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STRONG PRO-FIBRATIONS AND ANR OBJECTS IN PRO-CATEGORIES

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ABSTRACT. The notions of pro-fibration and approximate pro-fibration for morphisms in the pro-category pro- Top of topological spaces were introduced by S. Mardešić and T. B. Rushing . In this paper we introduce the notion of strong pro-fibration, which is a pro-fibration with some additional property, and the notion of ANR object in pro- Top, which is approximately an ANR-system, and we consider the full subcategory $\mathcal{ANR}$ of pro- Top whose objects are ANR objects. We prove that the category $\mathcal{ANR}$ satisfies most of the axioms for fibration category in the sense of H. J. Baues if fibrations are strong pro-fibrations and weak equivalences are morphisms inducing isomorphisms in the pro-homotopy category pro- $\mathrm{H}(\mathsf{Top})$ of topological spaces. We give various applications. First of all, we prove that every shape morphism is represented by a strong pro-fibration. Secondly, the fibre of a strong pro-fibration is well-defined in the category $\mathcal{ANR}$, and we obtain an isomorphism between the pro-homotopy groups of the base and total systems of a strong pro-fibration, and hence obtain the pro-homotopy sequence of a strong pro-fibration. Finally, we also show that there is a homotopy decomposition in the category $\mathcal{ANR}$.

1. INTRODUCTION

Shape category is based on the pro-category pro- HANR of the homotopy category of spaces having the homotopy type of an ANR (equivalently, a CW complex). In shape theory spaces are represented by inverse systems of good spaces such as ANR's (or CW complexes), and we are allowed to work on the inverse systems. However, from the viewpoint of homotopy theory, shape theory for general topological spaces is not completely analogous to homotopy theory for CW complexes. For example, the well-known Whitehead theorem which says that every weak homotopy equivalence between CW complexes is a homotopy equivalence does not hold in shape theory unless the spaces have finite shape dimensions [10, §. 5, Ch. II].

From the viewpoint of model category, the pro-category pro- $\mathrm{H}(\mathcal{C})$ of the homotopy category $\mathrm{H}(\mathcal{C})$ of a category $\mathcal{C}$ is not obtained as a model category on the pro-category pro- $\mathcal{C}$ [13]. To overcome this deficiency, D. A. Edwards and H. M. Hastings [3] defined homotopy globally in the pro-category (see also [14]).

In this paper we are interested in a homotopical algebraic approach to shape theory. Precisely, we introduce the notion of ANR object in the pro-category of topological spaces and the notion of strong pro-fibration between ANR objects. Then we prove that the full subcategory $\mathcal{ANR}$ whose objects are ANR objects satisfies most of the axioms for a fibration category in the sense of H. J. Baues [1]

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if fibrations are strong pro-fibrations and weak equivalences are morphisms inducing isomorphisms in the pro-homotopy category pro- $\mathrm{H}(\mathsf{Top})$ of topological spaces. Roughly speaking, an ANR object is a system which is approximately an ANR system. A strong pro-fibration is a pro-fibration, introduced by S. Mardešić and T. B. Rushing [8, §. 9], with some additional property.

We give various applications. First of all, we show that every shape morphism is represented by a strong pro-fibration. Secondly, the fibre of a strong pro-fibration is well-defined in the category $\mathcal{ANR}$. We obtain an isomorphism between the pro-homotopy groups of the base and total systems of a strong pro-fibration, and hence obtain the pro-homotopy sequence of a strong pro-fibration. Finally, we show that there is a homotopy decomposition in $\mathcal{ANR}$.

The notions of fibration category and cofibration category were introduced by H. J. Baues [1]. Those notions make the constructions of homotopy theory available in more contexts by simply weakening the assumptions and concentrating on either the fibrations or the cofibrations.

The usual model structure on the category Top of topological spaces and maps adopts Hurewicz fibration for fibration, i.e., maps with the homotopy lifting property (HLP) with respect to any spaces. But the HLP is not a very useful property for maps between spaces with bad local properties [8, Example 4]. S. Mardešić and T. B. Rushing [8] introduced the notion of shape fibration between compact metric spaces, extending the notion of approximate fibration, introduced by D. S. Coram and P. F. Duvall [2] (see also [7] for the definition of shape fibration between arbitrary topological spaces).

A shape fibration is a map in Top, and it is defined by a property called the approximate homotopy lifting property for system maps between ANR-systems. But it would be more appropriate to work directly on ANR-systems than on their limit spaces. One reason is that systems of noncompact spaces often appear, in which case their limits may be empty. S. Mardešić and T. B. Rushing [8, §. 9] called a system map a *pro-fibration* (resp., an *approximate pro-fibration*) if it has the homotopy lifting property (resp., approximate homotopy lifting property) with respect to all spaces. In this paper we define our strong pro-fibration as a pro-fibration with some reasonable additional property, and show that the category $\mathcal{ANR}$ with strong pro-fibrations satisfies most of the four axioms, called the composition axiom (F1), the pull-back axiom (F2), the factorization axiom (F3), and the axiom on cofibrant models (F4), for a fibration category. More precisely, we show that the category $\mathcal{ANR}$, where fibrations are strong pro-fibrations and weak equivalences are morphisms inducing isomorphisms in pro- $\mathrm{H}(\mathsf{Top})$, satisfies (F1), (F3), and a weak version of (F2). We are not able to verify (F4) at this moment, but it is unlikely to hold for the categroy $\mathcal{ANR}$.

The paper is organized as follows: In Section 2, we recall the definitions of basic properties of pro-categories and fibration categories. In Section 3, we introduce the notion of ANR object in pro- Top and obtain its characterization. In Section 4, we define the notion of strong pro-fibration in the category $\mathcal{ANR}$, and show its well-definedness. After verifying the composition axiom in Section 5, we verify the factorization axiom and show that every shape morphism is represented by a strong pro-fibration in Section 6, and discuss the pull-back axiom in Section 7. In Section 8, we define the fibre of a strong pro-fibration in $\mathcal{ANR}$ and obtain an isomorphism between the pro-homotopy groups of the base and total systems of a strong pro-fibration. Finally, in Section 9, we obtain a homotopy decomposition in $\mathcal{ANR}$.

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2. Pro-categories and fibration categories

First, we recall the definitions of pro-category pro-$\mathcal{C}$ and the category of inverse systems inv-$\mathcal{C}$ for any category $\mathcal{C}$. For more details, the reader is referred to [10, Chapter 1, §1.1].

Let $\mathcal{C}$ be any category. A *system* (resp., *tower*) in $\mathcal{C}$ means an inverse system (resp., inverse sequence) consisting of objects and morphisms in $\mathcal{C}$. A *system map* $(f, f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ between systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ in $\mathcal{C}$ consists of a function $f : M \to \Lambda$ and morphisms $f_\mu : X_{f(\mu)} \to Y_\mu$ in $\mathcal{C}$ for $\mu \in M$ such that for $\mu < \mu'$ there is $\lambda > f(\mu), f(\mu')$ with the following commutative diagram:

$$\begin{tikzcd}
 & & X_\lambda \arrow[dll, "p_{f(\mu)\lambda}"'] \arrow[dl, "p_{f(\mu)\lambda'}"] \\
X_{f(\mu)} \arrow[d, "f_\lambda"'] & X_{f(\mu')} \arrow[d, "f_{\lambda'}"] & \\
Y_\mu & Y_{\mu'} \arrow[l, "q_{\mu\mu'}"'] &
\end{tikzcd}$$

A *level map* $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ consists of morphisms $f_\lambda : X_\lambda \to Y_\lambda$ for $\lambda \in \Lambda$ in $\mathcal{C}$ with the following commutative diagram for $\lambda < \lambda'$:

$$\begin{tikzcd}
X_\lambda \arrow[d, "f_\lambda"'] & X_{\lambda'} \arrow[l, "p_{\lambda\lambda'}"'] \arrow[d, "f_{\lambda'}"] \\
Y_\lambda & Y_{\lambda'} \arrow[l, "q_{\lambda\lambda'}"']
\end{tikzcd}$$

The category inv-$\mathcal{C}$ consists of all systems in $\mathcal{C}$ and system maps.

We define an equivalence relation $\sim$ on inv-$\mathcal{C}$ by saying that two system maps $(f, f_\mu), (g, g_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ are equivalent, in notation, $(f, f_\mu) \sim (g, g_\mu)$, iff for each $\mu \in M$ there is $\lambda > f(\mu), g(\mu)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & X_\lambda \arrow[dll, "p_{f(\mu)\lambda}"'] \arrow[dl, "p_{g(\mu)\lambda}"] \\
X_{f(\mu)} \arrow[d, "f_\mu"'] & X_{g(\mu)} \arrow[dl, "g_\mu"] & \\
Y_\mu & &
\end{tikzcd}$$

The pro-category pro-$\mathcal{C}$ consists of all systems of $\mathcal{C}$ and morphisms are given by the formula

$$\text{pro-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y}) = \text{inv-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y}) / \sim .$$

A system is said to be *cofinite* provided the index set Λ is cofinite, i.e., each element of Λ has only finitely many predecessors. An *ANR-system* (resp., *ANR-tower*) means an inverse system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ (resp., a tower $X = (X_i, p_{i,i+1})$) such that all coordinates X_λ (resp., X_i) are ANR's (for metrizable spaces).

Throughout the paper, *space means topological space, map means continuous map, and open covering means normal open covering* unless otherwise stated.

Lemma 2.1. *For any category $\mathcal{C}$, we have the following statements:*

(1) *Given a 2-sink $\boldsymbol{B} \xrightarrow{\boldsymbol{g}} \boldsymbol{Y} \xleftarrow{\boldsymbol{f}} \boldsymbol{X}$ in* pro-$\mathcal{C}$, *there is a commutative diagram*

$$\begin{tikzcd} \boldsymbol{B} \arrow[r, "\boldsymbol{g}"] \arrow[d, "\boldsymbol{k}"'] & \boldsymbol{Y} \arrow[d, "\boldsymbol{j}"] & \boldsymbol{X} \arrow[l, "\boldsymbol{f}"'] \arrow[d, "\boldsymbol{i}"] \\ \boldsymbol{B}' \arrow[r, "\boldsymbol{g}'"'] & \boldsymbol{Y}' & \boldsymbol{X}' \arrow[l, "\boldsymbol{f}'"] \end{tikzcd}$$

with the following properties:

(a) *the systems $\boldsymbol{X}'$, $\boldsymbol{Y}'$, $\boldsymbol{B}'$ have the same index set,*

(b) *the morphisms $\boldsymbol{f}'$ and $\boldsymbol{g}'$ are represented by level maps,*

(c) *the coordinates and the bonding maps of $\boldsymbol{X}'$, $\boldsymbol{Y}'$, $\boldsymbol{B}'$ are those of $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{B}$, respectively, and*

(d) *$\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are isomorphisms.*

(2) *Given a commutative square*

$$\begin{tikzcd} \boldsymbol{A}_1 \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{h}_1"'] & \boldsymbol{A}_2 \arrow[d, "\boldsymbol{h}_2"] \\ \boldsymbol{B}_1 \arrow[r, "\boldsymbol{g}"'] & \boldsymbol{B}_2 \end{tikzcd}$$

in pro-$\mathcal{C}$, *there is a commutative diagram*

$$\begin{tikzcd} \boldsymbol{A}_1 \arrow[rr, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{k}_1"] \arrow[dd, "\boldsymbol{h}_1"'] & & \boldsymbol{A}_2 \arrow[dr, "\boldsymbol{k}_2"] \arrow[dd, "\boldsymbol{h}_2"' near start] & \\ & \boldsymbol{A}'_1 \arrow[rr, "\boldsymbol{f}'" near start] \arrow[dd, "\boldsymbol{h}'_1"' near start] & & \boldsymbol{A}'_2 \arrow[dd, "\boldsymbol{h}'_2"] \\ \boldsymbol{B}_1 \arrow[rr, "\boldsymbol{g}"' near end] \arrow[dr, "\boldsymbol{l}_1"] & & \boldsymbol{B}_2 \arrow[dr, "\boldsymbol{l}_1"] & \\ & \boldsymbol{B}'_1 \arrow[rr, "\boldsymbol{g}'"'] & & \boldsymbol{B}'_2 \end{tikzcd}$$

with the following properties:

(a) *the systems $\boldsymbol{A}'_1$, $\boldsymbol{A}'_2$, $\boldsymbol{B}'_1$, $\boldsymbol{B}'_2$ have the same index set,*

(b) *the morphisms $\boldsymbol{f}'$, $\boldsymbol{g}'$, $\boldsymbol{h}'_1$, $\boldsymbol{h}'_2$ are represented by level maps,*

(c) *the coordinates and the bonding maps of $\boldsymbol{A}'_1$, $\boldsymbol{A}'_2$, $\boldsymbol{B}'_1$, $\boldsymbol{B}'_2$ are those of $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, respectively, and*

(d) *$\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ are isomorphisms.*

Proof. The proof is based on the technique used by S. Mardešić and J. Segal [10, Theorem 3, Ch. I, §1.3]. For the complete proof, the reader is referred to [12]. $\square$

Let Top denote the category of spaces and maps, and let $\mathrm{H}(\mathsf{Top})$ denote the homotopy category of Top. For any map $f : X \to Y$ between spaces, let $[f]$ denote the homotopy class of f. Then every system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in Top induces a system $[\boldsymbol{X}] = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in $\mathrm{H}(\mathsf{Top})$, and each system map $(f, f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ in inv-Top induces a system map $(f, [f_\lambda]) : [\boldsymbol{X}] \to [\boldsymbol{Y}]$ in inv-$\mathrm{H}(\mathsf{Top})$. Moreover, each morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in pro-Top induces a morphism $[\boldsymbol{f}] : [\boldsymbol{X}] \to [\boldsymbol{Y}]$ in pro-$\mathrm{H}(\mathsf{Top})$.

A *fibration category* is a category $\mathcal{F}$ with the structure $(\mathcal{F}, fib, we)$ which satisfies axiom (F1), (F2), (F3), (F4) below. Here fib and we are classes of morphisms, called *fibrations* and *weak equivalences*, respectively. For more details, the reader is referred to [1].

(F1): *Composition axiom.* The isomorphisms in $\mathcal{F}$ are weak equivalences and fibrations. For any morphisms $f : X \to Y$ and $g : Y \to Z$, if any two of f, g, and gf are weak equivalences, so is the third. The composite of fibrations is a fibration.

(F2): *Pull-back axiom.* For any 2-sink $X \xrightarrow{f} Y \xleftarrow{g} B$ in $\mathcal{F}$ with f being a fibration, there is a pull-back diagram in $\mathcal{F}$

$$\begin{tikzcd} E \arrow[r, "\overline{g}"] \arrow[d, "\overline{f}"'] & X \arrow[d, "f"] \\ B \arrow[r, "g"] & Y \end{tikzcd}$$

and $\overline{f}$ is a fibration. Moreover, if f (resp., g) is a weak equivalence, so is $\overline{f}$ (resp., $\overline{g}$).

(F3): *Factorization axiom.* Each morphism $f : X \to Y$ admits a factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ A \arrow[ur, "h"'] & \end{tikzcd}$$

where g is a weak equivalence and h is a fibration.

(F4): *Axiom on cofibrant models.* Each object X in $\mathcal{F}$ admits a trivial fibration $RX \to X$ where RX is a cofibrant in $\mathcal{F}$, i.e., each trivial fibration $f : Q \to RX$ admits a morphism $s : RX \to Q$ such that $fs = 1_{RX}$.

For any metric space X, for any $\varepsilon > 0$, and for any $x \in X$ let $B_\varepsilon(x) = \{y \in X : \mathrm{d}(x, y) < \varepsilon\}$. For any $\varepsilon > 0$, two points y, y' of a metric space Y are *ε-near* provided $\mathrm{d}(y, y') < \varepsilon$, and any two maps $f, g : X \to Y$ of a space X into a metric space Y are *ε-near*, denoted $(f, g) < \varepsilon$, provided $f(x)$ and $g(x)$ are ε-near for each $x \in X$. More generally, for any open covering $\mathcal{V}$ of a space Y, two points y, y' of Y are *$\mathcal{V}$-near*, denoted $(y, y') < \mathcal{V}$, provided $y, y' \in V$ for some $V \in \mathcal{V}$, and any two maps $f, g : X \to Y$ are *$\mathcal{V}$-near*, denoted $(f, g) < \mathcal{V}$, provided $f(x)$ and $g(x)$ are $\mathcal{V}$-near for each $x \in X$. For any open covering $\mathcal{U}$ of a space X and for any subset A of X, let $\mathrm{st}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$, and write $A < \mathcal{U}$ if $A \subseteq U$ for some $U \in \mathcal{U}$. The *star covering* $\mathrm{st}\,\mathcal{U}$ of an open covering $\mathcal{U}$ of X is the open covering consisting of all subsets $\mathrm{st}(U, \mathcal{U})$, $U \in \mathcal{U}$. A covering $\mathcal{U}$ of X is said to be a *refinement* of a covering $\mathcal{V}$ of X, denoted $\mathcal{U} < \mathcal{V}$, provided each $U \in \mathcal{U}$ admits

$V \in \mathcal{V}$ such that $U \subseteq V$. For any coverings $\mathcal{U}_i$, $i = 1, 2, \dots, n$, of X, let $\overset{n}{\underset{i=1}{\wedge}} \mathcal{U}_i$ be the covering $\{U_1 \cap \dots \cap U_n : \ U_i \in \mathcal{U}_i, \ i = 1, 2, \dots, n\}$ of X.

3. ANR objects

A system $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ consisting of metric spaces X_λ is said to be an *ANR object* provided for each $\lambda \in \Lambda$ there exists $\lambda' \geq \lambda$ such that each map $f : A \to X_{\lambda'}$ from a closed subset A of a metric space X admits an open neighborhood U of A and a map $\tilde{f} : U \to X_\lambda$ such that $\tilde{f}|A = p_{\lambda\lambda'} f$.

$$\begin{tikzcd}
U \arrow[d, dashed, "\tilde{f}"'] & A \arrow[l, "\supseteq"'] \arrow[d, "f"] \\
X_\lambda & X_{\lambda'} \arrow[l, "p_{\lambda\lambda'}"']
\end{tikzcd}$$

Theorem 3.1. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be a system of metric spaces. Then the following are equivalent:*

(1) *$\boldsymbol{X}$ is an ANR object;*
(2) *Each $\lambda \in \Lambda$ admits $\lambda' \geq \lambda$ so that $p_{\lambda\lambda'}$ factors through an ANR.*

Proof. (1)$\Rightarrow$(2): For each $\lambda \in \Lambda$, let $\lambda' \geq \lambda$ be as in the definition of ANR object. Let $X_{\lambda'}$ be embedded into an ANR K as a closed subset. Then there exist an open neighborhood $U_{\lambda'}$ of $X_{\lambda'}$ in K and a map $\tilde{p}_{\lambda\lambda'} : U_{\lambda'} \to X_\lambda$ such that $\tilde{p}_{\lambda\lambda'}|X_{\lambda'} = p_{\lambda\lambda'}$. Then $U_{\lambda'}$ is an ANR as it is an open subset of an ANR K. This proves (2).

(2)$\Rightarrow$(1): For each $\lambda \in \Lambda$, take $\lambda' \geq \lambda$ as in (2). It is easy to see that for this λ' the property required for ANR object holds. $\square$

Theorem 3.1 immediately implies

Corollary 3.2. *Every ANR-system is an ANR object.*

Theorem 3.3. *Let $\boldsymbol{X} = (X_i, p_{i,i+1})$ be a sequence of metric spaces. Then the following are equivalent:*

(1) *$\boldsymbol{X}$ is an ANR object;*
(2) *$\boldsymbol{X}$ is isomorphic to an ANR-tower in* pro-Top.

