$\begin{vmatrix} e^{-\alpha}[\alpha/(\sigma+\alpha) + \alpha - \sqrt{G_1}\coth(2\sqrt{G_1})] & e^{\alpha}[\alpha/(\sigma+\alpha) - \alpha - \sqrt{G_1}\coth(2\sqrt{G_1})] \\ e^{\alpha}[\alpha/(\sigma-\alpha) + \alpha - \sqrt{G_3}\coth(2\sqrt{G_3})] & e^{-\alpha}[\alpha/(\sigma-\alpha) - \alpha + \sqrt{G_3}\coth(2\sqrt{G_3})] \end{vmatrix} = 0. \tag{33}$$

If we fix the values of N and σ, we can determine α from (33). This equation shows that if α is an eigenvalue, so is $-\alpha$, because if $\alpha \to -\alpha$, then $G_1 \to G_3$ and $G_3 \to G_1$, and we have (33) again. In a similar manner, if we put $-\sigma$ instead of σ, (33) does not change. Recalling that the phase velocity c of the wave is defined by α and σ as

$$c = \sigma/\alpha,$$

if we allow α to have a positive and negative value, we can define $\sigma \geqslant 0$ without losing generality. Positive values of α correspond to waves propagating in the positive x direction and vice versa.

Figure 1 shows the relationship between α_n, the wave number of the nth mode, and σ for $N = 10$, in which only the first four modes and positive wavenumbers are shown. The two asymptote lines $\alpha = \sigma \pm N^{1/2}$ are shown as dotted lines for interest, and all the $\alpha - \sigma$ curves approach these lines.

There are eigenvalues α and $-\alpha$ for fixed σ and N. Also, as seen in Fig. 1 for each α and $-\alpha$ of the same mode, there are two families of $|\alpha| > \sigma$ and of $|\alpha| < \sigma$. We shall call the family of $|\alpha| > \sigma$ the l family and that of $|\alpha| < \sigma$ the s family. Then there are four waves, in general, for each mode, two of the l family and the other two of the s family. But of these four, only two are distinct, that is, the two are different only in sign from the other two. We note here that for the l family wave $|c| < 1$, and for the s family wave $|c| > 1$.

VI. GRAPHICAL METHOD FOR THE RESONANT TRIADS

If the three wavenumbers and the corresponding frequencies satisfy the secular equation (33) and the resonant conditions (1), the resonant interaction will occur among these three waves. Instead of solving the simultaneous equations, we present here a graphical method of solution of the equations.

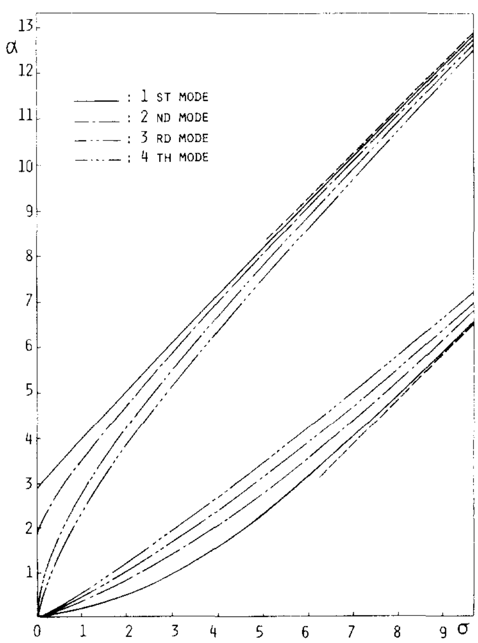

FIG. 1. Variation of α vs σ for $N = 10$. The first four modes are shown. The lines of the upper group represent the $\alpha_l - \sigma$ curves and the lines of the lower group represent the $\alpha_s - \sigma$ curves.

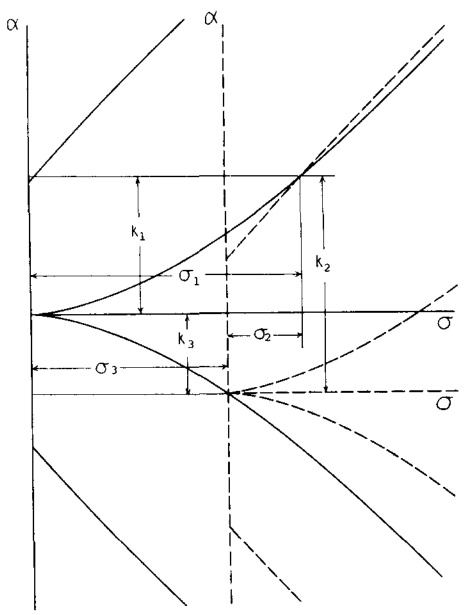

FIG. 2. Three waves satisfying the resonant conditions.

