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The accuracy of the lattice Boltzmann method (LBM) for describing the behavior of a gas in the continuum limit is systematically investigated. The asymptotic analysis for small Knudsen numbers is carried out to derive the corresponding fluid-dynamics-type equations, and the errors of the LBM are estimated by comparing them with the correct fluid-dynamics-type equations. We discuss the following three important cases: (I) the Mach number of the flow is much smaller than the Knudsen number, (II) the Mach number is of the same order as the Knudsen number, and (III) the Mach number is finite. From the von Karman relation, the above three cases correspond to the flows of (I) small Reynolds number, (II) finite Reynolds number, and (III) large Reynolds number, respectively. The analysis is made with the information only of the fundamental properties of the lattice Boltzmann models without stepping into their detailed form. The results are therefore applicable to various lattice Boltzmann models that satisfy the fundamental properties used in the analysis.

I. INTRODUCTION

The kinetic-equation approach is frequently used as a replacement of the traditional approach to solve the fluid-dynamics-type equations for describing the behavior of a gas [1–8]. The merits of this approach are good applicability to parallel computing, simplicity of the code, linear convection term, high resolution of shock capturing, etc. However, to solve the kinetic equation, the molecular velocity space must be considered in addition to the physical space, so that the numerical cost naturally becomes larger than the traditional numerical method. In order to avoid this demerit, molecular velocities of discrete type are often used. The lattice Boltzmann method (LBM) [6–8] is one of them. In fact, this method has already developed into an alternative and promising numerical method for simulating gas flows.

The point of LBM is to properly choose the collision operator and discrete molecular velocities such that the macroscopic variables obtained from the solution of the kinetic equation can satisfy the desired set of fluid-dynamics-type equations in the limit of $\varepsilon \to 0$, where ε is the Knudsen number. Carrying out the asymptotic analysis for small ε of the kinetic equation, therefore, we obtain the desired set of fluid-dynamics-type equations at the leading order of ε, and at some higher orders, the sets with deviation terms appear. These terms contribute to the error of the LBM. The LBM is always accompanied with this type of error.

It is imperative therefore to estimate this type of error properly before we use the LBM as a numerical method for simulating flows. To this end, we must investigate the governing equations of the macroscopic variables obtained from the solution of the kinetic equation of the LBM. The so-called Chapman-Enskog expansion with respect to small ε has been frequently used to derive the set of fluid-dynamics-type equations and to estimate its error [9–11]. However, in this expansion, the correct ordering of sizes of the variables is not made beforehand, so that the magnitude of the macroscopic variables in the derived sets is not clear. Moreover, the macroscopic variables are not expanded in the analysis, so that the ordering of each term in the derived set of equations is ambiguous.

In previous studies, the systematic error estimate was made by Inamuro *et al.* [12] and Junk *et al.* [13] (see also Junk and Yang [14]). Inamuro *et al.* first made an asymptotic analysis for small ε of the kinetic equation devised by Sone [15–17] where the correct ordering of sizes of the variables is made beforehand and the macroscopic variables are expanded as well as the distribution function. Then, the ordering of each term in the derived set of equations for the macroscopic variables was clearly arranged. They made the analysis for the flow of finite Reynolds number on the basis of Qian's isothermal LBM model [18]. According to their analysis, the incompressible Navier-Stokes (NS) set of equations is properly derived at the leading-order set in ε, and the relative error of the LBM solution to that of the incompressible NS set was systematically derived to be $O(\varepsilon^2)$. Junk *et al.* later made more sophisticated analysis for the same parameter region Re~1. By using general conditions of isothermal LBM models, e.g., symmetry of the discrete velocity set and several moment constraints, they derived the same results without stepping into particular form of individual LBM models. Thus, the above error estimate was found to be model independent and applicable to any isothermal LBM model.

However, they both considered the flow regime of Re ~ 1 only. Moreover, the LBM has three basic different models, i.e., the compressible, thermal, and isothermal models [18–22], and works of Inamuro *et al.* and Junk *et al.* are only for isothermal model. It is therefore necessary to estimate the accuracy of these three basic models not only in the flow regime of Re~1, but also in the other flow regimes, such as Re≪1 and Re≫1. Now the purpose of the present study is summarized as follows: we make the asymptotic analysis for small ε of the kinetic equation of the above three basic LBM

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models and derive the corresponding fluid-dynamics-type equations. The three important flow regimes are considered: (I) the Mach number of the flow is much smaller than ε, (II) the Mach number is of the same order as ε, and (III) the Mach number is finite. From the von Karman relation, the above three cases correspond to the flows of (I) Re $\ll 1$, (II) Re ~ 1, and (III) Re $\gg 1$, respectively. Then, we compare the derived set of fluid-dynamics-type equations with the correct set of equations that can describe the behavior of a gas in each flow regime: the Stokes set of equations for case (I) [17,23], the incompressible NS set of equations for case (II) [15,17], and the Euler and the viscous-boundary-layer set (when solution includes no discontinuities) or the weak form of the Euler set (when solution includes discontinuities) for case (III) [17]. In each regime, three typical LBM models, i.e., the compressible, thermal, and isothermal models, are treated individually. Thus, we can easily estimate the accuracy of any LBM models for describing the behavior of a gas in each flow regime. It is after these error estimates are properly done that the LBM, which is widely used among fluid dynamics researchers, becomes a real valid numerical tool for the flow simulation.

It should be noted again that the error estimate was previously made only for the isothermal model [12–14]. For the other LBM models, the present study treats their error estimates. Thus, the following analyses of the present study, which are based on the minimalistic set of assumptions, are systematic error estimates applicable to a wide variety of LBM models and in all the flow regimes (I)–(III).

The outline of this paper is as follows. In Sec. II, the above three basic lattice Boltzmann models are briefly introduced together with their fundamental properties. Their nondimensional expressions are given in Sec. III, and we estimate these errors by the asymptotic analysis in Secs. IV–VI. Cases (I), (II), and (III) are treated in Secs. IV–VI, respectively. For each case, accuracies of the above three basic models are investigated. The obtained results are then summarized in Sec. VII and some concluding remarks follow.

II. LATTICE BOLTZMANN METHOD

Let t, x_α ($\alpha = 1, 2, \ldots, D$, where D is the number of dimensions), $c_{i\alpha}$ ($i = 1, 2, \ldots, N$, where N is the total number of discrete molecular velocities), and $f_i(t, x_\alpha)$ be, respectively, the time, the space coordinates, the molecular velocity of the ith particle in the x_α direction, and the velocity distribution function of the ith particle. The macroscopic variables ρ (density), u_α (x_α component of flow velocity), and T (temperature) are defined as

$$\rho = \sum_{i=1}^{N} f_i, \tag{1a}$$

$$\rho u_\alpha = \sum_{i=1}^{N} f_i c_{i\alpha}, \tag{1b}$$

$$\rho(DRT + u_\alpha^2) = \sum_{i=1}^{N} f_i c_{i\alpha}^2. \tag{1c}$$

It should be noted here that throughout this paper the subscripts $\alpha, \beta, \ldots$ and $\beta_1, \beta_2, \ldots$ are used to represent the number of space coordinates to which the summation convention is applied. In contrast, the subscripts i and j represent the kind of molecules, and the summation convention is not applied to these subscripts. The pressure p is related to ρ and T by the equation of state:

$$p = \rho RT, \tag{2}$$

where R is the specific gas constant.

Consider a kinetic equation of the Bhatnagar-Gross-Krook (BGK) type [24], which is simple and widely used in the numerical computation of the LBM [18–22]:

$$\frac{\partial f_i}{\partial t} + c_{i\beta} \frac{\partial f_i}{\partial x_\beta} = \frac{f_i^{\mathrm{eq}} - f_i}{\sigma}, \tag{3}$$

where σ (the relaxation time) is a given constant and f_i^{eq} (the local equilibrium velocity distribution function) is a given function of the macroscopic variables ρ, u_α, and T. In the usual LBM, the following discretized form of Eq. (3) is often used:

$$\frac{f_i(t + \Delta t, x_\alpha + c_{i\alpha} \Delta t) - f_i(t, x_\alpha)}{\Delta t} = \frac{f_i^{\mathrm{eq}} - f_i}{\sigma}, \tag{4}$$

where Δt is the discrete time step of $O(\sigma)$. When a solution is smooth and includes no discontinuities such as shock waves and contact discontinuities, Eq. (4) is obtained by a first-order upwind discretization of the left-hand side of Eq. (3) with the spatial width $\Delta x_\alpha = c_{i\alpha} \Delta t$. To specify the difference between Eqs. (3) and (4), the distribution function f_i defined by Eq. (4), which is given only on lattice nodes, is extended to the whole domain (t, x_α) by a smooth function. Then we Taylor expand the smooth extended distribution function $f_i(t + \Delta t, x_\alpha + c_{i\alpha} \Delta t)$ on the left-hand side of Eq. (4) with respect to t around (t, x_α), which leads to

$$\frac{\partial f_i}{\partial t} + c_{i\beta} \frac{\partial f_i}{\partial x_\beta} + \left(\frac{1}{2} \frac{\partial^2 f_i}{\partial t^2} + c_{i\alpha} \frac{\partial^2 f_i}{\partial t \, \partial x_\alpha} + \frac{1}{2} c_{i\alpha} c_{i\beta} \frac{\partial^2 f_i}{\partial x_\alpha \, \partial x_\beta} \right)$$
$$\times \Delta t + \cdots$$
$$= \frac{f_i^{\mathrm{eq}} - f_i}{\sigma}. \tag{5}$$

The differences between Eqs. (3) and (4) are represented by the terms multiplied by Δt and their subsequent terms on the left-hand side of Eq. (5). In fact, the contribution of these terms to the fluid-dynamics-type equations appears only on their coefficients. Specifically, the set of fluid-dynamics-type equations derived from Eq. (4) can be obtained by modifying the coefficients of the fluid-dynamics-type equations derived from Eq. (3) in the following way: if the fluid-dynamics-type equations derived from Eq. (3) only have terms whose total number of differentiations in t and x_α is one, no modifications are required. If they include terms whose total number of differentiations is two (a second derivative or product of

two first derivatives), multiply $1 - \Delta t/2\sigma$ ($\Delta t < 2\sigma$) to these terms. If they include terms whose total number of differentiations is three, then multiply $1 - \Delta t/\sigma + (\Delta t/\sigma)^2/6$ to these terms.

When a solution includes discontinuities such as shocks and contact discontinuities, the extended distribution function f_i also includes discontinuities, and Eq. (4) does not converge to the kinetic equation (3) as $\Delta t \to 0$ but to the following integral relation:

$$\int \int \left[\left(\frac{\partial \theta}{\partial t} + c_{i\alpha} \frac{\partial \theta}{\partial x_\alpha} \right) f_i + \frac{f_i^{\text{eq}} - f_i}{\sigma} \theta \right] dt d\boldsymbol{x} = 0, \qquad (6)$$

where $\theta(t, x_\alpha)$ is any smooth test function of t and x_α which vanishes for $t + |x_\alpha|$ large enough, and integration is carried out over the whole t-x_α plane. In fact, multiplying Eq. (4) by $\theta(t, x_\alpha)$, integrating over the whole t-x_α plane, and using integration by parts, we easily find that the resulting equation converges to Eq. (6) as Δt and Δx_α tend to zero (see the Appendix of Ref. [25] for the derivation). The fundamental property of the fluid-dynamics-type equations is not affected by the use of Eq. (3) or Eq. (6) instead of Eq. (4) [26,27]. Thus, we use Eq. (3) or Eq. (6) as a basic kinetic equation in the following analysis. The fluid-dynamics-type equations derived from Eq. (4) are easily obtained by applying the above-mentioned modifications.

Now return to the explanation of the lattice Boltzmann models. There are three different kinds of basic models: (i) compressible model [19,28], (ii) thermal model [20,21], and (iii) isothermal model [18,22]. Their differences can be represented by the different constraints imposed on f_i^{eq} as follows:

(i) Compressible model:

$$\rho = \sum_{i=1}^{N} f_i^{\text{eq}}, \qquad (7a)$$

$$\rho u_\alpha = \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha}, \qquad (7b)$$

$$p\delta_{\alpha\beta} + \rho u_\alpha u_\beta = \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha} c_{i\beta}, \qquad (7c)$$

$$p(u_\alpha \delta_{\beta\gamma} + u_\beta \delta_{\gamma\alpha} + u_\gamma \delta_{\alpha\beta}) + \rho u_\alpha u_\beta u_\gamma = \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha} c_{i\beta} c_{i\gamma}, \qquad (7d)$$

$$(D+2)pRT\delta_{\alpha\beta} + p[(D+4)u_\alpha u_\beta + u_\gamma^2 \delta_{\alpha\beta}] + \rho u_\alpha u_\beta u_\gamma^2$$
$$= \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha} c_{i\beta} c_{i\gamma}^2. \qquad (7e)$$

(ii) Thermal model: Eqs. (7a)–(7c) and

$$p(u_\alpha \delta_{\beta\gamma} + u_\beta \delta_{\gamma\alpha} + u_\gamma \delta_{\alpha\beta}) + O(u^3) = \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha} c_{i\beta} c_{i\gamma}, \qquad (8a)$$

$$(D+2)pRT\delta_{\alpha\beta} + O(u^2) = \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha} c_{i\beta} c_{i\gamma}^2, \qquad (8b)$$

where $O(u^2)$ and $O(u^3)$ indicate arbitrary second-order and third-order terms with respect to the flow velocity u_α.

(iii) Isothermal model: Eqs. (7a) and (7b) and

$$\rho(RT_0 \delta_{\alpha\beta} + u_\alpha u_\beta) = \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha} c_{i\beta}, \qquad (9a)$$

$$\rho RT_0 (u_\alpha \delta_{\beta\gamma} + u_\beta \delta_{\gamma\alpha} + u_\gamma \delta_{\alpha\beta}) = \sum_{i=1}^{N} f_i^{\text{eq}} c_{i\alpha} c_{i\beta} c_{i\gamma}, \qquad (9b)$$

where T_0 (reference temperature of the system) is a given constant.

An innumerable number of examples can be constructed to satisfy the above constraints. One or two possible examples for each model are summarized in the Appendix.

III. NONDIMENSIONAL EXPRESSIONS

The nondimensional variables and equations, which are used in the following analysis, are listed here. Let t_0, L, ρ_0, and T_0 be, respectively, the reference time, length, density, and temperature. Then, the nondimensional variables are defined as follows:

$$\hat{t} = \frac{t}{t_0}, \quad \hat{x}_\alpha = \frac{x_\alpha}{L}, \quad \hat{c}_{i\alpha} = \frac{c_{i\alpha}}{\sqrt{RT_0}}, \quad \hat{f}_i = \frac{f}{\rho_0}, \quad \hat{f}_i^{\text{eq}} = \frac{f_i^{\text{eq}}}{\rho_0},$$

$$\hat{\rho} = \frac{\rho}{\rho_0}, \quad \hat{u}_\alpha = \frac{u_\alpha}{\sqrt{RT_0}}, \quad \hat{T} = \frac{T}{T_0}, \quad \hat{p} = \frac{p}{\rho_0 RT_0}. \qquad (10)$$

In terms of these nondimensional variables, the definition of macroscopic variables (1a)–(1c) and (2) becomes

$$\hat{\rho} = \sum_{i=1}^{N} \hat{f}_i, \qquad (11a)$$

$$\hat{\rho} \hat{u}_\alpha = \sum_{i=1}^{N} \hat{f}_i \hat{c}_{i\alpha} \quad (\alpha = 1, \ldots, D), \qquad (11b)$$

$$\hat{\rho}(D\hat{T} + \hat{u}_\alpha^2) = \sum_{i=1}^{N} \hat{f}_i \hat{c}_{i\alpha}^2, \qquad (11c)$$

$$\hat{p} = \hat{\rho} \hat{T}. \qquad (11d)$$

The kinetic equation (3) of nondimensional form is

$$\mathrm{Sh}\frac{\partial \hat{f}_i}{\partial \hat{t}} + \hat{c}_{i\alpha}\frac{\partial \hat{f}_i}{\partial \hat{x}_\alpha} = \frac{\hat{f}_i^{\mathrm{eq}} - \hat{f}_i}{\varepsilon}, \quad (12a)$$

where Sh is the Strouhal number and ε is the Knudsen number defined by

$$\mathrm{Sh} = \frac{L}{t_0\sqrt{RT_0}}, \quad \varepsilon = \frac{\sigma\sqrt{RT_0}}{L}. \quad (12b)$$

The integral relation (6) of nondimensional form is

$$\int\int\left[\left(\mathrm{Sh}\frac{\partial \vartheta}{\partial \hat{t}} + \hat{c}_{i\alpha}\frac{\partial \vartheta}{\partial \hat{x}_\alpha}\right)\hat{f}_i + \frac{\hat{f}_i^{\mathrm{eq}} - \hat{f}_i}{\varepsilon}\vartheta\right]d\hat{t}d\hat{x} = 0, \quad (13)$$

where $\vartheta(\hat{t},\hat{x}_\alpha)$ is a nondimensional smooth test function of $\hat{t}$ and $\hat{x}_\alpha$ which vanishes for $\hat{t}+|\hat{x}_\alpha|$ large enough. Then $\hat{f}_i^{\mathrm{eq}}$ satisfies the following constraints from Eqs. (7a)–(7e), (8a), (8b), (9a), and (9b):

(i) Compressible model:

$$\hat{\rho} = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}, \quad (14a)$$

$$\hat{\rho}\hat{u}_\alpha = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}, \quad (14b)$$

$$\hat{p}\delta_{\alpha\beta} + \hat{\rho}\hat{u}_\alpha\hat{u}_\beta = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}, \quad (14c)$$

$$\hat{p}(\hat{u}_\alpha\delta_{\beta\gamma} + \hat{u}_\beta\delta_{\gamma\alpha} + \hat{u}_\gamma\delta_{\alpha\beta}) + \hat{\rho}\hat{u}_\alpha\hat{u}_\beta\hat{u}_\gamma = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}, \quad (14d)$$

$$(D+2)\hat{p}\hat{T}\delta_{\alpha\beta} + \hat{p}[(D+4)\hat{u}_\alpha\hat{u}_\beta + \hat{u}_\gamma^2\delta_{\alpha\beta}] + \hat{\rho}\hat{u}_\alpha\hat{u}_\beta\hat{u}_\gamma^2$$

$$= \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}^2. \quad (14e)$$

(ii) Thermal model: Eqs. (14a)–(14c) and

$$\hat{p}(\hat{u}_\alpha\delta_{\beta\gamma} + \hat{u}_\beta\delta_{\gamma\alpha} + \hat{u}_\gamma\delta_{\alpha\beta}) + O(\hat{u}^3) = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}, \quad (15a)$$

$$(D+2)\hat{p}\hat{T}\delta_{\alpha\beta} + O(\hat{u}^2) = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}^2, \quad (15b)$$

where $O(\hat{u}^2)$ and $O(\hat{u}^3)$ indicate arbitrary second-order and third-order terms with respect to the nondimensional flow velocity $\hat{u}_\alpha$.