Proof. (1)$\Rightarrow$(2): Let $i_0 = 1$. There is $i_1 > i_0$ such that $p_{i_0 i_1}$ factors into the composite $X_{i_0} \xleftarrow{h_1} V_1 \xleftarrow{g_1} X_{i_1}$ for some ANR V_1. Embed X_{i_1} into an ANR K_{i_1} as a closed subset, and we obtain an extension $\overline{g}_1 : U_1 \to V_1$ over some open neighborhood U_1 of X_{i_1}. Let $f_1 : U_1 \to X_{i_0}$ be the composite $h_1 \overline{g}_1$. By a similar argument, we obtain a sequence $\{i_k\}$, ANR's U_k, each of which is an open neighborhood of X_{i_k} in some ANR K_{i_k}, and maps $f_k : U_k \to X_{i_{k-1}}$, each of which is an extension of $p_{i_{k-1} i_k} : X_{i_k} \to X_{i_{k-1}}$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
& U_1 \arrow[dl, "f_1"'] & U_2 \arrow[dl, "f_2"'] & \cdots & U_{k-1} & U_k \arrow[dl, "f_k"'] & \\
X_{i_0} & X_{i_1} \arrow[l, "p_{i_0 i_1}"] \arrow[u, "\subseteq"'] & X_{i_2} \arrow[l, "p_{i_1 i_2}"] \arrow[u, "\subseteq"'] & \cdots \arrow[l] & X_{i_{k-1}} \arrow[l] \arrow[u, "\subseteq"'] & X_{i_k} \arrow[l, "p_{i_{k-1} i_k}"] \arrow[u, "\subseteq"'] & \cdots \arrow[l]
\end{tikzcd}$$

Define the map $q_{k,k+1} : U_{k+1} \to U_k$ as the composite $U_{k+1} \xrightarrow{f_{k+1}} X_{i_k} \hookrightarrow U_k$. Then we have an ANR-tower $\boldsymbol{Y} = (U_k, q_{k,k+1})$ and a level map $(f_k) : \boldsymbol{Y} \to \boldsymbol{X}'$ to the subsequence $\boldsymbol{X}' = (X_{i_k}, p_{i_k i_{k+1}})$ of $\boldsymbol{X}$. By Morita's lemma [10, Theorem 5, Ch. II, §. 2.2], (f_k) represents an isomorphism in pro-Top. Since $\boldsymbol{X}'$ is isomorphic to $\boldsymbol{X}$ in pro-Top, we have (2).

(2)⇒(1): Without loss of generality, we can assume that a level map $(f_i) : \boldsymbol{X} \to \boldsymbol{Y}$ represents an isomorphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in pro-Top [10, Theorem 3, Ch. I, §. 1.3], where $\boldsymbol{Y} = (Y_i, q_{i,i+1})$ is an ANR-sequence. For each i, there exist $i' \geq i$ and a map $r_{ii'} : X_i \to Y_{i'}$ so that the following diagram commutes:

$$\begin{tikzcd} X_i \arrow[d, "f_i"'] & X_{i'} \arrow[l, "p_{ii'}"'] \arrow[d, "f_{i'}"] \\ Y_i & Y_{i'} \arrow[l, "q_{ii'}"] \arrow[ul, "r_{ii'}"'] \end{tikzcd}$$

Suppose that $f : A \to X_{i'}$ is a map from a closed subset A of a metric space Z. Then $f_{i'} f : A \to Y_{i'}$ extends to a map $f' : U \to Y_{i'}$ over an open neighborhood U of A. Then $\tilde{f} = r_{ii'} f' : U \to X_i$ is an extension of $p_{ii'} f$. □

Theorem 3.4. *Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ be ANR objects. Then we have*

(1) *the system $\boldsymbol{X}^I = (X_\lambda^I, (p_{\lambda\lambda'})_*, \Lambda)$ is an ANR object, where $(p_{\lambda\lambda'})_* : X_{\lambda'}^I \to X_\lambda^I$ is the map defined by $(p_{\lambda\lambda'})_*(\alpha) = p_{\lambda\lambda'}\alpha$ for $\alpha \in X_{\lambda'}^I$,*
(2) *the system $\boldsymbol{X} \times \boldsymbol{Y} = (X_\lambda \times Y_\mu, p_{\lambda\lambda'} \times q_{\mu\mu'}, \Lambda \times M)$ is an ANR object, and*
(3) *if $\boldsymbol{Z} = (Z_\lambda, r_{\lambda\lambda'}, \Lambda)$ is a system such that Z_λ are open subsets of X_λ with $p_{\lambda\lambda'}(Z_{\lambda'}) \subseteq Z_\lambda$ for $\lambda \leq \lambda'$ and $r_{\lambda\lambda'} : Z_{\lambda'} \to Z_\lambda$ are restrictions of $p_{\lambda\lambda'}$, then $\boldsymbol{Z}$ is an ANR object.*

Proof. The proof is similar to the analogous assertions for ANR's. □

4. Pro-fibrations, strong pro-fibrations, and approximate pro-fibrations

For any system map $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ between systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, a pair $(\lambda, \mu) \in \Lambda \times M$ is *admissible* provided $\lambda > f(\mu)$. For any admissible pairs (λ, μ), (λ', μ'), write $(\lambda, \mu) \geq (\lambda', \mu')$ if $\lambda \geq \lambda'$ and $\mu \geq \mu'$. A system map $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to have the *homotopy lifting property* (HLP) with respect to a space Z provided it satisfies the following property:

(HLP) For any admissible pair $(\lambda, \mu) \in \Lambda \times M$ there exist an admissible pair $(\lambda', \mu') \geq (\lambda, \mu)$ such that whenever $h : Z \times 0 \to X_{\lambda'}$ and $H : Z \times I \to Y_{\mu'}$ are maps satisfying

$$f_{\mu'} p_{f(\mu')\lambda'} h = H_0, \tag{4.1}$$

then there is a map $\tilde{H} : Z \times I \to X_\lambda$ satisfying

$$\tilde{H}_0 = p_{\lambda\lambda'} h \tag{4.2}$$

and

$$f_\mu p_{f(\mu)\lambda} \tilde{H} = q_{\mu\mu'} H. \tag{4.3}$$

$$\begin{tikzcd}
 & & X_{f(\mu')} \arrow[ddl, "f_{\mu'}"'] & \\
X_{f(\mu)} \arrow[d, "f_\mu"'] & X_\lambda \arrow[l, "p_{f(\mu)\lambda}"'] & X_{\lambda'} \arrow[l, "p_{\lambda\lambda'}"'] \arrow[u, "p_{f(\mu')\lambda'}"'] & Z \times 0 \arrow[l, "h"'] \arrow[d, "\subseteq"] \\
Y_\mu & Y_{\mu'} \arrow[l, "q_{\mu\mu'}"] & & Z \times I \arrow[ll, "H"] \arrow[uull, dashed, "\tilde{H}"]
\end{tikzcd}$$

A system map $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ is said to have the *strong homotopy lifting property* (SHLP) with respect to a space Z provided it satisfies the following property:

(SHLP) For any admissible pair $(\lambda, \mu) \in \Lambda \times M$ there exist an admissible pair $(\lambda', \mu') \geq (\lambda, \mu)$ such that whenever $h : Z \times 0 \to X_{\lambda'}$ and $H : Z \times I \to Y_{\mu'}$ are maps satisfying (4.1), then there is a map $\tilde{H} : Z \times I \to X_\lambda$ satisfying (4.2) and (4.3) and the property that

$$\text{if } H \text{ is constant on } z \times I, \text{ then } \tilde{H} \text{ is constant on } z \times I. \tag{4.4}$$

A system map $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ has the *approximate homotopy lifting property* (AHLP) with respect to a space Z provided it satisfies the following property:

(AHLP) For any admissible pair $(\lambda, \mu) \in \Lambda \times M$ and for any open coverings $\mathcal{U}$ and $\mathcal{V}$ of X_λ and Y_μ, respectively, there exist an admissible pair $(\lambda', \mu') \geq (\lambda, \mu)$ and an open covering $\mathcal{V}'$ of $Y_{\mu'}$ such that whenever $h : Z \to X_{\lambda'}$ and $H : Z \times I \to Y_{\mu'}$ are maps satisfying

$$(f_{\mu'} p_{f(\mu')\lambda'} h, H_0) < \mathcal{V}', \tag{4.5}$$

then there is a map $\tilde{H} : Z \times I \to X_\lambda$ satisfying

$$(\tilde{H}_0, p_{\lambda\lambda'} h) < \mathcal{U}, \tag{4.6}$$

and

$$(f_\mu p_{f(\mu)\lambda} \tilde{H}, q_{\mu\mu'} H) < \mathcal{V}. \tag{4.7}$$

(λ', μ') in (HLP) , (SHLP) and (AHLP) is called a *lifting index* for (f, f_μ), and $\mathcal{V}'$ in (AHLP) is called a *lifting mesh* for (f, f_μ).

A map $f : X \to Y$ is said to have the HLP (resp., SHLP, AHLP) with respect to a space Z provided the system map $(f) : (X) \to (Y)$ between the rudimentary systems has the HLP (resp., SHLP, AHLP) with respect to Z.

We wish to extend the definitions of HLP, SHLP, and AHLP to morphisms in pro- Top.

Proposition 4.1. *Suppose that $(f, f_\lambda), (g, g_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ are system maps such that $(f, f_\lambda) \sim (g, g_\lambda)$. If (f, f_λ) has the HLP (resp., SHLP, AHLP) with respect to a space Z, then so does (g, g_λ).*

Proof. We first prove the proposition for the AHLP. Let $(\lambda, \mu) \in \Lambda \times M$ be an admissible pair for (g, g_μ), and let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of X_λ and Y_μ, respectively. Take $\lambda_1 \geq \lambda, f(\mu)$ such that

$$f_\mu p_{f(\mu)\lambda_1} = g_\mu p_{g(\mu)\lambda_1}. \tag{4.8}$$

Apply the AHLP for (f, f_μ) with the admissible pair (λ_1, μ) and the open coverings $p_{\lambda\lambda_1}^{-1}\mathcal{U}$ and $\mathcal{V}$, and we have an admissible pair $(\lambda', \mu') \geq (\lambda_1, \mu)$ and an open covering

$\mathcal{V}'$ of $Y_{\mu'}$ with the corresponding properties in (AHLP). Take $\lambda_1' \geq \lambda', g(\mu')$ such that

$$f_{\mu'} p_{f(\mu')\lambda_1'} = g_{\mu'} p_{g(\mu')\lambda_1'}. \tag{4.9}$$

Then (λ_1', μ') is a lifting index and $\mathcal{V}'$ is a lifting mesh for (g, g_λ). Indeed, suppose that $h : Z \times 0 \to X_{\lambda_1'}$ and $H : Z \times I \to Y_{\mu'}$ are maps such that

$$(g_{\mu'} p_{g(\mu')\lambda_1'} h, H_0) < \mathcal{V}'. \tag{4.10}$$

By (4.9) and (4.10),

$$(f_{\mu'} p_{f(\mu')\lambda_1'} h, H_0) < \mathcal{V}'.$$

Since $\mathcal{V}'$ is a lifting mesh for (f, f_μ), this implies that there exists a map $\tilde{H} : Z \times I \to X_{\lambda_1}$ with the following properties:

$$(p_{\lambda_1\lambda_1'} h, \tilde{H}_0) < p^{-1}_{\lambda\lambda_1}\mathcal{U}, \tag{4.11}$$

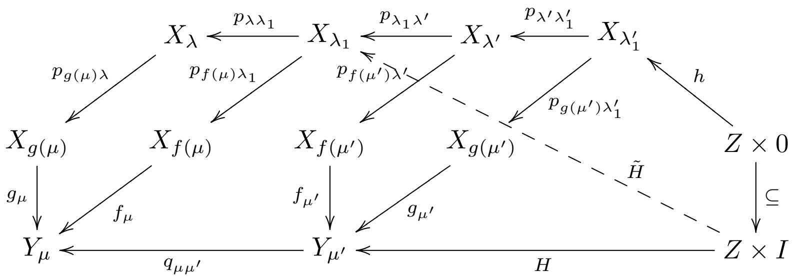
$$(f_\mu p_{f(\mu)\lambda_1} \tilde{H}, q_{\mu\mu'} H) < \mathcal{V}. \tag{4.12}$$

By (4.11), (4.12), and (4.8),

$$(p_{\lambda\lambda_1'} h, p_{\lambda\lambda_1} \tilde{H}_0) < \mathcal{U}, \tag{4.13}$$

$$(g_\mu p_{g(\mu)\lambda}(p_{\lambda\lambda_1} \tilde{H}), q_{\mu\mu'} H) < \mathcal{V}. \tag{4.14}$$

as required.



Similarly, we can prove the assertion for the HLP and SHLP similarly, replacing the nearness conditions (4.10) - (4.14) by the appropriate equality conditions. □

A morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in pro-Top is said to have the *homotopy lifting property* (*HLP*) (resp., *strong homotopy lifting property* (*SHLP*), *approximate homotopy lifting property* (*AHLP*)) with respect to a space Z provided $\boldsymbol{f}$ is represented by a system map with the HLP (resp., SHLP, AHLP) with respect to Z. In view of Proposition 4.1, this is equivalent to saying that any system map representing $\boldsymbol{f}$ has the HLP (resp., SHLP, AHLP) with respect to Z. Following S. Mardešić and T. B. Rushing [8, §. 9], we call a morphism in pro-Top having the HLP (resp., AHLP) with respect to all spaces a *pro-fibration* (resp., an *approximate pro-fibration*). We now call a morphism having the SHLP with respect to all spaces a *strong pro-fibration.*

Clearly, a strong pro-fibration is a pro-fibration. Although we do not know if we have a full implication from pro-fibration to approximate pro-fibration, we have

Theorem 4.2. *Let $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be a system map between ANR-systems $\boldsymbol{X}$ and $\boldsymbol{Y}$. If (f, f_μ) has the HLP with respect to a paracompact space Z, then it has the AHLP with respect to Z.*

Theorem 4.2 immediately follows from the following proposition, which is a generalization of [8, Proposition 2].

Proposition 4.3. *Let $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be a system map between systems $\boldsymbol{X}$ and $\boldsymbol{Y}$. If $\boldsymbol{Y}$ is an ANR-system, and if (f, f_μ) has the formally weaker lifting property (WAHLP) with respect to a paracompact space Z which is obtained by replacing (4.5) by (4.1), then it has the AHLP with respect to Z.*

Proof. The main idea of the proof is from [8, Proposition 2]. Suppose that (f, f_μ) has the WAHLP with respect to Z. Let $(\lambda, \mu) \in \Lambda \times M$ be an admissible pair, and let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of X_λ and Y_μ, respectively. Take an open covering $\mathcal{V}_1$ of Y_μ such that

$$\mathrm{st}(\mathrm{st}\,\mathcal{V}_1) < \mathcal{V}. \tag{4.15}$$

Apply (WAHLP) to (f, f_μ) with $\mathcal{U}$ and $\mathcal{V}_1$, and we obtain an admissible pair $(\lambda', \mu') \geq (\lambda, \mu)$ with the corresponding property in (WAHLP). There exists an open covering $\mathcal{V}'$ of $Y_{\mu'}$ so that

$$\text{any } \mathcal{V}'\text{-near maps into } Y_{\mu'} \text{ are } q_{\mu\mu'}^{-1}\mathcal{V}_1\text{-homotopic.} \tag{4.16}$$

To see that (λ', μ') and $\mathcal{V}'$ are respectively a lifting index and a liftig mesh, suppose that we have maps $h : Z \times 0 \to X_{\lambda'}$ and $H : Z \times I \to Y_{\mu'}$ such that

$$(f_{\mu'} p_{f(\mu')\lambda'} h, H_0) < \mathcal{V}'.$$

By (4.16), there exists a homotopy $H' : Z \times I \to Y_{\mu'}$ such that

$$H'_0 = f_{\mu'} p_{f(\mu')\lambda'} h,\ H'_1 = H_0, \text{ and}$$

$$q_{\mu\mu'} H' \text{ is a } \mathcal{V}_1\text{-homotopy.} \tag{4.17}$$

Define a homotopy $H'' : Z \times I \to Y_{\mu'}$ by

$$H''(z,t) = \begin{cases} H'(z, 2t) & 0 \leq t \leq \frac{1}{2}, \\ H(z, 2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases} \tag{4.18}$$

Then $H''_0 = f_{\mu'} p_{f(\mu')\lambda} h$, so (WAHLP) implies that there exists a homotopy $\tilde{H}'' : Z \times I \to X_\lambda$ such that

$$(f_\mu p_{f(\mu)\lambda} \tilde{H}'', q_{\mu\mu'} H'') < \mathcal{V}_1, \tag{4.19}$$

$$(p_{\lambda\lambda'} h, \tilde{H}''_0) < \mathcal{U}. \tag{4.20}$$

By (4.18) and (4.17), for each $z \in Z$,

$$q_{\mu\mu'} H'' \left(z \times \left[0, \frac{1}{2}\right] \right) < \mathcal{V}_1.$$

This together with (4.19) implies

$$f_\mu p_{f(\mu)\lambda} \tilde{H}'' \left(z \times \left[0, \frac{1}{2}\right] \right) < \mathrm{st}\,\mathcal{V}_1,$$

so by [6, 3.4] there exists a map $\varphi : Z \to (0, 1]$ such that

$$f_\mu p_{f(\mu)\lambda} \tilde{H}'' \left(z \times \left[0, \frac{1}{2}(\varphi(z)+1)\right] \right) < \mathrm{st}\,\mathcal{V}_1, \tag{4.21}$$

$$q_{\mu\mu'}H(z \times [0,\varphi(z)]) < \mathcal{V}_1. \tag{4.22}$$

Now we define a homotopy $\tilde{H} : Z \times I \to X_\lambda$ by

$$\tilde{H}(z,t) = \begin{cases} \tilde{H}''\left(z, \dfrac{t}{\varphi(z)}\right) & 0 \le t \le \dfrac{\varphi(z)}{2}, \\ \tilde{H}''\left(z, t + \dfrac{1}{2}(1-\varphi(z))\right) & \dfrac{\varphi(z)}{2} \le t \le \varphi(z), \\ \tilde{H}''\left(z, \dfrac{1}{2}(1+t)\right) & \varphi(z) \le t \le 1. \end{cases} \tag{4.23}$$

By (4.20) and (4.23),

$$(p_{\lambda\lambda'}h, \tilde{H}_0) < \mathcal{U}.$$

It remains to show

$$(f_\mu p_{f(\mu)\lambda}\tilde{H}, q_{\mu\mu'}H) < \mathcal{V}. \tag{4.24}$$

For $0 \le t \le \varphi(z)$, (4.23), (4.21), (4.19), (4.18), and (4.22) imply

$$(f_\mu p_{f(\mu)\lambda}\tilde{H}(z,t), q_{\mu\mu'}H(z,t)) < \mathrm{st}(\mathrm{st}\,\mathcal{V}_1). \tag{4.25}$$

For $\varphi(z) \le t \le 1$, by (4.23) and (4.18),

$$f_\mu p_{f(\mu)\lambda}\tilde{H}(z,t) = f_\mu p_{f(\mu)\lambda}\tilde{H}''\left(z, \frac{1}{2}(1+t)\right), \text{ and}$$

$$q_{\mu\mu'}H(z,t) = q_{\mu\mu'}H''\left(z, \frac{1}{2}(1+t)\right).$$

Those two equalities together with (4.19) imply

$$(f_\mu p_{f(\mu)\lambda}\tilde{H}(z,t), q_{\mu\mu'}H(z,t)) < \mathcal{V}_1. \tag{4.26}$$

By (4.25), (4.26), and (4.15), we have (4.24) as required. □

The following is also a generalization of [8, Proposition 1].

Proposition 4.4. *Let $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ be a system map between systems $\boldsymbol{X}$ and $\boldsymbol{Y}$. If $\boldsymbol{X}$ is an ANR-system, and if (f, f_μ) has the AHLP with respect to a paracompact space Z, then it has the formally stronger lifting property (SAHLP) with respect to Z which is obtained by replacing (4.6) by (4.2).*

Proof. This is proven for level maps in [6, 3.2], and the same technique applies to the case for general system maps. □

A map $f : X \to Y$ between spaces is a *shape fibration* provided f admits an AP-resolution $(\boldsymbol{p}, \boldsymbol{q}, (f, f_\mu))$ such that the system map (f, f_μ) has the AHLP with respect to all spaces [7]. The AHLP is invariant for AP-resolutions of f, but the HLP is not [8, Example 3]. However, in our theory, it suffices to consider the HLP since systems are chosen in advance.