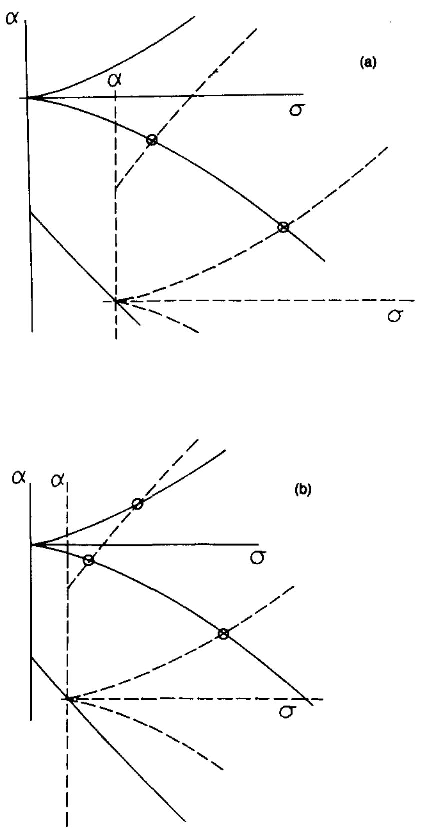

FIG. 3. Graphical explanation of the multiple resonances with (a) two intersections in case II, and (b) three intersections in case III.

We consider the resonant interaction among the waves which belong to the first mode. The first-mode wavenumbers $\pm\alpha_{l1}$, and $\pm\alpha_{s1}$ versus the frequency σ for $N = 10$ are shown by solid lines in Fig. 2. The dotted lines represent the same relations and the origin is located at some point on the line representing $-\alpha_{s1}$. When the origin is at an appropriate point, we can easily find the intersection of the α_{s1} line of the fixed graph (graph I) and the α_{l1} line of the superposed graph (graph II). Let the values of α and σ at the intersection be denoted by k_1 and σ_1, respectively, as measured in graph I; and by k_2 and σ_2, respectively, as measured in graph II; and let the values of those where the origin of graph II is located be denoted by k_3 and σ_3, respectively, in graph I. Then these values satisfy (33) and (1). The value of k_3 is negative in this case.

When the origin of graph II is located on the $-\alpha_{s1}$ line, only one intersection is found. Then one set of resonant triad exists, in which two of the three waves belong to the s family, which propagate in opposite directions, and the last belongs to the l family. We shall call it case I and the triad triad 1.

When the origin of graph II is on the $-\alpha_{l1}$ line, there are two or three intersections as shown in Figs. 3(a) and 3(b), respectively, which means that there are two or three resonant triads with one common set of wavenumbers and frequency. We shall call the former case case II and the latter case case III. If the origin of graph II is on the $+\alpha_{s1}$ line or the $+\alpha_{l1}$ line, the resonant triads consisting of the wavenumbers have opposite signs to the above. These types of resonant triads, their component waves, and their features are summarized in Table I.

In Table I, the upper or the lower signs of the wave numbers are taken for one triad. The wavenumber α_l or $-\alpha_l$ corresponding to k_3 and the frequency σ_3 have the same values and are common to the two triads for case II and to the three triads for case III, but the wavenumbers k_1 and k_2 and the frequencies σ_1 and σ_2 have different values. The first and the second triads in case II and case III are the same ones in nature and we shall call them triad 2 and triad 3, respectively. We shall also call the last one in case III, triad 4.

Martin, Simmons, and Wunsch[10] have reported the multiple resonance among waves of different modes in a stratified fluid at rest. But in case II and case III, the multiple resonance occurs among the waves of the same mode because of shear. We should note that in all the cases one wave propagates in the direction opposite to the other two, and one wave belongs to a different family from the others for each triad.

