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Green's function for a generalized two-dimensional fluid

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A Green's function for a generalized two-dimensional (2D) fluid in an unbounded domain (the so-called α turbulence system) is discussed. The generalized 2D fluid is characterized by a relationship between an advected quantity q and the stream function ψ: namely, $q=-(-\Delta)^{\alpha/2}\psi$. Here, α is a real number and q is referred to as the vorticity. In this study, the Green's function refers to the stream function produced by a delta-functional distribution of q, i.e., a point vortex with unit strength. The Green's function has the form $G^{(\alpha)}(\boldsymbol{r})\propto r^{\alpha-2}$, except when α is an even number, where r is the distance from the point vortex. This functional form is known as the Riesz potential. When α is a positive even number, the logarithmic correction to the Riesz potential has the form $G(\boldsymbol{r})^{(\alpha)}\propto r^{\alpha-2}\ln r$. In contrast, when α is a negative even number, $G^{(\alpha)}$ is given by the higher-order Laplacian of the delta function. The transition of the small-scale behavior of q at $\alpha=2$, a well-known property of forced and dissipative α turbulence, is explained in terms of the Green's function. Moreover, the azimuthal velocity around the point vortex is derived from the Green's function. The functional form of the azimuthal velocity indicates that physically realizable systems for the generalized 2D fluid exist only when $\alpha\leq 3$. The Green's function and physically realizable systems for an anisotropic generalized 2D fluid are presented as an application of the present study.

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I. INTRODUCTION

It is known that the governing equations for some geophysical two-dimensional (2D) fluids can be expressed by a unified form of the nonlinear advection equation,

$$\frac{\partial q}{\partial t}+J(\psi,q)=\mathcal{D}+\mathcal{F}, \tag{1a}$$

$$\hat{\psi}(\boldsymbol{k},t)=-|\boldsymbol{k}|^{-\alpha}\hat{q}(\boldsymbol{k},t), \tag{1b}$$

where $\psi(\boldsymbol{r},t)$ is the stream function, $q(\boldsymbol{r},t)$ is a scalar field advected by the velocity field $\boldsymbol{v}=\boldsymbol{e}_z\times\boldsymbol{\nabla}\psi$, $\boldsymbol{e}_z$ is a unit vector normal to the plane of motion, J is the 2D Jacobian, $\boldsymbol{k}=(k_x,k_y)$ is a 2D wave vector, α is a real number, and $\mathcal{D}$ and $\mathcal{F}$ represent dissipation and forcing terms, respectively. $\hat{q}(\boldsymbol{k},t)$ and $\hat{\psi}(\boldsymbol{k},t)$ are the Fourier transforms of q and ψ, respectively [1]. This system is referred to as a generalized 2D fluid system or α turbulence system. The governing equation (1) reduces to the vorticity equation for a 2D incompressible barotropic fluid [the so-called 2D Navier-Stokes (NS) system] for $\alpha=2$, and to the governing equation for the surface quasigeostrophic (SQG) system (describing the advection of temperature along a surface bounding a constant quasigeostrophic potential vorticity interior [2]) for $\alpha=1$. The Charney-Hasegawa-Mima (CHM) equation in the asymptotic model (AM) regime [3] and the shallow water quasigeostrophic potential vorticity equation at scales that are large compared with the radius of deformation [4] correspond to Eq. (1) in the case $\alpha=-2$.

In a manner similar to the 2D NS system, Eq. (1) has two quadratic inviscid invariants,

$$\mathcal{E}_\alpha\equiv-\frac{1}{2}\langle\psi q\rangle, \tag{2}$$

$$\mathcal{Q}_\alpha\equiv\frac{1}{2}\langle q^2\rangle, \tag{3}$$

where the angle brackets denote a spatial average. We will simply refer to q, $\mathcal{E}_\alpha$, and $\mathcal{Q}_\alpha$ as vorticity, energy, and enstrophy, respectively, although they have units $[q]=L^{2-\alpha}T^{-1}$, $[\mathcal{E}_\alpha]=L^{4-\alpha}T^{-2}$, and $[\mathcal{Q}_\alpha]=L^{4-2\alpha}T^{-2}$, where L and T are units of length and time, respectively.

Because the two inviscid invariants exist, one can infer that cascade phenomena of both invariants in wave number space are possible in turbulence governed by Eq. (1), for any value of α. Hence, the turbulent properties of a generalized 2D fluid have already been actively investigated by a number of researchers [1,5–12]. It is known that the enstrophy spectrum $Q_\alpha(k)$, defined by $\mathcal{Q}_\alpha\equiv\int_0^\infty Q_\alpha(k)dk$, exhibits some fascinating features in the enstrophy inertial range of forced and dissipative turbulence, governed by Eq. (1). For $0<\alpha<2$, the enstrophy spectrum in the enstrophy inertial range obeys the scaling form

$$Q_\alpha(k)\sim k^{-(7-2\alpha)/3}, \tag{4}$$

but takes the form

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$$Q_\alpha(k) \sim k^{-1} \tag{5}$$

for $\alpha > 2$ [1,5,7,9]. It is also known that the functional form of the spectrum (5) is characteristic for a passive scalar. These facts indicate that $\alpha=2$ is a transition point in the behavior of the advected quantity q, in terms of it being active or passive on a small scale. Furthermore, such a transition of the spectral slope is responsible for the dominance of enstrophy transfer by nonlocal triads, compared with the transfer by local triads that occurs in wave number space [1,7,9,12].

The relationship between the vorticity q and the stream function ψ, Eq. (1b), characterizes a generalized 2D fluid. When $\alpha > 0$, for example, the stream function (or, equivalently, the velocity) is a smoothed form of q. As α grows larger, the smoothing effect is enhanced, the velocity becomes more decoupled from the small-scale structure of the vorticity, and the problem becomes more spectrally nonlocal.