(iii) Isothermal model: Eqs. (14a) and (14b) and

$$\hat{\rho}(\delta_{\alpha\beta} + \hat{u}_\alpha\hat{u}_\beta) = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}, \quad (16a)$$

$$\hat{\rho}(\hat{u}_\alpha\delta_{\beta\gamma} + \hat{u}_\beta\delta_{\gamma\alpha} + \hat{u}_\gamma\delta_{\alpha\beta}) = \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}. \quad (16b)$$

From the next section, the asymptotic analysis for small ε of the kinetic equation (12a) or the integral relation (13) is made. We focus on the time scale t_0 of order L/U (U is a characteristic flow speed) or the time required to traverse the distance by the gas flow. Then the Strouhal number defined by Eq. (12b) is on the order of the Mach number Ma, or

$$\mathrm{Sh} = O(\mathrm{Ma}), \quad (17)$$

where Ma is defined by $U/\sqrt{\gamma RT_0}$ (γ is a specific-heat ratio). We consider here the case where the Mach number also represents the scales of the deviation of the system from a uniform state at rest. That is, the relative size of the deviation of thermodynamic variables (like temperature) to themselves is on the order of Ma (see Refs. [17,29] for examples of gas flows when this assumption does not hold). Then, the asymptotic behavior of the solution of the kinetic equation (12a) or the integral relation (13) for small ε is characterized only by the Mach number.

We discuss the following three important cases: (I) Ma $\ll \varepsilon$, (II) Ma $\sim \varepsilon$, and (III) Ma ~ 1. From the von Karman relation Ma $\sim$ Re ε, where Re is the Reynolds number of the system defined by

$$\mathrm{Re} = \frac{\rho_0 UL}{\mu_0} \quad (18)$$

(μ_0 is the reference viscosity of a gas), the above three cases correspond to the flow regimes of (I) Re $\ll 1$, (II) Re ~ 1, and (III) Re $\gg 1$, respectively. According to Sone, the correct sets of governing equations for describing the behavior of a gas in the continuum limit of the above flow regimes are (I) the Stokes set of equations [17,23], (II) the incompressible NS set of equations [15,17], and (III) the Euler and the viscous-boundary-layer set [17,30] or the weak form of the Euler set [31–33] of equations, respectively. We compare the fluid-dynamics-type equations derived from the LBM with the above correct sets of equations in each flow regime and estimate the error of the LBM. In the following, the analysis is made separately for the three different flow regimes (I)–(III). That is, case (I) is treated in Sec. IV, case (II) in Sec. V, and case (III) in Sec. VI. In each section, three kinds of different lattice Boltzmann models are treated in the following orders: (i) compressible model, (ii) thermal model, and (iii) isothermal model.

IV. ASYMPTOTIC ANALYSIS FOR SMALL ε WITH Ma $\ll \varepsilon$ (Re $\ll 1$)

A. Compressible model

We consider a situation where the distribution function $\hat{f}_i$ deviates slightly from the equilibrium distribution function $\hat{f}_i^{\mathrm{eq}(0)}$ of the uniform state at rest, so that the nonlinear terms of the deviation can be neglected [or $(\hat{f}_i - \hat{f}_i^{\mathrm{eq}(0)})^2 = O(\mathrm{Ma}^2)$ can be neglected]. Since Sh $= O(\mathrm{Ma})$ from Eq. (17) and $\partial\hat{f}_i/\partial\hat{t}$ is $O(\mathrm{Ma})$, the time-derivative term (or the first term) on the

left-hand side of the kinetic equation (12a) is also neglected. Under these assumptions, we make an asymptotic analysis for small ε. That is, we look for the solution whose scale of variation is on the order of unity with respect to the coordinates $\hat{t}$ and $\hat{x}_\alpha$, in the following series of ε:

$$\hat{f}_i = \hat{f}_i^{\mathrm{eq}(0)} + F_i^{(0)} + \varepsilon F_i^{(1)} + \varepsilon^2 F_i^{(2)} + \cdots, \quad (19)$$

where the terms with $F_i^{(m)}$ on the right-hand side represent the deviation of the velocity distribution function, whose nonlinear terms are neglected according to the assumption. The deviation of the macroscopic variables defined by

$$\omega = \hat{\rho} - 1, \quad \hat{u}_\alpha, \quad P = \hat{p} - 1, \quad \tau = \hat{T} - 1, \quad (20a)$$

is also expanded as

$$H = H^{(0)} + \varepsilon H^{(1)} + \varepsilon^2 H^{(2)} + \cdots, \quad (20b)$$

where H represents any of the deviation of the macroscopic variables ω, $\hat{u}_\alpha$, P, and τ. The component functions $H^{(m)}$ satisfy the following relations from Eqs. (11a)–(11d):

$$\omega^{(m)} = \sum_{i=1}^{N} F_i^{(m)}, \quad (21a)$$

$$\hat{u}_\alpha^{(m)} = \sum_{i=1}^{N} F_i^{(m)} \hat{c}_{i\alpha}, \quad (21b)$$

$$D(\omega^{(m)} + \tau^{(m)}) = \sum_{i=1}^{N} F_i^{(m)} \hat{c}_{i\alpha}^2, \quad (21c)$$

$$P^{(m)} = \omega^{(m)} + \tau^{(m)}, \quad (21d)$$

where the nonlinear terms of the deviation are neglected.

The equilibrium distribution function is also expanded as follows:

$$\hat{f}_i^{\mathrm{eq}} = \hat{f}_i^{\mathrm{eq}(0)} + F_i^{\mathrm{eq}(0)} + \varepsilon F_i^{\mathrm{eq}(1)} + \varepsilon^2 F_i^{\mathrm{eq}(2)} + \cdots. \quad (22)$$

Here, $F_i^{\mathrm{eq}(m)}$ satisfies the following constraints from Eqs. (14a)–(14e):

$$\omega^{(m)} = \sum_{i=1}^{N} F_i^{\mathrm{eq}(m)}, \quad (23a)$$

$$\hat{u}_\alpha^{(m)} = \sum_{i=1}^{N} F_i^{\mathrm{eq}(m)} \hat{c}_{i\alpha}, \quad (23b)$$

$$P^{(m)} \delta_{\alpha\beta} = \sum_{i=1}^{N} F_i^{\mathrm{eq}(m)} \hat{c}_{i\alpha} \hat{c}_{i\beta}, \quad (23c)$$

$$\hat{u}_\alpha^{(m)} \delta_{\beta\gamma} + \hat{u}_\beta^{(m)} \delta_{\gamma\alpha} + \hat{u}_\gamma^{(m)} \delta_{\alpha\beta} = \sum_{i=1}^{N} F_i^{\mathrm{eq}(m)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}, \quad (23d)$$

$$(D+2)(P^{(m)} + \tau^{(m)}) \delta_{\alpha\beta} = \sum_{i=1}^{N} F_i^{\mathrm{eq}(m)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}^2, \quad (23e)$$

where the nonlinear terms of the deviation are neglected.

Recalling that the time-derivative term can be neglected in Eq. (12a), we substitute Eqs. (19) and (22) into Eq. (12a) (without the time-derivative term). After arranging the same order terms in ε, we obtain the following series of linear equations for $F_i^{(m)}$:

$$L_i(F_j^{(0)}) = 0, \quad (24a)$$

$$L_i(F_j^{(m)}) = \hat{c}_{i\beta} \frac{\partial F_i^{(m-1)}}{\partial \hat{x}_\beta} \quad (m = 1, 2, \ldots), \quad (24b)$$

where $L_i(F_j^{(m)})$ is a linear operator expressed as

$$L_i(F_j^{(m)}) = F_i^{\mathrm{eq}(m)} - F_i^{(m)}$$

$$= \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \omega^{(0)}} \sum_{j=1}^{N} F_j^{(m)} + \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \hat{u}_\alpha^{(0)}} \sum_{j=1}^{N} F_j^{(m)} \hat{c}_{j\alpha}$$

$$+ \frac{1}{D} \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial P^{(0)}} \sum_{j=1}^{N} F_j^{(m)} \hat{c}_{j\alpha}^2 - F_i^{(m)}. \quad (25)$$

Here, the partial derivatives of $F_i^{\mathrm{eq}(0)}$ with respect to $\omega^{(0)}$, $\hat{u}_\alpha^{(0)}$ ($\alpha = 1, \ldots, D$), or $P^{(0)}$ are taken with the other $D+1$ variables among $\omega^{(0)}$, $\hat{u}_\alpha^{(0)}$, and $P^{(0)}$ being fixed.

The solution of the linear homogeneous equation (24a) is

$$F_i^{(0)} = F_i^{\mathrm{eq}(0)} = \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \omega^{(0)}} \omega^{(0)} + \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \hat{u}_\alpha^{(0)}} \hat{u}_\alpha^{(0)} + \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial P^{(0)}} P^{(0)}. \quad (26)$$

This is the linear combination of the following $D+2$ independent solutions:

$$\frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \omega^{(0)}}, \quad \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \hat{u}_\alpha^{(0)}} \quad (\alpha = 1, \ldots, D), \quad \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial P^{(0)}}. \quad (27)$$

Equation (24b) is an inhomogeneous equation whose homogeneous part is the same as that for Eq. (24a). Their common associated homogeneous equation has $D+2$ independent solutions (27). For this Eq. (24b) to have a solution, therefore, its inhomogeneous term must satisfy the following relation (solvability condition):

$$\sum_{i=1}^{N} g_i \hat{c}_{i\beta} \frac{\partial F_i^{(m-1)}}{\partial \hat{x}_\beta} = 0 \quad (m = 1, 2, \ldots), \quad (28a)$$

where

$$g_i = 1, \quad \hat{c}_{i\alpha}, \quad \text{or} \quad \hat{c}_{i\alpha}^2, \quad (28b)$$

since the left-hand side of Eq. (24b) or $L_i(F_j^{(m)})$ satisfies $\sum_{i=1}^{N} g_i L_i(F_j^{(m)}) = 0$ from Eqs. (21a)–(21d) and (23a)–(23c). When condition (28a) is satisfied, the solution of Eq. (24b) is given by

$$F_i^{(m)} = F_i^{\mathrm{eq}(m)} - \hat{c}_{i\beta}\frac{\partial F_i^{(m-1)}}{\partial \hat{x}_\beta} \quad (m=1,2,\ldots). \qquad (29)$$

Substituting Eqs. (26) and (29) into the solvability condition (28a), we can obtain the equations for the macroscopic variables in the following way. First, from Eq. (28a) ($m=1$) with $g_i = \hat{c}_{i\alpha}$,

$$\frac{\partial P^{(0)}}{\partial \hat{x}_\alpha} = 0. \qquad (30)$$

Next, from Eq. (28a) with $g_i = 1$, $\hat{c}_{i\alpha}$, and $\hat{c}_{i\alpha}^2$, we get the following set of equations for the set of variables $(\hat{u}_\alpha^{(m)}, P^{(m+1)}, \tau^{(m)})$:

$$\frac{\partial \hat{u}_\alpha^{(m)}}{\partial \hat{x}_\alpha} = 0, \qquad (31a)$$

$$\frac{\partial P^{(m+1)}}{\partial \hat{x}_\alpha} = \frac{\partial^2 \hat{u}_\alpha^{(m)}}{\partial \hat{x}_\beta^2} + R_\alpha^{(m)}, \qquad (31b)$$

$$\frac{\partial^2 \tau^{(m)}}{\partial \hat{x}_\alpha^2} = R_{D+1}^{(m)}, \qquad (31c)$$

with $m=0,1,2,\ldots$, where

$$P^{(m)} = \omega^{(m)} + \tau^{(m)}, \qquad (32)$$

$$R_\alpha^{(m)} = \begin{cases} 0 & (m=0) \\ \displaystyle\sum_{r=3}^{m+2} (-1)^r \frac{\partial^r \left(\displaystyle\sum_{i=1}^N F_i^{\mathrm{eq}(m+2-r)} \hat{c}_{i\alpha}\hat{c}_{i\beta_1}\hat{c}_{i\beta_2}\cdots\hat{c}_{i\beta_r} \right)}{\partial \hat{x}_{\beta_1}\partial \hat{x}_{\beta_2}\cdots\partial \hat{x}_{\beta_r}} & (m=1,2,\ldots), \end{cases} \qquad (33a)$$

$$R_{D+1}^{(m)} = \begin{cases} 0 & (m=0) \\ -\dfrac{\partial R_\alpha^{(m-1)}}{\partial \hat{x}_\alpha} + \displaystyle\sum_{r=3}^{m+2} \frac{(-1)^{r-1}}{D+2}\frac{\partial^r \left(\displaystyle\sum_{i=1}^N F_i^{\mathrm{eq}(m+2-r)} \hat{c}_{i\alpha}^2 \hat{c}_{i\beta_1}\hat{c}_{i\beta_2}\cdots\hat{c}_{i\beta_r} \right)}{\partial \hat{x}_{\beta_1}\partial \hat{x}_{\beta_2}\cdots\partial \hat{x}_{\beta_r}} & (m=1,2,\ldots). \end{cases} \qquad (33b)$$

The set of equations (31a)–(31c) is the desired set of fluid-dynamics-type equations derived from the kinetic equation (3) of the LBM. When $m=0$, the set of equations (31a)–(31c) is nothing but the Stokes set of equations, which is the correct set to describe the behavior a gas in the continuum limit with Ma $\ll \varepsilon$. From the next order, the inhomogeneous terms $R_\alpha^{(m)}$ and $R_{D+1}^{(m)}$ appear on the right-hand sides of Eqs. (31b) and (31c). These terms are defined by Eqs. (33a) and (33b) and they include the moments of $F_i^{\mathrm{eq}(q)}$ ($q=0,\ldots,m-1$) which are not prescribed at the outset in Eqs. (23a)–(23e). Noting that $\hat{u}_\alpha^{(m)}$ and $P^{(m+1)}$ are determined only by Eqs. (31a) and (31b) and $\tau^{(m)}$ is determined only by Eq. (31c) individually, we find the following results:

The relative error of the flow velocity and the pressure gradient is

$$O(\varepsilon^{M_0}) \quad \text{if } R_\alpha^{(M_0)} \text{ is not irrotational and } R_\alpha^{(m)}$$
$$(m \le M_0 - 1) \text{ is irrotational}, \qquad (34a)$$

where $R_\alpha^{(m)}$ ($m=1,2,\ldots,M_0$) is given by Eq. (33a). The relative error of the temperature is

$$O(\varepsilon^{M_1}) \quad \text{if } R_{D+1}^{(M_1)} \neq 0 \text{ and } R_{D+1}^{(m)} \ (m \le M_1 - 1) = 0,$$
$$(34b)$$

where $R_{D+1}^{(m)}$ ($m=1,2,\ldots,M_1$) is given by Eq. (33b).

In Eq. (34a), we used the fact that, if $R_\alpha^{(m)}$ is irrotational or $R_\alpha^{(m)} = \partial\Phi^{(m)}/\partial\hat{x}_\alpha$, this inhomogeneous term can be incorporated into the pressure term of the momentum equation (31b) by introducing $\bar{P}^{(m+1)} = P^{(m+1)} - \Phi^{(m)}$. Then the corresponding set of equations admits zero solution, and a deviation from the correct solution does not arise.

B. Thermal model

Both the analytical process and its results are completely the same as those of the compressible model, since the difference between the two models is represented by the nonlinear terms of the flow velocity in Eqs. (15a) and (15b), and these terms never appear in the above analysis where the nonlinear deviation terms are neglected. Thus, the set of equations for the macroscopic variables is given by the same set as that for the compressible model, or Eqs. (31a)–(31c). The order of the error is given by Eqs. (34a) and (34b).

C. Isothermal model

The analytical process is similar to that for the compressible model. Only its outline is given here. First, from Eqs. (14a), (14b), (16a), and (16b), the constraints for the moments of $F_i^{\mathrm{eq}(m)}$ are given by Eqs. (23a)–(23d) with $\tau^{(m)} = 0$ (or $P^{(m)} = \omega^{(m)}$). A series of equations for $F_i^{(m)}$ is given by

Eqs. (24a) and (24b). The common associated homogeneous equation for the inhomogeneous equations (24b) has no $D+2$ but $D+1$ independent solutions:

$$\frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \omega^{(0)}}, \quad \frac{\partial F_i^{\mathrm{eq}(0)}}{\partial \hat{u}_\alpha^{(0)}} \quad (\alpha = 1, \ldots, D), \tag{35}$$

since $F_i^{\mathrm{eq}(0)}$ depends only on $\omega^{(0)}$ and $\hat{u}_\alpha^{(0)}$. The solvability condition is therefore Eq. (28a) with $g_i = 1$ and $\hat{c}_{i\alpha}$ only. When the solvability condition is satisfied, the solution is given by Eq. (29). Substituting this solution and the leading-order solution $F_i^{(0)} = F_i^{\mathrm{eq}(0)}$ into the solvability condition, we get Eq. (30) and the sets of equations (31a) and (31b) for the variables $(u_\alpha^{(m)}, P^{(m+1)})$. This set [Eqs. (31a) and (31b)] is the same as that of the compressible model for the variables $(u_\alpha^{(m)}, P^{(m+1)})$. Thus, the relative error of the flow velocity and the pressure gradient is given by Eq. (34a). Junk and Yang [14] made the asymptotic analysis independently and obtained the result that the relative error of any isothermal model is $O(\varepsilon^2)$. It is because they used the symmetry condition of the discrete velocity set in their analysis, or $\Sigma_{i=1}^N \hat{c}_{i\beta_1} \cdots \hat{c}_{i\beta_{2r-1}} = 0$ (r is an integer), and this together with Eq. (30) (so that $P^{(0)} = 0$ without loss of generality) makes $R_\alpha^{(1)}$ irrotational, while $R_\alpha^{(2)}$ is not. The above symmetry condition was used by Junk and Yang because any existing LBM models possess this property. We avoided using this symmetry in view of a possibility that a new model may appear in the future without this symmetry.