For higher mode waves a similar discussion is applicable, and intermode resonance, the resonant interaction among waves of different modes, can also be investigated by the same method. But we concentrate our attention on the resonance of the waves of the same mode.

VII. THE INTERACTION EQUATION FOR A TRIAD

In order to obtain the interaction equations, we assume that ψ_0 has a form

$$\psi_0 = a_1(t_1, x_1)e_1 f_1 + a_2(t_1, x_1)e_2 f_2 + a_3(t_1, x_1)e_3 f_3 + \text{c.c.}, \quad (34)$$

with

$$e_j = \exp[i(k_j x_0 - \sigma_j t_0)], \quad j = 1,2, \text{ and } 3, \quad (35)$$

in which the amplitudes $a_1(t_1, x_1)$, $a_2(t_1, x_1)$, and $a_3(t_1, x_1)$ vary slowly with time and in space and are assumed to be functions of t_1 and x_1, and c.c. represents the complex conjugates. The functions f_1, f_2, and f_3 are identical to $f(y)$ of (26)

TABLE I. The features of the resonant triads.

Wavenumber and frequency	$k_1\sigma_1$	$k_2\sigma_2$	$k_2\sigma_2$		
Case I (one intersection)	$\pm\alpha_s\sigma_1$	$\pm\alpha_l\sigma_2$	$\mp\alpha_s\sigma_3$	(Triad 1)	
Case II (two intersections)	$\mp\alpha_s\sigma_1$	$\pm\alpha_s\sigma_2$	$\mp\alpha_l\sigma_3$	(Triad 2)	k_3 and σ_3 are common
	$\mp\alpha_s\sigma_1$	$\pm\alpha_l\sigma_2$	$\mp\alpha_l\sigma_3$	(Triad 3)	to the two triads
Case III (three intersections)	$\mp\alpha_s\sigma_1$	$\pm\alpha_s\sigma_2$	$\mp\alpha_l\sigma_3$	(Triad 2)	k_3 and σ_3 are common
	$\mp\alpha_s\sigma_1$	$\pm\alpha_l\sigma_2$	$\mp\alpha_l\sigma_3$	(Triad 3)	to the three triads
	$\pm\alpha_s\sigma_1$	$\pm\alpha_l\sigma_2$	$\mp\alpha_l\sigma_3$	(Triad 4)	

when α and σ are k_1 and σ_1, k_2 and σ_2, and k_3 and σ_3, respectively. From (23) and (34), we also have

$$\rho_0 = b_1 e_1 f_1 + b_2 e_2 f_2 + b_3 e_3 f_3 + \text{c.c.} \tag{36}$$

with

$$b_j = \beta k_j a_j/(\sigma_j - \bar{u}k_j), \quad j = 1,2, \text{ and } 3. \tag{37}$$

Substitution of (34) and (36) into (22), and a necessary condition for existence of a solution of ψ_1 to (22), we obtain the interaction equations

$$\frac{\partial a_1}{\partial t_1} + A_{11}\frac{\partial a_1}{\partial x_1} = iA_{12}\, a_2\, a_3, \tag{38a}$$

$$\frac{\partial a_2}{\partial t_1} + A_{21}\frac{\partial a_2}{\partial x_1} = iA_{22}\, a_3^*\, a_1, \tag{38b}$$

$$\frac{\partial a_3}{\partial t_1} + A_{31}\frac{\partial a_3}{\partial x_1} = iA_{32}\, a_1\, a_2^*, \tag{38c}$$

in which asterisks represent the complex conjugates and

$$A_{j1} = \int_{-3}^{3} A_{j4} f_j\, dy \left(\int_{-3}^{3} A_{j3} f_j\, dy\right)^{-1}, \tag{39}$$

and

$$A_{j2} = \int_{-3}^{3} A_{j5} f_j\, dy \left(\int_{-3}^{3} A_{j3} f_j\, dy\right)^{-1}. \tag{40}$$