5. Composition axiom

Theorem 5.1. *Every isomorphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *in* pro-Top *has the HLP, SHLP, and AHLP with respect to all spaces.*

Proof. Let $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ and $(g, g_\lambda) : \boldsymbol{Y} \to \boldsymbol{X}$ be system maps representing $\boldsymbol{f}$ and the inverse of $\boldsymbol{f}$, respectively. Let $(\lambda, \mu) \in \Lambda \times M$ be an admissible pair for (f, f_μ). By $(fg, g_\lambda f_{g(\lambda)}) \sim (1, 1_\lambda)$, there exists $\lambda_1 > \lambda, fg(\lambda)$ such that

$$g_\lambda f_{g(\lambda)} p_{fg(\lambda)\lambda_1} = p_{\lambda\lambda_1}. \tag{5.1}$$

By $(gf, f_\mu g_{f(\mu)}) \sim (1, 1_\mu)$ and the fact that (g, g_μ) is a system map, there exists $\mu' > \mu, g(\lambda), gf(\mu)$ such that

$$f_\mu g_{f(\mu)} q_{gf(\mu)\mu'} = q_{\mu\mu'}, \tag{5.2}$$

$$g_{f(\mu)} q_{gf(\mu)\mu'} = p_{f(\mu)\lambda} g_\lambda q_{g(\lambda)\mu'}. \tag{5.3}$$

By the fact that (f, f_μ) is a system map, there exists $\lambda' > f(\mu'), \lambda_1$ such that

$$f_{g(\lambda)} p_{fg(\lambda)\lambda'} = q_{g(\lambda)\mu'} f_{\mu'} p_{f(\mu')\lambda'}. \tag{5.4}$$

First, to show that (f, f_μ) has the AHLP with respect to a space Z, let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of X_λ and Y_μ, respectively, and let

$$\mathcal{V}' = q_{g(\lambda)\mu'}^{-1} g_\lambda^{-1} \mathcal{U}. \tag{5.5}$$

Then (λ', μ') is a lifting index, and $\mathcal{V}'$ is a lifting mesh for (f, f_μ). Indeed, suppose that $h : Z \times 0 \to X_{\lambda'}$ and $H : Z \times I \to Y_{\mu'}$ such that

$$(f_{\mu'} p_{f(\mu')\lambda'} h, H_0) < \mathcal{V}'. \tag{5.6}$$

Let $\tilde{H} = g_\lambda q_{g(\lambda)\mu'} H : Z \times I \to X_\lambda$. Consider the following diagram:

$$\begin{tikzcd}
 & X_\lambda \arrow[dl, "p_{f(\mu)\lambda}"'] & & X_{\lambda_1} \arrow[ll, "p_{\lambda\lambda_1}"'] \arrow[dl, "p_{fg(\lambda)\lambda_1}"'] & & X_{\lambda'} \arrow[ll, "p_{\lambda_1\lambda'}"'] \arrow[dl, "p_{f(\mu')\lambda'}"'] & \\
X_{f(\mu)} \arrow[d, "f_\mu"'] & & X_{fg(\lambda)} \arrow[d, "f_{g(\lambda)}"] & & X_{f(\mu')} \arrow[dd, "f_{\mu'}"'] & & Z \times 0 \arrow[ul, "h"'] \arrow[d, "\subseteq"] \\
Y_\mu & Y_{gf(\mu)} \arrow[ul, "g_{f(\mu)}"'] & Y_{g(\lambda)} \arrow[uul, "g_\lambda"] & & & & Z \times I \arrow[dll, "H"] \\
 & & & & Y_{\mu'} \arrow[ullll, "q_{\mu\mu'}"] \arrow[ulll, "q_{gf(\mu)\mu'}"'] \arrow[ull, "q_{g(\lambda)\mu'}"'] & &
\end{tikzcd}$$

Then, by (5.6), (5.5), (5.4), and (5.1),

$$(\tilde{H}_0, p_{\lambda\lambda'} h) < \mathcal{U}, \tag{5.7}$$

and by (5.2) and (5.3),

$$f_\mu p_{f(\mu)\lambda} \tilde{H} = q_{\mu\mu'} H, \tag{5.8}$$

which immediately implies

$$(f_\mu p_{f(\mu)\lambda} \tilde{H}, q_{\mu\mu'} H) < \mathcal{V}.$$

For the HLP, replacing (5.6) by the equality

$$f_{\mu'} p_{f(\mu')\lambda'} h = H_0,$$

we have the required properties

$$\tilde{H}_0 = p_{\lambda\lambda'}h$$

instead of (5.7), and (5.8). It is obvious that if H is constant on $z \times I$, then $\tilde{H}$ is constant on $z \times I$. Thus (f, f_μ) has the SHLP. $\square$

Theorem 5.2. *Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{g} : \boldsymbol{Y} \to \boldsymbol{Z}$ be morphisms in* pro-Top. *If $\boldsymbol{f}$ and $\boldsymbol{g}$ have the HLP (resp., SHLP, AHLP) with respect to a space A, then $\boldsymbol{gf}$ has the HLP (resp., SHLP, AHLP) with respect to A.*

Proof. We first prove the theorem for the AHLP. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$, and $\boldsymbol{Z} = (Z_\nu, r_{\nu\nu'}, N)$, and let $\boldsymbol{f}$ and $\boldsymbol{g}$ be represented by system maps (f, f_μ) and (g, g_ν), respectively. To show that the composition $(g, g_\nu)(f, f_\mu)$ has the AHLP with respect to A, let $(\lambda, \nu) \in \Lambda \times N$ be admissible pair for the composition $(g, g_\nu)(f, f_\mu)$, and let $\mathcal{U}$ and $\mathcal{W}$ be open coverings of X_λ and Z_ν, respectively. Let $\mathcal{W}_1$ be an open covering of Z_ν such that

$$\operatorname{st}\mathcal{W}_1 < \mathcal{W}. \tag{5.9}$$

Apply the AHLP to (f, f_μ) together with $\mathcal{U}$ and $g_\nu^{-1}\mathcal{W}_1$, and we have an admissible pair $(\lambda', \mu) \geq (\lambda, g(\nu))$ and an open covering $\mathcal{V}$ of Y_μ with the corresponding property. Then apply the AHLP to (g, g_ν) with $\mathcal{V}$ and $\mathcal{W}_1$, and we have an admissible pair $(\mu', \nu') \geq (\mu, \nu)$ and an open covering $\mathcal{W}'$ of $Z_{\nu'}$ with the corresponding property. By the fact that (f, f_μ) is a system map, there exists $\lambda'' \geq \lambda', fg(\nu'), f(\mu')$ such that

$$f_{g(\nu')}p_{fg(\nu')\lambda''} = q_{g(\nu')\mu'}f_{\mu'}p_{f(\mu')\lambda''}, \tag{5.10}$$

$$f_\mu p_{f(\mu)\lambda''} = q_{\mu\mu'}f_{\mu'}p_{f(\mu')\lambda''}. \tag{5.11}$$

We wish to show that (λ'', ν') is a lifting index and $\mathcal{W}'$ is a lifting mesh for the composition $(g, g_\nu)(f, f_\mu)$. Suppose that $h : A \times 0 \to X_{\lambda''}$ and $H : A \times I \to Z_{\nu'}$ are maps such that

$$(H_0, g_{\nu'}f_{g(\nu')}p_{fg(\nu')\lambda''}h) < \mathcal{W}'. \tag{5.12}$$

By (5.10) and (5.12),

$$(H_0, g_{\nu'}q_{g(\nu')\mu'}f_{\mu'}p_{f(\mu')\lambda''}h) < \mathcal{W}'.$$

Since $\mathcal{W}'$ is a lifting mesh for (g, g_ν), this implies that there exists a map $K : A \times I \to Y_\mu$ such that

$$(K_0, q_{\mu\mu'}f_{\mu'}p_{f(\mu')\lambda''}h) < \mathcal{V}, \tag{5.13}$$

$$(g_\nu q_{g(\nu)\mu}K, r_{\nu\nu'}H) < \mathcal{W}_1. \tag{5.14}$$

By (5.11) and (5.13),

$$(K_0, f_\mu p_{f(\mu)\lambda''}h) < \mathcal{V}.$$

Sinc $\mathcal{V}$ is a lifting mesh for (f, f_μ), this implies that there exists a map $L : Z \times I \to X_\lambda$ such that

$$(L_0, p_{\lambda\lambda''}h) < \mathcal{U},$$

$$(f_{g(\nu)}p_{fg(\nu)\lambda}L, q_{g(\nu)\mu}K) < g_\nu^{-1}\mathcal{W}_1. \tag{5.15}$$

By (5.15), (5.14), and (5.9),

$$(g_\nu f_{g(\nu)}p_{fg(\nu)\lambda}L, r_{\nu\nu'}H) < \mathcal{W}, \tag{5.16}$$

as required.

Replacing the nearness conditions in (5.12) - (5.16) by the corresponding equality conditions, we have the assertion for the HLP and also for the SHLP.

$$
\begin{tikzcd}
 & X_\lambda \arrow[dl, "p_{fg(\nu)\lambda}"'] & X_{\lambda'} \arrow[l, "p_{\lambda\lambda''}"'] \arrow[dl, "p_{f(\mu)\lambda'}"'] & & & X_{\lambda''} \arrow[lll, "p_{\lambda'\lambda''}"'] \arrow[dll, "p_{fg(\nu')\lambda''}"'] \arrow[dl, "p_{f(\mu')\lambda''}"] & \\
X_{fg(\nu)} \arrow[dd, "f_{g(\nu)}"'] & X_{f(\mu)} \arrow[dd, "f_\mu"'] & & X_{fg(\nu')} \arrow[dd, "f_{g(\nu')}"'] & X_{f(\mu')} \arrow[d, "f_{\mu'}"] & & A \times 0 \arrow[ul, "h"'] \arrow[ddd, "\subseteq"] \\
 & & & & Y_{\mu'} \arrow[dlll, "q_{\mu\mu'}"'] \arrow[dl, "q_{g(\nu')\mu'}"] & & \\
Y_{g(\nu)} \arrow[d, "g_\nu"'] & Y_\mu \arrow[l, "q_{g(\nu)\mu}"'] & & Y_{g(\nu')} \arrow[d, "g_{\nu'}"'] & & & \\
Z_\nu & & & Z_{\nu'} \arrow[lll, "r_{\nu\nu'}"] & & & A \times I \arrow[lll, "H"] \arrow[ulllll, dashed, "K"'] \arrow[uuuulllll, dashed, "L"']
\end{tikzcd}
$$

□

Let fib_s, fib, and fib_a denote the classes of strong pro-fibrations, pro-fibrations, and approximate pro-fibrations, respectively, and let we denote the class of morphisms in pro-Top inducing isomorphisms in pro-$\mathrm{H}(\mathsf{Top})$.

Theorems 5.1 and 5.2 immediately imply

Theorem 5.3. *$(\mathcal{ANR}, fib_s, we)$, $(\mathcal{ANR}, fib, we)$, and $(\mathcal{ANR}, fib_a, we)$ satisfy the composition axiom.*

6. Factorization axiom

Theorem 6.1. *Every morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathcal{ANR}$ admits a commutative diagram in $\mathcal{ANR}$*

$$
\begin{tikzcd}
\boldsymbol{X} \arrow[rr, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{g}"'] & & \boldsymbol{Y} \\
 & \boldsymbol{E} \arrow[ur, "\boldsymbol{h}"'] &
\end{tikzcd}
\tag{6.1}
$$

where $\boldsymbol{g}$ is a morphism inducing an isomorphism in pro-$\mathrm{H}(\mathsf{Top})$ *and $\boldsymbol{h}$ is a morphism with the HLP, SHLP, and AHLP with respect to all spaces.*

Proof. By [10, Theorems 2, 3, Ch. I, §. 1], Theorems 5.1 and 5.2, it suffices to assume that $\boldsymbol{f}$ is represented by a level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$, where $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ are ANR objects indexed by a cofinite directed set Λ. We define open coverings $\mathcal{V}_\lambda$ of Y_λ by induction on the number of predecessors of the elements of Λ. If $\lambda \in \Lambda$ has no predecessors, then we let $\mathcal{V}_\lambda$ be any open covering of X_λ. If we have defined open coverings $\mathcal{V}_\lambda$ for all $\lambda \in \Lambda$ that have at most $n-1$ predecessors, and if $\lambda \in \Lambda$ has n predecessors, then we define an open covering $\mathcal{V}_\lambda$ of Y_λ as follows: If λ_i, $i = 1, \ldots, n$, are the predecessors of λ, then for each i, we take an open covering $\mathcal{U}_i$ of Y_{λ_i} such that

$$\mathcal{U}_i < \mathcal{V}_{\lambda_i}, \text{ and} \tag{6.2}$$

(6.3) any two $\mathcal{U}_i$-near maps into Y_{λ_i} are $\mathcal{V}_{\lambda_i}$-homotopic.

We then take an open covering $\mathcal{V}_\lambda$ of Y_λ such that

$$\mathcal{V}_\lambda < \bigwedge_{i=1}^{n} q_{\lambda_i\lambda}^{-1}\mathcal{U}_i. \tag{6.4}$$

For each $\lambda \in \Lambda$, let

$$E_\lambda = \{(x,\omega) \in X_\lambda \times Y_\lambda^I : (f_\lambda(x), \omega(0)) < \mathcal{V}_\lambda\}.$$

Since $\mathcal{V}_{\lambda'} < q_{\lambda\lambda'}^{-1}\mathcal{V}_\lambda$ for $\lambda < \lambda'$ by (6.2) and (6.4), then there is a well-defined map

$$r_{\lambda\lambda'} : E_{\lambda'} \to E_\lambda : r_{\lambda\lambda'}(x,\omega) = (p_{\lambda\lambda'}(x), q_{\lambda\lambda'}\omega) \text{ for } \lambda < \lambda'.$$

For each $\lambda \in \Lambda$, we define maps

$$g_\lambda : X_\lambda \to E_\lambda : g_\lambda(x) = (x, e_{f_\lambda(x)}) \text{ for } x \in X_\lambda,$$

and

$$h_\lambda : E_\lambda \to Y_\lambda : h_\lambda(x,\omega) = \omega(1) \text{ for } (x,\omega) \in E_\lambda.$$

Here for any space W and for any $w_0 \in W$, let $e_{w_0} \in W^I$ denote the constant path defined by $e_{w_0}(t) = w_0$ for $t \in I$. Then there is a commutative diagram:

$$\begin{tikzcd}
X_\lambda \arrow[d, "f_\lambda"'] \arrow[dr, "g_\lambda"] & & X_{\lambda'} \arrow[ll, "p_{\lambda\lambda'}"'] \arrow[d, "f_{\lambda'}"] \arrow[dr, "g_{\lambda'}"] & \\
{} & E_\lambda \arrow[dl, "h_\lambda"] & & E_{\lambda'} \arrow[ll, "r_{\lambda\lambda'}"'] \arrow[dl, "h_{\lambda'}"] \\
Y_\lambda & & Y_{\lambda'} \arrow[ll, "q_{\lambda\lambda'}"] &
\end{tikzcd} \tag{6.5}$$

Thus we have a sytem $\boldsymbol{E} = (E_\lambda, r_{\lambda\lambda'}, \Lambda)$ which is an ANR object by Theorem 3.4, and have level maps $(g_\lambda) : \boldsymbol{X} \to \boldsymbol{E}$ and $(h_\lambda) : \boldsymbol{E} \to \boldsymbol{Y}$ which induce the commutative diagram (6.1) in $\mathcal{ANR}$.

Claim 1. $\boldsymbol{h}$ has the HLP, SHLP, AHLP with respect to all spaces.

It suffices to show that each h_λ has the SHLP with respect to any space Z. Consider the commutative diagram:

$$\begin{tikzcd}
E_\lambda \arrow[d, "h_\lambda"'] & Z \times 0 \arrow[l, "\varphi"'] \arrow[d, "\subseteq"] \\
Y_\lambda & Z \times I \arrow[l, "\Phi"] \arrow[ul, dashed, "\Psi"']
\end{tikzcd}$$

First, for any path $\alpha : I \to X$ on a metric space X, we define the *diameter* $|\alpha|$ of α as the diameter of the image of α, i.e.,

$$|\alpha| = \sup\{\mathrm{d}(\alpha(t), \alpha(t')) : t, t' \in I\}.$$

Note that the function $X^I \to \mathbb{R}_{\geq 0} : \alpha \mapsto |\alpha|$ is continuous, and $|\alpha| = 0$ iff α is a constant path.

If we write $\varphi(z) = (\varphi_1(z), \varphi_2(z)) \in X_\lambda \times Y_\lambda^I$, then the map $\Psi : Z \times I \to E_\lambda$: $\Psi(z,t) = (\Psi_1(z,t), \Psi_2(z,t)) \in X_\lambda \times Y_\lambda^I$ is defined as follows: Let $\Psi_1(z,t) = \varphi_1(z)$. For each $(z,t) \in Z \times I$ with $|\varphi_2(z)| \neq 0$ or $|\Phi_t(z)| \neq 0$, set

$$\mu(z,t) = \frac{|\varphi_2(z)|}{|\varphi_2(z)| + |\Phi_t(z)|}, \ \nu(z,t) = \frac{|\Phi_t(z)|}{|\varphi_2(z)| + |\Phi_t(z)|}.$$

Here, for each $(z,t) \in Z \times I$, $\Phi_t(z) \in Y^I$ is defined by

$$\Phi_t(z)(u) = \Phi(z, tu) \text{ for } u \in I.$$

Then for each $(z,t) \in Z \times I$, define the path $\Psi_2(z,t)$ by the following formula: for each $u \in I$,

$$\Psi_2(z,t)(u) = \begin{cases} \left\{ \begin{array}{l} \varphi_2(z)\left(\dfrac{u}{\mu(z,t)}\right) \\ \qquad (0 \le u \le \mu(z,t)) \\ \\ \Phi_t(z)\left(\dfrac{u-\mu(z,t)}{\nu(z,t)}\right) \\ \qquad (\mu(z,t) < u \le 1) \end{array} \right\} & \text{if } |\varphi_2(z)| \neq 0 \text{ and } |\Phi_t(z)| \neq 0, \\ \qquad \Phi_t(z)(u) & \text{if } |\varphi_2(z)| = 0 \text{ and } |\Phi_t(z)| \neq 0, \\ \qquad \varphi_2(z)(u) & \text{if } |\Phi_t(z)| = 0. \end{cases}$$

Using the same argument as in the proof of [11, Theorem A], in which a function between ANR's defined in the same way as above is shown to be continuous, we can show that the function $\Psi_2 : Z \times I \to Y_\lambda^I$ is continuous.
Claim 2. $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{E}$ induces an isomorphism in pro-$\mathrm{H}(\mathsf{Top})$.