V. ASYMPTOTIC ANALYSIS FOR SMALL ε WITH Ma $\sim \varepsilon$ (Re ~ 1)

A. Compressible model

Consider a case where the deviation of the distribution function $\hat{f}_i$ from the equilibrium distribution function of the uniform state at rest $\hat{f}_i^{\mathrm{eq}(0)}$ is on the order of ε. From Eq. (17) and Ma $\sim \varepsilon$, Sh can be put as

$$\mathrm{Sh} = B\varepsilon, \tag{36}$$

where B is a given constant of order unity. We then make an asymptotic analysis for small ε according to Refs. [15–17]. That is, we look for the solution whose scale of variation is on the order of unity with respect to the coordinates $\hat{t}$ and $\hat{x}_\alpha$, in the following power series of ε:

$$\hat{f}_i = \hat{f}_i^{\mathrm{eq}(0)} + \varepsilon \hat{f}_i^{(1)} + \varepsilon^2 \hat{f}_i^{(2)} + \cdots, \tag{37}$$

where the component function $\hat{f}_i^{(m)}$ is a quantity on the order of unity, and the series of the deviation starts from the first order of ε.

The deviation of the macroscopic variables defined by Eq. (20a) is also expanded:

$$H = \varepsilon H^{(1)} + \varepsilon^2 H^{(2)} + \varepsilon^2 H^{(3)} + \cdots, \tag{38}$$

where H represents any of the deviation of the macroscopic variables ω, $\hat{u}_\alpha$, P, and τ. The component functions $H^{(m)}$ satisfy the following relations from Eqs. (11a)–(11d):

$$\omega^{(1)} = \sum_{i=1}^N \hat{f}_i^{(1)}, \tag{39a}$$

$$\hat{u}_\alpha^{(1)} = \sum_{i=1}^N \hat{f}_i^{(1)} \hat{c}_{i\alpha}, \tag{39b}$$

$$D(\omega^{(1)} + \tau^{(1)}) = \sum_{i=1}^N \hat{f}_i^{(1)} \hat{c}_{i\alpha}^2, \tag{39c}$$

$$P^{(1)} = \omega^{(1)} + \tau^{(1)}, \tag{39d}$$

$$\omega^{(2)} = \sum_{i=1}^N \hat{f}_i^{(2)}, \tag{40a}$$

$$\hat{u}_\alpha^{(2)} + \omega^{(1)} \hat{u}_\alpha^{(1)} = \sum_{i=1}^N \hat{f}_i^{(2)} \hat{c}_{i\alpha}, \tag{40b}$$

$$D(\omega^{(2)} + \tau^{(2)} + \omega^{(1)}\tau^{(1)}) + (\hat{u}_\alpha^{(1)})^2 = \sum_{i=1}^N \hat{f}_i^{(2)} \hat{c}_{i\alpha}^2, \tag{40c}$$

$$P^{(2)} = \omega^{(2)} + \tau^{(2)} + \omega^{(1)}\tau^{(1)}, \tag{40d}$$

$$\cdots.$$

The equilibrium distribution function is also expanded as follows:

$$\hat{f}_i^{\mathrm{eq}} = \hat{f}_i^{\mathrm{eq}(0)} + \varepsilon \hat{f}_i^{\mathrm{eq}(1)} + \varepsilon^2 \hat{f}_i^{\mathrm{eq}(2)} + \cdots. \tag{41}$$

Here, $\hat{f}_i^{\mathrm{eq}(m)}$ ($m = 1, 2, \ldots$) satisfies the following constraints from Eqs. (14a)–(14e):

$$\omega^{(1)} = \sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(1)}, \tag{42a}$$

$$\hat{u}_\alpha^{(1)} = \sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(1)} \hat{c}_{i\alpha}, \tag{42b}$$

$$P^{(1)} \delta_{\alpha\beta} = \sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(1)} \hat{c}_{i\alpha} \hat{c}_{i\beta}, \tag{42c}$$

$$\hat{u}_\alpha^{(1)} \delta_{\beta\gamma} + \hat{u}_\beta^{(1)} \delta_{\gamma\alpha} + \hat{u}_\gamma^{(1)} \delta_{\alpha\beta} = \sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(1)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}, \tag{42d}$$

$$(D+2)(P^{(1)} + \tau^{(1)}) \delta_{\alpha\beta} = \sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(1)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}^2, \tag{42e}$$

$$\omega^{(2)} = \sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(2)}, \tag{43a}$$

$$\hat{u}_\alpha^{(2)} + \omega^{(1)}\hat{u}_\alpha^{(1)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(2)}\hat{c}_{i\alpha}, \qquad (43b)$$

$$P^{(2)}\delta_{\alpha\beta} + \hat{u}_\alpha^{(1)}\hat{u}_\beta^{(1)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(2)}\hat{c}_{i\alpha}\hat{c}_{i\beta}, \qquad (43c)$$

$$\hat{u}_\alpha^{(2)}\delta_{\beta\gamma} + \hat{u}_\beta^{(2)}\delta_{\gamma\alpha} + \hat{u}_\gamma^{(2)}\delta_{\alpha\beta} + P^{(1)}(\hat{u}_\alpha^{(1)}\delta_{\beta\gamma} + \hat{u}_\beta^{(1)}\delta_{\gamma\alpha} + \hat{u}_\gamma^{(1)}\delta_{\alpha\beta})$$
$$= \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(2)}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}, \qquad (43d)$$

$$(D+2)(P^{(2)} + \tau^{(2)} + P^{(1)}\tau^{(1)})\delta_{\alpha\beta} + (D+4)\hat{u}_\alpha^{(1)}\hat{u}_\beta^{(1)} + (\hat{u}_\gamma^{(1)})^2\delta_{\alpha\beta}$$
$$= \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(2)}\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}^2,$$

$$\cdots. \qquad (43e)$$

Substituting Eqs. (36), (37), and (41) into Eq. (12a) and arranging the same order terms in ε, we obtain the following series of equations for $\hat{f}_i^{(m)}$:

$$L_i(\hat{f}_j^{(1)}) = 0, \qquad (44a)$$

$$L_i(\hat{f}_j^{(m)}) = -Q_i^{(m)}(\hat{f}_j^{(1)}, \dots, \hat{f}_j^{(m-1)}) + B\frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{t}} + \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{x}_\beta}$$

$$(m=2,3,\dots), \qquad (44b)$$

where

$$L_i(\hat{f}_j^{(m)}) + Q_i^{(m)}(\hat{f}_j^{(1)}, \dots, \hat{f}_j^{(m-1)}) = \hat{f}_i^{\text{eq}(m)} - \hat{f}_i^{(m)}. \qquad (44c)$$

Here, $L_i(\hat{f}_j^{(m)})$ is the collision operator of the homogeneous part, which is specifically given by

$$L_i(\hat{f}_j^{(m)}) = \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \omega^{(1)}}\sum_{j=1}^{N}\hat{f}_j^{(m)} + \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \hat{u}_\alpha^{(1)}}\sum_{j=1}^{N}\hat{f}_j^{(m)}\hat{c}_{j\alpha}$$
$$+ \frac{1}{D}\frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial P^{(1)}}\sum_{j=1}^{N}\hat{f}_j^{(m)}\hat{c}_{j\alpha}^2 - \hat{f}_i^{(m)}, \qquad (45)$$

and $Q_i^{(m)}$ is the remaining collision operator of the inhomogeneous part composed of the lower-order component functions. Note that $L_i(\hat{f}_j^{(m)})$ and $Q_i^{(m)}$ have the following properties for their moments:

$$\sum_{i=1}^{N} g_i L_i(\hat{f}_j^{(m)}) = 0, \quad \sum_{i=1}^{N} g_i Q_i^{(m)} = 0, \qquad (46)$$

where g_i is given by Eq. (28b). The first relation of Eq. (46) is obtained by taking the moments of Eq. (45) with respect to g_i and using Eqs. (42a)–(42c). The second is obtained by taking the moments of Eq. (44c), where the moments on the right-hand side vanish.

Equation (44a) is a linear homogeneous equation. The solution is

$$\hat{f}_i^{(1)} = \hat{f}_i^{\text{eq}(1)} = \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \omega^{(1)}}\omega^{(1)} + \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \hat{u}_\alpha^{(1)}}\hat{u}_\alpha^{(1)} + \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial P^{(1)}}P^{(1)}. \quad (47)$$

This is the linear combination of the following $D+2$ independent solutions:

$$\frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \omega^{(1)}}, \quad \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \hat{u}_\alpha^{(1)}} \quad (\alpha=1,\dots,D), \quad \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial P^{(1)}}. \qquad (48)$$

Equation (44b) is an inhomogeneous equation whose homogeneous part is the same as that for Eq. (44a). For this Eq. (44b) to have a solution, its inhomogeneous term must satisfy the following relation (solvability condition):

$$\sum_{i=1}^{N} g_i\left(B\frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{t}} + \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{x}_\beta}\right) = 0 \quad (m=2,3,\dots), \quad (49)$$

since $L_i(\hat{f}_j^{(m)})$ and $Q_i^{(m)}$ satisfy Eqs. (46). It is noted that the first term on the left-hand side of Eq. (49) is zero when $m=2$. When condition (49) is satisfied, the solution of Eq. (44b) is given by

$$\hat{f}_i^{(m)} = \hat{f}_i^{\text{eq}(m)} - B\frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{t}} - \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{x}_\beta} \quad (m=2,3,\dots).$$

$$(50)$$

Substituting Eqs. (47) and (50) into the solvability condition (49), we can get the equations for the macroscopic variables in the following way. First, from the solvability condition (49) with $m=2$,

$$\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\alpha} = 0, \qquad (51a)$$

$$\frac{\partial P^{(1)}}{\partial \hat{x}_\alpha} = 0. \qquad (51b)$$

The two relations in the solvability condition with $g_i=1$ and $g_i=\hat{c}_{i\alpha}^2$ degenerate into the same Eq. (51a).

From the solvability condition (49) with $m=3$,

$$B\frac{\partial \omega^{(1)}}{\partial \hat{t}} + \hat{u}_\alpha^{(1)}\frac{\partial \omega^{(1)}}{\partial \hat{x}_\alpha} + \frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\alpha} = 0, \qquad (52a)$$

$$B\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{t}} + \hat{u}_\beta^{(1)}\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta} + \frac{\partial P^{(2)}}{\partial \hat{x}_\alpha} = \frac{\partial^2 \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta^2}, \qquad (52b)$$

$$B\left(\frac{\partial \tau^{(1)}}{\partial \hat{t}} - \frac{2}{D+2}\frac{\partial P^{(1)}}{\partial \hat{t}}\right) + \hat{u}_\alpha^{(1)}\frac{\partial \tau^{(1)}}{\partial \hat{x}_\alpha} = \frac{\partial^2 \tau^{(1)}}{\partial \hat{x}_\alpha^2}. \qquad (52c)$$

From the solvability condition (49) with $m=4$,

$$B\frac{\partial \omega^{(2)}}{\partial \hat{t}} + \hat{u}_\alpha^{(1)}\frac{\partial \omega^{(2)}}{\partial \hat{x}_\alpha} + \hat{u}_\alpha^{(2)}\frac{\partial \omega^{(1)}}{\partial \hat{x}_\alpha} + \frac{\partial \hat{u}_\alpha^{(3)}}{\partial \hat{x}_\alpha} + \omega^{(1)}\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\alpha} = 0,$$

$$(53a)$$

$$B\left(\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{t}} + \omega^{(1)}\frac{\partial u_\alpha^{(1)}}{\partial \hat{t}}\right) + \hat{u}_\beta^{(1)}\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\beta} + (\hat{u}_\beta^{(2)} + \omega^{(1)}\hat{u}_\beta^{(1)})\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta}$$
$$+ \frac{\partial}{\partial \hat{x}_\alpha}\left(P^{(3)} - 2\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\alpha}\right)$$
$$= \frac{\partial}{\partial \hat{x}_\beta}\left(\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\beta} + P^{(1)}\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta}\right) + \hat{R}_\alpha, \tag{53b}$$

$$B\left(\frac{\partial \tau^{(2)}}{\partial \hat{t}} + \omega^{(1)}\frac{\partial \tau^{(1)}}{\partial \hat{t}}\right) + \hat{u}_\alpha^{(1)}\frac{\partial \tau^{(2)}}{\partial \hat{x}_\alpha} + (\hat{u}_\alpha^{(2)} + \omega^{(1)}\hat{u}_\alpha^{(1)})\frac{\partial \tau^{(1)}}{\partial \hat{x}_\alpha}$$
$$- \frac{2}{D+2}\left(B\frac{\partial P^{(2)}}{\partial \hat{t}} + \hat{u}_\alpha^{(1)}\frac{\partial P^{(2)}}{\partial \hat{x}_\alpha}\right)$$
$$= \frac{\partial}{\partial \hat{x}_\alpha}\left(\frac{\partial \tau^{(2)}}{\partial \hat{x}_\alpha} + P^{(1)}\frac{\partial \tau^{(1)}}{\partial \hat{x}_\alpha}\right) + \frac{1}{D+2}\left(\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{u}_\beta^{(1)}}{\partial \hat{x}_\alpha}\right)^2$$
$$+ \hat{R}_{D+1}, \tag{53c}$$

where

$$\hat{R}_\alpha = -\frac{\partial^3\left(\sum_{i=1}^N \hat{f}_i^{\text{eq}(1)}\hat{c}_{i\alpha}\hat{c}_{i\beta_1}\hat{c}_{i\beta_2}\hat{c}_{i\beta_3}\right)}{\partial \hat{x}_{\beta_1}\,\partial \hat{x}_{\beta_2}\,\partial \hat{x}_{\beta_3}} \quad (\alpha = 1,\ldots,D), \tag{54a}$$

$$\hat{R}_{D+1} = -\frac{1}{D+2}\frac{\partial^3\left(\sum_{i=1}^N \hat{f}_i^{\text{eq}(1)}\hat{c}_{i\alpha}^2\hat{c}_{i\beta_1}\hat{c}_{i\beta_2}\hat{c}_{i\beta_3}\right)}{\partial \hat{x}_{\beta_1}\,\partial \hat{x}_{\beta_2}\,\partial \hat{x}_{\beta_3}}. \tag{54b}$$

We can obtain the higher-order sets in a similar way.

The series of equations obtained from the solvability condition (49), together with the equation of state, constitutes systems of equations that determine the component functions $\omega^{(m)}$, $\hat{u}_\alpha^{(m)}$, $\tau^{(m)}$, and $\hat{P}^{(m)}$ of the macroscopic variables ω, $\hat{u}_\alpha$, τ, and $\hat{P}$. First, $P^{(1)}$ is constant from Eq. (51b). Next, $\omega^{(1)}$, $\hat{u}_\alpha^{(1)}$, $\tau^{(1)}$, and $P^{(2)}$ are determined by Eqs. (51a), (52b), and (52c). Then, $\omega^{(2)}$, $\hat{u}_\alpha^{(2)}$, $\tau^{(2)}$, and $P^{(3)}$ are determined by Eqs. (52a), (53b), and (53c). The higher-order sets are also determined in a similar way.

We find that the lowest-order set [Eqs. (51a), (52b), and (52c)] corresponds to the incompressible Navier-Stokes (NS) set of equations, which correctly describes the behavior of a gas in the continuum limit with Re $\sim$ 1 [17]. The incompressible NS set is given, in its nondimensional form, as

$$\frac{\partial \hat{u}_\alpha}{\partial \hat{x}_\alpha} = 0, \tag{55a}$$

$$\text{Sh}\frac{\partial \hat{u}_\alpha}{\partial \hat{t}} + \hat{u}_\beta\frac{\partial \hat{u}_\alpha}{\partial \hat{x}_\beta} + \frac{\partial P}{\partial \hat{x}_\alpha} = \frac{U}{\sqrt{RT_0}}\frac{1}{\text{Re}}\frac{\partial^2 \hat{u}_\alpha}{\partial \hat{x}_\beta^2}, \tag{55b}$$

$$\text{Sh}\left(\frac{\partial \tau}{\partial \hat{t}} - \frac{R}{C_p}\frac{\partial P}{\partial \hat{t}}\right) + \hat{u}_\alpha\frac{\partial \tau}{\partial \hat{x}_\alpha} = \frac{U}{\sqrt{RT_0}}\frac{1}{\text{Pr Re}}\frac{\partial^2 \tau}{\partial \hat{x}_\alpha^2}, \tag{55c}$$

with

$$P = \omega + \tau + \omega\tau, \tag{55d}$$

where C_p is the specific heat at constant pressure and Pr is the Prandtl number defined by

$$\text{Pr} = \frac{C_p\mu}{\lambda} \tag{56}$$

(λ is the thermal conductivity). Considering the ordering of each term in the leading-order set [Eqs. (51a), (52b), and (52c)] and noting Eq. (36), we find that this leading-order set corresponds to the incompressible NS set [or Eqs. (55a)–(55c)] whose parameters C_p/R, Re, and Pr are given by

$$\frac{C_p}{R} = \frac{D+2}{2}, \quad \text{Re} = \frac{U}{\varepsilon\sqrt{RT_0}} = \frac{UL}{\sigma RT_0}, \quad \text{Pr} = 1. \tag{57}$$

Proceeding to the next-order set of equations (52a), (53b), and (53c), however, we find that it includes inhomogeneous terms composed of the leading-order solution. For example, the continuity equation (52a) includes the inhomogeneous terms $B\partial\omega^{(1)}/\partial\hat{t} + \hat{u}_\alpha^{(1)}\partial\omega^{(1)}/\partial\hat{x}_\alpha$, which are nonzero in general. Thus, the solution at this order becomes nonzero and produces a deviation from the correct solution of the leading-order set. The relative error of the compressible LBM model is therefore $O(\varepsilon)$.