To discuss the coupling between the vorticity and velocity fields quantitatively, a study of the Green's function for Eq. (1b) (hereafter referred to as the Green's function for a generalized 2D fluid) will be quite helpful, because the Green's function includes fundamental properties of the flow field generated by a velocity source (the vorticity q). However, to date, the Green's function for a generalized 2D fluid has only been investigated for the cases $\alpha=1$ and 2. In the present research, we therefore undertake a more comprehensive examination of the Green's function for a generalized 2D fluid.

Because Eq. (1b) can be rewritten as

$$(-\Delta)^{\alpha/2}\psi = -q, \tag{6}$$

the Green's function for a generalized 2D fluid is the Green's function for the fractional Laplacian $(-\Delta)^{\alpha/2}$ in 2D space. The Green's function for the fractional Laplacian of any order has been already known mathematically. However, mathematically rigorous derivation of the Green's function for the fractional Laplacian of any order requires knowledge of both the functional analysis and the generalized functions. Accordingly, for the purposes of keeping the present paper reasonably self-contained and the readers' convenience, we present a pedagogical derivation of the Green's function of the fractional Laplacian in 2D space for all values of α.

Another objective of the present study is to determine whether physically realizable 2D fluids exist for all values of α. Although Eq. (1) was originally proposed as a generalization of the 2D NS system, physically realizable fluid systems are known to exist for $\alpha=1$ and -2. This naturally raises the question of whether there are 2D fluid systems the governing equations of which can be expressed by Eq. (1) for values other than $\alpha=2$, 1, and -2, or whether physically realizable 2D fluid systems exist for all values of α. Because this problem has not yet been examined, we attempt a solution using the Green's function obtained in the present research.

This paper is organized as follows. In Sec. II, we present a formulation of the problem. The Green's function for a generalized 2D fluid is derived in Sec. III. The velocity field calculated from the Green's function is described in Sec. IV. We discuss the transition of the small-scale behavior of q from active to passive in terms of the Green's function, and the existence of physically realizable 2D fluids is investigated in Sec. V. The transition of the small-scale behavior of q at $\alpha=2$ has been explained in terms of the coupling coefficients of the triad interactions in the spectral form of the governing equation [12]. However, we present another explanation of such transition using the Green's function. A system similar to that of Eq. (1), but anisotropic in nature, exists in geophysical contexts [13,14]. We shall refer to such a system as an anisotropic generalized 2D fluid. As an application of the present study, the Green's function and the existence of a physically realizable system for an anisotropic generalized 2D fluid are also considered in Sec. V. We summarize our results in Sec. VI.

II. FORMULATION

In this study, we consider fluid motion in an unbounded domain. Hereafter, the time argument is omitted for the sake of brevity. We define the 2D Fourier transform of a quantity $A(\boldsymbol{r})$ as

$$\hat{A}(\boldsymbol{k}) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy A(\boldsymbol{r}) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{7}$$

Likewise, the inverse Fourier transform of $\hat{A}$ is defined by

$$A(\boldsymbol{r}) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \hat{A}(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}}. \tag{8}$$

The Green's function is the stream function produced by a point vortex with unit strength. Without loss of generality, we place the point vortex at the origin of coordinates,

$$q(\boldsymbol{r}) = \delta(\boldsymbol{r}) = \delta(x)\delta(y). \tag{9}$$

That is, we calculate the Green's function $G^{(\alpha)}(\boldsymbol{r};0)$ in standard notation. Hereafter, we simply write $G^{(\alpha)}(\boldsymbol{r};0)$ as $G^{(\alpha)}(\boldsymbol{r})$. Thus the formal expression of the Green's function is

$$G^{(\alpha)}(\boldsymbol{r}) = -(-\Delta)^{-\alpha/2}\delta(\boldsymbol{r}). \tag{10}$$

The Green's function for the point vortex located at $\boldsymbol{r}_0$, $G^{(\alpha)}(\boldsymbol{r};\boldsymbol{r}_0)$, is obtained from $G^{(\alpha)}(\boldsymbol{r})$ by substituting the argument $\boldsymbol{r}-\boldsymbol{r}_0$ in place of $\boldsymbol{r}$. Because the Fourier transform of Eq. (9) is $\hat{q}(\boldsymbol{k}) = \frac{1}{2\pi}$, the Fourier transform of Eq. (10) is

$$\hat{G}^{(\alpha)}(\boldsymbol{k}) = -\frac{1}{2\pi}\frac{1}{|\boldsymbol{k}|^\alpha}. \tag{11}$$

Thus, from Eqs. (8) and (11), the Green's function for the present problem is expressed by

$$G^{(\alpha)}(\mathbf{r}) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{|\boldsymbol{k}|^\alpha}. \tag{12}$$

Because $\hat{G}^{(\alpha)}(\boldsymbol{k})$ satisfies the relationship $\hat{G}^{(\alpha)}(\boldsymbol{k})^* = \hat{G}^{(\alpha)}(-\boldsymbol{k})$, where the asterisk denotes the complex conjugate, $G^{(\alpha)}(\boldsymbol{r})$ is a real function. Introducing polar coordinates in the physical and Fourier spaces, Eq. (12) yields

$$G^{(\alpha)}(\boldsymbol{r}) = -\frac{1}{8\pi^2}\int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dk \frac{e^{ikr\cos(\theta-\phi)}}{|k|^{\alpha-1}}, \quad (13)$$

where $r \equiv |\boldsymbol{r}|$, $k \equiv |\boldsymbol{k}|$, and ϕ and θ denote the azimuthal angles in the physical and Fourier spaces, respectively. The Green's function for the present version of the problem is isotropic in the physical space, because the point vortex is placed at the origin of coordinates. Therefore, without loss of generality, we can set $\phi=0$ in Eq. (13). Moreover, letting $z=\cos\theta$, and using the fact that the integrand is an even function with respect to z, Eq. (13) becomes

$$G^{(\alpha)}(\boldsymbol{r}) = -\frac{1}{2\pi^2}\int_0^1 dz \frac{1}{\sqrt{1-z^2}}\left\{\int_{-\infty}^{\infty} dk \frac{e^{ikrz}}{|k|^{\alpha-1}}\right\}. \quad (14)$$

In the next section, we explicitly carry out the integrations in Eq. (14).