It suffices to show that for $\lambda \le \lambda'$ there is a map $\beta_{\lambda\lambda'} : E_{\lambda'} \to X_\lambda$ which makes the following diagram commute up to homotopy (Morita's lemma [10, Theorem 5, Ch. II, §. 2]):

$$\begin{tikzcd} X_\lambda \arrow[d, "g_\lambda"'] & X_{\lambda'} \arrow[l, "p_{\lambda\lambda'}"'] \arrow[d, "g_{\lambda'}"] \\ E_\lambda & E_{\lambda'} \arrow[l, "r_{\lambda\lambda'}"] \arrow[ul, dashed, "\beta_{\lambda\lambda'}"'] \end{tikzcd}$$

Let $\gamma_\lambda : E_\lambda \to X_\lambda$ be the restriction of the projection map $X_\lambda \times Y_\lambda^I$ onto X_λ and define the map $\beta_{\lambda\lambda'} : E_{\lambda'} \to X_\lambda$ by $\beta_\lambda = \gamma_\lambda r_{\lambda\lambda'}$. Then obviously $\beta_{\lambda\lambda'} g_{\lambda'} = p_{\lambda\lambda'}$. It remains to show $g_\lambda \beta_{\lambda\lambda'} \simeq r_{\lambda\lambda'}$. Indeed, $g_\lambda \beta_{\lambda\lambda'}(x,\omega) = (p_{\lambda\lambda'}(x), e_{f_\lambda p_{\lambda\lambda'}(x)})$ and $r_{\lambda\lambda'}(x,\omega) = (p_{\lambda\lambda'}(x), q_{\lambda\lambda'}\omega)$ for $(x,\omega) \in E_{\lambda'}$. Define a homotopy $K : E_{\lambda'} \times I \to E_\lambda$ by

$$K(x,\omega,t) = (p_{\lambda\lambda'}(x), q_{\lambda\lambda'}\omega_t) \text{ for } (x,\omega,t) \in E_{\lambda'} \times I,$$

where for each $t \in I$, $\omega_t \in Y_{\lambda'}^I$ is defined by $\omega_t(s) = \omega(st)$ for $s \in I$. For each $(x,\omega) \in E_{\lambda'}$, $(f_{\lambda'}(x), \omega(0)) < \mathcal{V}_{\lambda'}$, so by (6.4) and (6.3), there is a $\mathcal{V}_\lambda$-homotopy $L : E_\lambda \times I \to Y_\lambda$ such that

$$L(x,\omega,0) = q_{\lambda\lambda'} f_{\lambda'}(x) \text{ and } L(x,\omega,1) = q_{\lambda\lambda'}\omega(0) \text{ for } (x,\omega) \in E_{\lambda'}.$$

So there is a well-defined map $H : E_{\lambda'} \times I \to E_\lambda$ defined by

$$H(x,\omega,t) = (p_{\lambda\lambda'}(x), e_{L(x,\omega,t)}) \text{ for } (x,\omega,t) \in E_{\lambda'} \times I.$$

Combining H and K, we have a homotopy between $g_\lambda \beta_{\lambda\lambda'}$ and $r_{\lambda\lambda'}$. □

Remark 6.2. (1) In Theorem 6.1, if the systems $\boldsymbol{X}$ and $\boldsymbol{Y}$ are ANR-systems (resp., ANR-towers), we can choose $\boldsymbol{E}$ as an ANR-system (resp., ANR-tower).

(2) In Theorem 6.1, if $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism in the pro-cateogry pro- Top_* of pointed spaces and base point preserving maps, then we can choose $\boldsymbol{g}$ and $\boldsymbol{h}$ as morphisms in pro- Top_* so that $\boldsymbol{g}$ induces an isomorphism in pro- $\mathrm{H}(\mathsf{Top})_*$. Here pro- $\mathrm{H}(\mathsf{Top})_*$ is the pro-homotpy category of Top_* where the homotopies preserve base points. In this case, the analog of (1) also holds.

Theorem 6.1 immediately implies

Theorem 6.3. *$(\mathcal{ANR}, fib_s, we)$, $(\mathcal{ANR}, fib, we)$, and $(\mathcal{ANR}, fib_a, we)$ satisfy the factorization axiom.*

More generally, we have the following version of the factorization axiom, where the naturality is guaranteed:

Theorem 6.4. *Every map $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathcal{ANR}$ admits a commutative diagram in $\mathcal{ANR}$*

(6.6)
$$\begin{tikzcd} \boldsymbol{X} \arrow[rr, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{g}"'] & & \boldsymbol{Y} \\ & \boldsymbol{E_f} \arrow[ur, "\boldsymbol{h}"'] & \end{tikzcd}$$

where $\boldsymbol{g}$ is a morphism inducing an isomorphism in pro- $\mathrm{H}(\mathsf{Top})$ *and $\boldsymbol{h}$ has the HLP, SHLP, and AHLP with respect to all spaces. Moreover, for every commutative diagram in $\mathcal{ANR}$*

(6.7)
$$\begin{tikzcd} \boldsymbol{X} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{\alpha}"'] & \boldsymbol{Y} \arrow[d, "\boldsymbol{\beta}"] \\ \boldsymbol{X'} \arrow[r, "\boldsymbol{f'}"'] & \boldsymbol{Y'} \end{tikzcd}$$

there is a morphism $\boldsymbol{\gamma} : \boldsymbol{E_f} \to \boldsymbol{E_{f'}}$ in $\mathcal{ANR}$ which makes the following diagram commute:

(6.8)
$$\begin{tikzcd} \boldsymbol{X} \arrow[rr, "\boldsymbol{f}"] \arrow[dr, "\boldsymbol{g}"'] \arrow[dd, "\boldsymbol{\alpha}"'] & & \boldsymbol{Y} \arrow[dd, "\boldsymbol{\beta}"] \\ & \boldsymbol{E_f} \arrow[ur, "\boldsymbol{h}"'] \arrow[dd, "\boldsymbol{\gamma}" near start] & \\ \boldsymbol{X'} \arrow[rr, "\boldsymbol{f'}" near start] \arrow[dr, "\boldsymbol{g'}"'] & & \boldsymbol{Y'} \\ & \boldsymbol{E_{f'}} \arrow[ur, "\boldsymbol{h'}"'] & \end{tikzcd}$$

Proof. Let $\boldsymbol{f}$ be represented by a level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$, where $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ are indexed by a cofinite set Λ. Let $\mathcal{V}_\lambda$, $\lambda \in \Lambda$, be as in the proof of Theorem 6.1.

Let Σ be the set of the pairs $(\lambda, \mathcal{V})$ of elements λ of Λ and open coverings $\mathcal{V}$ of Y_λ, which is ordered by $(\lambda, \mathcal{V}) < (\lambda', \mathcal{V}')$ iff $\lambda = \lambda'$ and $\mathcal{V}' < \mathcal{V}$, or $\lambda < \lambda'$ and $\mathcal{V}' < q_{\lambda\lambda'}^{-1}\mathcal{V}$. For each $(\lambda, \mathcal{V}) \in \Sigma$, let

$$E_{(\lambda,\mathcal{V})} = \{(x, \omega) \in X_\lambda \times Y_\lambda^I : (f_\lambda(x), \omega(0)) < \mathcal{V} \wedge \mathcal{V}_\lambda\}.$$

For $(\lambda, \mathcal{V}) < (\lambda', \mathcal{V}')$, there is a well-defined map

$$r_{(\lambda,\mathcal{V})(\lambda',\mathcal{V}')} : E_{(\lambda',\mathcal{V}')} \to E_{(\lambda,\mathcal{V})} : r_{(\lambda,\mathcal{V})(\lambda',\mathcal{V}')}(x, \omega) = (p_{\lambda\lambda'}(x), q_{\lambda\lambda'}\omega).$$

We define the maps

$$g_{(\lambda,\mathcal{V})} : X_\lambda \to E_{(\lambda,\mathcal{V})} : g_{(\lambda,\mathcal{V})}(x) = (x, e_{f_\lambda(x)}) \text{ for each } (\lambda, \mathcal{V}),$$

and

$$h_\lambda : E_{(\lambda,\{Y_\lambda\})} \to Y_\lambda : h_\lambda(x, \omega) = \omega(1) \text{ for each } \lambda \in \Lambda.$$

Then we have the following commutative diagrams for $(\lambda, \mathcal{V}) < (\lambda', \mathcal{V}')$:

$$\begin{tikzcd}
X_\lambda \arrow[d, "g_{(\lambda,\mathcal{V})}"'] & & X_{\lambda'} \arrow[ll, "p_{\lambda\lambda'}"'] \arrow[d, "g_{(\lambda',\mathcal{V}')}"] \\
E_{(\lambda,\mathcal{V})} & & E_{(\lambda',\mathcal{V}')} \arrow[ll, "r_{(\lambda,\mathcal{V})(\lambda',\mathcal{V}')}"]
\end{tikzcd}$$

Moreover, the following diagram commutes for $\lambda < \lambda'$:

$$\begin{tikzcd}
X_\lambda \arrow[dd, "f_\lambda"'] \arrow[dr, "g_\lambda"] & & X_{\lambda'} \arrow[ll, "p_{\lambda\lambda'}"'] \arrow[dd, "f_{\lambda'}"'] \arrow[dr, "g_{\lambda'}"] & \\
& E_{(\lambda,\{Y_\lambda\})} \arrow[dl, "h_\lambda"] & & E_{(\lambda',\{Y_{\lambda'}\})} \arrow[ll, "r_{(\lambda,\{Y_\lambda\})(\lambda',\{Y_{\lambda'}\})}"'] \arrow[dl, "h_{\lambda'}"] \\
Y_\lambda & & Y_{\lambda'} \arrow[ll, "q_{\lambda\lambda'}"] &
\end{tikzcd}$$

Thus we have the system $\boldsymbol{E_f} = (E_{(\lambda,\mathcal{V})}, r_{(\lambda,\mathcal{V})(\lambda',\mathcal{V}')}, \Sigma)$ which is an ANR object, and have system maps $(g, g_{(\lambda,\mathcal{V})}) : \boldsymbol{X} \to \boldsymbol{E_f}$ and $(h, h_\lambda) : \boldsymbol{E_f} \to \boldsymbol{Y}$, where $g : \Sigma \to \Lambda : g(\lambda, \mathcal{V}) = \lambda$, and $h : \Lambda \to \Sigma : h(\lambda) = (\lambda, \{Y_\lambda\})$, so that the system maps represent the morphisms which make diagram (6.6) commute. We can similarly prove Claims 1 and 2 in the proof of Theorem 6.1.

To prove the naturality, suppose that we have a commutative diagram (6.7). By Lemma 2.1 (2), we can assume that $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{X}' = (X'_\lambda, p'_{\lambda\lambda'}, \Lambda), \boldsymbol{Y}' = (Y'_\lambda, q'_{\lambda\lambda'}, \Lambda)$ have the same index set Λ and the maps $\boldsymbol{f}, \boldsymbol{f}', \boldsymbol{\alpha}, \boldsymbol{\beta}$ are represented by level maps $(f_\lambda), (f'_\lambda), (\alpha_\lambda), (\beta_\lambda)$, respectively, and that the following diagram commutes:

$$\begin{tikzcd}
X_\lambda \arrow[r, "f_\lambda"] \arrow[d, "\alpha_\lambda"'] & Y_\lambda \arrow[d, "\beta_\lambda"] \\
X'_\lambda \arrow[r, "f'_\lambda"'] & Y'_\lambda
\end{tikzcd}$$

Let $\boldsymbol{E_{f'}} = (E'_{(\lambda,\mathcal{V})}, r'_{(\lambda,\mathcal{V})(\lambda',\mathcal{V}')}, \Sigma')$ be the ANR object which is induced by the morphism $\boldsymbol{f}'$. We define a function $\gamma : \Sigma' \to \Sigma : \gamma(\lambda,\mathcal{V}) = (\lambda,\mathcal{W})$, where $\mathcal{W}$ is an open covering of Y_λ such that

$$\mathcal{W} < \beta_\lambda^{-1}(\mathcal{V}'_\lambda \wedge \mathcal{V}).$$

Here $\mathcal{V}'_\lambda$ is an open covering of Y'_λ with the property corresponding to (6.4). For each $(\lambda,\mathcal{V}) \in \Sigma$, we define the map

$$\gamma_{(\lambda,\mathcal{V})} : E_{\gamma(\lambda,\mathcal{V})} \to E'_{(\lambda,\mathcal{V})} : \gamma_{(\lambda,\mathcal{V})}(x,\omega) = (\alpha_\lambda(x), \beta_\lambda\omega).$$

Then for $(\lambda,\mathcal{V}) < (\lambda',\mathcal{V}')$, we have the following commutative diagram:

$$\begin{tikzcd}
E_{\gamma(\lambda,\mathcal{V})} \arrow[d, "\gamma_{(\lambda,\mathcal{V})}"'] & E_{\gamma(\lambda',\mathcal{V}')} \arrow[l, "r_{\gamma(\lambda,\mathcal{V})\gamma(\lambda',\mathcal{V}')}"'] \arrow[d, "\gamma_{(\lambda',\mathcal{V}')}"] \\
E'_{(\lambda,\mathcal{V})} & E'_{(\lambda',\mathcal{V}')} \arrow[l, "r_{(\lambda,\mathcal{V})(\lambda',\mathcal{V}')}"]
\end{tikzcd}$$

Thus we have the system map $(\gamma, \gamma_{(\lambda,\mathcal{V})}) : \boldsymbol{E_f} \to \boldsymbol{E_{f'}}$ and the following commutative diagrams for $(\lambda,\mathcal{V}) \in \Sigma'$ and $\lambda \in \Lambda$, respectively:

$$\begin{tikzcd}
X_\lambda \arrow[r, "g_{\gamma(\lambda,\mathcal{V})}"] \arrow[d, "\alpha_\lambda"'] & E_{\gamma(\lambda,\mathcal{V})} \arrow[d, "\gamma_{(\lambda,\mathcal{V})}"] \\
X'_\lambda \arrow[r, "g'_{\gamma(\lambda,\mathcal{V})}"'] & E'_{(\lambda,\mathcal{V})}
\end{tikzcd}
\qquad
\begin{tikzcd}
E'_{\gamma(\lambda,\mathcal{V})} \arrow[r, "h_\lambda"] \arrow[d, "\gamma_{(\lambda,\mathcal{V})}"'] & Y_\lambda \arrow[d, "\beta_\lambda"] \\
E'_{(\lambda,\mathcal{V})} \arrow[r, "h'_\lambda"'] & Y'_\lambda
\end{tikzcd}$$

Here $g'_{\gamma(\lambda,\mathcal{V})}$ and h'_λ are the maps corresponding to $g_{\gamma(\lambda,\mathcal{V})}$ and h_λ, respectively. Thus we have the commutative diagram (6.8). □

Remark 6.5. The properties analogous to (1) and (2) in Remark 6.2 hold for Theorem 6.4.

Next, we use the factorization axiom to prove

Theorem 6.6. *Every shape morphism between compact metric spaces is represented by a strong pro-fibration in* pro-ANR.

Before proving Theorem 6.6, we need several lemmas.

Lemma 6.7. *Every ANR-system $\boldsymbol{X} = (X_i, p_{i,i+1})$ admits an ANR-sequence $\boldsymbol{X}' = (X'_{i+1}, p'_{i,i+1})$ with all bonding maps being Hurewicz fibrations and a level map $(\varphi_i) : \boldsymbol{X} \to \boldsymbol{X}'$ consisting of homotopy equivalences.*

Proof. Let $X'_1 = X_1$, and let $\varphi_1 : X_1 \to X'_1$ be the identity map. Then, for $i = 2, 3, \ldots$, we inductively apply [11, Theorem 2.1] to $\varphi_{i-1}p_{i-1,i} : X_i \to X'_{i-1}$. Note that [11, Theorem 2.1] says that every map between ANR's is the composition of a homotopy equivalence and a map having the strong homotopy lifting property with respect to arbitrary spaces in the sense of [11]. Also, note that a map having the strong homotopy lifting property with respect to arbitrary spaces is a Hurewicz fibration. Then we obtain an ANR X'_i and a factorization $X_i \xrightarrow{\varphi_i} X'_i \xrightarrow{p'_{i-1,i}} X'_{i-1}$ of $\varphi_{i-1}p_{i-1,i}$ where φ_i is a homotopy equivalence, and $p'_{i-1,i}$ is a Hurewicz fibration. □

Lemma 6.8. *Suppose that $\boldsymbol{X} = (X_i, p_{i,i+1})$ and $\boldsymbol{Y} = (Y_i, q_{i,i+1})$ are ANR-towers. If the bonding maps $q_{i,i+1}$ are Hurewicz fibrations, then every system map $(f, [f_i]) : [\boldsymbol{X}] \to [\boldsymbol{Y}]$ is equivalent to a system map induced by a system map $(f', f'_i) : \boldsymbol{X} \to \boldsymbol{Y}$.*

Proof. For $i = 1$, let $f'(1) = f(1)$ and $f'_1 = f_1 : X_{f'(1)} \to Y_1$. Assume that we have defined integers $f'(1) \leq f'(2) \leq \cdots \leq f'(n-1)$ with $f'(i) \geq f(i)$ and maps $f'_i : X_{f'(i)} \to Y_i$, $i = 2, 3, \ldots, n-1$, with the following properties:

$$f'_i \simeq f_i p_{f(i)f'(i)}, \tag{6.9}$$

$$f'_{i-1} p_{f'(i-1)f'(i)} = q_{i-1,i} f'_i. \tag{6.10}$$

Take an integer $f'(n) \geq f(n), f'(n-1)$ such that

$$f_{n-1} p_{f(n-1)f'(n)} = q_{n-1,n} f_n p_{f(n)f'(n)}.$$

This together with (6.9) implies

$$f'_{n-1} p_{f'(n-1)f'(n)} \simeq q_{n-1,n} f_n p_{f(n)f'(n)}.$$

Since $q_{n-1,n}$ is a Hurewicz fibration, this then implies that there is a map $f'_n : X_{f'(n)} \to Y_n$ such that

$$f'_{n-1} p_{f'(n-1)f'(n)} = q_{n-1,n} f'_n.$$

Thus we obtain the system map $(f', f'_i) : \boldsymbol{X} \to \boldsymbol{Y}$. By (6.9), the induced map $(f', [f'_i]) : [\boldsymbol{X}] \to [\boldsymbol{Y}]$ is equivalent to $(f, [f_i])$ as required.

$$\begin{tikzcd}
& X_{f'(n-1)} \arrow[dl, "p_{f(n-1)f'(n-1)}"'] \arrow[ddl, "f'_{n-1}"] & & X_{f'(n)} \arrow[ll, "p_{f'(n-1)f'(n)}"'] \arrow[dl, "p_{f(n)f'(n)}"'] \arrow[ddl, dashed, "f'_n"] \\
X_{f(n-1)} \arrow[d, "f_{n-1}"'] & & X_{f(n)} \arrow[d, "f_n"'] & \\
Y_{n-1} & & Y_n \arrow[ll, "q_{n-1,n}"] &
\end{tikzcd}$$

□

Using Lemmas 6.7 and 6.8, we can easily show

Lemma 6.9. *Every shape morphism $F : X \to Y$ between compact metric spaces is induced by a system map $(f, f_i) : \boldsymbol{X} \to \boldsymbol{Y}$ between ANR-systems, where the bonding maps of $\boldsymbol{Y}$ are fibrations.*

Proof of Theorem 6.6. Theorem 6.6 immediately follows from Lemma 6.9 and Theorem 6.1.

7. Pull-back axiom

Suppose that we are given a 2-sink $\boldsymbol{B} \xrightarrow{\boldsymbol{g}} \boldsymbol{Y} \xleftarrow{\boldsymbol{f}} \boldsymbol{X}$ in $\mathcal{ANR}$ with $\boldsymbol{f}$ being a strong pro-fibration. By Lemma 2.1, we have the following commutative diagram in $\mathcal{ANR}$:

(7.1)
$$\begin{tikzcd} \boldsymbol{B} \arrow[r, "\boldsymbol{g}"] \arrow[d, "\boldsymbol{k}"'] & \boldsymbol{Y} \arrow[d, "\boldsymbol{j}"] & \boldsymbol{X} \arrow[l, "\boldsymbol{f}"'] \arrow[d, "\boldsymbol{i}"] \\ \boldsymbol{B}' \arrow[r, "\boldsymbol{g}'"'] & \boldsymbol{Y}' & \boldsymbol{X}' \arrow[l, "\boldsymbol{f}'"] \end{tikzcd}$$

where $\boldsymbol{X}'$, $\boldsymbol{Y}'$, $\boldsymbol{B}'$ have the same index set, the coordinates and the bonding maps of $\boldsymbol{X}'$, $\boldsymbol{Y}'$, $\boldsymbol{B}'$ are those of $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{B}$, respectively, the morphisms $\boldsymbol{f}'$ and $\boldsymbol{g}'$ are represented by level maps, and the morphisms $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are isomorphisms. Let $\boldsymbol{X}' = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{Y}' = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$, and $\boldsymbol{B}' = (B_\lambda, s_{\lambda\lambda'}, \Lambda)$, and let $(f_\lambda) : \boldsymbol{X}' \to \boldsymbol{Y}'$ and $(g_\lambda) : \boldsymbol{B}' \to \boldsymbol{Y}'$ represent $\boldsymbol{f}'$ and $\boldsymbol{g}'$, respectively. For each $\lambda \in \Lambda$, let

$$E_\lambda = \{(x,b) \in X_\lambda \times B_\lambda : f_\lambda(x) = g_\lambda(b)\},$$

and for $\lambda \le \lambda'$, define a map $r_{\lambda\lambda'} : E_{\lambda'} \to E_\lambda$ by

$$r_{\lambda\lambda'}(x,b) = (p_{\lambda\lambda'}(x), s_{\lambda\lambda'}(b)) \text{ for } (x,b) \in E_{\lambda'}.$$

Then we have the system $\boldsymbol{E} = (E_\lambda, r_{\lambda\lambda'}, \Lambda)$ and the commutative diagram in inv- Top:

(7.2)
$$\begin{tikzcd} \boldsymbol{E} \arrow[r, "(k_\lambda)"] \arrow[d, "(h_\lambda)"'] & \boldsymbol{X}' \arrow[d, "(f_\lambda)"] \\ \boldsymbol{B}' \arrow[r, "(g_\lambda)"'] & \boldsymbol{Y}' \end{tikzcd}$$

Here, for each $\lambda \in \Lambda$, $k_\lambda : E_\lambda \to X_\lambda$ and $h_\lambda : E_\lambda \to B_\lambda$ are the restrictions of the projection maps from $X_\lambda \times B_\lambda$ onto X_λ and B_λ, respectively.