Here

$$A_{13} = -(\sigma_1 - \bar{u}k_1)(f_1'' - k_1^2 f_1),$$

$$A_{14} = -u(\sigma_1 - \bar{u}k_1)(f_1'' - k_1 f_1) + (\sigma_1 - \bar{u}k_1)^2 k_1 f_1 - Nk_1 f_1,$$

$$\begin{aligned} A_{15} = {} & (\sigma 1 - \bar{u}k_1)[k_2 f_3'(f_2'' - k_2^2 f_2) + k_3 f_2'(f_3'' - k_3 f_3) \\ & - k_2 f_2(f_3''' - k_3 f_3') - k_3 f_3(f_2''' - k_2 f_2')] \\ & - Nk_1\left[\left(\frac{k_2}{\sigma_2 - \bar{u}k_2} - \frac{k_3}{\sigma_3 - \bar{u}k_3}\right)(k_3 f_2' f_3 - k_2 f_3' f_2) \right. \\ & \left. + \left(\frac{k_2}{(\sigma_2 - \bar{u}k_2)^2} + \frac{k_3}{(\sigma_3 - \bar{u}k_3)^2}\right) k_2 k_3 \bar{u}' f_2 f_3\right], \end{aligned} \tag{41}$$

and A_{23}, A_{24}, and A_{25} are obtained from A_{13}, A_{14}, and A_{15}, respectively, by putting

$$(k_1,\sigma_1) \to (k_2,\sigma_2), (k_2,\sigma_2) \to (-k_3, -\sigma_3), (k_3,\sigma_3) \to (k_1,\sigma_1),$$
$$f_1 \to f_2, f_2 \to f_3, f_3 \to f_1;$$

and A_{33}, A_{34}, and A_{35} are obtained from A_{13}, A_{14}, and A_{15}, respectively, by putting

$$(k_1,\sigma_1) \to (k_3,\sigma_3), (k_2,\sigma_2) \to (k_1,\sigma_1), (k_3,\sigma_3) \to (-k_2, -\sigma_2),$$
$$f_1 \to f_3, f_2 \to f_1, f_3 \to f_2.$$

We note that A_{j1} and A_{j2} ($j = 1,2,3$) are constants, but A_{j3}, A_{j4}, and A_{j5} are functions of y.

When the resonant interaction by (38) is the decay type in which there are two types of wave, that is, waves of increasing amplitude and of decreasing amplitude, simultaneously, (38) has the cnoidal wave solution or the solitary wave solution and infinitely many conservative quantities exist. Kaup[11] has solved this interaction equation by the inverse scattering method. We shall consider here the multiple resonances found in the stratified flow model presented above from the standpoint of explosive and decay instabilities.

For simplicity, a_j in (38) is assumed to be a function of t_1 only, then we have

$$\frac{d|a_1|^2}{dt_1} = K_1 \frac{d|a_2|^2}{dt_1}, \quad \frac{d|a_3|^2}{dt_1} = K_2 \frac{d|a_1|^2}{dt_1}, \tag{42a}$$

with

$$K_1 = -\frac{A_{21}}{A_{22}}, \quad K_2 = -\frac{A_{23}}{A_{21}}. \tag{42b}$$

Obviously, in the case that both K_1 and K_2 are positive, the interaction is the explosive type and otherwise the decay type. On the basis of wave energy, when one of the three waves of the triad has an energy of a sign different from the other two and the frequency of the wave is the largest in the absolute value, explosive instability occurs.

The wave energy is expressed by

$$E(k,\sigma) = \frac{1}{4}\sigma \frac{\partial D}{\partial \sigma}|a|^2, \tag{43}$$

where a is the amplitude of the wave and

$$D(k,\sigma) = 0$$

is the linear dispersion relation. In the present case (only the first-mode waves considered), the α_l waves have negative energy and the α_s wave have positive energy. Referring to Fig. 3, explosive instability takes place in triad 2 and triad 3 in Table I. Both triads are, however, members of the multiple resonance which consists of two or three resonant triads where the k_3 wave is common. Then if we solve (38) for each triad individually, the k_3 wave will be multivalued. Therefore, the interaction of five or seven waves must be considered from the beginning in these cases.