Once the Green's function has been obtained, the stream function and velocity field are calculated from

$$\psi(\boldsymbol{r}) = \int G^{(\alpha)}(\boldsymbol{r};\boldsymbol{r}_0) q(\boldsymbol{r}_0) d\boldsymbol{r}_0, \quad (15)$$

$$\boldsymbol{v}(\boldsymbol{r}) = \int \boldsymbol{e}_z \times \boldsymbol{\nabla} G^{(\alpha)}(\boldsymbol{r};\boldsymbol{r}_0) q(\boldsymbol{r}_0) d\boldsymbol{r}_0 = -\int \boldsymbol{e}_z \times \boldsymbol{\nabla}_0 G^{(\alpha)}(\boldsymbol{r};\boldsymbol{r}_0) q(\boldsymbol{r}_0) d\boldsymbol{r}_0, \quad (16)$$

respectively. Here, $\boldsymbol{\nabla}_0$ indicates the gradient with respect to the variable $\boldsymbol{r}_0$. The last expression of Eq. (16) is obtained from the fact that the Green's function $G^{(\alpha)}(\boldsymbol{r};\boldsymbol{r}_0)$ is a function of $|\boldsymbol{r}-\boldsymbol{r}_0|$. The azimuthal velocity around the point vortex can be calculated from

$$v_\phi^{(\alpha)}(\boldsymbol{r}) = \frac{\partial G^{(\alpha)}(\boldsymbol{r})}{\partial r}. \quad (17)$$

III. GREEN'S FUNCTIONS

In this section, the derivation of the Green's function for a generalized 2D fluid is outlined. In Eq. (14), the integral with respect to k is the Fourier transform of an algebraic function, while the integral with respect to z can be expressed in terms of the gamma function. Gamma function formulas necessary for the present study are listed in the Appendix A. The Fourier transform of an algebraic function can be classified according to three types, depending on the value of the exponent of the algebraic function, as summarized in Appendix B. In what follows, let n be a non-negative integer, and let m be a natural number. The results of integrating Eq. (14) are then classified into the following three cases: excluding the values $\alpha = \pm 2n$,

$$G^{(\alpha)}(\boldsymbol{r}) = \Psi(\alpha) r^{\alpha-2}, \quad (18)$$

for $\alpha = 2m$,

$$G^{(2m)}(\boldsymbol{r}) = \Psi_{\mathrm{L}}(2m) r^{2m-2}(\ln r + C), \quad (19)$$

where C is an arbitrary constant, and for $\alpha=-2n$,

$$G^{(-2n)}(\boldsymbol{r}) = -(-\Delta)^n \delta(\boldsymbol{r}). \quad (20)$$

The coefficients $\Psi(\alpha)$ and $\Psi_{\mathrm{L}}(2m)$ can be written in terms of the gamma function, as follows:

$$\Psi(\alpha) = -\frac{1}{2^\alpha \pi} \frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}, \quad (21)$$

$$\Psi_{\mathrm{L}}(2m) = \frac{(-1)^{m+1}}{2^{2m-1}\pi\{\Gamma(m)\}^2}. \quad (22)$$

Detailed derivations of Eqs. (18) and (19) are presented in Appendixes C and D, respectively. The functional form of Eq. (18) was originally derived by Riesz [15], and is frequently referred to as the Riesz potential [16,17].

In practical calculations involving Eq. (21), it is advantageous to rewrite the expression in terms of the gamma function with a positive argument. Negative arguments occur in Eq. (21) when $\alpha<0$ or $\alpha>2$. Using the formula (A2) and (21) can be rewritten as

$$\Psi(\alpha) = -\frac{1}{2^\alpha \pi^2}\left\{\Gamma\left(\frac{2-\alpha}{2}\right)\right\}^2 \sin\frac{\alpha\pi}{2} \quad (23)$$

or

$$\Psi(\alpha) = -\frac{1}{2^\alpha \left\{\Gamma\left(\frac{\alpha}{2}\right)\right\}^2 \sin\frac{\alpha\pi}{2}}. \quad (24)$$

Equations (23) and (24) are useful expressions for $\Psi(\alpha)$ when $\alpha \leq 0$ and $\alpha > 2$, respectively.

We confirm the validity of the above results by calculating existing Green's functions. For $\alpha=2$, Eq. (19) coupled with Eq. (22) reduces to

$$G^{(2)}(\boldsymbol{r}) = \frac{1}{2\pi}\ln r + C. \quad (25)$$

This is the well-known Green's function for the 2D Laplace-Poisson equation (the 2D NS system). On the other hand, when $\alpha=1$, Eqs. (18) and (21) give

$$G^{(1)}(\boldsymbol{r}) = -\frac{1}{2\pi r}. \quad (26)$$

This is consistent with the Green's function for the SQG system discussed in [2].

We note in passing that for $\alpha=0$, the stream function is proportional to the vorticity. In this case, the nonlinear term in the governing Eq. (1a) vanishes. Thus, this case is meaningless.

Let us explicitly examine the dependence of $\Psi(\alpha)$ and $\Psi_{\mathrm{L}}(\alpha)$ on α. Figure 1 is a plot of $\Psi(\alpha)$ over the interval $-3 \leq \alpha \leq 7$. Numerical values of the gamma function are computed using the gammln subroutine in [18]. As this figure shows, $\Psi(1) = -\frac{1}{2\pi}$, while $\Psi(-1) = \Psi(3) = \frac{1}{2\pi}$. Moreover, $\Psi(\alpha)$ diverges as $\alpha \to 2m$, as Eq. (24) indicates. For $\alpha>4$,

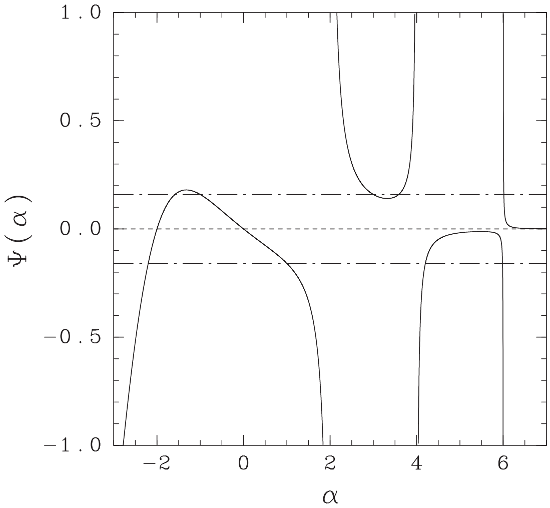

FIG. 1. Coefficient $\Psi(\alpha)$ as a function of α. The dash-dot lines indicate the reference lines with the values of $\pm\frac{1}{2\pi}$.