Lemma 7.1. *A level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ has the SHLP with respect to a space Z iff it has the following property:*

(SHLP)$_L$: *Each $\lambda \in \Lambda$ admits $\lambda' \ge \lambda$ such that whenever $h : Z \to X_{\lambda'}$ and $H : Z \times I \to Y_{\lambda'}$ are maps satisfying*

$$f_{\lambda'} h = H_0,$$

there is a map $\tilde{H} : Z \times I \to X_\lambda$ satisfying

$$\tilde{H}_0 = p_{\lambda\lambda'} h,$$

$$f_\lambda \tilde{H} = q_{\lambda\lambda'} H,$$

and the condition that

(7.3) *whenever H is constant on $z \times I$, $\tilde{H}$ is constant on $z \times I$.*

Proof. This is basically proven in [5, Lemma 4.3]. Note here that conditions (4.4) in (SHLP) and (7.3) in (SHLP)$_L$ are automatically induced from each other in the corresponding implications. □

λ' in (SHLP)$_L$ is called a *lifting index* for (f_λ).

Lemma 7.2. *$\boldsymbol{E}$ is an ANR object.*

Proof. Let $\lambda \in \Lambda$, and let $\lambda_1 \geq \lambda$ be a lifting index for (f_λ) (Lemma 7.1). Using the fact that $\boldsymbol{X}'$, $\boldsymbol{Y}'$, and $\boldsymbol{B}'$ are ANR objects, we take $\lambda_2 \geq \lambda_1$ such that $q_{\lambda_1\lambda_2} = \gamma'\gamma$ for some maps $\gamma : Y_{\lambda_2} \to R$ and $\gamma' : R \to Y_{\lambda_1}$ where R is an ANR, and in turn take $\lambda_3 \geq \lambda_2$ such that $p_{\lambda_2\lambda_3} = \alpha'\alpha$ and $s_{\lambda_2\lambda_3} = \beta'\beta$ for some maps $\alpha : X_{\lambda_3} \to P$, $\alpha' : P \to X_{\lambda_2}$, $\beta : B_{\lambda_3} \to Q$, and $\beta' : Q \to B_{\lambda_2}$ where P and Q are ANR's.

Let $\varphi : C \to E_{\lambda_3}$ be a map from a closed subset C of a metric space Z. We show that $r_{\lambda\lambda_3}\varphi$ extends over some open neighborhood of C. Since P and Q are ANR's, $\alpha k_{\lambda_3}\varphi$ and $\beta h_{\lambda_3}\varphi$ respectively extend to maps $\psi_1 : U' \to P$ and $\psi_2 : U' \to Q$ for some open neighborhood U' of C. Let $\mathcal{V}$ be an open covering of R such that

(7.4) any two $\mathcal{V}$-near maps f_1, f_2 into R are homotopic via a homotopy which is constant on $x \times I$ whenever $f_1(x) = f_2(x)$,

and take an open covering $\mathcal{V}'$ of R such that

$$\operatorname{st}\mathcal{V}' < \mathcal{V}. \tag{7.5}$$

Let

$$U = \operatorname{st}\left(C, (\psi_1^{-1}\alpha'^{-1}f_{\lambda_2}^{-1}\gamma^{-1}\mathcal{V}') \wedge (\psi_2^{-1}\beta'^{-1}g_{\lambda_2}^{-1}\gamma^{-1}\mathcal{V}')\right). \tag{7.6}$$

Then

$$f_{\lambda_1}p_{\lambda_1\lambda_2}\alpha'\psi_1|U \simeq g_{\lambda_1}s_{\lambda_1\lambda_2}\beta'\psi_2|U. \tag{7.7}$$

Indeed, (7.6) implies that each $x \in U$ admits $x' \in C$ and $V_1, V_2 \in \mathcal{V}'$ so that

$$\gamma f_{\lambda_2}\alpha'\psi_1(x), \gamma f_{\lambda_2}\alpha'\psi_1(x') \in V_1 \text{ and } \gamma g_{\lambda_2}\beta'\psi_2(x), \gamma g_{\lambda_2}\beta'\psi_2(x') \in V_2. \tag{7.8}$$

Then, tracing diagram (7.13) below, we have

$$\gamma f_{\lambda_2}\alpha'\psi_1(x') = \gamma q_{\lambda_2\lambda_3}f_{\lambda_3}k_{\lambda_3}\varphi(x') = \gamma q_{\lambda_2\lambda_3}g_{\lambda_3}h_{\lambda_3}\varphi(x') = \gamma g_{\lambda_2}\beta'\psi_2(x'). \tag{7.9}$$

By (7.8), (7.9), and (7.5),

$$(\gamma f_{\lambda_2}\alpha'\psi_1|U, \gamma g_{\lambda_2}\beta'\psi_2|U) < \mathcal{V}.$$

By (7.4), this implies $\gamma f_{\lambda_2}\alpha'\psi_1|U \simeq \gamma g_{\lambda_2}\beta'\psi_2|U$. Applying γ' to both sides, we have

$$q_{\lambda_1\lambda_2}f_{\lambda_2}\alpha'\psi_1|U \simeq q_{\lambda_1\lambda_2}g_{\lambda_2}\beta'\psi_2|U.$$

But, tracing diagram (7.13), we have

$$q_{\lambda_1\lambda_2}f_{\lambda_2}\alpha'\psi_1|U = f_{\lambda_1}p_{\lambda_1\lambda_2}\alpha'\psi_1|U,$$
$$q_{\lambda_1\lambda_2}g_{\lambda_2}\beta'\psi_2|U = g_{\lambda_1}s_{\lambda_1\lambda_2}\beta'\psi_2|U,$$

and hence we have (7.7). Since $\lambda_1 \geq \lambda$ is a lifting index for (f_λ), (7.7) implies the existence of a map $\xi : U \to X_\lambda$ such that

$$f_\lambda\xi = q_{\lambda\lambda_1}g_{\lambda_1}s_{\lambda_1\lambda_2}\beta'\psi_2|U, \tag{7.10}$$
$$p_{\lambda\lambda_2}\alpha'\psi_1|U \simeq \xi, \text{ and}$$

(7.11)
$$p_{\lambda\lambda_2}\alpha'\psi_1(x) = \xi(x) \text{ whenever } f_{\lambda_1}p_{\lambda_1\lambda_2}\alpha'\psi_1(x) = g_{\lambda_1}s_{\lambda_1\lambda_2}\beta'\psi_2(x) \text{ for } x \in U.$$

But

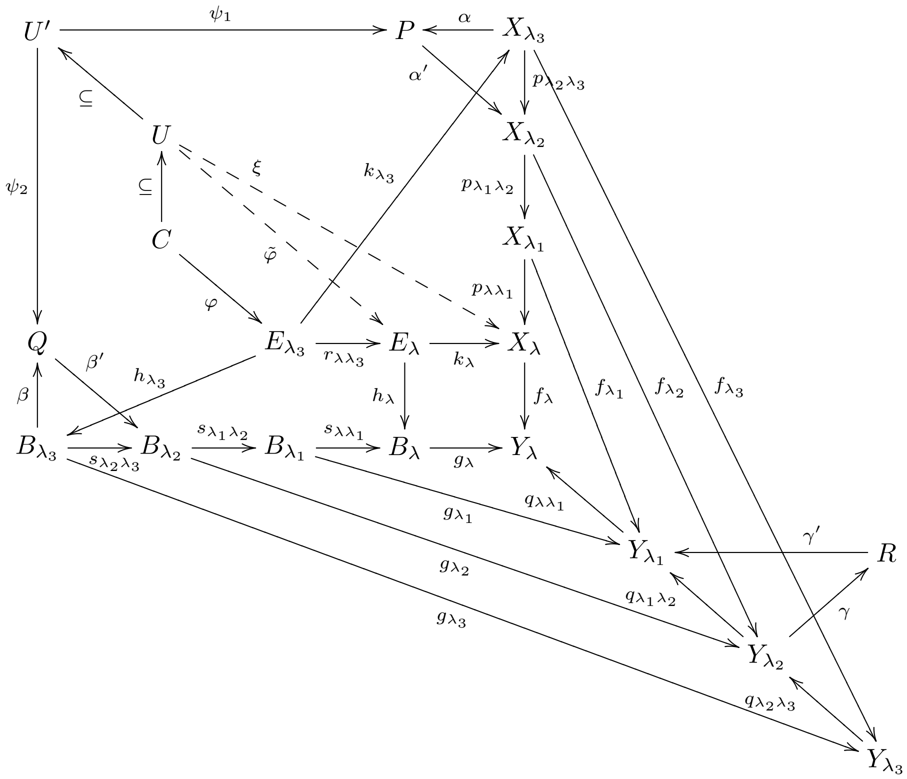
$$g_\lambda s_{\lambda\lambda_2}\beta'\psi_2|U = q_{\lambda\lambda_1}g_{\lambda_1}s_{\lambda_1\lambda_2}\beta'\psi_2|U. \tag{7.12}$$

By (7.10) and (7.12), ξ and $s_{\lambda\lambda_2}\beta'\psi_2|U$ define a map $\tilde{\varphi} : U \to E_\lambda$ such that

$$k_\lambda\tilde{\varphi} = \xi,$$
$$h_\lambda\tilde{\varphi} = s_{\lambda\lambda_2}\beta'\psi_2|U.$$

On the other hand, if $x \in C$, then $\xi(x) = p_{\lambda\lambda_2}\alpha'\psi_1(x)$ by (7.11). So, $\xi|C$ and $s_{\lambda\lambda_2}\beta'\psi_2|C$ determine the map $r_{\lambda\lambda_3}\varphi$. Thus $\tilde{\varphi}|C = r_{\lambda\lambda_3}\varphi$.

(7.13)



$\square$

Diagrams (7.1) and (7.2) induce the following commutative diagram in pro- Top:

(7.14)
$$\begin{tikzcd}
\boldsymbol{E} \arrow[r, "\boldsymbol{k}"] \arrow[d, "\boldsymbol{h}"'] & \boldsymbol{X} \arrow[d, "\boldsymbol{f}"] \\
\boldsymbol{B} \arrow[r, "\boldsymbol{g}"'] & \boldsymbol{Y}
\end{tikzcd}$$

Theorem 7.3. *Diagram (7.14) is a pull-back diagram in $\mathcal{ANR}$.*

Proof. Suppose that we have the following commutative diagram in $\mathcal{ANR}$:

(7.15)
$$\begin{tikzcd}
\boldsymbol{C} \arrow[dr, dashed, "\boldsymbol{\rho}"] \arrow[drr, "\boldsymbol{\varphi}"] \arrow[ddr, "\boldsymbol{\psi}"'] & & \\
& \boldsymbol{E} \arrow[r, "\boldsymbol{k}"'] \arrow[d, "\boldsymbol{h}"] & \boldsymbol{X} \arrow[d, "\boldsymbol{f}"] \\
& \boldsymbol{B} \arrow[r, "\boldsymbol{g}"'] & \boldsymbol{Y}
\end{tikzcd}$$

We must find a unique map $\boldsymbol{\rho}$ which makes this diagram commute. Without loss of generality, we can assume that $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$, and $\boldsymbol{E}$ have the same index set, say $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{B} = (B_\lambda, s_{\lambda\lambda'}, \Lambda)$, and $\boldsymbol{E} = (E_\lambda, r_{\lambda\lambda'}, \Lambda)$, and that $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{k}$ are represented by level maps (f_λ), (g_λ), (h_λ), and (k_λ), respectively. Let $\boldsymbol{C} = (C_\mu, t_{\mu\mu'}, M)$, and let $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ be represented by system maps $(\varphi, \varphi_\lambda)$ and (ψ, ψ_λ), respectively.

For each $\lambda \in \Lambda$, there is $\rho(\lambda) \geq \varphi(\lambda), \psi(\lambda)$ such that

$$f_\lambda \varphi_\lambda t_{\varphi(\lambda)\rho(\lambda)} = g_\lambda \psi_\lambda t_{\psi(\lambda)\rho(\lambda)}.$$

There is a function $\rho : \Lambda \to M$ and for each $\lambda \in \Lambda$ there is a unique map $\rho_\lambda : C_{\rho(\lambda)} \to E_\lambda$ which makes the following diagram commute:

$$\begin{tikzcd}
C_{\rho(\lambda)} \arrow[r, "t_{\rho(\lambda)\varphi(\lambda)}"] \arrow[d, "t_{\rho(\lambda)\psi(\lambda)}"'] \arrow[dr, dashed, "\rho_\lambda"] & C_{\varphi(\lambda)} \arrow[dr, "\varphi_\lambda"] & \\
C_{\psi(\lambda)} \arrow[dr, "\psi_\lambda"'] & E_\lambda \arrow[r, "k_\lambda"] \arrow[d, "h_\lambda"'] & X_\lambda \arrow[d, "f_\lambda"] \\
 & B_\lambda \arrow[r, "g_\lambda"'] & Y_\lambda
\end{tikzcd}$$

That $(\rho, \rho_\lambda) : \boldsymbol{C} \to \boldsymbol{E}$ is a system map follows from the fact that for $\lambda_1 \leq \lambda_2$ there is $\mu \geq \rho(\lambda_1), \rho(\lambda_2)$ so that

$$\varphi_{\lambda_1} t_{\varphi(\lambda_1)\mu} = p_{\lambda_1\lambda_2} \varphi_{\lambda_2} t_{\varphi(\lambda_2)\mu},$$

$$\psi_{\lambda_1} t_{\psi(\lambda_1)\mu} = s_{\lambda_1\lambda_2} \psi_{\lambda_2} t_{\psi(\lambda_2)\mu}.$$

This shows the existence of $\boldsymbol{\rho}$.

$$\begin{tikzcd}
C_\mu \arrow[rr, "t_{\rho(\lambda_2)\mu}"] \arrow[d, "t_{\rho(\lambda_1)\mu}"'] & & C_{\rho(\lambda_2)} \arrow[r, "t_{\varphi(\lambda_2)\rho(\lambda_2)}"] \arrow[d, "t_{\rho(\lambda_2)\psi(\lambda_2)}"'] \arrow[dr, "\rho_{\lambda_2}"] & C_{\varphi(\lambda_2)} \arrow[drr, "\varphi_{\lambda_2}"] & & \\
C_{\rho(\lambda_1)} \arrow[r, "t_{\varphi(\lambda_1)\rho(\lambda_1)}"] \arrow[dr, "t_{\psi(\lambda_1)\rho(\lambda_1)}"'] \arrow[drr, "\rho_{\lambda_1}"] & C_{\varphi(\lambda_1)} \arrow[drrr, "\varphi_{\lambda_1}"] & C_{\psi(\lambda_2)} \arrow[ddr, "\psi_{\lambda_2}"'] & E_{\lambda_2} \arrow[rr, "k_{\lambda_2}"] \arrow[dl, "r_{\lambda_1\lambda_2}"'] \arrow[dd, "h_{\lambda_2}"] & & X_{\lambda_2} \arrow[dl, "p_{\lambda_1\lambda_2}"'] \arrow[dd, "f_{\lambda_2}"] \\
 & C_{\psi(\lambda_1)} \arrow[ddr, "\psi_{\lambda_1}"'] & E_{\lambda_1} \arrow[rr, "k_{\lambda_1}"] \arrow[dd, "h_{\lambda_2}"'] & & X_{\lambda_1} \arrow[dd, "f_{\lambda_1}"] & \\
 & & & B_{\lambda_2} \arrow[rr, "g_{\lambda_2}"] \arrow[dl, "s_{\lambda_1\lambda_2}"'] & & Y_{\lambda_2} \arrow[dl, "q_{\lambda_1\lambda_2}"] \\
 & & B_{\lambda_1} \arrow[rr, "g_{\lambda_1}"'] & & Y_{\lambda_1} &
\end{tikzcd}$$

To show the uniqueness of $\boldsymbol{\rho}$, let $\boldsymbol{\rho}' : \boldsymbol{C} \to \boldsymbol{E}$ be another map that makes diagram (7.15) commute, where $\boldsymbol{\rho}$ is replaced by $\boldsymbol{\rho}'$. Let $\boldsymbol{\rho}'$ be represented by (ρ', ρ'_λ). Then for each $\lambda \in \Lambda$ there is $\mu \geq \varphi(\lambda), \psi(\lambda), \rho(\lambda), \rho'(\lambda)$ such that

$$k_\lambda \rho_\lambda t_{\rho(\lambda)\mu} = \varphi_\lambda t_{\varphi(\lambda)\mu} = k_\lambda \rho'_\lambda t_{\rho'(\lambda)\mu},$$

$$h_\lambda \rho_\lambda t_{\rho(\lambda)\mu} = \psi_\lambda t_{\psi(\lambda)\mu} = h_\lambda \rho'_\lambda t_{\rho'(\lambda)\mu}.$$

This means $\rho_\lambda t_{\rho(\lambda)\mu} = \rho'_\lambda t_{\rho'(\lambda)\mu}$, so $\boldsymbol{\rho} = \boldsymbol{\rho}'$ as required. □

A morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is said to have property (F) provided every system map $(f, f_\mu) : \boldsymbol{X} \to \boldsymbol{Y}$ representing $\boldsymbol{f}$ has the following property:

(F): for any admissible pairs $(\lambda, \mu) < (\lambda', \mu')$ and for any space Z, if $h : Z \to X_{\lambda'}$ and $H : Z \times I \to X_\lambda$ are maps such that

$$H_0 = p_{\lambda\lambda'} h,$$
$$f_\mu p_{f(\mu)\lambda} H_0 = f_\mu p_{f(\mu)\lambda} H_1,$$

then there is a map $\tilde{H} : Z \times I \to X_{\lambda'}$ such that

$$\tilde{H}_0 = h,$$
$$p_{\lambda\lambda'} \tilde{H} = H,$$
$$f_{\mu'} p_{f(\mu')\lambda'} \tilde{H}_0 = f_{\mu'} p_{f(\mu')\lambda'} \tilde{H}_1.$$

Note that if a morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is represented by system maps (f, f_μ) and (f', f'_μ), and if (f, f_μ) has property (F), so does (f', f'_μ). Moreover, if a level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ has property (F) iff it has the following property:

(F)$_L$: for $\lambda < \lambda'$ and for any space Z, if $h : Z \to X_{\lambda'}$ and $H : Z \times I \to X_\lambda$ are maps such that

$$H_0 = p_{\lambda\lambda'} h,$$
$$f_\lambda H_0 = f_\lambda H_1,$$

then there is a map $\tilde{H} : Z \times I \to X_{\lambda'}$ such that

$$\tilde{H}_0 = h,$$
$$p_{\lambda\lambda'} \tilde{H} = H,$$
$$f_{\lambda'} \tilde{H}_0 = f_{\lambda'} \tilde{H}_1.$$

If $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is a system with each bonding map $p_{\lambda\lambda'}$ being a Hurewicz fibration, and if $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ is a system map with each bonding map $q_{\lambda\lambda'}$ being injective, then every level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ has property (F). In particular, every level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ between systems with each boding map being both a Hurewicz fibration and a cofibration (e.g., a homeomorphism) has property (F).