B. Thermal model

Both the analytical process and its results are almost the same as those of the compressible model, since the difference between the two models is represented by the nonlinear terms of the flow velocity in Eqs. (15a) and (15b), and these terms appear only at the higher orders of ε. Thus, the leading-order set of equations for the variables $\omega^{(1)}$, $\hat{u}_\alpha^{(1)}$, $\tau^{(1)}$, and $P^{(2)}$ is the same as that of the compressible model [or Eqs. (51a), (52b), and (52c)], which is the incompressible NS set. The next-order set of equations for the variables $\omega^{(2)}$, $\hat{u}_\alpha^{(2)}$, $\tau^{(2)}$, and $P^{(3)}$ is also the same as that of the compressible model [or Eqs. (52a), (53b), and (53c)] except for the energy equation (53c). This set evidently includes nonzero inhomogeneous terms since Eq. (52a) includes $B\partial\omega^{(1)}/\partial\hat{t} + \hat{u}_\alpha^{(1)}\partial\omega^{(1)}/\partial\hat{x}_\alpha$. Thus, the solution at this order becomes nonzero and the relative error of the thermal model is $O(\varepsilon)$.

C. Isothermal model

Only the outline of the analysis is given. From Eqs. (14a), (14b), (16a), and (16b), the constraints for the moments of the component function $\hat{f}_i^{\text{eq}(m)}$ $(m=1,2,\ldots)$ are Eqs. (42a)–(42d) and (43a)–(43d), and their higher orders with $\tau^{(m)}=0$ (or $P^{(m)}=\omega^{(m)}$). The equations for $\hat{f}_i^{(m)}$ are Eqs. (44a) and (44b). The common associated homogeneous equation for the inhomogeneous equations (44b) has no $D+2$ but $D+1$ independent solutions:

$$\frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \omega^{(1)}}, \quad \frac{\partial \hat{f}_i^{\text{eq}(1)}}{\partial \hat{u}_\alpha^{(1)}} \quad (\alpha = 1,\ldots,D), \tag{58}$$

since $\hat{f}_i^{\text{eq}(1)}$ depends only on $\omega^{(1)}$ and $\hat{u}_\alpha^{(1)}$. Thus, the solvability condition is Eq. (49) with only $g_i=1$ and $\hat{c}_{i\alpha}$. When the

solvability condition is satisfied, the solution is given by Eq. (50). Substituting this solution and the leading-order solution $\hat{f}_i^{(1)} = \hat{f}_i^{\mathrm{eq}(1)}$ into the solvability condition, we get the equations for the macroscopic variables in the following way. First, from the solvability condition (49) with $m=2$ and $\tau^{(1)}=0$, we get

$$\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\alpha} = 0, \tag{59a}$$

$$\frac{\partial P^{(1)}}{\partial \hat{x}_\alpha} = 0, \tag{59b}$$

$$\tau^{(1)} = 0. \tag{59c}$$

Here, $P^{(1)}$ is independent of $\hat{x}_\alpha$ from Eq. (59b). From Eq. (59c) and the equation of state (39d), we find that $\omega^{(1)}$ is also independent of $\hat{x}_\alpha$. In an open-domain problem, where the time-independent density is specified, for example, at infinity, $\omega^{(1)}$ can be put to zero by choosing the reference state at infinity. In a closed-domain problem, where the total mass in the domain is specified, $\omega^{(1)}$ can be put to zero by choosing the average density as the reference density. Thus, we can put

$$\omega^{(1)} = P^{(1)} = 0, \tag{60}$$

without losing generality.

From the solvability condition (49) with $m=3$ and $\tau^{(2)}=0$, we have

$$\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\alpha} = 0, \tag{61a}$$

$$B\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{t}} + \hat{u}_\beta^{(1)}\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta} + \frac{\partial P^{(2)}}{\partial \hat{x}_\alpha} = \frac{\partial^2 \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta^2}, \tag{61b}$$

$$\tau^{(2)} = 0. \tag{61c}$$

From the solvability condition (49) with $m=4$ and the definition of $\tau^{(3)}=0$,

$$B\frac{\partial \omega^{(2)}}{\partial \hat{t}} + \hat{u}_\alpha^{(1)}\frac{\partial \omega^{(2)}}{\partial \hat{x}_\alpha} + \frac{\partial \hat{u}_\alpha^{(3)}}{\partial \hat{x}_\alpha} = 0, \tag{62a}$$

$$B\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{t}} + \hat{u}_\beta^{(1)}\frac{\partial \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\beta} + \hat{u}_\beta^{(2)}\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta} + \frac{\partial P^{(3)}}{\partial \hat{x}_\alpha} = \frac{\partial^2 \hat{u}_\alpha^{(2)}}{\partial \hat{x}_\beta^2} + \hat{R}_\alpha, \tag{62b}$$

$$\tau^{(3)} = 0, \tag{62c}$$

where $\hat{R}_\alpha$ is given by Eq. (54a). We can obtain the higher-order sets in a similar way.

The component functions $\hat{u}_\alpha^{(m)}$ and $P^{(m)}$ ($\omega^{(m)}$ is determined by the equation of state and $\tau^{(m)}=0$) of the macroscopic variables $\hat{u}_\alpha$ and P are determined in the following way: first, $P^{(1)}=0$ from Eq. (60). Next, $\hat{u}_\alpha^{(1)}$ and $P^{(2)}$ are determined by Eqs. (59a) and (61b). Then, $\hat{u}_\alpha^{(2)}$ and $P^{(3)}$ are

determined by Eqs. (61a) and (62b). The higher-order sets are also determined in a similar way.

Considering the ordering of each term in the leading-order set [Eqs. (59a) and (61b)], we find that this set corresponds to the incompressible NS set (55a) and (55b) [without the energy equation (55c)], whose parameter Re is given by Eq. (57). Proceeding to the next-order set of equations (61a) and (62b), however, we find that the inhomogeneous term $\hat{R}_\alpha$ appears on the right-hand side of Eq. (62b). If $\hat{R}_\alpha$ is irrotational, $\hat{R}_\alpha$ is represented by $\hat{R}_\alpha = \partial \Phi/\partial \hat{x}_\alpha$, and this term can be incorporated into the pressure term of the momentum equation by introducing $\bar{P}^{(3)} = P^{(3)} - \Phi$. This set of equations (61a) and (62b) therefore admits zero solution when $\hat{R}_\alpha$ is irrotational, while it has nonzero solution when $\hat{R}_\alpha$ is not irrotational. The next-order set evidently includes inhomogeneous terms, for instance, its continuity equation (62a) includes the inhomogeneous terms $B\partial \omega^{(2)}/\partial \hat{t} + \hat{u}_\alpha^{(1)}\partial \omega^{(2)}/\partial \hat{x}_\alpha$, and the solution at this order becomes nonzero. Thus, the relative error of the thermal model is given by

$$O(\varepsilon^2) \quad \text{if } \hat{R}_\alpha \text{ is irrotational,}$$

$$O(\varepsilon) \quad \text{otherwise,} \tag{63}$$

where $\hat{R}_\alpha$ is given by Eq. (54a). Junk *et al.* [13] made the asymptotic analysis independently and obtained the result that the relative error of any isothermal model is $O(\varepsilon^2)$. It is because they used the symmetry condition $\Sigma_{i=1}^N \hat{c}_{i\beta_1}\cdots\hat{c}_{i\beta_{2r-1}} = 0$ (r is an integer) of the discrete velocity set, and this together with Eq. (60) makes $\hat{R}_\alpha$ irrotational. We avoided using this symmetry in the analysis.

VI. ASYMPTOTIC ANALYSIS FOR SMALL ε WITH Ma$\sim$1 (Re$\gg$1)

We consider here two typical flow patterns for Ma$\sim$1. First is a flow around a simple boundary without any shocks and contact discontinuities. Second is a flow without any boundary but with shocks and contact discontinuities. In the former case, the thin viscous boundary layer is formed near the boundary. In the layer, the variation of the state is anisotropic, while it is isotropic outside the layer called the Euler region. The two regions (Euler region and viscous boundary layer) therefore should be treated separately in the analysis. The Euler region is treated in Sec. VI A, and the viscous boundary layer is treated in Sec. VI B [17,30]. In the latter case, where the shock waves and contact discontinuities appear, the solution includes discontinuities, and they must be taken into account correctly in the analysis. Since the usual LBM scheme (4) is not consistent with the kinetic equation (3) itself, but with the integral relation (6), the (nondimensional) integral relation (13) is used as a basic equation. Then the macroscopic variables obtained from the integral relation are found to satisfy, at their leading order in ε, the weak form of the Euler equations, which can correctly describe the behavior of a gas with shocks and contact discontinuities [31–33]. This last case is treated in Sec. VI C.

A. Euler region

1. Compressible model

Consider a case where the deviation of the distribution function $\hat{f}_i$ is on the order of unity. We then look for the smooth solution whose scale of variation is on the order of unity with respect to the coordinates $\hat{t}$ and $\hat{x}_\alpha$, in a power series of ε [34]:

$$\hat{f}_i = \hat{f}_i^{(0)} + \varepsilon \hat{f}_i^{(1)} + \varepsilon^2 \hat{f}_i^{(2)} + \cdots, \tag{64}$$

where the component function $\hat{f}_i^{(m)}$ is a quantity on the order of unity.

Macroscopic variables are also expanded:

$$\hat{h} = \hat{h}^{(0)} + \varepsilon \hat{h}^{(1)} + \varepsilon^2 \hat{h}^{(2)} + \cdots, \tag{65}$$

where $\hat{h}$ represents any of the macroscopic variables $\hat{\rho}$, $\hat{u}_\alpha$, $\hat{T}$, and $\hat{p}$. The component functions $\hat{h}^{(m)}$ satisfy the following relations from Eqs. (11a)–(11d):

$$\hat{\rho}^{(0)} = \sum_{i=1}^{N} \hat{f}_i^{(0)}, \tag{66a}$$

$$\hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} = \sum_{i=1}^{N} \hat{f}_i^{(0)} \hat{c}_{i\alpha}, \tag{66b}$$

$$D\hat{\rho}^{(0)} \hat{T}^{(0)} + \hat{\rho}^{(0)} (\hat{u}_\alpha^{(0)})^2 = \sum_{i=1}^{N} \hat{f}_i^{(0)} \hat{c}_{i\alpha}^2, \tag{66c}$$

$$\hat{p}^{(0)} = \hat{\rho}^{(0)} \hat{T}^{(0)}, \tag{66d}$$

$$\hat{\rho}^{(1)} = \sum_{i=1}^{N} \hat{f}_i^{(1)}, \tag{67a}$$

$$\hat{\rho}^{(1)} \hat{u}_\alpha^{(0)} + \hat{\rho}^{(0)} \hat{u}_\alpha^{(1)} = \sum_{i=1}^{N} \hat{f}_i^{(1)} \hat{c}_{i\alpha}, \tag{67b}$$

$$D(\hat{\rho}^{(0)} \hat{T}^{(1)} + \rho^{(1)} T^{(0)}) + 2\hat{\rho}^{(0)} \hat{u}_\alpha^{(1)} \hat{u}_\alpha^{(0)} + \hat{\rho}^{(1)} (\hat{u}_\alpha^{(0)})^2 = \sum_{i=1}^{N} \hat{f}_i^{(1)} \hat{c}_{i\alpha}^2, \tag{67c}$$

$$\hat{p}^{(1)} = \hat{\rho}^{(0)} \hat{T}^{(1)} + \rho^{(1)} T^{(0)},$$

$$\cdots. \tag{67d}$$

The equilibrium distribution function is also expanded as follows:

$$\hat{f}_i^{\text{eq}} = \hat{f}_i^{\text{eq}(0)} + \varepsilon \hat{f}_i^{\text{eq}(1)} + \varepsilon^2 \hat{f}_i^{\text{eq}(2)} + \cdots. \tag{68}$$

Here, $\hat{f}_i^{\text{eq}(m)}$ satisfies the following constraints from Eqs. (14a)–(14e):

$$\hat{\rho}^{(0)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(0)}, \tag{69a}$$

$$\hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(0)} \hat{c}_{i\alpha}, \tag{69b}$$

$$\hat{p}^{(0)} \delta_{\alpha\beta} + \hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(0)} \hat{c}_{i\alpha} \hat{c}_{i\beta}, \tag{69c}$$

$$\hat{p}^{(0)} (\hat{u}_\alpha^{(0)} \delta_{\beta\gamma} + \hat{u}_\beta^{(0)} \delta_{\gamma\alpha} + \hat{u}_\gamma^{(0)} \delta_{\alpha\beta}) + \hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} \hat{u}_\gamma^{(0)}$$
$$= \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(0)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}, \tag{69d}$$

$$(D+2)\hat{p}^{(0)} \hat{T}^{(0)} \delta_{\alpha\beta} + \hat{p}^{(0)}[(D+4)\hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} + (\hat{u}_\gamma^{(0)})^2 \delta_{\alpha\beta}]$$
$$+ \hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} (\hat{u}_\gamma^{(0)})^2$$
$$= \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(0)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}^2, \tag{69e}$$

$$\hat{\rho}^{(1)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(1)}, \tag{70a}$$

$$\hat{\rho}^{(0)} \hat{u}_\alpha^{(1)} + \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(1)} \hat{c}_{i\alpha}, \tag{70b}$$

$$\hat{p}^{(1)} \delta_{\alpha\beta} + \hat{\rho}^{(0)} (\hat{u}_\alpha^{(1)} \hat{u}_\beta^{(0)} + \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(1)}) + \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}(1)} \hat{c}_{i\alpha} \hat{c}_{i\beta}, \tag{70c}$$

$$\hat{p}^{(0)}(\hat{u}_\alpha^{(1)} \delta_{\beta\gamma} + \hat{u}_\beta^{(1)} \delta_{\gamma\alpha} + \hat{u}_\gamma^{(1)} \delta_{\alpha\beta}) + \hat{p}^{(1)}(\hat{u}_\alpha^{(0)} \delta_{\beta\gamma} + \hat{u}_\beta^{(0)} \delta_{\gamma\alpha}$$
$$+ \hat{u}_\gamma^{(0)} \delta_{\alpha\beta}) + \hat{\rho}^{(0)}(\hat{u}_\alpha^{(1)} \hat{u}_\beta^{(0)} \hat{u}_\gamma^{(0)} + \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(1)} \hat{u}_\gamma^{(0)} + \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} \hat{u}_\gamma^{(1)})$$
$$+ \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} \hat{u}_\gamma^{(0)}$$
$$= \sum \hat{f}_i^{\text{eq}(1)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}, \tag{70d}$$

$$(D+2)(\hat{p}^{(1)} \hat{T}^{(0)} + \hat{p}^{(0)} \hat{T}^{(1)}) \delta_{\alpha\beta} + \hat{p}^{(0)}[(D+4)(\hat{u}_\alpha^{(1)} \hat{u}_\beta^{(0)}$$
$$+ \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(1)}) + 2\hat{u}_\gamma^{(0)} \hat{u}_\gamma^{(1)} \delta_{\alpha\beta}] + \hat{p}^{(1)}[(D+4)\hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)}$$
$$+ (\hat{u}_\gamma^{(0)})^2 \delta_{\alpha\beta}] + \hat{\rho}^{(0)}[\hat{u}_\alpha^{(1)} \hat{u}_\beta^{(0)} (\hat{u}_\gamma^{(0)})^2 + \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(1)} (\hat{u}_\gamma^{(0)})^2$$
$$+ 2\hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} \hat{u}_\gamma^{(0)} \hat{u}_\gamma^{(1)}] + \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} (\hat{u}_\gamma^{(0)})^2$$
$$= \sum \hat{f}_i^{\text{eq}(1)} \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}^2,$$

$$\cdots. \tag{70e}$$

Substituting Eqs. (64) and (68) into Eq. (12a) and noting that Sh is on the order of unity from Eq. (17) with Ma ~ 1, we arrange the same order terms in ε. We then obtain the following series of equations for $\hat{f}_i^{(m)}$:

$$\hat{f}_i^{\text{eq}(0)} - \hat{f}_i^{(0)} = 0, \tag{71a}$$

$$L_i(\hat{f}_j^{(m)}) = -Q_i^{(m)}(\hat{f}_j^{(0)}, \dots, \hat{f}_j^{(m-1)}) + \text{Sh}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{t}}$$

$$+ \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{x}_\beta} \quad (m=1,2,\dots), \tag{71b}$$

where

$$L_i(\hat{f}_j^{(m)}) + Q_i^{(m)}(\hat{f}_j^{(0)}, \dots, \hat{f}_j^{(m-1)}) = \hat{f}_i^{\text{eq}(m)} - \hat{f}_i^{(m)}, \tag{71c}$$

and $L_i(\hat{f}_j^{(m)})$ is the collision operator of the homogeneous part, which is specifically given by

$$L_i(\hat{f}_j^{(m)}) = \frac{\partial \hat{f}_i^{\text{eq}(0)}}{\partial \hat{\rho}^{(0)}}\sum_{j=1}^{N}\hat{f}_j^{(m)} + \frac{\partial \hat{f}_i^{\text{eq}(0)}}{\partial(\hat{\rho}^{(0)}\hat{u}_\alpha^{(0)})}\sum_{j=1}^{N}\hat{f}_j^{(m)}\hat{c}_{j\alpha}$$

$$+ \frac{\partial \hat{f}_i^{\text{eq}(0)}}{\partial[D\hat{p}^{(0)} + \hat{\rho}^{(0)}(\hat{u}_\alpha^{(0)})^2]}\sum_{j=1}^{N}\hat{f}_j^{(m)}\hat{c}_{j\alpha}^2 - \hat{f}_i^{(m)}. \tag{72}$$

Here, the derivatives of $\hat{f}_i^{\text{eq}(0)}$ with respect to $\hat{\rho}^{(0)}$, $\hat{\rho}^{(0)}\hat{u}_\alpha^{(0)}$ ($\alpha=1,\dots,D$), or $D\hat{p}^{(0)} + \hat{\rho}^{(0)}(\hat{u}_\alpha^{(0)})^2$ are taken with the other $D+1$ variables among $\hat{\rho}^{(0)}$, $\hat{\rho}^{(0)}\hat{u}_\alpha^{(0)}$, and $D\hat{p}^{(0)} + \hat{\rho}^{(0)}(\hat{u}_\alpha^{(0)})^2$ being fixed. $Q_i^{(m)}$ is the remaining collision operator of the inhomogeneous part composed of the lower-order component functions. $L_i(\hat{f}_j^{(m)})$ and $Q_i^{(m)}$ have the following properties for their moments:

$$\sum_{i=1}^{N} g_i L_i(\hat{f}_j^{(m)}) = 0, \quad \sum_{i=1}^{N} g_i Q_i^{(m)} = 0, \tag{73}$$

where g_i is given by Eq. (28b).