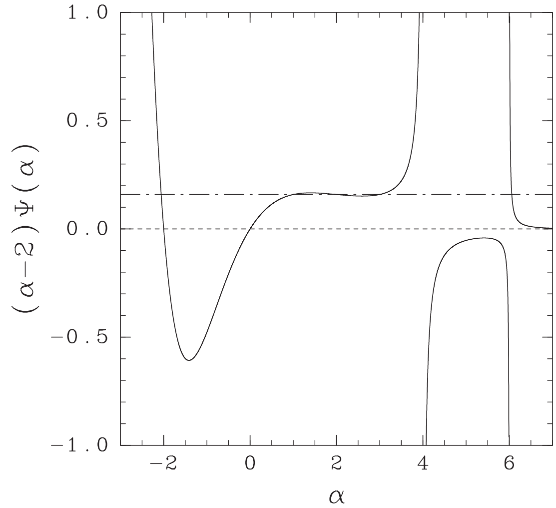

FIG. 3. Dependence of $(\alpha-2)\Psi(\alpha)$ on α. The dash-dot line indicates the reference line with the value of $\frac{1}{2\pi}$.

except in the vicinity of $\alpha=2m+2$, $\Psi(\alpha)$ is quite small [for example, $\Psi(5)=\frac{1}{18\pi}$, $\Psi(7)=\frac{1}{450\pi}$]. In contrast, $\Psi(\alpha)$ converges to zero as $\alpha\to-2n$. At the values of α where $\Psi(\alpha)$ diverges, both algebraic and logarithmic dependence on r appear in the Green's function. On the other hand, at the values of α where $\Psi(\alpha)$ converges to zero, the Green's function is given by the higher-order Laplacian of the delta function. In this sense, the functional form of the Green's function is discontinuous at $\alpha=\pm 2n$. Figure 2 is a plot of $\Psi_{\rm L}(\alpha)$, given by Eq. (22). $\Psi_{\rm L}(\alpha)$ has a maximum value at $\alpha=2$, and its magnitude decreases uniformly toward zero as α increases.

IV. AZIMUTHAL VELOCITY AROUND A POINT VORTEX

In this section, we calculate the azimuthal velocity $v_\phi^{(\alpha)}$ around a point vortex located at the origin of coordinates $\boldsymbol{r}=0$. For $\alpha\neq\pm 2n$, Eqs. (17) and (18) give

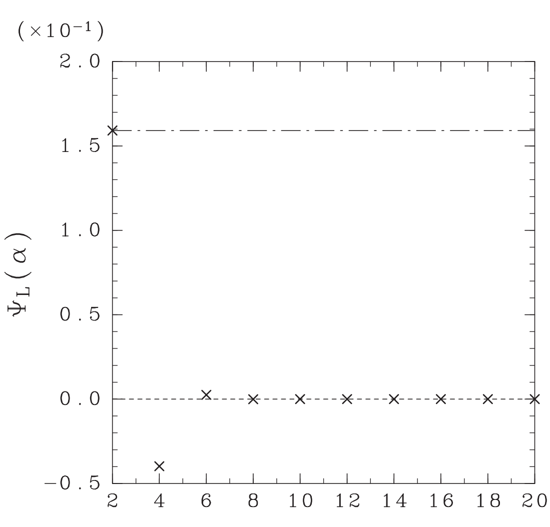

FIG. 2. Coefficient $\Psi_{\rm L}(\alpha)$ as a function of α. The dash-dot line indicates the reference line with the value of $\frac{1}{2\pi}$.

$$v_\phi^{(\alpha)} = (\alpha-2)\Psi(\alpha)r^{\alpha-3}. \tag{27}$$

For $\alpha=1$ and 3, the coefficient $(\alpha-2)\Psi(\alpha)$ takes the value $\frac{1}{2\pi}$. Note that one can write

$$(\alpha-2)\Psi(\alpha) = \frac{1}{2^{\alpha-1}\pi}\frac{\Gamma\left(\dfrac{2-\alpha}{2}+1\right)}{\Gamma\left(\dfrac{\alpha}{2}\right)} \tag{28}$$

for $0<\alpha<2$, with the aid of Eqs. (21) and (A1).

On the other hand, for $\alpha=2m$, Eqs. (17) and (19) give

$$v_\phi^{(2m)} = \Psi_{\rm L}(2m)r^{2m-3} + (2m-2)\Psi_{\rm L}(2m)r^{2m-3}\{\ln r + C\}. \tag{29}$$

For $\alpha=2$, Eq. (29) reduces to $v_\phi^{(2)}=\frac{1}{2\pi r}$. Although Eq. (27) coupled with Eq. (28) cannot be applied when $\alpha=2$, the case reduces to $v_\phi^{(2)}=\frac{1}{2\pi r}$, which is consistent with Eq. (29) with $\alpha=2$. Thus, the velocity given by Eq. (27) with Eq. (28) is valid even for $\alpha=2$.

Figure 3 is a plot of the coefficient $(\alpha-2)\Psi(\alpha)$ over the interval $-3\leq\alpha\leq 7$. As stated above, $(\alpha-2)\Psi(\alpha)$ is exactly equal to $\frac{1}{2\pi}$ for $\alpha=1,2,3$, while it is approximately equal to $\frac{1}{2\pi}$ for $1\lesssim\alpha\lesssim 3$. As $\alpha\to 2m+2$, $(\alpha-2)\Psi(\alpha)$ tends to diverge. In contrast, as $\alpha\to-2n$, $(\alpha-2)\Psi(\alpha)$ converges to zero.