Next we wish to prove

Theorem 7.4. *In the pull-back diagram (7.14), if $\boldsymbol{f}$ has property (F) with respect to $\boldsymbol{f}$ and if $\boldsymbol{f}$ induces an isomorphism in* pro-H(Top)*, then so does $\boldsymbol{h}$.*

We do not know at this moment if the theorem holds without the condition that $\boldsymbol{f}$ has property (F). Before proving the theorem, we obtain the following two lemmas.

Lemma 7.5. *Suppose that a level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ between systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ has the HLP with respect to all spaces. Let $\lambda_1 \geq \lambda$ be a lifting index for (f_λ). Then if $\lambda_2 \geq \lambda_1$ and $\alpha : X_{\lambda_2} \to X_{\lambda_1}$ is a map such that*

(7.16)
$$\begin{tikzcd} X_{\lambda_1} \arrow[d, "f_{\lambda_1}"'] & X_{\lambda_2} \arrow[l, "\alpha"'] \arrow[d, "f_{\lambda_2}"] \\ Y_{\lambda_1} & Y_{\lambda_2} \arrow[l, "q_{\lambda_1\lambda_2}"] \end{tikzcd}$$

commutes, and for any $\lambda_1' \geq \lambda_2$ *there exists a map* $\alpha' : X_{\lambda_2'} \to X_{\lambda_1'}$ *with* $\alpha' \simeq p_{\lambda_1'\lambda_2'}$ *and the following commutative diagrams:*

$$\begin{tikzcd} X_{\lambda_1'} \arrow[d, "f_{\lambda_1'}"'] & X_{\lambda_2'} \arrow[l, "\alpha'"'] \arrow[d, "f_{\lambda_2'}"] \\ Y_{\lambda_1'} & Y_{\lambda_2'} \arrow[l, "q_{\lambda_1'\lambda_2'}"] \end{tikzcd} \tag{7.17}$$

$$\begin{tikzcd} X_{\lambda_2} \arrow[d, "\alpha"'] & X_{\lambda_2'} \arrow[l, "p_{\lambda_2\lambda_2'}"'] \arrow[d, "\alpha'"] \\ X_{\lambda_1} & X_{\lambda_1'} \arrow[l, "p_{\lambda_1\lambda_1'}"] \end{tikzcd} \tag{7.18}$$

then there exist $\lambda_3 \geq \lambda_2$ *and a map* $\beta : X_{\lambda_3} \to X_{\lambda_2}$ *such that*

(1) *the diagram*

$$\begin{tikzcd} X_{\lambda_2} \arrow[d, "f_{\lambda_2}"'] & X_{\lambda_3} \arrow[l, "\beta"'] \arrow[d, "f_{\lambda_3}"] \\ Y_{\lambda_2} & Y_{\lambda_3} \arrow[l, "q_{\lambda_2\lambda_3}"] \end{tikzcd} \tag{7.19}$$

commutes, and

(2) $p_{\lambda\lambda_1}\alpha\beta \simeq p_{\lambda\lambda_3}$ *via a homotopy* $h : X_{\lambda_3} \times I \to X_\lambda$ *such that the diagram*

$$\begin{tikzcd} X_{\lambda} \arrow[d, "f_{\lambda}"'] & X_{\lambda_3} \arrow[l, "h_t"'] \arrow[d, "f_{\lambda_3}"] \\ Y_{\lambda} & Y_{\lambda_3} \arrow[l, "q_{\lambda\lambda_3}"] \end{tikzcd} \tag{7.20}$$

commutes for $t \in I$.

Proof. Fix $\lambda \in \Lambda$, and let $\lambda_1 \geq \lambda$ be a lifting index for λ. Let $\alpha : X_{\lambda_2} \to X_{\lambda_1}$ be a map as in the hypothesis. Let $\lambda_1' \geq \lambda_2$ be a lifting index for λ_2, and choose a map $\alpha' : X_{\lambda_2'} \to X_{\lambda_1'}$ with $\alpha' \simeq p_{\lambda_1'\lambda_2'}$ and the commutative diagrams (7.17) and (7.18). Let $L : X_{\lambda_2'} \times I \to X_{\lambda_1'}$ be a homotopy such that $L_0 = \alpha'$ and $L_1 = p_{\lambda_1'\lambda_2'}$. Since λ_1' is a lifting index for λ_2, there is a map $G : X_{\lambda_2'} \times I \to X_{\lambda_2}$ which makes the following diagram commute:

$$\begin{tikzcd} X_{\lambda_2} \arrow[d, "f_{\lambda_2}"'] & X_{\lambda_1'} \arrow[l, "p_{\lambda_2\lambda_1'}"'] \arrow[d, "f_{\lambda_1'}"'] & X_{\lambda_2'} \times 0 \arrow[l, "p_{\lambda_1'\lambda_2'}"'] \arrow[d, "\subseteq"] \\ Y_{\lambda_2} & Y_{\lambda_1'} \arrow[l, "q_{\lambda_2\lambda_1'}"] & X_{\lambda_2'} \times I \arrow[l, "f_{\lambda_1'}L"] \arrow[llu, dashed, "G"'] \end{tikzcd} \tag{7.21}$$

Define a map $F : X_{\lambda'_2} \times I \to X_{\lambda_1}$ by

$$F(x,t) = \begin{cases} \alpha G(x, 1-2t) & 0 \le t \le \dfrac{1}{2}, \\ p_{\lambda_1 \lambda'_1} L(x, 2t-1) & \dfrac{1}{2} \le t \le 1, \end{cases}$$

and a map $k : X_{\lambda'_2} \times I \times I \to Y_{\lambda_1}$ by

$$k(x,t,T) = \begin{cases} q_{\lambda_1 \lambda_2} f_{\lambda_2} G(x, 1-2t(1-T)) & 0 \le t \le \dfrac{1}{2}, \\ q_{\lambda_1 \lambda'_1} f_{\lambda'_1} L(x, 1-2(1-t)(1-T)) & \dfrac{1}{2} \le t \le 1. \end{cases}$$

Note here that the maps F and k are well-defined by the commutativities of diagrams (7.18) and (7.21), respectively. Then, since λ_1 is a lifting index for λ, there is a map $K : X_{\lambda'_2} \times I \times I \to X_\lambda$ which makes the following diagram commute:

$$\begin{tikzcd} X_\lambda \arrow[d, "f_\lambda"'] & X_{\lambda_1} \arrow[l, "p_{\lambda\lambda_1}"'] \arrow[d, "f_{\lambda_1}"'] & X_{\lambda'_2} \times I \times 0 \arrow[l, "F"'] \arrow[d, "\subseteq"] \\ Y_\lambda & Y_{\lambda_1} \arrow[l, "q_{\lambda\lambda_1}"] & X_{\lambda'_2} \times I \times I \arrow[l, "k"] \arrow[ull, dashed, "K"'] \end{tikzcd}$$

Note here that the commutativity of the right square follows from that of (7.16).

Now let $\lambda_3 = \lambda'_2$, and let $\beta = G_1 : X_{\lambda_3} \to X_{\lambda_2}$. Then $p_{\lambda\lambda_1}\alpha\beta = K_{0,0}$, and $p_{\lambda\lambda_3} = K_{1,0}$. Moreover, for $0 \le t \le 1$ and $0 \le T \le 1$,

$$f_\lambda K_{0,T} = q_{\lambda\lambda_3} f_{\lambda_3}, \; f_\lambda K_{t,1} = q_{\lambda\lambda_3} f_{\lambda_3}, \; f_\lambda K_{1,T} = f_\lambda p_{\lambda\lambda_3}.$$

So, the homotopy $h : X_{\lambda_3} \times I \to X_\lambda$ given by

$$h(x,t) = \begin{cases} K(x,0,3t) & 0 \le t \le \dfrac{1}{3}, \\ K(x, 3t-1, 1) & \dfrac{1}{3} \le t \le \dfrac{2}{3}, \\ K(x, 1, 3-3t) & \dfrac{2}{3} \le t \le 1, \end{cases}$$

gives the required homotopy from $p_{\lambda\lambda_1}\alpha\beta$ to $p_{\lambda\lambda_3}$. □

Lemma 7.6. *If a level map $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ between systems $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ has the HLP with respect to all spaces and if it induces an isomorphism in* pro-$\mathrm{H}(\mathsf{Top})$*, then each $\lambda \in \Lambda$ admits $\lambda' \ge \lambda$ and a map $\varphi_{\lambda\lambda'} : Y_{\lambda'} \to X_\lambda$ such that*

$$f_\lambda \varphi_{\lambda\lambda'} = q_{\lambda\lambda'}, \tag{7.22}$$

$$p_{\lambda\lambda'} \simeq \varphi_{\lambda\lambda'} f_{\lambda'}. \tag{7.23}$$

$$\begin{tikzcd} X_\lambda \arrow[d, "f_\lambda"'] & X_{\lambda'} \arrow[l, "p_{\lambda\lambda'}"'] \arrow[d, "f_{\lambda'}"] \\ Y_\lambda & Y_{\lambda'} \arrow[l, "q_{\lambda\lambda'}"] \arrow[ul, dashed, "\varphi_{\lambda\lambda'}"'] \end{tikzcd}$$

Proof. Fix $\lambda \in \Lambda$, and let $\lambda'' \geq \lambda$ be a lifting index for λ. Then by Morita's lemma [10, Theorem 5, Ch. II, §. 2], there exist $\lambda' \geq \lambda''$ and a map $\varphi'_{\lambda''\lambda'} : Y_{\lambda'} \to X_{\lambda''}$ such that

$$f_{\lambda''}\varphi'_{\lambda''\lambda'} \simeq q_{\lambda''\lambda'}, \tag{7.24}$$

$$p_{\lambda''\lambda'} \simeq \varphi'_{\lambda''\lambda'} f_{\lambda'}. \tag{7.25}$$

By (7.24) and the choice of λ'', there is a map $\varphi_{\lambda\lambda'} : Y_{\lambda'} \to X_\lambda$ such that equality (7.22) and the relation

$$\varphi_{\lambda\lambda'} \simeq p_{\lambda\lambda''}\varphi'_{\lambda''\lambda'}. \tag{7.26}$$

hold. (7.26) and (7.25) imply (7.23).

$$\begin{tikzcd}
X_\lambda \arrow[d, "f_\lambda"'] & X_{\lambda''} \arrow[l, "p_{\lambda\lambda''}"'] \arrow[d, "f_{\lambda''}"'] & X_{\lambda'} \arrow[l, "p_{\lambda''\lambda'}"'] \arrow[d, "f_{\lambda'}"] \\
Y_\lambda & Y_{\lambda''} \arrow[l, "q_{\lambda\lambda''}"] & Y_{\lambda'} \arrow[l, "q_{\lambda''\lambda'}"] \arrow[ul, "\varphi'_{\lambda''\lambda'}"'] \arrow[ull, dashed, "\varphi_{\lambda\lambda'}"]
\end{tikzcd}$$

□

Proof of Theorem 7.4. Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$, $\boldsymbol{B} = (B_\lambda, s_{\lambda\lambda'}, \Lambda)$, and $\boldsymbol{E} = (E_\lambda, r_{\lambda\lambda'}, \Lambda)$, and let $\boldsymbol{f}$, $\boldsymbol{g}$, $\boldsymbol{h}$, and $\boldsymbol{k}$ be represented by level maps (f_λ), (g_λ), (h_λ), and (k_λ), respectively. Let $\lambda \in \Lambda$, and let $\lambda_1 \geq \lambda$ be a lifting index for (f_λ). Since $\boldsymbol{f}$ induces an isomorphism in pro-$\mathrm{H}(\mathsf{Top})$, by Lemma 7.6, there is $\lambda_2 \geq \lambda_1$ and a map $\varphi_{\lambda_1\lambda_2} : Y_{\lambda_2} \to X_{\lambda_1}$ such that

$$\varphi_{\lambda_1\lambda_2} f_{\lambda_2} \simeq p_{\lambda_1\lambda_2},$$

$$f_{\lambda_1}\varphi_{\lambda_1\lambda_2} = q_{\lambda_1\lambda_2}. \tag{7.27}$$

Again by Lemma 7.6, for any $\lambda'_1 \geq \lambda_2$ there exist $\lambda'_2 \geq \lambda'_1$ and a map $\varphi'_{\lambda'_1\lambda'_2} : Y_{\lambda'_2} \to X_{\lambda'_1}$ such that

$$\varphi'_{\lambda'_1\lambda'_2} f_{\lambda'_2} \simeq p_{\lambda'_1\lambda'_2},$$

$$f_{\lambda'_1}\varphi'_{\lambda'_1\lambda'_2} = q_{\lambda'_1\lambda'_2}.$$

By $(\mathrm{F})_L$ there is a map $\varphi_{\lambda'_1\lambda'_2} : X_{\lambda'_2} \to X_{\lambda'_1}$ such that

$$\varphi_{\lambda'_1\lambda'_2} \simeq \varphi'_{\lambda'_1\lambda'_2} f_{\lambda'_2},$$

$$f_{\lambda'_1}\varphi_{\lambda'_1\lambda'_2} = q_{\lambda'_1\lambda'_2} f_{\lambda'_2},$$

$$p_{\lambda_1\lambda'_1}\varphi_{\lambda'_1\lambda'_2} = \varphi_{\lambda_1\lambda_2} q_{\lambda_2\lambda'_2} f_{\lambda'_2}.$$

Then for the map $\alpha = \varphi_{\lambda_1\lambda_2} f_{\lambda_2}$ the hypothesis of Lemma 7.5 is fulfilled with $\alpha' = \varphi'_{\lambda'_1\lambda'_2} f_{\lambda'_2}$.

$$\begin{tikzcd}
X_{\lambda_1} \arrow[d, "f_{\lambda_1}"'] & X_{\lambda_2} \arrow[l, "p_{\lambda_1\lambda_2}"'] \arrow[d, "f_{\lambda_2}"] & X_{\lambda'_1} \arrow[l, "p_{\lambda_2\lambda'_1}"'] \arrow[d, "f_{\lambda'_1}"'] & X_{\lambda'_2} \arrow[l, "p_{\lambda'_1\lambda'_2}"'] \arrow[d, "f_{\lambda'_2}"] \\
Y_{\lambda_1} & Y_{\lambda_2} \arrow[l, "q_{\lambda_1\lambda_2}"] \arrow[ul, "\varphi_{\lambda_1\lambda_2}"'] & Y_{\lambda'_1} \arrow[l, "q_{\lambda_2\lambda'_1}"] & Y_{\lambda'_2} \arrow[l, "q_{\lambda'_1\lambda'_2}"] \arrow[ul, "\varphi'_{\lambda'_1\lambda'_2}"']
\end{tikzcd}$$

So, by Lemma 7.5, there exist $\lambda_3 \geq \lambda_2$ and a map $t_{\lambda_2\lambda_3} : X_{\lambda_3} \to X_{\lambda_2}$ with the following two properties:

(1) the following diagram commutes:

(7.28)
$$\begin{tikzcd}
X_{\lambda_2} \arrow[d, "f_{\lambda_2}"'] & X_{\lambda_3} \arrow[l, "t_{\lambda_2\lambda_3}"'] \arrow[d, "f_{\lambda_3}"] \\
Y_{\lambda_2} & Y_{\lambda_3} \arrow[l, "q_{\lambda_2\lambda_3}"]
\end{tikzcd}$$

(2) there is a homotopy $H : X_{\lambda_3} \times I \to X_\lambda$ such that $H_0 = p_{\lambda\lambda_1}\varphi_{\lambda_1\lambda_2} f_{\lambda_2} t_{\lambda_2\lambda_3}$, $H_1 = p_{\lambda\lambda_3}$, and the following diagram commutes for $t \in I$:

(7.29)
$$\begin{tikzcd}
X_{\lambda} \arrow[d, "f_{\lambda}"'] & X_{\lambda_3} \arrow[l, "H_t"'] \arrow[d, "f_{\lambda_3}"] \\
Y_{\lambda} & Y_{\lambda_3} \arrow[l, "q_{\lambda\lambda_3}"]
\end{tikzcd}$$

Consider the following diagram:
(7.30)
$$\begin{tikzcd}
E_{\lambda_3} \arrow[rrrrrrr, "k_{\lambda_3}"] \arrow[dr, "r_{\lambda_2\lambda_3}"] \arrow[ddddddd, "h_{\lambda_3}"'] & & & & & & & X_{\lambda_3} \arrow[dl, "t_{\lambda_2\lambda_3}"'] \arrow[ddddddd, "f_{\lambda_3}"] \\
& E_{\lambda_2} \arrow[rrrrr, "k_{\lambda_2}"] \arrow[dr, "r_{\lambda_1\lambda_2}"] \arrow[ddddd, "h_{\lambda_2}"'] & & & & & X_{\lambda_2} \arrow[dl, "p_{\lambda_1\lambda_2}"'] \arrow[ddddd, "f_{\lambda_2}"] & \\
& & E_{\lambda_1} \arrow[rrr, "k_{\lambda_1}"] \arrow[dr, "r_{\lambda\lambda_1}"] \arrow[ddd, "h_{\lambda_1}"'] & & & X_{\lambda_1} \arrow[dl, "p_{\lambda\lambda_1}"'] \arrow[ddd, "f_{\lambda_1}"'] & & \\
& & & E_\lambda \arrow[r, "k_\lambda"] \arrow[d, "h_\lambda"] & X_\lambda \arrow[d, "f_\lambda"] & & & \\
& & & B_\lambda \arrow[r, "g_\lambda"] & Y_\lambda & & & \\
& & B_{\lambda_1} \arrow[ur, "s_{\lambda\lambda_1}"'] \arrow[rrr, "g_{\lambda_1}"] & & & Y_{\lambda_1} \arrow[ul, "q_{\lambda\lambda_1}"] & & \\
& B_{\lambda_2} \arrow[ur, "s_{\lambda_1\lambda_2}"'] \arrow[rrrrr, "g_{\lambda_2}"] & & & & & Y_{\lambda_2} \arrow[ul, "q_{\lambda_1\lambda_2}"] \arrow[uuuul, "\varphi_{\lambda_1\lambda_2}"'] & \\
B_{\lambda_3} \arrow[ur, "s_{\lambda_3\lambda_2}"'] \arrow[rrrrrrr, "g_{\lambda_3}"] \arrow[uuuurrr, dashed, "\psi_{\lambda\lambda_3}"] & & & & & & & Y_{\lambda_3} \arrow[ul, "q_{\lambda_2\lambda_3}"]
\end{tikzcd}$$

It suffices to find a map $\psi_{\lambda\lambda_3} : B_{\lambda_3} \to E_\lambda$ such that

(7.31)
$$h_\lambda \psi_{\lambda\lambda_3} \simeq s_{\lambda\lambda_3},$$

(7.32)
$$r_{\lambda\lambda_3} \simeq \psi_{\lambda\lambda_3} h_{\lambda_3}.$$

By (7.27),

$$g_\lambda s_{\lambda\lambda_3} = f_\lambda p_{\lambda\lambda_1} \varphi_{\lambda_1\lambda_2} q_{\lambda_2\lambda_3} g_{\lambda_3}.$$

This determines a unique map $\psi_{\lambda\lambda_3} : B_{\lambda_3} \to E_\lambda$ such that

$$h_\lambda \psi_{\lambda\lambda_3} = s_{\lambda\lambda_3}, \tag{7.33}$$

$$k_\lambda \psi_{\lambda\lambda_3} = p_{\lambda\lambda_1}\varphi_{\lambda_1\lambda_2} q_{\lambda_2\lambda_3} g_{\lambda_3}. \tag{7.34}$$

(7.33) immediately implies (7.31), so it remains to show (7.32) holds. Indeed,

$$h_\lambda \psi_{\lambda\lambda_3} h_{\lambda_3} = h_\lambda r_{\lambda\lambda_3}. \tag{7.35}$$