The solution of Eq. (71a) is

$$\hat{f}_i^{(0)} = \hat{f}_i^{\text{eq}(0)}. \tag{74}$$

Equation (71b) for $\hat{f}_i^{(m)}$ is inhomogeneous, and for Eq. (71b) to have a solution, its inhomogeneous term must satisfy the following relation (solvability condition):

$$\sum_{i=1}^{N} g_i\left(\text{Sh}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{t}} + \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{x}_\beta}\right) = 0 \quad (m=1,2,\dots),$$

$$\tag{75}$$

since $L_i(\hat{f}_j^{(m)})$ and $Q_i^{(m)}$ satisfy Eqs. (73). When condition (75) is satisfied, the solution of Eq. (71b) is given by

$$\hat{f}_i^{(m)} = \hat{f}_i^{\text{eq}(m)} - \text{Sh}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{t}} - \hat{c}_{i\beta}\frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{x}_\beta} \quad (m=1,2,\dots).$$

$$\tag{76}$$

Substituting Eqs. (74) and (76) into the solvability condition (75), we can get the equations for the macroscopic variables in the following way. From the solvability condition (75) with $m=1$,

$$\text{Sh}\frac{\partial \hat{\rho}^{(0)}}{\partial \hat{t}} + \frac{\partial \hat{\rho}^{(0)}\hat{u}_\alpha^{(0)}}{\partial \hat{x}_\alpha} = 0, \tag{77a}$$

$$\hat{\rho}^{(0)}\left(\text{Sh}\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{t}} + \hat{u}_\beta^{(0)}\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta}\right) + \frac{\partial \hat{p}^{(0)}}{\partial \hat{x}_\alpha} = 0, \tag{77b}$$

$$\frac{D}{2}\hat{\rho}^{(0)}\left(\text{Sh}\frac{\partial \hat{T}^{(0)}}{\partial \hat{t}} + \hat{u}_\beta^{(0)}\frac{\partial \hat{T}^{(0)}}{\partial \hat{x}_\beta}\right) + \hat{p}^{(0)}\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\alpha} = 0, \tag{77c}$$

From the solvability condition (75) with $m=2$,

$$\text{Sh}\frac{\partial \hat{\rho}^{(1)}}{\partial \hat{t}} + \frac{\partial(\hat{\rho}^{(0)}\hat{u}_\alpha^{(1)} + \hat{\rho}^{(1)}\hat{u}_\alpha^{(0)})}{\partial \hat{x}_\alpha} = 0, \tag{78a}$$

$$\text{Sh}\left(\hat{\rho}^{(0)}\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{t}} + \hat{\rho}^{(1)}\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{t}}\right) + \hat{\rho}^{(0)}\hat{u}_\beta^{(0)}\frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta} + (\hat{\rho}^{(1)}\hat{u}_\beta^{(0)}$$

$$+ \hat{\rho}^{(0)}\hat{u}_\beta^{(1)})\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{p}^{(1)}}{\partial \hat{x}_\alpha}$$

$$= \frac{\partial}{\partial \hat{x}_\beta}\left[\hat{p}^{(0)}\left(\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{u}_\beta^{(0)}}{\partial \hat{x}_\alpha} - \frac{2}{3}\frac{\partial \hat{u}_\gamma^{(0)}}{\partial \hat{x}_\gamma}\delta_{\alpha\beta}\right)\right.$$

$$\left. + 2\hat{p}^{(0)}\left(\frac{1}{3} - \frac{1}{D}\right)\frac{\partial \hat{u}_\gamma^{(0)}}{\partial \hat{x}_\gamma}\delta_{\alpha\beta}\right], \tag{78b}$$

$$\frac{D}{2}\left[\text{Sh}\left(\hat{\rho}^{(0)}\frac{\partial \hat{T}^{(1)}}{\partial \hat{t}} + \hat{\rho}^{(1)}\frac{\partial \hat{T}^{(0)}}{\partial \hat{t}}\right) + \hat{\rho}^{(0)}\hat{u}_\beta^{(0)}\frac{\partial \hat{T}^{(1)}}{\partial \hat{x}_\beta} + (\hat{\rho}^{(0)}\hat{u}_\beta^{(1)}\right.$$

$$\left. + \hat{\rho}^{(1)}\hat{u}_\beta^{(0)})\frac{\partial \hat{T}^{(0)}}{\partial \hat{x}_\beta}\right] + \hat{p}^{(0)}\frac{\partial \hat{u}_\beta^{(1)}}{\partial \hat{x}_\beta} + \hat{p}^{(1)}\frac{\partial \hat{u}_\beta^{(0)}}{\partial \hat{x}_\beta}$$

$$= \frac{D+2}{2}\frac{\partial}{\partial \hat{x}_\alpha}\left(\hat{p}^{(0)}\frac{\partial \hat{T}^{(0)}}{\partial \hat{x}_\alpha}\right) + \frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta}\left[\hat{p}^{(0)}\left(\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{u}_\beta^{(0)}}{\partial \hat{x}_\alpha}\right.\right.$$

$$\left.\left. - \frac{2}{3}\frac{\partial \hat{u}_\gamma^{(0)}}{\partial \hat{x}_\gamma}\delta_{\alpha\beta}\right) + 2\hat{p}^{(0)}\left(\frac{1}{3} - \frac{1}{D}\right)\frac{\partial \hat{u}_\gamma^{(0)}}{\partial \hat{x}_\gamma}\delta_{\alpha\beta}\right]. \tag{78c}$$

We can obtain the higher-order sets in a similar way.

The above series of equations, together with the equation of state, constitutes systems of equations that determine the component functions $\hat{\rho}^{(m)}$, $\hat{u}_\alpha^{(m)}$, $\hat{T}^{(m)}$, and $\hat{p}^{(m)}$ of the macroscopic variables $\hat{\rho}$, $\hat{u}_\alpha$, $\hat{T}$, and $\hat{p}$. First, $\hat{\rho}^{(0)}$, $\hat{u}_\alpha^{(0)}$, $\hat{T}^{(0)}$, and $\hat{p}^{(0)}$ are determined by Eqs. (77a)–(77c). Next, $\hat{\rho}^{(1)}$, $\hat{u}_\alpha^{(1)}$, $\hat{T}^{(1)}$, and $\hat{p}^{(1)}$ are determined by Eqs. (78a)–(78c). The higher-order sets are also determined in a similar way.

We find that the leading-order set (79a)–(79c) corresponds to the compressible Euler set of equations, which correctly describes the flow with a finite Mach number of a gas outside the boundary layer. The compressible Euler set is given, in its nondimensional form, as

$$\text{Sh}\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{\partial \hat{\rho}\hat{u}_\alpha}{\partial \hat{x}_\alpha} = 0, \tag{79a}$$

$$\hat{\rho}\left(\mathrm{Sh}\frac{\partial \hat{u}_\alpha}{\partial \hat{t}} + \hat{u}_\beta \frac{\partial \hat{u}_\alpha}{\partial \hat{x}_\beta}\right) + \frac{\partial \hat{p}}{\partial \hat{x}_\alpha} = 0, \qquad (79\text{b})$$

$$\frac{C_v}{R}\hat{\rho}\left(\mathrm{Sh}\frac{\partial \hat{T}}{\partial \hat{t}} + \hat{u}_\alpha \frac{\partial \hat{T}}{\partial \hat{x}_\alpha}\right) + \hat{p}\frac{\partial \hat{u}_\alpha}{\partial \hat{x}_\alpha} = 0, \qquad (79\text{c})$$

with

$$\hat{p} = \hat{\rho}\hat{T}, \qquad (79\text{d})$$

where C_v is the specific heat at constant volume. Comparing the leading-order set (77a)–(77c) obtained from the LBM with the compressible Euler set [or Eqs. (79a)–(79c)], we find that the former set corresponds to the latter set whose specific heat at constant volume is given by

$$C_v = \frac{D}{2}R. \qquad (80)$$

Proceeding to the next-order set of equations (78a)–(78c), however, we find that it evidently includes inhomogeneous terms. For example, the momentum equation (78b) includes the viscous terms composed of the leading-order solution on its right-hand side. Thus, the solution at this order becomes nonzero and contributes to a deviation from the correct solution of the leading-order set. The relative error is therefore $O(\varepsilon)$.

2. Thermal model

The analytical process is almost the same as that of the compressible model, since the differences between the two models are represented only by the nonlinear terms of the flow velocity in Eqs. (15a) and (15b). However, we find that these differences appear at the leading-order set of fluid-dynamics-type equations. Specifically, the energy equation (77c) includes an additional third-order term of the flow velocity which is absent in the compressible Euler set. Thus, this model cannot describe the flow with a finite Mach number. This model is not treated any more in the subsequent analysis of the present section.

3. Isothermal model

This model gives $\hat{T}^{(0)}=1$ at the leading order, and it is evident that the energy equation is not derived at this order properly. Thus, the isothermal model cannot describe the flow with a finite Mach number. This model is not treated any more in the subsequent analysis of the present section.

B. Viscous boundary layer

Compressible model

In the viscous boundary layer, the variation of the state is strongly anisotropic. The length scale of the variation in the direction normal to the boundary is shrunk by the factor of $1/\sqrt{\mathrm{Re}}$ than that in the direction parallel to the boundary. In order to extract the effect of this anisotropy on the accuracy of the LBM and to avoid complexities, the boundary is assumed to be a plane. Under this assumption, we rotate and

transfer the spatial coordinates so as to locate the $\hat{x}_3=0$ plane on that of the boundary.

Following the above estimate of the length scale of variation, we introduce the stretched coordinate y in the $\hat{x}_3$ direction, which is normal to the boundary:

$$y = \hat{x}_3/\delta, \qquad (81)$$

where

$$\delta = \sqrt{\varepsilon}. \qquad (82)$$

We look for the solution whose scale of variation is on the order of unity with respect to $\hat{t}$, $\hat{x}_\alpha$ ($\alpha=1,\ldots,D-1$), and y in a power series of δ [17,30]:

$$\hat{f}_i = \hat{f}_i^{(0)} + \delta\hat{f}_i^{(1)} + \delta^2\hat{f}_i^{(2)} + \cdots. \qquad (83)$$

Macroscopic variables are also expanded in terms of δ:

$$\hat{h} = \hat{h}^{(0)} + \delta\hat{h}^{(1)} + \delta^2\hat{h}^{(2)} + \cdots, \qquad (84)$$

where $\hat{h}$ represents any of the macroscopic variables $\hat{\rho}$, $\hat{u}_\alpha$, $\hat{T}$, and $\hat{p}$. The component functions $\hat{h}^{(m)}$ satisfy the same relations as Eqs. (66a)–(66d) and (67a)–(67d), and their higher orders. The equilibrium distribution function is also expanded as follows:

$$\hat{f}_i^{\mathrm{eq}} = \hat{f}_i^{\mathrm{eq}(0)} + \delta\hat{f}_i^{\mathrm{eq}(1)} + \delta^2\hat{f}_i^{\mathrm{eq}(2)} + \cdots. \qquad (85)$$

The component function $\hat{f}_i^{\mathrm{eq}(m)}$ satisfies the same constraints as Eqs. (69a)–(69e) and (70a)–(70e), and their higher orders.

Before proceeding to the analysis, we note that, hereafter in this section, we use double indices of α (or β) with $\Sigma_{\alpha=1}^{D-1}$ sign as representing the summation for α (or β) $=1,\ldots,D-1$, e.g., $\Sigma_{\alpha=1}^{D-1}(\hat{u}_\alpha^{(0)})^2=(\hat{u}_1^{(0)})^2$ for $D=2$, and $\Sigma_{\alpha=1}^{D-1}(\hat{u}_\alpha^{(0)})^2=(\hat{u}_1^{(0)})^2+(\hat{u}_2^{(0)})^2$ for $D=3$. Now we return to the main analysis. Substituting Eqs. (83) and (85) into Eq. (12a) and noting that Sh is on the order of unity, we arrange the same order terms in δ. We then obtain the following series of equations for $\hat{f}_i^{(m)}$:

$$\hat{f}_i^{\mathrm{eq}(0)} - \hat{f}_i^{(0)} = 0, \qquad (86\text{a})$$

$$L_i(\hat{f}_j^{(m)}) = -Q_i^{(m)}(\hat{f}_j^{(0)},\ldots,\hat{f}_j^{(m-1)}) + \mathrm{Sh}\frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{t}} + \sum_{\alpha=1}^{D-1} \hat{c}_{i\alpha}\frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{x}_\alpha}$$
$$+ \hat{c}_{i3}\frac{\partial \hat{f}_i^{(m-1)}}{\partial y} \quad (m=1,2,\ldots), \qquad (86\text{b})$$

where $\hat{f}_i^{(-1)}=0$, and

$$L_i(\hat{f}_j^{(m)}) + Q_i^{(m)}(\hat{f}_j^{(0)},\ldots,\hat{f}_j^{(m-1)}) = \hat{f}_i^{\mathrm{eq}(m)} - \hat{f}_i^{(m)}. \qquad (86\text{c})$$

Here, $L_i(\hat{f}_j^{(m)})$ is the collision operator of the homogeneous part and $Q_i^{(m)}$ is the remaining collision operator of the inhomogeneous part composed of the lower-order component functions, both of which have properties (73) for their moments.

The solution of Eq. (86a) is

$$\hat{f}_i^{(0)} = \hat{f}_i^{\text{eq}(0)}. \tag{87}$$

For Eq. (86b) to have a solution, its inhomogeneous term must satisfy the following relation (solvability condition):

$$\sum_{i=1}^{N} g_i \left(\text{Sh} \frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{t}} + \sum_{\alpha=1}^{D-1} \hat{c}_{i\alpha} \frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{x}_\alpha} + \hat{c}_{i3} \frac{\partial \hat{f}_i^{(m-1)}}{\partial y} \right) = 0$$

$$(m = 1, 2, \ldots), \tag{88}$$

since $L_i(\hat{f}_j^{(m)})$ and $Q_i^{(m)}$ satisfy Eqs. (73). When condition (88) is satisfied, the solution of Eq. (86b) is given by

$$\hat{f}_i^{(m)} = \hat{f}_i^{\text{eq}(m)} - \text{Sh} \frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{t}} - \sum_{\alpha=1}^{D-1} \hat{c}_{i\alpha} \frac{\partial \hat{f}_i^{(m-2)}}{\partial \hat{x}_\alpha} - \hat{c}_{i3} \frac{\partial \hat{f}_i^{(m-1)}}{\partial y}$$

$$(m = 1, 2, \ldots). \tag{89}$$

Substituting Eqs. (87) and (89) into the solvability condition (88), we can get the equations for the macroscopic variables in the following way. First, from the solvability condition (88) with $m=1$,

$$\frac{\partial \hat{\rho}^{(0)} \hat{u}_3^{(0)}}{\partial y} = 0, \tag{90a}$$

$$\frac{\partial \hat{p}^{(0)}}{\partial y} = 0, \tag{90b}$$

and the other D relations are reduced to identities with the aid of the following Eq. (91a). From Eqs. (90a) and (90b) and the boundary condition $\hat{u}_3^{(0)}=0$ at $\hat{x}_3=0$, we get

$$\hat{u}_3^{(0)} = 0, \tag{91a}$$

$$\hat{p}^{(0)} = \hat{p}^{(0)}(\hat{t}, \hat{x}_\alpha). \tag{91b}$$

From the solvability condition (88) with $m=2$,

$$\text{Sh} \frac{\partial \hat{\rho}^{(0)}}{\partial \hat{t}} + \sum_{\alpha=1}^{D-1} \frac{\partial \hat{\rho}^{(0)} \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\alpha} + \frac{\partial \hat{\rho}^{(0)} \hat{u}_3^{(1)}}{\partial y} = 0, \tag{92a}$$

$$\hat{\rho}^{(0)} \left(\text{Sh} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{t}} + \sum_{\beta=1}^{D-1} \hat{u}_\beta^{(0)} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta} + \hat{u}_3^{(1)} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} \right) + \frac{\partial \hat{p}^{(0)}}{\partial \hat{x}_\alpha}$$

$$= \frac{\partial}{\partial y} \left(\hat{p}^{(0)} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} \right) \quad (\alpha = 1, \ldots, D-1), \tag{92b}$$

$$\frac{\partial \hat{p}^{(1)}}{\partial y} = 0, \tag{92c}$$

$$\frac{D}{2} \hat{\rho}^{(0)} \left(\text{Sh} \frac{\partial \hat{T}^{(0)}}{\partial \hat{t}} + \sum_{\beta=1}^{D-1} \hat{u}_\beta^{(0)} \frac{\partial \hat{T}^{(0)}}{\partial \hat{x}_\beta} + \hat{u}_3^{(1)} \frac{\partial \hat{T}^{(0)}}{\partial y} \right)$$

$$+ \hat{p}^{(0)} \left(\sum_{\alpha=1}^{D-1} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\alpha} + \frac{\partial \hat{u}_3^{(1)}}{\partial y} \right)$$

$$= \frac{\partial}{\partial y} \left(\frac{D+2}{2} \hat{p}^{(0)} \frac{\partial \hat{T}^{(0)}}{\partial y} \right) + \sum_{\alpha=1}^{D-1} \hat{p}^{(0)} \left(\frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} \right)^2. \tag{92d}$$