V. DISCUSSION

First, we discuss the transition of the small-scale behavior of the vorticity q at $\alpha=2$ in terms of the Green's function obtained in this study. As stated in Sec. I, the slope of the enstrophy spectrum in the enstrophy inertial range of forced and dissipated generalized 2D turbulence is discontinuous at $\alpha=2$. For $\alpha<2$, the slope of the enstrophy spectrum depends on α, whereas it remains constant for $\alpha>2$. For $\alpha>2$, the enstrophy spectrum in the enstrophy inertial range has characteristic of the variance spectrum of a passive scalar. This indicates that for $\alpha > 2$, the advected quantity q behaves as if it were a passive quantity, on the small scale. The transition of small-scale behavior of q at $\alpha=2$ was explained in terms of the magnitude of the coupling coefficients in the spectral form of the governing equation by Tran *et al.* [12]. Their explanation is independent of the amplitude of the three modal members. Moreover, the coupling coefficients of the triad interaction have the same dimension with the Green's function in the present study. These facts imply that the transition of small-scale behavior of q at $\alpha=2$ can be explained in terms of the Green's function. We introduce a decomposition of the velocity field by separating the domain of integration in Eq. (16) into a local region of radius $|\boldsymbol{r}_0-\boldsymbol{r}|\leq R$ and a nonlocal region that accounts for the remainder of the domain [19–21]. That is, we divide the velocity field as follows:

$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{R}} + \boldsymbol{v}_{\mathrm{B}}, \tag{30}$$

$$\boldsymbol{v}_{\mathrm{R}}(\boldsymbol{r}) \equiv -\int_{|\boldsymbol{r}_0-\boldsymbol{r}|\leq R} \boldsymbol{e}_z \times \boldsymbol{\nabla}_0 G^{(\alpha)}(\boldsymbol{r};\boldsymbol{r}_0) q(\boldsymbol{r}_0) d\boldsymbol{r}_0, \tag{31}$$

$$\boldsymbol{v}_{\mathrm{B}}(\boldsymbol{r}) \equiv -\int_{|\boldsymbol{r}_0-\boldsymbol{r}|> R} \boldsymbol{e}_z \times \boldsymbol{\nabla}_0 G^{(\alpha)}(\boldsymbol{r};\boldsymbol{r}_0) q(\boldsymbol{r}_0) d\boldsymbol{r}_0. \tag{32}$$

Here, R is set according to the maximum scale of the enstrophy inertial range (if the smallest wave number of the enstrophy inertial range is $k_{\min}$, we let $R \sim k_{\min}^{-1}$). Then, $\boldsymbol{v}_{\mathrm{R}}(\boldsymbol{r})$ is the velocity induced by q within a scale less than or equal to the enstrophy inertial range, centered at $\boldsymbol{r}$, while $\boldsymbol{v}_{\mathrm{B}}(\boldsymbol{r})$ is produced by q at values of $\boldsymbol{r}$ far beyond the enstrophy inertial range. If $\boldsymbol{v}_{\mathrm{B}}$ is dominant over $\boldsymbol{v}_{\mathrm{R}}$, the behavior of the advection of q in the enstrophy inertial range can be considered passive. We present a qualitative explanation of the α dependence of the predominance of $\boldsymbol{v}_{\mathrm{R}}$ over $\boldsymbol{v}_{\mathrm{B}}$ (or vice versa) in terms of the Green's function. We can assume that the vorticity q is statistically homogeneous and isotropic in the enstrophy inertial range, $|\boldsymbol{r}-\boldsymbol{r}_0|\leq R$. Thus, in a statistical sense, the integrand of Eq. (16) is a function of $r_0 \equiv |\boldsymbol{r}_0|$. We can then make use of the estimate

$$\boldsymbol{v}_{\mathrm{R}} \sim \int_{r_0\leq R} \boldsymbol{e}_z \times \boldsymbol{\nabla}_0 G^{(\alpha)}(r_0) q(r_0) r_0 dr_0. \tag{33}$$

Equation (33) indicates that the velocity $\boldsymbol{v}_{\mathrm{R}}$ is given by a weighted integral of the vorticity q, with the weight function $r_0\boldsymbol{e}_z \times \boldsymbol{\nabla}_0 G^{(\alpha)}(r_0)$. Moreover, the weight function can be estimated by

$$|r_0\boldsymbol{e}_z \times \boldsymbol{\nabla}_0 G^{(\alpha)}(r_0)| \sim r_0^{\alpha-2}, \tag{34}$$

except when $\alpha=\pm 2n$. Equation (33), coupled with Eq. (34), states that the weighted integral of q gives more weight to small scales when $\alpha < 2$, and to large scales when $\alpha > 2$. This indicates that $\boldsymbol{v}_{\mathrm{R}}$ is the dominant part of $\boldsymbol{v}$ when $\alpha < 2$, and $\boldsymbol{v}_{\mathrm{B}}$ is the dominant part of $\boldsymbol{v}$ when $\alpha > 2$. Thus, we conclude that the small-scale behavior of q is active for $\alpha < 2$ and passive for $\alpha > 2$, and the transition of such behavior occurs at $\alpha=2$.

We next discuss the existence of physically realizable 2D fluid systems, using the azimuthal velocity around a point vortex that was derived from the Green's function. As stated in the Sec. I, it is known that the governing equations of three systems (the 2D NS, SQG, and CHM-AM systems) can be expressed by Eq. (1). This raises the question of whether there are other 2D fluid systems the governing equations of which can be expressed by Eq. (1). Equation (27) indicates that the azimuthal velocity around a point vortex $v_\phi^{(\alpha)}$ is a monotonically increasing function of r for $\alpha > 3$; i.e., when $\alpha > 3$, the azimuthal velocity around a velocity source increases with the distance from the source. This is physically incongruous because it is natural to expect that the velocity will decrease (or remain constant at the worst) as the distance from the source increases. Thus, we conclude that physically realizable 2D fluid systems exist only for $\alpha \leq 3$.