VIII. INTERACTION EQUATION FOR MULTIPLE RESONANCE

The multiple resonance in case II, that is, that of two resonant triads, will be considered and we assume that the amplitude a's are the functions of t_1 only. Then the interaction condition is written as

$$k_1 = k_2 + k_3, \quad \sigma_1 = \sigma_2 + \sigma_3, \tag{44a}$$

and

$$k_4 = k_5 + k_3, \quad \sigma_4 = \sigma_5 + \sigma_3. \tag{44b}$$

Assuming that ψ_0 has a form

$$\psi_0 = a_1 e_1 f_1 + a_2 e_2 f_2 + a_3 e_3 f_3 + a_4 e_4 f_4 + a_5 e_5 f_5 + \text{c.c.} \tag{45}$$

instead of that in (34), we obtain the interaction equations

$$\frac{da_1}{dt_1} = iQ_1 a_2 a_3, \tag{46a}$$

$$\frac{da_2}{dt_1} = iQ_2 a_1 a_3^*, \tag{46b}$$

$$\frac{da_3}{dt_1} = iQ_3 a_1 a_2^* + iQ_6 a_4 a_5^*, \tag{46c}$$

$$\frac{da_4}{dt_1} = iQ_4 a_5 a_3, \tag{46d}$$

$$\frac{da_5}{dt_1} = iQ_5 a_4 a_3^*. \tag{46e}$$

Equation (46c) implies that this equation is given by adding the right-hand sides of the two equations for individual triads.

From (46), we obtain

$$\frac{d|a_1|^2}{dt_1} = K_1 \frac{d|a_2|^2}{dt_1}, \tag{47a}$$

$$\frac{d|a_3|^2}{dt_1} = K_2 \frac{d|a_1|^2}{dt_1} + K_3 \frac{d|a_4|^2}{dt_1}, \tag{47b}$$

$$\frac{d|a_4|^2}{dt_1} = K_4 \frac{d|a_5|^2}{dt_1}, \tag{47c}$$

where

$$K_1 = -\frac{Q_2}{Q_1}, \quad K_2 = -\frac{Q_3}{Q_1}, \quad K_3 = -\frac{Q_6}{Q_4}, \quad K_4 = -\frac{Q_5}{Q_4}.$$

The following are easily verified from (47). When all of K_1, K_2, K_3, and K_4 are positive, each triad is the explosive type. In the case of

$$\frac{d|a_3|^2}{dt_1} > K_2 \frac{d|a_1|^2}{dt_1}, \quad \frac{d|a_3|^2}{dt_1} > K_3 \frac{d|a_4|^2}{dt_1}, \tag{48}$$

all of $d|a_j|^2/dt_1$ ($j = 1,2,...,5$) are positive or negative and thus the multiple resonance itself is the explosive type; otherwise, $d|a_1|^2/dt_1$ and $d|a_2|^2/dt_1$ have a sign different from that of $d|a_4|^2/dt_1$ and $d|a_5|^2/dt_1$, and the resonance becomes the decay type as a whole. In the latter case, it is obvious that some energy transfer takes place from one triad to the other. The condition of (48) in general is dependent on time and then the term $d|a_3|^2/dt_1$ will possibly change signs. Therefore, the multiple resonance can be switched from the explosive type to the decay type by time or vice versa.

On the other hand, when only K_3 is negative and all the other coefficients in (47) are positive, the triad of the k_1, k_2 and k_3 wave is the explosive type and the triad of the k_4, k_5, and k_3 wave is the decay type. In the case of

$$\left| K_2 \frac{d|a_1|^2}{dt_1} \right| > \left| K_3 \frac{d|a_4|^2}{dt_1} \right|, \tag{49}$$

all the amplitudes of the five waves increase and explosive instability occurs, otherwise decay instability occurs. It is shown that this type of multiple resonance can also be switched from the discussion presented above. In the case of multiple resonance, that which consists of two decay type triads cannot be explosive.

The multiple resonance of more than three triads can be investigated in the same manner. Actually, the resonant interaction presented here is the multiple resonance which consists of an infinite number of triads if we include higher mode waves, and the energy exchange may take place among those triads.

IX. CONCLUSIONS

Resonant triads of internal waves of the same mode exist in a stratified shear flow described by (25). In each triad, one wave must belong to a family different from that of the other two as shown in Table I. Also, one wave is traveling in the direction opposite that of the others.

Some multiple resonances are also found to exist, and interaction equations for the multiple resonances are derived easily from ordinary interaction equations for each triad. Either the explosive or the decay instabilities can occur in the multiple resonances, which consist of only the explosive type triads or those of both the explosive and the decay type triads. But in the multiple resonances consisting of the decay type triad only, the explosive instability cannot take place.

Energy transfers are predicted to exist among the triads through the multiple resonances.

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