From the solvability condition (88) with $m=3$,

$$\text{Sh} \frac{\partial \hat{\rho}^{(1)}}{\partial \hat{t}} + \sum_{\alpha=1}^{D-1} \frac{\partial (\hat{\rho}^{(0)} \hat{u}_\alpha^{(1)} + \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)})}{\partial \hat{x}_\alpha} + \frac{\partial (\hat{\rho}^{(0)} \hat{u}_3^{(2)} + \hat{\rho}^{(1)} \hat{u}_3^{(1)})}{\partial y} = 0, \tag{93a}$$

$$\text{Sh} \left(\hat{\rho}^{(0)} \frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{t}} + \hat{\rho}^{(1)} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{t}} \right) + \sum_{\beta=1}^{D-1} \left[\hat{\rho}^{(0)} \hat{u}_\beta^{(0)} \frac{\partial \hat{u}_\alpha^{(1)}}{\partial \hat{x}_\beta} + (\hat{\rho}^{(0)} \hat{u}_\beta^{(1)} \right.$$

$$\left. + \hat{\rho}^{(1)} \hat{u}_\beta^{(0)}) \frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta} \right] + \hat{\rho}^{(0)} \hat{u}_3^{(1)} \frac{\partial \hat{u}_\alpha^{(1)}}{\partial y}$$

$$+ (\hat{\rho}^{(0)} \hat{u}_3^{(2)} + \hat{\rho}^{(1)} \hat{u}_3^{(1)}) \frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} + \frac{\partial \hat{p}^{(1)}}{\partial \hat{x}_\alpha}$$

$$= \frac{\partial}{\partial y} \left(\hat{p}^{(0)} \frac{\partial \hat{u}_\alpha^{(1)}}{\partial y} + \hat{p}^{(1)} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} \right) + \bar{R}_\alpha \quad (\alpha = 1, \ldots, D-1), \tag{93b}$$

$$\hat{\rho}^{(0)} \left(\text{Sh} \frac{\partial \hat{u}_3^{(1)}}{\partial \hat{t}} + \sum_{\beta=1}^{D-1} \hat{u}_\beta^{(0)} \frac{\partial \hat{u}_3^{(1)}}{\partial \hat{x}_\beta} + \hat{u}_3^{(1)} \frac{\partial \hat{u}_3^{(1)}}{\partial y} \right) + \frac{\partial \hat{p}^{(2)}}{\partial y}$$

$$= \sum_{\alpha=1}^{D-1} \frac{\partial}{\partial \hat{x}_\alpha} \left(2\hat{p}^{(0)} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} \right) + \frac{\partial}{\partial y} \left(3\hat{p}^{(0)} \frac{\partial \hat{u}_3^{(1)}}{\partial y} \right) + \bar{R}_3, \tag{93c}$$

$$\frac{D}{2} \left\{ \text{Sh} \left(\hat{\rho}^{(0)} \frac{\partial \hat{T}^{(1)}}{\partial \hat{t}} + \hat{\rho}^{(1)} \frac{\partial \hat{T}^{(0)}}{\partial \hat{t}} \right) + \sum_{\beta=1}^{D-1} \left[\hat{\rho}^{(0)} \hat{u}_\beta^{(0)} \frac{\partial \hat{T}^{(1)}}{\partial \hat{x}_\beta} + (\hat{\rho}^{(0)} \hat{u}_\beta^{(1)} \right. \right.$$

$$\left. + \hat{\rho}^{(1)} \hat{u}_\beta^{(0)}) \frac{\partial \hat{T}^{(0)}}{\partial \hat{x}_\beta} \right] + \hat{\rho}^{(0)} \hat{u}_3^{(1)} \frac{\partial \hat{T}^{(1)}}{\partial y} + (\hat{\rho}^{(0)} \hat{u}_3^{(2)}$$

$$\left. + \hat{\rho}^{(1)} \hat{u}_3^{(1)}) \frac{\partial \hat{T}^{(0)}}{\partial y} \right\} + \hat{p}^{(0)} \left(\sum_{\beta=1}^{D-1} \frac{\partial \hat{u}_\beta^{(1)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{u}_3^{(2)}}{\partial y} \right) + \hat{p}^{(1)}$$

$$\times \left(\sum_{\beta=1}^{D-1} \frac{\partial \hat{u}_\beta^{(0)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{u}_3^{(1)}}{\partial y} \right)$$

$$= \frac{D+2}{2} \frac{\partial}{\partial y} \left(\hat{p}^{(0)} \frac{\partial \hat{T}^{(1)}}{\partial y} + \hat{p}^{(1)} \frac{\partial \hat{T}^{(0)}}{\partial y} \right)$$

$$+ \sum_{\alpha=1}^{D-1} \left[2\hat{p}^{(0)} \frac{\partial \hat{u}_\alpha^{(1)}}{\partial y} \frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} + \hat{p}^{(1)} \left(\frac{\partial \hat{u}_\alpha^{(0)}}{\partial y} \right)^2 \right] + \bar{R}_4, \tag{93d}$$

where

$$\bar{R}_\alpha = -\frac{\partial^3}{\partial y^3}\left(\sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(0)} \hat{c}_{i\alpha}\hat{c}_{i3}^3\right), \qquad (94a)$$

$$\bar{R}_3 = -\frac{\partial^3}{\partial y^3}\left(\sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(0)} \hat{c}_{i3}^4\right), \qquad (94b)$$

$$\bar{R}_4 = -\sum_{\alpha=1}^{D-1} \bar{R}_\alpha \hat{u}_\alpha^{(0)} - \frac{1}{2}\frac{\partial^3}{\partial y^3}\left[\sum_{i=1}^N \hat{f}_i^{\mathrm{eq}(0)} \hat{c}_{i3}^3\left(\sum_{\alpha=1}^{D-1} \hat{c}_{i\alpha}^2 + \hat{c}_{i3}^2\right)\right]. \qquad (94c)$$

We can obtain the higher-order sets in a similar way.

The series of equations obtained from the solvability condition (88), together with the equation of state, constitutes systems of equations that determine the component functions $\hat{\rho}^{(m)}$, $\hat{u}_\alpha^{(m)}$ ($\alpha=1,\dots,D-1$), $\hat{u}_3^{(m)}$, $\hat{T}^{(m)}$, and $\hat{p}^{(m)}$ of the macroscopic variables $\hat{\rho}$, $\hat{u}_\alpha$, $\hat{u}_3$, $\hat{T}$, and $\hat{p}$. First, $\hat{u}_3^{(0)}=0$ from Eq. (91a). Next, $\hat{\rho}^{(0)}$, $\hat{u}_\alpha^{(0)}$, $\hat{u}_3^{(1)}$, $\hat{T}^{(0)}$, and $\hat{p}^{(0)}$ are determined by Eqs. (92a), (92b), (92d), and (91b). Then, $\hat{\rho}^{(1)}$, $\hat{u}_\alpha^{(1)}$, $\hat{u}_3^{(2)}$, $\hat{T}^{(1)}$, and $\hat{p}^{(1)}$ are determined by Eqs. (93a), (93b), (93d), and (92c). The higher-order sets are also determined in a similar way.

We see that the leading-order set of equations (92a), (92b), (92d), and (91b) corresponds to the viscous-boundary-layer set of equations that correctly describes the flow with a finite Mach number adjacent to the boundary. The viscous-boundary-layer set is given, in its nondimensional form, as

$$\mathrm{Sh}\frac{\partial\hat{\rho}}{\partial\hat{t}} + \sum_{\alpha=1}^{D-1}\frac{\partial\hat{\rho}\hat{u}_\alpha}{\partial\hat{x}_\alpha} + \frac{\partial\hat{\rho}\hat{u}_3}{\partial\hat{x}_3} = 0, \qquad (95a)$$

$$\hat{\rho}\left(\mathrm{Sh}\frac{\partial\hat{u}_\alpha}{\partial\hat{t}} + \sum_{\beta=1}^{D-1}\hat{u}_\beta\frac{\partial\hat{u}_\alpha}{\partial\hat{x}_\beta} + \hat{u}_3\frac{\partial\hat{u}_\alpha}{\partial\hat{x}_3}\right) + \frac{\partial\hat{p}}{\partial\hat{x}_\alpha}$$
$$= \frac{\partial}{\partial\hat{x}_3}\left(\frac{\mu}{L\sqrt{RT_0}}\frac{\partial\hat{u}_\alpha}{\partial\hat{x}_3}\right) \quad (\alpha=1,\dots,D-1), \quad (95b)$$

$$\frac{\partial\hat{p}}{\partial\hat{x}_3} = 0, \qquad (95c)$$

$$\frac{C_v}{R}\hat{\rho}\left(\mathrm{Sh}\frac{\partial\hat{T}}{\partial\hat{t}} + \sum_{\alpha=1}^{D-1}\hat{u}_\alpha\frac{\partial\hat{T}}{\partial\hat{x}_\alpha} + \hat{u}_3\frac{\partial\hat{T}}{\partial\hat{x}_3}\right) + \hat{p}\left(\sum_{\alpha=1}^{D-1}\frac{\partial\hat{u}_\alpha}{\partial\hat{x}_\alpha} + \frac{\partial\hat{u}_3}{\partial\hat{x}_3}\right)$$
$$= \frac{\partial}{\partial\hat{x}_3}\left(\frac{\lambda}{LR\sqrt{RT_0}}\frac{\partial\hat{T}}{\partial\hat{x}_3}\right) + \frac{\mu}{L\sqrt{RT_0}}\sum_{\alpha=1}^{D-1}\left(\frac{\partial\hat{u}_\alpha}{\partial\hat{x}_3}\right)^2, \qquad (95d)$$

with

$$\hat{p} = \hat{\rho}\hat{T}, \qquad (95e)$$

where C_v is the specific heat at constant volume, μ is the viscosity, and λ is the thermal conductivity. Considering the ordering of each term in Eqs. (92a), (92b), (92d), and (91b), we find that this set corresponds to the viscous-boundary-layer set (95a)–(95d) whose specific heat at constant volume

C_v and transport coefficients μ and λ are given by

$$C_v = \frac{D}{2}R, \quad \mu = \delta^2 L\sqrt{RT_0}\hat{p} = \sigma RT_0\hat{p},$$

$$\lambda = \delta^2\frac{(D+2)LR\sqrt{RT_0}}{2}\hat{p} = \frac{D+2}{2}\sigma R^2 T_0\hat{p}. \qquad (96)$$

Proceeding to the next-order set (in terms of δ) [Eqs. (93a), (93b), (93d), and (92c)], however, we find that the inhomogeneous terms $\bar{R}_\alpha$ ($\alpha=1,\dots,D-1$) and $\bar{R}_4$ appear on the right-hand sides of Eqs. (93b) and (93d). If all of them vanish, this set admits zero solution, and a deviation from the correct solution does not arise. If $\bar{R}_\alpha$ or $\bar{R}_4$ does not vanish, this set has nonzero solution and produces an error. As for the next-order set, it includes inhomogeneous terms [see, for instance, its momentum equation in the y direction (93c)] and the solution at this order is nonzero. Thus, the relative error is

$$O(\delta^2) \; [=O(\varepsilon)] \quad \text{if } \bar{R}_\alpha = \bar{R}_4 = 0 \quad (\alpha=1,\dots,D-1),$$

$$O(\delta) \; [=O(\varepsilon^{1/2})] \quad \text{otherwise.} \qquad (97a)$$

It is noted that the above boundary-layer analysis has been made after rotating the spatial coordinate so as to locate the $\hat{x}_3=0$ plane on the boundary surface. Thus, we should rewrite the above error estimate in the more general form as

$$O(\varepsilon) \quad \text{if the components of } \bar{R}_\alpha \text{ parallel to the}$$
$$\text{boundary surface vanish and } \bar{R}_4 = 0,$$
$$O(\varepsilon^{1/2}) \quad \text{otherwise.} \qquad (97b)$$

Thermal and isothermal models are not considered here as already mentioned at the end of Sec. VIA.

C. Flows with shocks and contact discontinuities

Compressible model

We consider the nondimensional integral relation (13) as the basic equation of the kinetic-equation system because the usual finite-difference scheme (4) is not consistent with the kinetic equation (3) itself but with the integral relation (6) when the solution includes discontinuities such as shocks and contact discontinuities. We look for the solution of the integral relation (13) whose scale of variation other than discontinuities is on the order of unity with respect to $\hat{t}$ and $\hat{x}_\alpha$, in a power series of ε:

$$\hat{f}_i = \hat{f}_i^{(0)} + \varepsilon\hat{f}_i^{(1)} + \varepsilon^2\hat{f}_i^{(2)} + \cdots, \qquad (98)$$

where the component function $\hat{f}_i^{(m)}$ is a quantity on the order of unity.

Macroscopic variables are also expanded:

$$\hat{h} = \hat{h}^{(0)} + \varepsilon\hat{h}^{(1)} + \varepsilon^2\hat{h}^{(2)} + \cdots, \qquad (99)$$

where $\hat{h}$ represents any of the macroscopic variables $\hat{\rho}$, $\hat{u}_\alpha$, $\hat{T}$, and $\hat{p}$. The component functions $\hat{h}^{(m)}$ satisfy the same rela-

tions as Eqs. (66a)–(66d) and (67a)–(67d) and their higher orders. The equilibrium distribution function is also expanded as

$$\hat{f}_i^{\text{eq}} = \hat{f}_i^{\text{eq}(0)} + \varepsilon \hat{f}_i^{\text{eq}(1)} + \varepsilon^2 \hat{f}_i^{\text{eq}(2)} + \cdots, \quad (100)$$

where $\hat{f}_i^{\text{eq}(m)}$ satisfies the same constraints as Eqs. (69a)–(69e) and (70a)–(70e) and their higher orders.

Substituting Eqs. (98) and (100) into Eq. (13) and arranging the same order terms in ε, we obtain the following series of equations for $\hat{f}_i^{(m)}$:

$$\int \int (\hat{f}_i^{\text{eq}(0)} - \hat{f}_i^{(0)}) \vartheta d\hat{t} d\hat{\boldsymbol{x}} = 0, \quad (101a)$$

$$\int \int L_i(\hat{f}_j^{(m)}) \vartheta d\hat{t} d\hat{\boldsymbol{x}}$$
$$= -\int \int \left[Q_i^{(m)}(\hat{f}_j^{(0)}, \ldots, \hat{f}_j^{(m-1)}) \vartheta \right.$$
$$+ \left(\text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} + \hat{c}_{i\alpha} \frac{\partial \vartheta}{\partial \hat{x}_\alpha} \right) \hat{f}_i^{(m-1)} \right]$$
$$\times d\hat{t} d\hat{\boldsymbol{x}} \quad (m = 1, 2, \ldots), \quad (101b)$$

where

$$L_i(\hat{f}_j^{(m)}) + Q_i^{(m)}(\hat{f}_j^{(0)}, \ldots, \hat{f}_j^{(m-1)}) = \hat{f}_i^{\text{eq}(m)} - \hat{f}_i^{(m)}, \quad (101c)$$

and $L_i(\hat{f}_j^{(m)})$ is the collision operator of the homogeneous part, which is specifically given by Eq. (72). $L_i(\hat{f}_j^{(m)})$ and $Q_i^{(m)}$ possess properties (73) for their moments.

From Eq. (101a), we get

$$\hat{f}_i^{(0)} = \hat{f}_i^{\text{eq}(0)}, \quad (102)$$

since ϑ is arbitrary. For Eq. (101b) to have a solution, its inhomogeneous term must satisfy the following relation (solvability condition):

$$\int \int \left\{ \sum_{i=1}^N g_i \left(\text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} + \hat{c}_{i\alpha} \frac{\partial \vartheta}{\partial \hat{x}_\alpha} \right) \hat{f}_i^{(m-1)} \right\} d\hat{t} d\hat{\boldsymbol{x}} = 0 \quad (m$$
$$= 1, 2, \ldots), \quad (103)$$

since $L_i(\hat{f}_j^{(m)})$ and $Q_i^{(m)}$ satisfy Eqs. (73) [g_i is defined by Eq. (28b)]. When condition (103) is satisfied, we have a solution of Eq. (101b) in the regions where the solution is smooth as

$$\hat{f}_i^{(m)} = \hat{f}_i^{\text{eq}(m)} - \text{Sh} \frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{t}} - \hat{c}_{i\beta} \frac{\partial \hat{f}_i^{(m-1)}}{\partial \hat{x}_\beta} \quad (m = 1, 2, \ldots), \quad (104)$$

where integration by parts is applied to Eq. (101b).

Substituting Eqs. (102) and (104) into the solvability condition (103) and noting that there is no contribution from the solution $\hat{f}_i^{(m)}$ at its discontinuities, we can get the equations for the macroscopic variables in the following way. From the solvability condition (103) with $m = 1$,

$$\int \int \left(\text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \hat{\rho}^{(0)} + \frac{\partial \vartheta}{\partial \hat{x}_\alpha} \hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} \right) d\hat{t} d\hat{\boldsymbol{x}} = 0, \quad (105a)$$

$$\int \int \left\{ \text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} + \frac{\partial \vartheta}{\partial \hat{x}_\beta} (\hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} + \hat{p}^{(0)} \delta_{\alpha\beta}) \right\} d\hat{t} d\hat{\boldsymbol{x}} = 0, \quad (105b)$$

$$\int \int \left\{ \text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \hat{\rho}^{(0)} \frac{D\hat{T}^{(0)} + \hat{u}_\alpha^{(0)2}}{2} + \frac{\partial \vartheta}{\partial \hat{x}_\beta} \left(\hat{\rho}^{(0)} \frac{D\hat{T}^{(0)} + \hat{u}_\alpha^{(0)2}}{2} \right. \right.$$
$$\left. \left. + \hat{p}^{(0)} \right) \hat{u}_\beta^{(0)} \right\} d\hat{t} d\hat{\boldsymbol{x}} = 0. \quad (105c)$$

From the solvability condition (103) with $m = 2$,

$$\int \int \left[\text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \hat{\rho}^{(1)} + \frac{\partial \vartheta}{\partial \hat{x}_\alpha} (\hat{\rho}^{(0)} \hat{u}_\alpha^{(1)} + \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)}) \right] d\hat{t} d\hat{\boldsymbol{x}} = 0, \quad (106a)$$

$$\int \int \left\{ \text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} (\hat{\rho}^{(0)} \hat{u}_\alpha^{(1)} + \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)}) + \frac{\partial \vartheta}{\partial \hat{x}_\beta} \left[\hat{\rho}^{(0)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(1)} \right. \right.$$
$$+ \hat{\rho}^{(0)} \hat{u}_\alpha^{(1)} \hat{u}_\beta^{(0)} + \hat{\rho}^{(1)} \hat{u}_\alpha^{(0)} \hat{u}_\beta^{(0)} + \hat{p}^{(1)} \delta_{\alpha\beta} - \hat{p}^{(0)}$$
$$\left. \left. \times \left(\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{u}_\beta^{(0)}}{\partial \hat{x}_\alpha} - \frac{2}{D} \frac{\partial \hat{u}_\beta^{(0)}}{\partial \hat{x}_\alpha} \delta_{\alpha\beta} \right) \right] \right\} d\hat{t} d\hat{\boldsymbol{x}} = 0, \quad (106b)$$

$$\int \int \left\{ \text{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \left(\hat{\rho}^{(0)} \frac{D\hat{T}^{(1)} + 2\hat{u}_\alpha^{(0)} \hat{u}_\alpha^{(1)}}{2} + \hat{\rho}^{(1)} \frac{D\hat{T}^{(0)} + \hat{u}_\alpha^{(0)2}}{2} \right) \right.$$
$$+ \frac{\partial \vartheta}{\partial \hat{x}_\beta} \left[\left(\hat{\rho}^{(0)} \frac{D\hat{T}^{(0)} + \hat{u}_\alpha^{(0)2}}{2} + \hat{p}^{(0)} \right) \hat{u}_\beta^{(1)} \right.$$
$$+ \left(\hat{\rho}^{(0)} \frac{D\hat{T}^{(1)} + 2\hat{u}_\alpha^{(0)} \hat{u}_\alpha^{(1)}}{2} + \hat{\rho}^{(1)} \frac{D\hat{T}^{(0)} + \hat{u}_\alpha^{(0)2}}{2} + \hat{p}^{(1)} \right) \hat{u}_\beta^{(0)}$$
$$- \frac{D+2}{2} \hat{p}^{(0)} \frac{\partial \hat{T}^{(0)}}{\partial \hat{x}_\beta} - \hat{p}^{(0)} \hat{u}_\alpha^{(0)} \left(\frac{\partial \hat{u}_\alpha^{(0)}}{\partial \hat{x}_\beta} + \frac{\partial \hat{u}_\beta^{(0)}}{\partial \hat{x}_\alpha} \right)$$
$$\left. \left. - \frac{2}{D} \frac{\partial \hat{u}_\gamma^{(0)}}{\partial \hat{x}_\gamma} \delta_{\alpha\beta} \right) \right] \right\} d\hat{t} d\hat{\boldsymbol{x}} = 0, \quad (106c)$$

where Eqs. (105a)–(105c) are utilized. We can obtain the higher-order sets in a similar way.