Finally, as an application of the present results, we discuss the Green's function and the existence of a physically realizable system for the following:

$$\frac{\partial q}{\partial t} + J\left(\frac{\partial \varphi}{\partial x}, q\right) = \mathcal{D} + \mathcal{F}, \tag{35a}$$

$$\hat{q}(\boldsymbol{k},t) = -|\boldsymbol{k}|^{\alpha}\hat{\varphi}(\boldsymbol{k},t), \tag{35b}$$

The above system is regarded as an anisotropic extension of Eq. (1). It is known that the governing equations for 2D thermal convection of a Boussinesq fluid at an infinite Prandtl number and 2D low-frequency plasma dynamics of the E and F regions of the terrestrial ionosphere can be reduced to Eq. (35). For example, the systems studied by Gruzinov *et al.* [14] and Weinstein *et al.* [13] correspond to Eq. (35) with $\alpha=2$ and 4, respectively. We note, in passing, that the system studied by Weinstein *et al.* [13] has sometimes been misread as Eq. (1) with $\alpha=3$ [5,8]. Because anisotropy is crucial for the system studied by Weinstein *et al.* [13], one cannot interpret it as a special case of Eq. (1), which represents an isotropic system. The y direction in Eq. (35) is the direction of the temperature gradients or the electron density of the basic state. Although boundaries should exist in the y direction, we now consider Eq. (35) in an unbounded domain. Using the results of the present study, we can then calculate the Green's function for Eq. (35). The quantity φ corresponding to the delta-functional distribution of q is just the Green's function derived in Sec. III. Thus, the Green's function for Eq. (35), $G_{\mathrm{AI}}^{(\alpha)}$, is given by the partial derivative with respect to x of the Green's function for Eq. (1); i.e., $G_{\mathrm{AI}}^{(\alpha)} = \frac{\partial G^{(\alpha)}}{\partial x}$. Except when $\alpha=\pm 2n$, the Green's function for Eq. (35) is then given by

$$G_{\mathrm{AI}}^{(\alpha)}(\boldsymbol{r}) = (\alpha-2)\Psi(\alpha) x r^{\alpha-4}. \tag{36}$$

Moreover, we obtain the x and y components of the velocity associated with Eq. (36) from

$$v_x = -\frac{\partial G_{\mathrm{AI}}^{(\alpha)}}{\partial y} = -(\alpha-2)(\alpha-4)\Psi(\alpha) x y r^{\alpha-6}, \tag{37a}$$

$$v_y = \frac{\partial G_{\mathrm{AI}}^{(\alpha)}}{\partial x} = (\alpha-2)\Psi(\alpha)r^{\alpha-4}\left\{1+(\alpha-4)\frac{x^2}{r^2}\right\}, \quad (37b)$$

respectively. Equation (37b) shows that v_y is an increasing function of r for $\alpha>4$. Thus, we conclude that a physically realizable system for Eq. (35) should have $\alpha\leq 4$, because the velocity v_y should not be a monotonically increasing function of the distance from the velocity source.

We interpret the above result in terms of the governing equations for 2D thermal convection of a Boussinesq fluid, consisting of the thermodynamical equation governing the temperature [which has the same form as Eq. (1a), where q is interpreted as the temperature] and the vorticity equation

$$\frac{\partial}{\partial t}\Delta\psi + J(\psi,\Delta\psi) = \tilde{\alpha}g\frac{\partial q}{\partial x} - \nu_p(-\Delta)^p(\Delta\psi), \quad (38)$$

where $\Delta\psi$ is the vorticity in the ordinary sense, $\tilde{\alpha}$ is the thermal expansion rate, g is the acceleration due to gravity, p is the degree of hyperviscosity, and ν_p is the viscosity coefficient. When the inertial term of Eq. (38) is negligible (i.e., the Reynolds number is low), if we introduce the function φ, defined by $\frac{\partial\varphi}{\partial x}=\psi$, together with suitable nondimensionalization, Eqs. (1a) and (38) reduce to Eq. (35) with $\alpha=2(p+1)$. Therefore, we can interpret Eq. (35) as the governing equation for 2D thermal convection at a low Reynolds number. As stated above, the degree of hyperviscosity p relates to the parameter α; normal viscosity $(p=1)$ corresponds to $\alpha=4$, and hyperviscosity $(p>1)$ to $\alpha>4$. It is well known that hyperviscosity is often adopted in numerical calculations to realize a high Reynolds number state at low numerical resolutions. Moreover, hyperviscosity is suitable for the physical modeling of viscosity at high Reynolds numbers. Thus, Eq. (35a) for $\alpha>4$, which describes the dynamics of 2D thermal convection at a low Reynolds number by using hyperviscosity, is physically unreasonable.

VI. SUMMARY

We have discussed the Green's functions and the associated velocity fields for both isotropic and anisotropic generalized 2D fluids in an unbounded domain. The functional form of the Green's function for an isotropic generalized 2D fluid depends on the value of α. As long as $\alpha\neq\pm 2n$, where n is non-negative integer, the Green's function is given by the Riesz potential, which is an algebraic function of the distance from the point vortex, with an exponent depending on α. For $\alpha=2m$, where m is a natural number, the logarithmic correction to the Riesz potential appears. When $\alpha=-2n$, the Green's function is given by the nth Laplacian of the delta function. As a function of α, the Green's function is discontinuous at $\alpha=\pm 2n$. In contrast, the azimuthal velocity around the point vortex is a continuous function of α at $\alpha=2$, corresponding to the 2D NS system. Using the functional form of the Green's function, we presented a qualitative discussion of the transition of the small-scale behavior of q at $\alpha=2$, which is a well-known property of isotropic generalized 2D turbulence in the enstrophy inertial range. We also discussed the existence of physically realizable systems for isotropic and anisotropic generalized 2D fluids.