Moreover,

$$k_\lambda r_{\lambda\lambda_3} = p_{\lambda\lambda_3} k_{\lambda_3}, \tag{7.36}$$

and by (7.34) and the commutativity of (7.28),

$$k_\lambda \psi_{\lambda\lambda_3} h_{\lambda_3} = p_{\lambda\lambda_1}\varphi_{\lambda_1\lambda_2} q_{\lambda_2\lambda_3} g_{\lambda_3} h_{\lambda_3} = p_{\lambda\lambda_1}\varphi_{\lambda_1\lambda_2} q_{\lambda_2\lambda_3} f_{\lambda_3} k_{\lambda_3} = p_{\lambda\lambda_1}\varphi_{\lambda_1\lambda_2} f_{\lambda_2} t_{\lambda_2\lambda_3} k_{\lambda_3}. \tag{7.37}$$

Then the map $\Psi : E_{\lambda_3} \times I \to E_\lambda$ defined by

$$\Psi(x, b, t) = (H(k_{\lambda_3}(x, b), t), h_\lambda r_{\lambda\lambda_3}(x, b))$$

gives the homotopy

$$p_{\lambda\lambda_1}\varphi_{\lambda_1\lambda_2} f_{\lambda_2} t_{\lambda_2\lambda_3} \simeq p_{\lambda\lambda_3}.$$

The map Ψ is well-defined by the commutativity of (7.29) . Thus (7.35), (7.36), (7.37), and the homotopy Ψ give (7.32). This completes the proof of the assertion. □

Next, we show

Theorem 7.7. *In the pull-back diagram (7.14), if* $\boldsymbol{g}$ *induces an isomorphism in* pro-$\mathrm{H}(\mathsf{Top})$*, then so does* $\boldsymbol{k}$*.*

Before proving the theorem, we introduce some notations and obtain two lemmas. For any 2-sink $B \xrightarrow{g} Y \xleftarrow{f} X$, let

$$E_{f,g} = \{(x, b, \varphi) \in X \times B \times Y^I : \ \varphi(0) = f(x), \ \varphi(1) = g(b)\},$$

and for any map $f : X \to Y$, let

$$E_f = \{(x, \varphi) \in X \times Y^I : \ \varphi(0) = f(x)\}.$$

For $\lambda \le \lambda'$, there exist well-defined maps

$$t_{\lambda\lambda'} : E_{f_{\lambda'}, g_{\lambda'}} \to E_{f_\lambda, g_\lambda} : t_{\lambda\lambda'}(x, b, \varphi) = (p_{\lambda\lambda'}(x), s_{\lambda\lambda'}(b), q_{\lambda\lambda'}\varphi),$$

and

$$u_{\lambda\lambda'} : E_{f_{\lambda'}} \to E_{f_\lambda} : u_{\lambda\lambda'}(x, \varphi) = (p_{\lambda\lambda'}(x), q_{\lambda\lambda'}\varphi).$$

Thus we have systems $\boldsymbol{E}_{(f_\lambda),(g_\lambda)} = (E_{f_\lambda, g_\lambda}, t_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{E}_{(f_\lambda)} = (E_{f_\lambda}, u_{\lambda\lambda'}, \Lambda)$. For each $\lambda \in \Lambda$, there is a map $\rho_\lambda : E_\lambda \to E_{f_\lambda, g_\lambda}$ which is defined by $\rho_\lambda(x, b) = (x, b, e_{f(x)})$, and (ρ_λ) forms a level map $(\rho_\lambda) : \boldsymbol{E} \to \boldsymbol{E}_{(f_\lambda),(g_\lambda)}$. Here, for each $y \in Y$, let $e_y \in Y^I$ denote the constant path at y.

Lemma 7.8. $(\rho_\lambda) : \boldsymbol{E} \to \boldsymbol{E}_{(f_\lambda),(g_\lambda)}$ *induces an isomorphism in* pro-$\mathrm{H}(\mathsf{Top})$.

Proof. This follows from the fact that each $\rho_\lambda : E_\lambda \to E_{f_\lambda, g_\lambda}$ is a homotopy equivalence. □

Suppose that we have a commutative diagram

$$\begin{tikzcd}
\boldsymbol{B} \arrow[r, "(g_\lambda)"] \arrow[d, "(\beta_\lambda)"'] & \boldsymbol{Y} \arrow[d, "(1_{Y_\lambda})"'] & \boldsymbol{X} \arrow[l, "(f_\lambda)"'] \arrow[d, "(\alpha_\lambda)"] \\
\boldsymbol{B}' \arrow[r, "(g'_\lambda)"'] & \boldsymbol{Y} & \boldsymbol{X}' \arrow[l, "(f'_\lambda)"]
\end{tikzcd}$$

where $\boldsymbol{B}' = (B'_\lambda, s'_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{X}' = (X'_\lambda, p'_{\lambda\lambda'}, \Lambda)$. Then there is a well-defined level map $(\Phi_\lambda) : \boldsymbol{E}_{(f_\lambda),(g_\lambda)} \to \boldsymbol{E}_{(f'_\lambda),(g'_\lambda)}$, where the map $\Phi_\lambda : E_{(f_\lambda),(g_\lambda)} \to E_{(f'_\lambda),(g'_\lambda)}$ is defined by $\Phi_\lambda(x, b, \varphi) = (\alpha_\lambda(x), \beta_\lambda(b), \varphi)$ for $(x, b, \varphi) \in E_{(f_\lambda),(g_\lambda)}$. We write $\boldsymbol{E}_{(f'_\lambda),(g'_\lambda)} = (E_{(f'_\lambda),(g'_\lambda)}, t'_{\lambda\lambda'}, \Lambda)$.

Lemma 7.9. *If* (α_λ) *and* (β_λ) *induce isomorphisms in* pro-$\mathrm{H}(\mathsf{Top})$*, so does* (Φ_λ).

Proof. Fix $\lambda \in \Lambda$. Then by Morita's lemma, there exist $\lambda' \geq \lambda$ and maps $\alpha'_{\lambda\lambda'} : X'_{\lambda'} \to X_\lambda$ and $\beta'_{\lambda\lambda'} : B'_{\lambda'} \to B_\lambda$ such that the two triangles in the following diagrams commute up to homotopy:

$$\begin{tikzcd}
X_\lambda \arrow[d, "\alpha_\lambda"'] & X_{\lambda'} \arrow[l, "p_{\lambda\lambda'}"'] \arrow[d, "\alpha_{\lambda'}"] \\
X'_\lambda & X'_{\lambda'} \arrow[l, "p'_{\lambda\lambda'}"] \arrow[ul, dashed, "\alpha'_{\lambda\lambda'}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B_\lambda \arrow[d, "\beta_\lambda"'] & B_{\lambda'} \arrow[l, "s_{\lambda\lambda'}"'] \arrow[d, "\beta_{\lambda'}"] \\
B'_\lambda & B'_{\lambda'} \arrow[l, "s'_{\lambda\lambda'}"] \arrow[ul, dashed, "\beta'_{\lambda\lambda'}"']
\end{tikzcd}$$

Let $\sigma : X'_{\lambda'} \times I \to X'_\lambda$ be a homotopy with $\sigma_0 = \alpha_\lambda \alpha'_{\lambda\lambda'}$ and $\sigma_1 = p'_{\lambda\lambda'}$, and let $\tau : B'_{\lambda'} \times I \to B'_\lambda$ be a homotopy with $\tau_0 = \beta_\lambda \beta'_{\lambda\lambda'}$ and $\tau_1 = s'_{\lambda\lambda'}$. Also, let $\theta : X_{\lambda'} \times I \to X_\lambda$ be a homotopy with $\theta_0 = \alpha'_{\lambda\lambda'} \alpha_{\lambda'}$ and $\theta_1 = p_{\lambda\lambda'}$, and let $\omega : B_{\lambda'} \times I \to B_\lambda$ be a homotopy with $\omega_0 = \beta'_{\lambda\lambda'} \beta_{\lambda'}$ and $\omega_1 = s_{\lambda\lambda'}$. We define a map $\Phi'_{\lambda\lambda'} : E_{f'_{\lambda'}, g'_{\lambda'}} \to E_{f_\lambda, g_\lambda}$ by

$$\Phi'_{\lambda\lambda'}(x, b, \varphi) = (\alpha'_{\lambda\lambda'}(x), \beta'_{\lambda\lambda'}(b), \overline{\varphi}) \text{ for } (x, b, \varphi) \in E_{f'_{\lambda'}, g'_{\lambda'}}$$

where $\overline{\varphi} \in Y_\lambda^I$ is defined by

$$\overline{\varphi}(t) = \begin{cases} f'_\lambda \sigma(x, 3t) & 0 \leq t \leq \dfrac{1}{3}, \\ q_{\lambda\lambda'} \varphi(3t - 1) & \dfrac{1}{3} \leq t \leq \dfrac{2}{3}, \\ g'_\lambda \tau(b, 3 - 3t) & \dfrac{2}{3} \leq t \leq 1. \end{cases}$$

Then the two triangles in the following commutative diagram commute up to homotopy:

$$\begin{tikzcd}
E_{f_\lambda, g_\lambda} \arrow[d, "\Phi_\lambda"'] & E_{f_{\lambda'}, g_{\lambda'}} \arrow[l, "t_{\lambda\lambda'}"'] \arrow[d, "\Phi_{\lambda'}"] \\
E_{f'_\lambda, g'_\lambda} & E_{f'_{\lambda'}, g'_{\lambda'}} \arrow[l, "t'_{\lambda\lambda'}"] \arrow[ul, dashed, "\Phi'_{\lambda\lambda'}"']
\end{tikzcd}$$

Indeed, $\Phi_\lambda \Phi'_{\lambda\lambda'}(x, b, \varphi) = (\alpha_\lambda \alpha'_{\lambda\lambda'}(x), \beta_\lambda \beta'_{\lambda\lambda'}(b), \overline{\varphi})$. So the homotopy $\Phi_\lambda \Phi'_{\lambda\lambda'} \simeq t'_{\lambda\lambda'}$ is given by the map $H : E_{f'_{\lambda'}, g'_{\lambda'}} \times I \to E_{f'_\lambda, g'_\lambda}$ where:

$$H(x, b, \varphi) = (\sigma(x, b, \varphi), \tau(b, t), \Gamma(\varphi, t)),$$

and the map $\Gamma : Y^I_{\lambda'} \times I \to Y^I_\lambda$ is defined as follows:

$$\Gamma(\varphi, T)(t) = \begin{cases} f'_\lambda \sigma(x, 3t+T) & 0 \le t < \frac{1}{3}(1-T), \\ q_{\lambda\lambda'}\varphi\left(\frac{3t+T-1}{2T+1}\right) & \frac{1}{3}(1-T) \le t \le \frac{1}{3}(T+2), \\ g'_\lambda \tau(b, 3-3t+T) & \frac{1}{3}(T+2) < t \le 1. \end{cases}$$

Moreover, $\Phi'_{\lambda\lambda'}\Phi_{\lambda'}(x, b, \varphi) = (\alpha'_{\lambda\lambda'}\alpha_{\lambda'}(x), \beta'_{\lambda\lambda'}\beta_{\lambda'}(b), \overline{\varphi})$. The homotopy $\Phi'_{\lambda\lambda'}\Phi_{\lambda'} \simeq t_{\lambda\lambda'}$ is given by the map $L : E_{f_{\lambda'}, g_{\lambda'}} \times I \to E_{f_\lambda, g_\lambda}$ where:

$$L(x, b, \varphi) = (\theta(x, t), \omega(b, t), \Theta(\varphi, t)),$$

and the map $\Theta : Y^I_{\lambda'} \times I \to Y^I_\lambda$ is defined as follows:

$$\Theta(\varphi, T)(t) = \begin{cases} f'_\lambda \sigma(x, 3t+T) & 0 \le t < \frac{1}{3}(1-T), \\ q_{\lambda\lambda'}\varphi\left(\frac{3t+T-1}{2T+1}\right) & \frac{1}{3}(1-T) \le t \le \frac{1}{3}(T+2), \\ g'_\lambda \tau(b, 3-3t+T) & \frac{1}{3}(T+2) < t \le 1. \end{cases}$$

□

Proof of Theorem 7.7. Consider the commutative diagram:

$$\begin{tikzcd} \boldsymbol{B} \arrow[r, "(g_\lambda)"] \arrow[d, "(g_\lambda)"'] & \boldsymbol{Y} \arrow[d, "(1_{Y_\lambda})"'] & \boldsymbol{X} \arrow[l, "(f_\lambda)"'] \arrow[d, "(1_{X_\lambda})"] \\ \boldsymbol{Y} \arrow[r, "(1_{Y_\lambda})"'] & \boldsymbol{Y} & \boldsymbol{X} \arrow[l, "(f_\lambda)"] \end{tikzcd}$$

Note that the following diagrams induce pull-back diagrams in $\mathcal{ANR}$:

$$\begin{tikzcd} \boldsymbol{E} \arrow[r, "(\overline{g}_\lambda)"] \arrow[d, "(\overline{f}_\lambda)"'] & \boldsymbol{X} \arrow[d, "(f_\lambda)"] \\ \boldsymbol{B} \arrow[r, "(g_\lambda)"'] & \boldsymbol{Y} \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{X} \arrow[r, "(1_{X_\lambda})"] \arrow[d, "(f_\lambda)"'] & \boldsymbol{X} \arrow[d, "(f_\lambda)"] \\ \boldsymbol{Y} \arrow[r, "(1_{Y_\lambda})"'] & \boldsymbol{Y} \end{tikzcd}$$

Then there exist level maps $(\rho_\lambda) : \boldsymbol{E} \to \boldsymbol{E}_{(f_\lambda),(g_\lambda)}$, $(\rho'_\lambda) : \boldsymbol{X} \to \boldsymbol{E}_{(f_\lambda),(1_{Y_\lambda})}$, and $(\Phi_\lambda) : \boldsymbol{E}_{(f_\lambda),(g_\lambda)} \to \boldsymbol{E}_{(f_\lambda),(1_{Y_\lambda})}$, and the following diagram commutes:

$$\begin{tikzcd} \boldsymbol{E} \arrow[r, "(\overline{g}_\lambda)"] \arrow[d, "(\rho_\lambda)"'] & \boldsymbol{X} \arrow[d, "(\rho'_\lambda)"] \\ \boldsymbol{E}_{(f_\lambda),(g_\lambda)} \arrow[r, "(\Phi_\lambda)"'] & \boldsymbol{E}_{(f_\lambda),(1_{Y_\lambda})} \end{tikzcd}$$

By Lemmas 7.8 and 7.9, the level maps (ρ_λ), (ρ'_λ), and (Φ_λ) induce isomorphisms in pro-$\mathrm{H}(\mathsf{Top})$, and hence $(\overline{g}_\lambda)$ induces an isomorphism in pro-$\mathrm{H}(\mathsf{Top})$.

□

Finally in this section, we prove

Theorem 7.10. *In the pull-back diagram (7.14), the morphism $\boldsymbol{h}$ is a strong pro-fibration.*

Proof. The assertion immediately follows from the fact that the pro-fibration $\boldsymbol{f}$ is represented by the level map (f_λ) with the $(\mathrm{SHLP})_L$ with respect to all spaces. $\square$

8. Homotopy groups of the base and total systems of strong pro-fibration

Suppose that $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\boldsymbol{Y} = (Y_\lambda, q_{\lambda\lambda'}, \Lambda)$ are ANR objects, and let $\boldsymbol{f} : (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ be a morphism represented by a level map $(f_\lambda) : (\boldsymbol{X}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$. Let $\boldsymbol{B} = (B_\lambda, q_{\lambda\lambda'}|B_{\lambda'}, \Lambda)$ be a system in $\mathcal{ANR}$ consisting of subsets B_λ of Y_λ containing the base points $y_{0\lambda}$ of Y_λ. Then we have the morphism $\boldsymbol{g} : (\boldsymbol{B}, \boldsymbol{y}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ represented by the level map $(g_\lambda) : (\boldsymbol{B}, \boldsymbol{y}_0) \to (\boldsymbol{Y}, \boldsymbol{y}_0)$ consisting of the inclusion maps $g_\lambda : B_\lambda \hookrightarrow Y_\lambda$. Suppose that $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a strong pro-fibration in $\mathcal{ANR}$, and let $\boldsymbol{A} = (f_\lambda^{-1}(B_\lambda), p_{\lambda\lambda'}|f_{\lambda'}^{-1}(B_{\lambda'}), \Lambda)$. Then we have the pull-back diagram in $\mathcal{ANR}$:

(8.1)
$$\begin{tikzcd}
\boldsymbol{A} \arrow[r, "\boldsymbol{g}'"] \arrow[d, "\boldsymbol{f}'"'] & \boldsymbol{X} \arrow[d, "\boldsymbol{f}"] \\
\boldsymbol{B} \arrow[r, "\boldsymbol{g}"'] & \boldsymbol{Y}
\end{tikzcd}$$

where $\boldsymbol{f}'$ and $\boldsymbol{g}'$ are the maps represented by the level map $(f_\lambda|A_\lambda)$ and the level map (g'_λ) consisting of the inclusion maps $g'_\lambda : A_\lambda \hookrightarrow X_\lambda$, respectively.

The main result of this section is Theorem 8.1. This is a generalization of the results by S. Mardešić and R. T. Rushing [9, Theorem 2] and Q. Haxhibeqiri [6, Theorem 5.7] for ANR-systems. The fibre of a strong pro-fibration is well-defined in ANR, and as a consequence of Theorem 8.1, we have the sequence of a strong pro-fibration.

Theorem 8.1. *The morphism* $\boldsymbol{f} : (\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0) \to (\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{y}_0)$ *induces isomorphisms* $\boldsymbol{f}_\sharp : \pi_q(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}_0) \to \pi_q(\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{y}_0)$.

Before proving the thereom, we first prove the following lemma:

Lemma 8.2. *The level map* $(f_\lambda) : \boldsymbol{X} \to \boldsymbol{Y}$ *has the following properties:*

(1) *Each* $\lambda \in \Lambda$ *admits* $\lambda' \geq \lambda$ *with the following property: any maps* $g : I^n \times 0 \cup \dot{I}^n \times I \to X_{\lambda'}$ *and* $H : I^n \times I \to Y_{\lambda'}$ *such that*
$$f_{\lambda'} g = H | I^n \times 0 \cup \dot{I}^n \times I$$
admit a map $G : I^n \times I \to X_\lambda$ *such that*
$$f_\lambda G = q_{\lambda\lambda'} H, \;\; G | I^n \times 0 \cup \dot{I}^n \times I = p_{\lambda\lambda'} g.$$

(2) *For any finite dimensional polyhedral pair* (Z, C)*, each* $\lambda \in \Lambda$ *admits* $\lambda' \geq \lambda$ *with the following property: any maps* $g : Z \times 0 \cup C \times I \to X_{\lambda'}$ *and* $H : Z \times I \to Y_{\lambda'}$ *such that*
$$f_{\lambda'} g = H | Z \times 0 \cup C \times I$$
admit a map $G : Z \times I \to X_\lambda$ *such that*
$$f_\lambda G = q_{\lambda\lambda'} H, \;\; G | Z \times 0 \cup C \times I = p_{\lambda\lambda'} g.$$

(3) *If (Z, C) is a finite dimensional polyhedral pair such that C is a strong deformation retract of Z, then each $\lambda \in \Lambda$ admits $\lambda' \geq \lambda$ with the following property: any maps $g : C \to X_{\lambda'}$ and $H : Z \to Y_{\lambda'}$ such that*

$$H|C = f_{\lambda'} g$$

admit a map $G : Z \to X_\lambda$ such that

$$f_\lambda G = q_{\lambda\lambda'} H, \; G|C = p_{\lambda\lambda'} g.$$

$$\begin{tikzcd}
X_\lambda \arrow[d, "f_\lambda"'] & X_{\lambda'} \arrow[l, "p_{\lambda\lambda'}"'] \arrow[d, "f_{\lambda'}"] & B \arrow[l, "g"'] \arrow[d, "\subseteq"] \\
Y_\lambda & Y_{\lambda'} \arrow[l, "q_{\lambda\lambda'}"] & A \arrow[l, "H"] \arrow[ull, dashed, "G"']
\end{tikzcd}$$

where $(A, B) = (I^n \times I, I^n \times 0 \cup \dot{I}^n \times I)$, $(Z \times I, Z \times 0 \cup C \times I)$, (Z, C).