The above series of equations, together with the equation of state, constitutes systems of equations that determine the component functions $\hat{\rho}^{(m)}$, $\hat{u}_\alpha^{(m)}$, $\hat{T}^{(m)}$, and $\hat{p}^{(m)}$ of the macroscopic variables $\hat{\rho}$, $\hat{u}_\alpha$, $\hat{T}$, and $\hat{p}$. First, $\hat{\rho}^{(0)}$, $\hat{u}_\alpha^{(0)}$, $\hat{T}^{(0)}$, and $\hat{p}^{(0)}$ are determined by Eqs. (105a)–(105c). Next, $\hat{\rho}^{(1)}$, $\hat{u}_\alpha^{(1)}$, $\hat{T}^{(1)}$, and $\hat{p}^{(1)}$ are determined by Eqs. (106a)–(106c). The higher-order sets are also determined in a similar way.

We find that the leading-order set (105a)–(105c) corresponds to the weak form of the Euler set, which correctly describes the flow with a finite Mach number of a gas includ-

TABLE I. The relative errors of the solution of the LBM (compressible, thermal, and isothermal models) for describing the behavior of a gas in the continuum limit of three different flow regimes $\mathrm{Ma} \ll \varepsilon$ ($\mathrm{Re} \ll 1$), $\mathrm{Ma} \sim \varepsilon$ ($\mathrm{Re} \sim 1$), and $\mathrm{Ma} \sim 1$ ($\mathrm{Re} \gg 1$). The symbol – indicates that the corresponding lattice Boltzmann model cannot describe the behavior of a gas in the corresponding flow regime. Note that the error estimate for the temperature in the flow regime $\mathrm{Ma} \ll \varepsilon$ ($\mathrm{Re} \ll 1$) is not applied to the Isothermal model.

	$\mathrm{Ma} \ll \varepsilon$ ($\mathrm{Re} \ll 1$)	$\mathrm{Ma} \sim \varepsilon$ ($\mathrm{Re} \sim 1$)	$\mathrm{Ma} \sim 1$ ($\mathrm{Re} \gg 1$)
Compressible model	$O(\varepsilon^{M_0})$ if $R_\alpha^{(M_0)}$ is not irrotational and $R_\alpha^{(m)}$ $(m \le M_0 - 1)$ is irrotational (for the flow velocity and the pressure gradient)	$O(\varepsilon)$	$\langle$Euler region$\rangle$ $O(\varepsilon)$ $\langle$Viscous boundary layer$\rangle$ $\begin{cases} O(\varepsilon) \text{ if the components of } \bar{R}_\alpha \\ \text{parallel to the boundary surface} \\ \text{vanish and } \bar{R}_4 = 0 \\ O(\varepsilon^{1/2}) \text{ otherwise.} \end{cases}$ $\langle$Flow with shocks or contact discontinuities$\rangle O(\varepsilon)$
Thermal model	$O(\varepsilon^{M_1})$ if $R_{D+1}^{(M_1)} \ne 0$ and $R_{D+1}^{(m)}$ $(m \le M_1 - 1) = 0$ (for the temperature)		
Isothermal model		$\begin{cases} O(\varepsilon^2) \text{ if } \hat{R}_\alpha \text{ is irrotational} \\ O(\varepsilon) \text{ otherwise.} \end{cases}$	

ing shock waves and contact discontinuities. Specifically, a solution of the weak form of the Euler set satisfies the Euler equations themselves in the regions where the solution is smooth, and it satisfies, at its discontinuities, the correct jump conditions derived from the conservation form of the compressible Euler equations [31–33] or the Rankine-Hugoniot relations [35] (strictly speaking, the subsidiary entropy condition also must be satisfied across a shock wave, but numerical examination [25,28] indicates that this entropy condition is satisfied automatically). The weak form of the Euler set is given, in its nondimensional form, as

$$\int \int \left(\mathrm{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \hat{\rho} + \frac{\partial \vartheta}{\partial \hat{x}_\alpha} \hat{\rho} \hat{u}_\alpha \right) d\hat{t} d\hat{x} = 0, \qquad (107a)$$

$$\int \int \left\{ \mathrm{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \hat{\rho} \hat{u}_\alpha + \frac{\partial \vartheta}{\partial \hat{x}_\beta} (\hat{\rho} \hat{u}_\alpha \hat{u}_\beta + \hat{p} \delta_{\alpha\beta}) \right\} d\hat{t} d\hat{x} = 0, \qquad (107b)$$

$$\int \int \left\{ \mathrm{Sh} \frac{\partial \vartheta}{\partial \hat{t}} \hat{\rho} \left(\frac{C_v}{R} \hat{T} + \frac{\hat{u}_\alpha^2}{2} \right) + \frac{\partial \vartheta}{\partial \hat{x}_\beta} \right.$$
$$\left. \times \left[\hat{\rho} \left(\frac{C_v}{R} \hat{T} + \frac{\hat{u}_\alpha^2}{2} \right) + \hat{p} \right] \hat{u}_\beta \right\} d\hat{t} d\hat{x} = 0, \quad (107c)$$

with

$$\hat{p} = \hat{\rho} \hat{T}, \qquad (107d)$$

where C_v is the specific heat at constant volume. Comparing the leading-order set (105a)–(105c) obtained from the LBM with the weak form of the Euler set [or Eqs. (107a)–(107c)], we find that the former corresponds to the latter whose specific heat at constant volume C_v is given by Eq. (80). Proceeding to the next-order set of equations (106a)–(106c), however, we find that it evidently includes inhomogeneous

terms composed of the leading-order solution. For example, the momentum equation (106b) includes the inhomogeneous terms representing viscosity of fluid. The solution at this order is therefore nonzero, and it becomes a deviation from the correct solution of the leading-order set or the weak form of the Euler set. Thus, the relative error is $O(\varepsilon)$. Thermal and isothermal models are not considered here as already mentioned at the end of Sec. VI A.

VII. SUMMARY AND CONCLUDING REMARKS

In the present study, the relative errors of the solution of the LBM are derived for describing the behavior of a gas in the continuum limit of three different flow regimes $\mathrm{Ma} \ll \varepsilon$ ($\mathrm{Re} \ll 1$), $\mathrm{Ma} \sim \varepsilon$ ($\mathrm{Re} \sim 1$), and $\mathrm{Ma} \sim 1$ ($\mathrm{Re} \gg 1$). The error is defined as a difference between the macroscopic variables obtained from the solution of the LBM and the solution of the correct fluid-dynamics-type equations, or the Stokes set for $\mathrm{Ma} \ll \varepsilon$ ($\mathrm{Re} \ll 1$), the incompressible NS set for $\mathrm{Ma} \sim \varepsilon$ ($\mathrm{Re} \sim 1$), and the Euler and the viscous-boundary-layer set or the weak form of the Euler set for $\mathrm{Ma} \sim 1$ ($\mathrm{Re} \gg 1$). The obtained results of the error estimates are summarized as follows (see also Table I):

(I) The case of $\mathrm{Ma} \ll \varepsilon$ ($\mathrm{Re} \ll 1$). Any basic lattice Boltzmann models (compressible, thermal, and isothermal models) have the following errors: the relative error of the flow velocity and the pressure gradient is

$$O(\varepsilon^{M_0}) \quad \text{if } R_\alpha^{(M_0)} \text{ is not irrotational and } R_\alpha^{(m)}$$
$$(m \le M_0 - 1) \text{ is irrotational}, \qquad (108a)$$

where $R_\alpha^{(m)}$ $(\alpha = 1, \ldots, D; \ m = 1, \ldots, M_0)$ is given by Eq. (33a). The relative error of the temperature is

$$O(\varepsilon^{M_1}) \quad \text{if } R_{D+1}^{(M_1)} \ne 0 \text{ and } R_{D+1}^{(m)} \ (m \le M_1 - 1) = 0, \qquad (108b)$$

where $R_{D+1}^{(m)}$ $(m = 1, \ldots, M_1)$ is given by Eq. (33b).

(II) The case of Ma $\sim \varepsilon$ (Re ~ 1). The relative error for the compressible model and the thermal model is $O(\varepsilon)$. The relative error for the isothermal model is

$$O(\varepsilon^2) \quad \text{if } \hat{R}_\alpha \text{ is irrotational,}$$

$$O(\varepsilon) \quad \text{otherwise,} \tag{109}$$

where $\hat{R}_\alpha$ ($\alpha = 1, \ldots, D$) is given by Eq. (54a).

(III) The case of Ma ~ 1 (Re $\gg 1$). Only the compressible model can describe flows in this regime.

(i) When a flow does not include shocks or contact discontinuities, the relative error in the Euler region is $O(\varepsilon)$, and that in the viscous boundary layer is

$$O(\varepsilon) \quad \text{if the components of } \bar{R}_\alpha \text{ parallel to the}$$

$$\text{boundary surface vanish and } \bar{R}_4 = 0,$$

$$O(\varepsilon^{1/2}) \quad \text{otherwise.} \tag{110}$$

where $\bar{R}_\alpha$ ($\alpha = 1, \ldots, D$) and $\bar{R}_4$ are given by Eqs. (94a)–(94c).

(ii) When a flow includes shocks or contact discontinuities, the relative error is $O(\varepsilon)$.

The above error estimates are also summarized in Table I. Application of these error estimates to the specific lattice Boltzmann models is shown in the Appendix.

From the above arrangements about the errors of the LBM, we find that the errors of the LBM can be controlled to some extent. For example, any lattice Boltzmann model can reduce its error from $O(\varepsilon)$ to any higher order when describing the flow with Ma $\ll \varepsilon$ (Re $\ll 1$). The isothermal model can reduce its error from $O(\varepsilon)$ to $O(\varepsilon^2)$ when describing the flow with Ma $\sim \varepsilon$ (Re ~ 1). The compressible model can reduce its error from $O(\varepsilon^{1/2})$ to $O(\varepsilon)$ when describing the flow with Ma ~ 1 (Re $\gg 1$) in the viscous boundary layer. The above arrangements will be of help not only in evaluating the magnitude of the error for using the LBM, but also in constructing the lattice Boltzmann model of the better accuracy.

In the present study, we have focused on the accuracy of the LBM, and no account is given of its defects. In fact, some parameters of the fluid-dynamics-type equations derived from the kinetic equation of the LBM cannot be given freely. For instance, C_v/R in Eq. (79c) is fixed to the value

given by Eq. (80). In the past, Yan *et al.* [36] and Kataoka and Tsutahara [25] suggested the models that overcome this defect and described the flow of a gas with various specific-heat ratios in the Euler region. However, these models cannot describe the flow in the viscous boundary layer. The authors [28] also suggested a new model that can describe the flow both in the Euler region and the viscous boundary layer of a gas with various specific-heat ratios. The error estimate of this model is specifically given in the Appendix. The other problem is the dependence of the transport coefficient on the pressure in the equations of the viscous boundary layer, while in the real gas it depends only on the temperature. This defect is easy to overcome by multiplying density on the right-hand side of the kinetic equation (3) [37]. In the present analysis, this modification is not introduced since we are interested in the accuracy of the basic lattice Boltzmann model, and this modification does not affect the order of the errors presented above.

Finally, we must give some comments about the initial and the boundary conditions. In the present study, the effects of the initial and the boundary conditions are not taken into account. That is, the error due to the initial condition and the boundary condition of the velocity distribution function is assumed to be much smaller than that due to the kinetic equation itself. When we numerically solve a given initial-boundary-value problem by the LBM, therefore, we must also estimate the errors of the initial and the boundary conditions appropriately in addition to the error given above. This matter is treated as a future problem.

APPENDIX: SPECIFIC LATTICE BOLTZMANN MODELS AND THEIR ACCURACY

Several specific lattice Boltzmann models were proposed in the past. Here, the compressible model proposed by Chen *et al.* [19] and that by the authors [28], the thermal model by Alexander *et al.* [20], and the isothermal model by Qian *et al.* [18] of their two-dimensional versions ($D = 2$) are given in the nondimensional form, and their errors are estimated on the basis of the results summarized in Sec. VII of the present study.

1. Chen's model (compressible model)

$\hat{c}_{i\alpha}$ ($\alpha = 1, 2$; $i = 1, 2, \ldots, 16$) is given by

$$(\hat{c}_{i1}, \hat{c}_{i2}) = \begin{cases} (\cos\theta_i, \sin\theta_i) & \text{for } i = 1, 2, 3, 4 \\ 2(\cos\theta_i, \sin\theta_i) & \text{for } i = 5, 6, 7, 8 \\ \sqrt{2}[\cos(\theta_i + \pi/4), \sin(\theta_i + \pi/4)] & \text{for } i = 9, 10, 11, 12 \\ 2\sqrt{2}[\cos(\theta_i + \pi/4), \sin(\theta_i + \pi/4)] & \text{for } i = 13, 14, 15, 16, \end{cases} \tag{A1}$$

where $\theta_i = \pi i/2$. $\hat{f}_i^{\text{eq}}$ is given by

$$\hat{f}_i^{\text{eq}} = \hat{\rho}(E_i + F_i \hat{u}_\alpha \hat{c}_{i\alpha} + G_i \hat{u}_\alpha \hat{u}_\beta \hat{c}_{i\alpha} \hat{c}_{i\beta} + H_i \hat{u}_\alpha \hat{u}_\beta \hat{u}_\gamma \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}) \quad \text{for } i = 1, 2, \ldots, 16, \tag{A2}$$

where

$$
E_i = \begin{cases}
\dfrac{8}{15} - \dfrac{2}{3}\hat{T} + \dfrac{\hat{T}^2}{3} + \left(-\dfrac{2}{3} + \dfrac{5}{6}\hat{T}\right)\hat{u}_\alpha^2 + \dfrac{\hat{u}_\alpha^2 \hat{u}_\beta^2}{8} & \text{for } i = 1,2,3,4 \\[2mm]
-\dfrac{1}{30} + \dfrac{\hat{T}}{24} + \dfrac{\hat{T}^2}{24} + \left(\dfrac{1}{24} - \dfrac{\hat{T}}{12}\right)\hat{u}_\alpha^2 - \dfrac{\hat{u}_\alpha^2 \hat{u}_\beta^2}{64} & \text{for } i = 5,6,7,8 \\[2mm]
-\dfrac{4}{15} + \dfrac{2}{3}\hat{T} - \dfrac{5}{12}\hat{T}^2 + \left(\dfrac{1}{6} - \dfrac{7}{24}\hat{T}\right)\hat{u}_\alpha^2 - \dfrac{\hat{u}_\alpha^2 \hat{u}_\beta^2}{32} & \text{for } i = 9,10,11,12 \\[2mm]
\dfrac{1}{60} - \dfrac{\hat{T}}{24} + \dfrac{\hat{T}^2}{24} + \left(-\dfrac{1}{96} + \dfrac{\hat{T}}{96}\right)\hat{u}_\alpha^2 & \text{for } i = 13,14,15,16,
\end{cases}
$$

$$
F_i = \begin{cases}
\dfrac{2}{3} - \hat{T} - \dfrac{\hat{u}_\beta^2}{2} & \text{for } i = 1,2,3,4 \\[2mm]
-\dfrac{1}{24} + \dfrac{\hat{T}}{8} & \text{for } i = 5,6,7,8 \\[2mm]
\dfrac{\hat{T}}{4} - \dfrac{\hat{u}_\beta^2}{8} & \text{for } i = 9,10,11,12 \\[2mm]
0 & \text{for } i = 13,14,15,16,
\end{cases}
$$

$$
G_i = \begin{cases}
\dfrac{2}{3} - \hat{T} - \dfrac{\hat{u}_\gamma^2}{6} & \text{for } i = 1,2,3,4 \\[2mm]
-\dfrac{1}{96} + \dfrac{\hat{T}}{16} + \dfrac{\hat{u}_\gamma^2}{96} & \text{for } i = 5,6,7,8 \\[2mm]
\dfrac{1}{6} - \dfrac{\hat{T}}{8} - \dfrac{\hat{u}_\gamma^2}{48} & \text{for } i = 9,10,11,12 \\[2mm]
-\dfrac{1}{384} + \dfrac{\hat{T}}{128} + \dfrac{\hat{u}_\gamma^2}{768} & \text{for } i = 13,14,15,16,
\end{cases}
\qquad
H_i = \begin{cases}
\dfrac{1}{3} & \text{for } i = 1,2,3,4 \\[2mm]
\dfrac{1}{96} & \text{for } i = 5,6,7,8 \\[2mm]
\dfrac{1}{8} & \text{for } i = 9,10,11,12 \\[2mm]
0 & \text{for } i = 13,14,15,16.
\end{cases}
\tag{A3}
$$

$\hat{c}_{i\alpha}$ and $\hat{f}_i^{\text{eq}}$ given above satisfy their constraints (14a)–(14e) for the compressible model. Now, we estimate the error of this model based on the results summarized in Sec. VII [or Eqs. (108a), (108b), (109), and (110)].