In this study, we have discussed the Green's function for an isotropic generalized 2D fluid in an unbounded domain. In numerical simulations of an isotropic generalized 2D fluid, doubly periodic boundary conditions are commonly adopted [1,5,7,9,11]. Under such boundary conditions, the Green's function remains to be derived. Therefore, the derivation of the Green's function for a generalized 2D fluid in doubly periodic boundary conditions is one of the subjects of future studies. Once the Green's function has been determined, it should be possible to construct a numerical point vortex model of a generalized 2D fluid and to examine the motion of point vortices. These topics also await further research.

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APPENDIX A: GAMMA FUNCTION FORMULAS

(i) The difference equation satisfied by the gamma function,

$$\Gamma(x+1) = x\Gamma(x). \quad (A1)$$

(ii) Relationship between the gamma function and a trigonometric function,

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}. \quad (A2)$$

(iii) Duplication formula for the gamma function,

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2x). \quad (A3)$$

(iv) Formula for the integral of an algebraic function,

$$\int_0^1 \frac{x^{p-1}}{\sqrt{1-x^q}}dx = \frac{\sqrt{\pi}}{q}\frac{\Gamma\left(\frac{p}{q}\right)}{\Gamma\left(\frac{p}{q}+\frac{1}{2}\right)}, \quad (A4)$$

where p and q are real numbers satisfying $\frac{p}{q}>0$.

APPENDIX B: FOURIER TRANSFORM OF AN ALGEBRAIC FUNCTION

In this study, we define the inverse Fourier transform of the algebraic function $|k|^{-p}$ by

$$\mathcal{F}^{-1}[|k^{-p}|] \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|k|^{-p}e^{ikx}dk \tag{B1}$$

where p is a real number. The integral in Eq. (B1) converges for $0<p<1$. However, if one introduces generalized functions, Eq. (B1) can be evaluated for all values of p [22], and the results are summarized as follows:

$$\mathcal{F}^{-1}[|k|^{-p}] = \sqrt{\frac{2}{\pi}}\sin\left(\frac{p\pi}{2}\right)\frac{\Gamma(1-p)}{|x|^{1-p}}, \quad (p \neq -2n, 2m-1) \tag{B2a}$$

$$\mathcal{F}^{-1}[|k|^{2n}] = (-1)^n\sqrt{2\pi}\frac{d^{2n}}{dx^{2n}}\delta(x), \tag{B2b}$$

$$\mathcal{F}^{-1}[|k|^{-2m+1}] = (-1)^m\sqrt{\frac{2}{\pi}}\frac{|x|^{2m-2}}{\Gamma(2m-1)}(\ln|x|+C). \tag{B2c}$$

Here, C is an arbitrary constant, originating from the fact that $|k|^{-1}$ contains an arbitrary multiple of $\delta(k)$.

APPENDIX C: DERIVATION OF EQ. (18)

Using Eq. (A2), the right hand side of Eq. (B2a) can be rewritten as

$$\sin\left(\frac{p\pi}{2}\right)\frac{\Gamma(1-p)}{|x|^{1-p}} = \frac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{1-p}{2}\right)}{2\Gamma(p)|x|^{1-p}}. \tag{C1}$$

Thus, Eq. (14) reduces to

$$\begin{aligned}G^{(\alpha)}(\boldsymbol{r}) &= -\frac{1}{2\pi^2}\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma(\alpha-1)}\left\{\int_0^1 dz\frac{z^{\alpha-2}}{\sqrt{1-z^2}}\right\}r^{\alpha-2}\\ &= -\frac{1}{4\pi^{3/2}}\frac{\Gamma\left(\frac{\alpha-1}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma(\alpha-1)}r^{\alpha-2}\\ &= -\frac{1}{2^\alpha\pi}\frac{\Gamma\left(\frac{2-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}r^{\alpha-2}.\end{aligned} \tag{C2}$$

for $\alpha>\frac{1}{2}$ and $\alpha\neq 2m$. Integration with respect to k is performed using Eqs. (B2a) and (C1), and integration with respect to z is accomplished using Eq. (A4). The derivation of the last expression from the second one is carried out with the aid of Eq. (A3). This is just Eq. (18) combined with Eq. (21).

Note that because the integral with respect to z in the first expression is convergent for $\alpha>\frac{1}{2}$, and the first expression of Eq. (C2) is derived using Eq. (B2a), the result Eq. (C2) is valid only for $\alpha>\frac{1}{2}$ and $\alpha\neq 2m$. We derive the Green's function for $\alpha\leq\frac{1}{2}$ from the Green's function for $\alpha>\frac{1}{2}$, using the following relationship:

$$-(-\Delta)^{\alpha/2}G^{(\alpha)}(\boldsymbol{r}) = -(-\Delta)^{(\alpha-2m)/2}\{(-\Delta)^m G^{(\alpha)}(\boldsymbol{r})\} = \delta(\boldsymbol{r}), \tag{C3}$$

where $\alpha>\frac{1}{2}$. Applying the 2D Laplacian to Eq. (C2) and using Eq. (A1), one obtains

$$-\Delta G^{(\alpha)}(\boldsymbol{r}) = -(\alpha-2)^2\Psi(\alpha)r^{\alpha-4} = \Psi(\alpha-2)r^{\alpha-4}. \tag{C4}$$

The last expression in the above equation is just Eq. (18) with $\alpha-2$ in place of α (that is, $-\Delta G^{(\alpha)}=G^{(\alpha-2)}$). This indicates that by repeatedly applying $(-\Delta)$ to Eq. (C2), one obtains the Green's function for $\alpha\leq\frac{1}{2}$, except when $\alpha=-2n$, and the formula (C2) is valid for all real values of α except $\alpha=\pm 2n$. The desired result follows.