(I) $\text{Ma} \ll \varepsilon$ ($\text{Re} \ll 1$). Expanding the above equilibrium distribution function (A2) in the same way as Eq. (22), we obtain its component function $F_i^{\text{eq}(m)}$ explicitly. Substituting this $F_i^{\text{eq}(m)}$ into Eq. (33a), we get

$$
R_\alpha^{(1)} = 0, \tag{A4}
$$

$$
R_\alpha^{(2)} \text{ is not irrotational} \quad (\alpha = 1,2). \tag{A5}
$$

Thus, the relative error of the flow velocity and the pressure gradient is $O(\varepsilon^2)$. Substituting $F_i^{\text{eq}(m)}$ into Eq. (33b), we get

$$
R_3^{(1)} \neq 0. \tag{A6}
$$

Thus, the relative error of the temperature is $O(\varepsilon)$.

(II) $\text{Ma} \sim \varepsilon$ ($\text{Re} \sim 1$). The relative error is $O(\varepsilon)$.

(III) $\text{Ma} \sim 1$ ($\text{Re} \gg 1$). The relative error in the Euler region is $O(\varepsilon)$.

The relative error in the viscous boundary layer is determined by obtaining the specific values of $\bar{R}_\alpha$ ($\alpha = 1,2$) and $\bar{R}_4$ defined in Eqs. (94a)–(94c). To this end, we expand the equilibrium distribution function represented by Eq. (A2) in a series of δ in the same way as Eq. (85). We then obtain an explicit form of its component function $\hat{f}_i^{\text{eq}(m)}$. Substituting this $\hat{f}_i^{\text{eq}(m)}$ into Eqs. (94a)–(94c), we find that the components of $\bar{R}_\alpha$ parallel to the boundary surface and $\bar{R}_4$ are always zero. Thus, from Eq. (110), the relative error in the viscous boundary layer is $O(\varepsilon)$. The relative error for flows with shocks and contact discontinuities is $O(\varepsilon)$.

2. Kataoka and Tsutahara's model (compressible model)

$\hat{c}_{i\alpha}$ ($\alpha = 1,2$; $i = 1,2,\ldots,16$) is given by

$$(\hat{c}_{i1}, \hat{c}_{i2}) = \begin{cases} (\cos\theta_i, \sin\theta_i) & \text{for } i = 1,2,3,4 \\ 6(\cos\theta_i, \sin\theta_i) & \text{for } i = 5,6,7,8 \\ \sqrt{2}[\cos(\theta_i + \pi/4), \sin(\theta_i + \pi/4)] & \text{for } i = 9,10,11,12 \\ \dfrac{3}{\sqrt{2}}[\cos(\theta_i + \pi/4), \sin(\theta_i + \pi/4)] & \text{for } i = 13,14,15,16, \end{cases} \tag{A7}$$

where $\theta_i = \pi i/2$. $\hat{f}_i^{\text{eq}}$ is given by

$$\hat{f}_i^{\text{eq}} = \hat{\rho}(E_i + F_i \hat{u}_\alpha \hat{c}_{i\alpha} + G_i \hat{u}_\alpha \hat{u}_\beta \hat{c}_{i\alpha} \hat{c}_{i\beta} + H_i \hat{u}_\alpha \hat{u}_\beta \hat{u}_\gamma \hat{c}_{i\alpha} \hat{c}_{i\beta} \hat{c}_{i\gamma}) \quad \text{for } i = 1,2,\ldots,16, \tag{A8}$$

where

$$E_i = \begin{cases} \dfrac{b-2}{25}\hat{T} + \dfrac{-36 + (b+4)\hat{T} + \hat{u}_\beta^2}{115}\hat{u}_\alpha^2 & \text{for } i = 1,2,3,4 \\[3mm] \dfrac{1}{96} - \dfrac{121b+408}{86400}\hat{T} + \dfrac{b+2}{1728}\hat{T}^2 + \dfrac{-799 + (19b+306)\hat{T} + 19\hat{u}_\beta^2}{397440}\hat{u}_\alpha^2 & \text{for } i = 5,6,7,8 \\[3mm] \dfrac{81}{160} + \dfrac{-229b+8}{3200}\hat{T} + \dfrac{b+2}{320}\hat{T}^2 + \dfrac{-117 + (9b+38)\hat{T} + 9\hat{u}_\beta^2}{640}\hat{u}_\alpha^2 & \text{for } i = 9,10,11,12 \\[3mm] -\dfrac{4}{15} + \dfrac{89b+222}{2700}\hat{T} - \dfrac{b+2}{270}\hat{T}^2 + \dfrac{26 - (2b+9)\hat{T} - 2\hat{u}_\beta^2}{270}\hat{u}_\alpha^2 & \text{for } i = 13,14,15,16, \end{cases}$$

$$F_i = \begin{cases} \dfrac{2(b-2)}{25}\hat{T} & \text{for } i = 1,2,3,4 \\[3mm] \dfrac{-2b+29}{32400}\hat{T} - \dfrac{\hat{u}_\beta^2}{2592} & \text{for } i = 5,6,7,8 \\[3mm] \dfrac{9}{40} + \dfrac{-14b+3}{400}\hat{T} + \dfrac{\hat{u}_\beta^2}{80} & \text{for } i = 9,10,11,12 \\[3mm] -\dfrac{2}{45} + \dfrac{2(7b+11)}{2025}\hat{T} - \dfrac{7}{810}\hat{u}_\beta^2 & \text{for } i = 13,14,15,16, \end{cases}$$

$$G_i = \begin{cases} \dfrac{72 - 2(b+4)\hat{T} - 2\hat{u}_\gamma^2}{115} & \text{for } i = 1,2,3,4 \\[3mm] \dfrac{-29 + 4(b+4)\hat{T} + 4\hat{u}_\gamma^2}{298080} & \text{for } i = 5,6,7,8 \\[3mm] \dfrac{9 - (b+4)\hat{T} - \hat{u}_\gamma^2}{160} & \text{for } i = 9,10,11,12 \\[3mm] \dfrac{-4 + (b+4)\hat{T} + \hat{u}_\gamma^2}{810} & \text{for } i = 13,14,15,16, \end{cases} \qquad H_i = \begin{cases} 0 & \text{for } i = 1,2,3,4 \\[3mm] \dfrac{1}{46656} & \text{for } i = 5,6,7,8 \\[3mm] -\dfrac{3}{320} & \text{for } i = 9,10,11,12 \\[3mm] \dfrac{8}{3645} & \text{for } i = 13,14,15,16. \end{cases} \tag{A9}$$

Here, b is a given constant related to the specific-heat ratio γ by $b = 2/(\gamma - 1)$. $\hat{c}_{i\alpha}$ and $\hat{f}_i^{\text{eq}}$ given above satisfy constraints (14a)–(14d) for the usual compressible model and the following constraints:

$$(b+2)\hat{p} + \hat{\rho}\hat{u}_\alpha^2 = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}}(\hat{c}_{i\alpha}^2 + \hat{\eta}_i^2), \tag{A10}$$

$$[(b+2)\hat{p} + \hat{\rho}\hat{u}_\alpha^2]\hat{u}_\beta = \sum_{i=1}^{N} \hat{f}_i^{\text{eq}}(\hat{c}_{i\alpha}^2 + \hat{\eta}_i^2)\hat{c}_{i\beta}, \tag{A11}$$

$$(b+2)\hat{p}\hat{T}\delta_{\alpha\beta} + \hat{p}[(b+4)\hat{u}_\alpha\hat{u}_\beta + \hat{u}_\gamma^2\delta_{\alpha\beta}] + \hat{\rho}\hat{u}_\alpha\hat{u}_\beta\hat{u}_\gamma^2$$

$$= \sum_{i=1}^{N} \hat{f}_i^{\mathrm{eq}}\hat{c}_{i\alpha}\hat{c}_{i\beta}(\hat{c}_{i\gamma}^2 + \hat{\eta}_i^2), \tag{A12}$$

where

$$\hat{\eta}_i = \begin{cases} 5/2 & \text{for } i = 1,2,3,4 \\ 0 & \text{for } 5 \leq i \leq 16 \end{cases} \tag{A13}$$

is an independent variable introduced to satisfy Eqs. (A10)–(A12). This is a lattice Boltzmann model whose specific-heat ratio can be chosen according to our convenience, while the previous usual model gives the unphysical value of $\gamma=2$ (when $D=2$) only. When $b=2$, the above constraints (14a)–(14d) and (A10)–(A12) reduce to Eqs. (14a)–(14e) for the usual model. The asymptotic analyses for small ε of this LBM model can be made in a similar way as those presented in Secs. IV–VI. The same results (108a), (108b), (109), and (110) are derived of their orders of the relative error, and they are summarized as follows:

(I) Ma $\ll \varepsilon$ (Re $\ll 1$). Expanding the above equilibrium distribution function (A8) in the same way as Eq. (22), we obtain its component function $F_i^{\mathrm{eq}(m)}$ explicitly. Substituting this $F_i^{\mathrm{eq}(m)}$ into Eqs. (33a) and (33b), we get

$$R_\alpha^{(1)} \text{ is not irrotational} \quad (\alpha = 1,2), \tag{A14}$$

$$R_3^{(1)} \neq 0. \tag{A15}$$

Thus, the relative error of this model is $O(\varepsilon)$ for any macroscopic variables.

(II) Ma $\sim \varepsilon$ (Re ~ 1). The relative error is $O(\varepsilon)$.

(III) Ma ~ 1 (Re $\gg 1$). The relative error in the Euler region is $O(\varepsilon)$.

The relative error in the viscous boundary layer is determined by obtaining the specific values of $\bar{R}_\alpha$ ($\alpha=1,2$) and $\bar{R}_4$ defined in Eqs. (94a)–(94c). To this end, we expand the equilibrium distribution function represented by Eq. (A8) in a series of δ in the same way as Eq. (85). We then obtain an explicit form of its component function $\hat{f}_i^{\mathrm{eq}(m)}$. Substituting this $\hat{f}_i^{\mathrm{eq}(m)}$ into Eqs. (94a)–(94c), we find that the components of $\bar{R}_\alpha$ parallel to the boundary surface and $\bar{R}_4$ are always zero. Thus, from Eq. (110), the relative error in the viscous boundary layer is $O(\varepsilon)$. The relative error for flows with shocks and contact discontinuities is $O(\varepsilon)$.

3. Alexander's model (thermal model)

$\hat{c}_{i\alpha}$ ($\alpha=1,2; i=1,2,\ldots,13$) is given by

$$(\hat{c}_{i1},\hat{c}_{i2}) = \begin{cases} (0,0) & \text{for } i = 1 \\ (\cos\theta_i,\sin\theta_i) & \text{for } i = 2,3,\ldots,7 \\ 2(\cos\theta_i,\sin\theta_i) & \text{for } i = 8,9,\ldots,13, \end{cases} \tag{A16}$$

where $\theta_i = \pi i/3$. $\hat{f}_i^{\mathrm{eq}}$ is given by

$$\hat{f}_i^{\mathrm{eq}} = \hat{\rho}(E_i + F_i\hat{u}_\alpha\hat{c}_{i\alpha} + G_i\hat{u}_\alpha\hat{u}_\beta\hat{c}_{i\alpha}\hat{c}_{i\beta}$$
$$+ H_i\hat{u}_\alpha\hat{u}_\beta\hat{u}_\gamma\hat{c}_{i\alpha}\hat{c}_{i\beta}\hat{c}_{i\gamma}) \quad \text{for } i = 1,2,\ldots,13, \tag{A17}$$

where

$$E_i = \begin{cases} 1 - \dfrac{5}{2}\hat{T} + 2\hat{T}^2 + \left(-\dfrac{5}{4}+2\hat{T}\right)\hat{u}_\alpha^2 & \text{for } i=1 \\[2mm] \dfrac{4}{9}(\hat{T}-\hat{T}^2) - \dfrac{2}{9}(1-\hat{T})\hat{u}_\alpha^2 & \text{for } i=2,3,\ldots,7 \\[2mm] -\dfrac{\hat{T}}{36} + \dfrac{\hat{T}^2}{9} + \left(\dfrac{1}{72}-\dfrac{\hat{T}}{18}\right)\hat{u}_\alpha^2 & \text{for } i=8,9,\ldots,13, \end{cases}$$

$$F_i = \begin{cases} \dfrac{4}{9}(1-\hat{T}) & \text{for } i=2,3,\ldots,7 \\[2mm] -\dfrac{1}{36} + \dfrac{\hat{T}}{9} & \text{for } i=8,9,\ldots,13, \end{cases}$$

$$G_i = \begin{cases} \dfrac{8}{9} - \dfrac{4}{3}\hat{T} & \text{for } i=2,3,\ldots,7 \\[2mm] -\dfrac{1}{72} + \dfrac{\hat{T}}{12} & \text{for } i=8,9,\ldots,13, \end{cases}$$

$$H_i = \begin{cases} -\dfrac{4}{27} & \text{for } i=2,3,\ldots,7 \\[2mm] \dfrac{1}{108} & \text{for } i=8,9,\ldots,13. \end{cases} \tag{A18}$$

$\hat{c}_{i\alpha}$ and $\hat{f}_i^{\mathrm{eq}}$ given above satisfy their constraints (14a)–(14c), (15a), and (15b) for the thermal model. Now, we estimate the error of this model based on the results summarized at the beginning of Sec. VII [or Eqs. (108a), (108b), (109), and (110)].

(I) Ma $\ll \varepsilon$ (Re $\ll 1$). Expanding the above equilibrium distribution function (A17) in the same way as Eq. (22), we obtain an explicit form of its component function $F_i^{\mathrm{eq}(m)}$. Substituting this $F_i^{\mathrm{eq}(m)}$ into Eq. (33a), we get

$$R_\alpha^{(1)} = 0, \tag{A19}$$

$$R_\alpha^{(2)} \text{ is not irrotational} \quad (\alpha = 1, 2). \qquad (A20)$$

Thus, the relative error of the flow velocity and the pressure gradient is $O(\varepsilon^2)$. Substituting $F_i^{\text{eq}(m)}$ into Eq. (33b), we get

$$R_3^{(1)} = R_3^{(2)} = 0, \quad R_3^{(3)} \neq 0. \qquad (A21)$$

Thus, the relative error of the temperature is $O(\varepsilon^3)$.

(II) Ma $\sim \varepsilon$ (Re ~ 1). The relative error is $O(\varepsilon)$.

(III) Ma ~ 1 (Re ≥ 1). The thermal model cannot describe the flow of Ma ~ 1.

4. Qian's model (isothermal model)

$\hat{c}_{i\alpha}$ ($\alpha = 1, 2; i = 1, 2, \ldots, 9$) is given by

$$(\hat{c}_{i1}, \hat{c}_{i2})$$

$$= \begin{cases} (0,0) & \text{for } i = 1 \\ \sqrt{3}(\cos\theta_i, \sin\theta_i) & \text{for } i = 2, 3, 4, 5 \\ \sqrt{6}[\cos(\theta_i + \pi/4), \sin(\theta_i + \pi/4)] & \text{for } i = 6, 7, 8, 9, \end{cases}$$
$$(A22)$$

where $\theta_i = \pi i / 2$. $\hat{f}_i^{\text{eq}}$ is given by

$$\hat{f}_i^{\text{eq}} = E_i \hat{\rho} \left(1 - \tfrac{1}{2}\hat{u}_\alpha^2 + \hat{u}_\alpha \hat{c}_{i\alpha} + \tfrac{1}{2}\hat{u}_\alpha \hat{u}_\beta \hat{c}_{i\alpha} \hat{c}_{i\beta} \right) \quad \text{for } i = 1, 2, \ldots, 9,$$
$$(A23)$$

where

$$E_1 = 4/9, \quad E_i = 1/9 \quad \text{for } i = 2, 3, 4, 5,$$

$$E_i = 1/36 \quad \text{for } i = 6, 7, 8, 9. \qquad (A24)$$

$\hat{c}_{i\alpha}$ and $\hat{f}_i^{\text{eq}}$ given above satisfy their constraints (14a), (14b), (16a), and (16b) for the isothermal model. Now, we estimate the error of this model based on the results summarized at the beginning of Sec. VII [or Eqs. (108a), (108b), (109), and (110)]. The errors of the flow velocity and the pressure gradient are to be considered.

(I) Ma $\ll \varepsilon$ (Re $\ll 1$). Expanding the above equilibrium distribution function (A23) in the same way as Eq. (22), we obtain its component function $F_i^{\text{eq}(m)}$. Substituting this $F_i^{\text{eq}(m)}$ into Eq. (33a), we get

$$R_\alpha^{(1)} = 0, \qquad (A25)$$

$$R_\alpha^{(2)} \text{ is not irrotational} \quad (\alpha = 1, 2). \qquad (A26)$$

Thus, the relative error is $O(\varepsilon^2)$.

(II) Ma $\sim \varepsilon$ (Re ~ 1). Expanding the equilibrium distribution function represented by Eq. (A23) in a series of ε in the same way as Eq. (41), we obtain an explicit form of its component function $\hat{f}_i^{\text{eq}(m)}$. Substituting this $\hat{f}_i^{\text{eq}(m)}$ into Eq. (54a), we get $\hat{R}_\alpha = 0$, which is evidently irrotational. Thus, the relative error is $O(\varepsilon^2)$ from Eq. (109). This result agrees with the error estimate by Inamuro *et al.* [12].

(III) Ma ~ 1 (Re ≥ 1). The isothermal model cannot describe the flow of Ma ~ 1.

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