In the case $\alpha=-2n+1$, the validity of the result Eq. (C2) is proved in the following way. For $\alpha=-2n+1$, an application of Eq. (B2b) to Eq. (18) yields

$$\begin{aligned}G^{(-2n+1)}(\boldsymbol{r}) &= \frac{(-1)^{n+1}}{\pi}\int_0^1\frac{1}{\sqrt{1-z^2}}\frac{d^{2n}}{d(rz)^{2n}}\delta(rz)dz\\ &= \frac{(-1)^{n+1}}{\pi}\left\{\int_0^1\left(\frac{d^{2n}}{dz^{2n}}\frac{1}{\sqrt{1-z^2}}\right)\delta(rz)dz\right\}r^{-2n}\\ &= \frac{(-1)^{n+1}}{2\pi}\left[\frac{d^{2n}}{dz^{2n}}\frac{1}{\sqrt{1-z^2}}\right]_{z=0}r^{-2n-1}\\ &= \frac{(-1)^{n+1}2^{2n-1}}{\pi^2}\left\{\Gamma\left(\frac{2n+1}{2}\right)\right\}^2 r^{-2n-1},\end{aligned} \tag{C5}$$

where integration by parts is used in the derivation of the second expression, and Eq. (E1) in the derivation of the last expression. Note that by inserting $\alpha=-2n+1$ into Eq. (C2), and making use of Eq. (A2), one obtains Eq. (C5). Thus, the validity of Eq. (C2) is proved in the case $\alpha=-2n+1$.

APPENDIX D: DERIVATION OF EQ. (19)

For $\alpha=2m$, applying Eq. (B2c) to Eq. (14), one obtains

$$\begin{aligned}G^{(2m)}(\boldsymbol{r}) &= \frac{(-1)^{m+1}}{\pi^2\Gamma(2m-1)}\int_0^1\frac{|rz|^{2m-2}}{\sqrt{1-z^2}}(\ln|rz|+C)dz\\ &= \frac{(-1)^{m+1}}{\pi^2\Gamma(2m-1)}\left\{\left(\int_0^1\frac{z^{2m-2}}{\sqrt{1-z^2}}dz\right)(\ln r+C)\right.\\ &\quad\left.+\int_0^1\frac{z^{2m-2}\ln z}{\sqrt{1-z^2}}dz\right\}r^{2m-2}.\end{aligned} \tag{D1}$$

The first integral with respect to z in Eq. (D1) can be evaluated with the aid of Eq. (A4). In contrast, one readily finds that the second integral in Eq. (D1) has a negative value, because the integrand is negative definite over the interval of

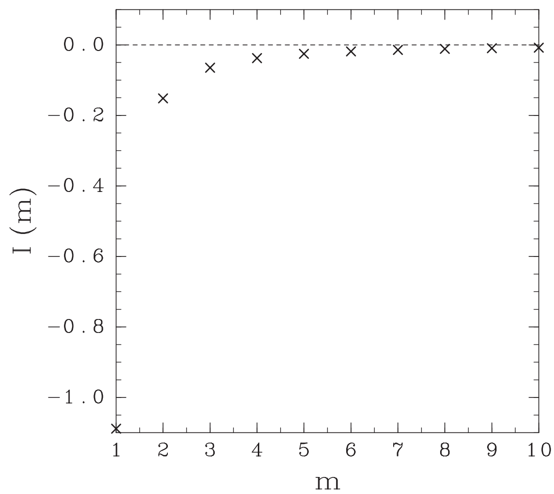

FIG. 4. $I(m)$ vs m.

integration. (See Appendix F.) However, the definite value of this integral is unimportant, because it can be combined with the arbitrary constant C. Thus, one obtains

$$G^{(2m)}(\boldsymbol{r}) = \frac{(-1)^{m+1} r^{2m-2}}{\pi^2 \Gamma(2m-1)} \times \frac{\sqrt{\pi}\Gamma\left(\frac{2m-1}{2}\right)}{2\Gamma(m)}(\ln r + C)$$
$$= \frac{(-1)^{m+1}}{2^{2m-1}\pi\{\Gamma(m)\}^2} r^{2m-2}(\ln r + C). \tag{D2}$$

Here, the duplication formula for the gamma function [Eq. (A3)] is used.

APPENDIX E: nTH DERIVATIVE OF $\frac{1}{\sqrt{1-x^2}}$

One can derive the following formula:

$$\left[\frac{d^{2n}}{dx^{2n}} \frac{1}{\sqrt{1-x^2}}\right]_{x=0} = \frac{2^{2n}\Gamma\left(\frac{2n+1}{2}\right)^2}{\pi}. \tag{E1}$$

Let $y(x) = \frac{1}{\sqrt{1-x^2}}$, and denote the nth derivative of $y(x)$ by $y^{(n)}(x)$. Then, $y^{(n)}(x)$ satisfies the following equation:

$$(1-x^2)y^{(n+2)}(x) - (2n+3)xy^{(n+1)}(x) - (n+1)y^{(n)}(x) = 0. \tag{E2}$$

Equation (E2) can be proved by mathematical induction. From Eq. (E2), one obtains the recurrence formula

$$y^{(2n)}(0) = (2n-1)^2 y^{(2n-2)}(0). \tag{E3}$$

Applying Eq. (E3) successively, one can derive Eq. (E1).

$$y^{(2n)}(0) = \{(2n-1)(2n-3)\}^2 y^{(2n-4)}(0) = \cdots \cdots$$
$$= \{(2n-1)(2n-3)\cdots(3)(1)\}^2 y^{(0)}(0)$$
$$= \left\{2^n \frac{2n-1}{2}\frac{2n-3}{2}\cdots\frac{3}{2}\frac{1}{2}\right\}^2 = \frac{2^{2n}\Gamma\left(\frac{2n+1}{2}\right)^2}{\pi}. \tag{E4}$$

APPENDIX F: ON THE INTEGRAL $I(m) = \int_0^1 \frac{x^{2m-2}\ln x}{\sqrt{1-x^2}}dx$

It is known that for $m=1$,

$$I(1) = \int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx = -\frac{\pi}{2}\ln 2. \tag{F1}$$

For other values of m, we evaluate $I(m)$ numerically via the trapezoidal rule for numerical integration, dividing the interval $0 < x < 1$ into 10^6 bins. The results are shown in Fig. 4. $I(m)$ attains a minimum value at $m=1$, and approaches zero asymptotically as m increases.

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