Proof. The first assertion immediately follows from the facts that $(I^n \times I, I^n \times 0 \cup \dot{I}^n \times I)$ is homeomorphic to $(I^n \times I, I^n \times 0)$ and that (f_λ) has property $(\mathrm{SHLP})_L$ (Lemma 7.1).

For the second part, let (K, L) be a triangulation of the pair (Z, C). Let $W_{-1} = Z \times 0 \cup C \times I$, and $W_q = |K| \times 0 \cup |K^q \cup L| \times I$ for each $q \geq 0$,. By induction, we obtain a sequence $\lambda = \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_0 \leq \lambda_{-1} = \lambda'$ and maps $G_q : W_q \to X_{\lambda_q}$ for $q = -1, 0, \ldots, n$ such that

$$\begin{aligned}
&G_{-1} = g,\\
&G_q|W_{q-1} = G_{q-1} \; (0 \leq q \leq n),\\
&f_{\lambda_q} G_q = H|W_q \; (-1 \leq q \leq n),
\end{aligned}$$

where $n = \dim Z$. Then $G = G_n : W_n = Z \times 0 \cup C \times I \to X_\lambda$ is the desired map. Indeed, using (1), we inductively take $\lambda = \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_0 \leq \lambda_{-1} = \lambda'$ so that whenever $k : I^q \times 0 \cup \dot{I}^q \times I \to X_{\lambda_{q-1}}$ and $K : I^q \times I \to Y_{\lambda_{q-1}}$ are maps such that

$$K|I^q \times 0 \cup \dot{I}^q \times I = f_{\lambda_{q-1}} k,$$

there is a map $\tilde{K} : I^q \times I \to X_{\lambda_q}$ such that

$$\tilde{K}|I^q \times 0 \cup \dot{I}^q \times I = p_{\lambda_q \lambda_{q-1}} k,$$

$f_{\lambda_q} \tilde{K} = q_{\lambda_q \lambda_{q-1}} K$. Assuming we have defined maps G_q for $q < m \leq n$, we define the map G_m as follows: for each m-simplex $\sigma \in K \setminus L$, define maps

$$g_\sigma : |\sigma| \times 0 \cup |\dot{\sigma}| \times I \to X_{\lambda_{m-1}} : g_\sigma = G_{m-1}||\sigma| \times 0 \cup |\dot{\sigma}| \times I, \text{ and}$$

$$H_\sigma : |\sigma| \times I \to Y_{\lambda_{m-1}} : H_\sigma = H||\sigma| \times I.$$

Then $H_\sigma||\sigma| \times 0 \cup |\dot{\sigma}| \times I = f_{\lambda_{m-1}} g_\sigma$. So, there is a map $G_\sigma : |\sigma| \times I \to X_{\lambda_m}$ such that

$$\begin{aligned}
G_\sigma||\sigma| \times 0 \cup |\dot{\sigma}| \times I &= p_{\lambda_m \lambda_{m-1}} g_\sigma,\\
f_{\lambda_m} G_\sigma &= q_{\lambda_m \lambda_{m-1}} H_\sigma.
\end{aligned}$$

Define the map $G_m : W_m \to X_{\lambda_m}$ by $G_m||\sigma| \times I = G_\sigma$.

For the third assertion, let $D : Z \times I \to Z$ be a strong deformation retract of Z to C, i.e., $D_0 = 1_Z$, $D_1 = ir$ where $r : Z \to C$ is a retraction and $i : C \hookrightarrow Z$ is the inclusion map. For each $\lambda \in \Lambda$, choose $\lambda' \geq \lambda$ as in (2). For any maps $g : C \to X_{\lambda'}$

and $H : Z \to Y_{\lambda'}$ such that $H|C = f_{\lambda'}g$, define the map $g' : Z \times 1 \cup C \times I \to X_{\lambda'}$ as the composite

$$Z \times 1 \cup C \times I \xrightarrow{D|Z\times 1\cup C\times I} C \xrightarrow{g} X_{\lambda'}$$

and the map $H' : Z \times I \to Y_{\lambda'}$ as the composite

$$Z \times I \xrightarrow{D} Z \xrightarrow{H} Y_{\lambda'}.$$

Then, by (2), we obtain a map $G' : Z \times I \to X_\lambda$ such that

$$G'|Z \times 1 \cup C \times I = p_{\lambda\lambda'}g',$$
$$f_\lambda G' = q_{\lambda\lambda'}H'.$$

Then $G = G'_0 : Z \to X_\lambda$ is the desired map. $\square$

Proof. Theorem 8.1. We will show that each $\lambda \in \Lambda$ admits $\lambda' \geq \lambda$ and a map $\Psi : \pi_q(Y_{\lambda'}, B_{\lambda'}, y_{0\lambda_2}) \to \pi_q(X_\lambda, A_\lambda, x_{0\lambda})$ which makes the following diagram commute:

(8.2)
$$\begin{tikzcd}
\pi_q(X_\lambda, A_\lambda, x_{0\lambda}) \arrow[d, "(f_\lambda)_\sharp"'] & \pi_q(X_{\lambda'}, A_{\lambda'}, x_{0\lambda'}) \arrow[l, "(p_{\lambda\lambda'})_\sharp"'] \arrow[d, "(f_{\lambda'})_\sharp"] \\
\pi_q(Y_\lambda, B_\lambda, y_{0\lambda}) & \pi_q(Y_{\lambda'}, B_{\lambda'}, y_{0\lambda'}) \arrow[l, "(q_{\lambda\lambda'})_\sharp"] \arrow[ul, dashed, "\Psi"']
\end{tikzcd}$$

Let $\lambda \in \Lambda$. Choose a lifting index $\lambda_1 \geq \lambda$, and in turn for this λ_1, choose a lifting index $\lambda_2 \geq \lambda_1$. Suppose that $\alpha : (I^n, \dot{I}^n, p_0) \to (Y_{\lambda_2}, B_{\lambda_2}, y_{0\lambda_2})$ represents an element of $\pi_q(Y_{\lambda_2}, B_{\lambda_2}, y_{0\lambda_2})$. Since $\{p_0\}$ is a strong deformation retract of I^n, by Lemma 8.2 (3), there is a map $H' : I^n \to X_{\lambda_1}$ which makes the following diagram commute:

$$\begin{tikzcd}
X_{\lambda_1} \arrow[d, "f_{\lambda_1}"'] & X_{\lambda_2} \arrow[l, "p_{\lambda_1\lambda_2}"'] \arrow[d, "f_{\lambda_2}"] & \{p_0\} \arrow[l] \arrow[d, "\subseteq"] \\
Y_{\lambda_1} & Y_{\lambda_2} \arrow[l, "q_{\lambda_1\lambda_2}"] & I^n \arrow[l, "\alpha"] \arrow[ull, "H'"']
\end{tikzcd}$$

Since $\alpha(\dot{I}^n) \subseteq B_{\lambda_2}$, $f_{\lambda_1}H'_1(\dot{I}^n) = q_{\lambda_1\lambda_2}\alpha(\dot{I}^n) \subseteq B_{\lambda_1}$, and so $H'(\dot{I}^n) \subseteq A_{\lambda_1}$. We now define the map $\psi(\alpha) : (I^n, \dot{I}^n, p_0) \to (X_{\lambda_1}, A_{\lambda_1}, x_{0\lambda})$ by $\psi(\alpha) = p_{\lambda\lambda_1}H'$. Note that $f_\lambda\psi(\alpha) = q_{\lambda\lambda_2}\alpha$. Using the fact that $I^n \times \{0,1\} \cup \{p_0\} \times I$ is a strong deformation retract of $I^n \times I$ and Lemma 8.2 (3), we can show that there is a well-defined homomorphism (function if $q = 1$) $\Psi : \pi_q(Y_{\lambda_2}, B_{\lambda_2}, y_{0\lambda_2}) \to \pi_q(X_\lambda, A_\lambda, x_{0\lambda})$: $\Psi([\alpha]) = [\psi(\alpha)]$ which makes diagram (8.2) with $\lambda' = \lambda_2$ commute. $\square$

For any strong pro-fibration $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ in $\mathcal{ANR}$, the *fibre* $\boldsymbol{F}$ of $\boldsymbol{f}$ is the ANR object which is defined by the following pull-back diagram (Lemma 7.2):

$$\begin{tikzcd}
\boldsymbol{F} \arrow[r] \arrow[d] & \boldsymbol{X} \arrow[d] \\
* \arrow[r] & \boldsymbol{Y}
\end{tikzcd}$$

By Theorem 8.1 and the exactness of the pro-homotopy sequence of the pair $(\boldsymbol{X}, \boldsymbol{F}, \boldsymbol{x}_0)$, we have the *sequence of strong pro-fibration*

$$\cdots \to \pi_q(\boldsymbol{F}, \boldsymbol{x}_0) \to \pi_q(\boldsymbol{X}, \boldsymbol{x}_0) \to \pi_q(\boldsymbol{Y}, \boldsymbol{y}_0) \to \pi_{q-1}(\boldsymbol{F}, \boldsymbol{x}_0) \to \cdots,$$

which is exact.

9. Applications to homotopy decomposition

In this section, we assume that all spaces have base points and that all maps and homotopies preserve base points. Let pro- $\mathrm{H}(\mathsf{Top})_*$ denote the pro-homotopy category of pointed spaces and base point preserving maps.

Here are the main results of this section.

Theorem 9.1. *Every ANR-tower $\boldsymbol{X}$ admits a sequence of ANR-towers*

$$P_1(\boldsymbol{X}) \xleftarrow{\boldsymbol{f}_1} P_2(\boldsymbol{X}) \xleftarrow{\boldsymbol{f}_2} \cdots \longleftarrow P_n(\boldsymbol{X}) \xleftarrow{\boldsymbol{f}_n} \cdots \tag{9.1}$$

and morphisms $\boldsymbol{i}_n : \boldsymbol{X} \to P_n(\boldsymbol{X})$ $(n = 1, 2, \ldots)$ in pro- Top_* *with the following properties:*

1. *$\boldsymbol{f}_n$ are strong pro-fibrations,*
2. *$\pi_q(P_n(\boldsymbol{X})) \approx 0$ for $q > n$,*
3. *$\boldsymbol{i}_n$ induce isomorphisms $(\boldsymbol{i}_n)_\sharp : \pi_q(\boldsymbol{X}) \to \pi_q(P_n(\boldsymbol{X}))$ for $q \leq n$, and*
4. *the following diagram commutes in* pro- $\mathrm{H}(\mathsf{Top})_*$*:*

$$\begin{tikzcd}
{[\boldsymbol{X}]} \arrow[d, "{[\boldsymbol{i}_1]}"] \arrow[dr, "{[\boldsymbol{i}_2]}"] \arrow[drrr, "{[\boldsymbol{i}_n]}"] & & & & \\
{[P_1(\boldsymbol{X})]} & {[P_2(\boldsymbol{X})]} \arrow[l, "{[\boldsymbol{f}_1]}"] & \cdots \arrow[l, "{[\boldsymbol{f}_2]}"] & {[P_n(\boldsymbol{X})]} \arrow[l] & \cdots\cdots \arrow[l]
\end{tikzcd}$$

Theorem 9.2. *Every ANR-tower $\boldsymbol{X}$ admits a sequence of ANR objects*

$$\overline{P}_0(\boldsymbol{X}) \xleftarrow{\overline{\boldsymbol{f}}_0} \overline{P}_1(\boldsymbol{X}) \xleftarrow{\overline{\boldsymbol{f}}_1} \cdots \overline{P}_n(\boldsymbol{X}) \xleftarrow{\overline{\boldsymbol{f}}_n} \cdots$$

with the following properties:

1. *$\overline{\boldsymbol{f}}_n$ are strong pro-fibrations,*
2. *$\pi_q(\overline{P}_n(\boldsymbol{X})) \approx \left\{ \begin{array}{cc} \pi_q(\boldsymbol{X}) & q > n \\ 0 & q \leq n \end{array} \right\}$ for $n \geq 1$, and*
3. *$\overline{P}_0(\boldsymbol{X})$ is isomorphic to $\boldsymbol{X}$ in* pro- $\mathrm{H}(\mathsf{Top})_*$.

Every CW complex X and $n \geq 1$ admit the n-th Postnikov section $X^{[n]}$ of X and the inclusion map $j^n_X : X \to X^{[n]}$ (see [4]). So, $X^{[n]}$ is a CW complex satisfying

$$\pi_q(X^{[n]}) \approx \left\{ \begin{array}{cc} \pi_q(X) & q \leq n, \\ 0 & q < n. \end{array} \right.$$

For $m \leq n$, every map $f : X \to Y$ between CW complexes induces a map (unique up to homotopy) $f_{n,m} : X^{[n]} \to Y^{[m]}$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j^n_X"'] & Y \arrow[d, "j^n_Y"] \\
X^{[n]} \arrow[r, "f_{n,m}"'] & Y^{[m]}
\end{tikzcd}$$

Write $f^{[n]}$ for $f_{n,n}$. Thus every CW-tower $\boldsymbol{X} = (X_i, p_{i,i+1})$ induces a CW-tower $\boldsymbol{X}^{[n]} = (X^{[n]}_i, p^{[n]}_{i,i+1})$, and every system map $(f, f_i) : \boldsymbol{X} \to \boldsymbol{Y}$ between CW-towers induces a system map $(f, [(f_i)_{n,m}]) : [\boldsymbol{X}^{[n]}] \to [\boldsymbol{Y}^{[m]}]$. So for each CW-tower $\boldsymbol{X}$

there exists a commutative diagram:
(9.2)

$$\begin{tikzcd}
{[\boldsymbol{X}]} \arrow[d, "{([j_i^1])}"] \arrow[dr, "{([j_i^2])}"] \arrow[drrr, "{([j_i^n])}"] & & & & \\
{[\boldsymbol{X}^{[1]}]} & {[\boldsymbol{X}^{[2]}]} \arrow[l, "{([(g_i)_{2,1}])}"'] & \cdots \arrow[l, "{([(g_i)_{3,2}])}"'] & {[\boldsymbol{X}^{[n]}]} \arrow[l, "{([(g_i)_{n,n-1}])}"'] & \cdots \arrow[l, "{([(g_i)_{n+1,n}])}"']
\end{tikzcd}$$

where $([j_i^n])$ is the level map induced by the inclusion maps $j_i^n : X_i \hookrightarrow X_i^{[n]}$, and $([(g_i)_{n+1,n}])$ is the level map induced by the identity maps $g_i : X_i \to X_i$. Then

$$\pi_q(\boldsymbol{X}^{[n]}) \approx \begin{cases} \pi_q(\boldsymbol{X}) & q \leq n, \\ 0 & q > n. \end{cases}$$

Lemma 9.3. *Every CW-tower (resp., ANR-tower) $\boldsymbol{X}$ admits an ANR-tower (resp., a CW-tower) $\boldsymbol{X}'$ and a level map $([\varphi_i]) : [\boldsymbol{X}] \to [\boldsymbol{X}']$ consisting of homotopy classes of homotopy equivalences φ_i .*

Proof. We can easily prove the lemma, using the fact that every CW complex (resp., ANR) has the homotopy type of an ANR (resp., a CW complex). □

Proof of Theorem 9.1. By Lemma 9.3 there exist a CW-tower $\boldsymbol{Y}$ and a level map $([\varphi_i]) : \boldsymbol{X} \to \boldsymbol{Y}$ in pro-$\mathrm{H}(\mathsf{Top})_*$ consisting of homotopy equivalences. For this CW-tower $\boldsymbol{Y}$ there exist CW-towers $\boldsymbol{Y}^{[n]}$ $(n = 1, 2, \ldots)$ with the commutative diagram (9.2) with $\boldsymbol{X}$ and $\boldsymbol{X}^{[n]}$ being replaced by $\boldsymbol{Y}$ and $\boldsymbol{Y}^{[n]}$, respectively. By Lemma 9.3 and the pointed version of Lemma 6.7, for each n, there exist an ANR-tower $\boldsymbol{X}_n$ with the bonding maps being Hurewicz fibrations and a level map (a_i^n) : $\boldsymbol{Y}^{[n]} \to \boldsymbol{X}_n$ consisting of homotopy equivalences. Let b_i^n be the homotopy inverse of a_i^n. Since the bonding maps of $\boldsymbol{X}_n$ are Hurewicz fibrations, there is a level map $(h_i^n) : \boldsymbol{X}_{n+1} \to \boldsymbol{X}_n$ which induces the level map $([a_i^n(g_i)_{n+1,n}b_i^{n+1}]) : [\boldsymbol{X}_{n+1}] \to [\boldsymbol{X}_n]$. So we have the following commutative diagram in pro-$\mathrm{H}(\mathsf{Top})_*$:

$$\begin{tikzcd}
{[\boldsymbol{Y}^{[n+1]}]} \arrow[r, "{([a_i^{n+1}])}"] \arrow[d, "{([(g_i)_{n+1,n}])}"'] & {[\boldsymbol{X}_{n+1}]} \arrow[d, "{([h_i^n])}"] \\
{[\boldsymbol{Y}^{[n]}]} \arrow[r, "{([a_i^n])}"'] & {[\boldsymbol{X}_n]}
\end{tikzcd}$$

Now let $\boldsymbol{h}_n : \boldsymbol{X}_{n+1} \to \boldsymbol{X}_n$ be the morphism represented by (h_i^n), let $P_1(\boldsymbol{X}) = \boldsymbol{X}_1$, and let $\boldsymbol{c}_1 : \boldsymbol{X}_1 \to P_1(\boldsymbol{X})$ be the morphism induced by the identity maps. Then by Remark 6.2, we inductively obtain ANR-towers $P_n(\boldsymbol{X})$ $(n = 1, 2, 3, \ldots)$ and the commutative diagrams in pro-Top_*

$$\begin{tikzcd}
\boldsymbol{X}_{n+1} \arrow[r, "\boldsymbol{c}_{n+1}"] \arrow[d, "\boldsymbol{h}_n"'] & P_{n+1}(\boldsymbol{X}) \arrow[d, "\boldsymbol{f}_n"] \\
\boldsymbol{X}_n \arrow[r, "\boldsymbol{c}_n"'] & P_n(\boldsymbol{X})
\end{tikzcd}$$

where $\boldsymbol{c}_n$ are morphisms which induce isomorphisms in pro-$\mathrm{H}(\mathsf{Top})_*$, and $\boldsymbol{f}_n$ are strong pro-fibrations. Thus we have a sequence of ANR-towers (9.1) with properties

(1) and (2). Since the bonding maps of $\boldsymbol{X}_n$ are Hurewicz fibrations, by Lemma 6.8, the morphism $\boldsymbol{d}_n : [\boldsymbol{X}] \to [\boldsymbol{X}^{[n]}]$ represented by the composite of the level maps

$$[\boldsymbol{X}] \xrightarrow{([\varphi_i])} [\boldsymbol{Y}] \xrightarrow{([j_i^n])} [\boldsymbol{Y}^{[n]}] \xrightarrow{([a_i^n])} [\boldsymbol{X}_n]$$

is induced by a level map $(d_i^n) : \boldsymbol{X} \to \boldsymbol{X}_n$. Let $\boldsymbol{i}_n : \boldsymbol{X} \to P_n(\boldsymbol{X})$ be the composite

$$\boldsymbol{X} \xrightarrow{\boldsymbol{d}_n} \boldsymbol{X}_n \xrightarrow{\boldsymbol{c}_n} P_n(\boldsymbol{X}).$$

Then sequence (9.1) and the morphisms $\boldsymbol{i}_n$ have properties (3) and (4). This completes the proof of Theorem 9.1.

Proof of Theorem 9.2. By Remark 6.5, for each $n \geq 1$, there is a commutative diagram in $\mathcal{ANR}$:

$$\begin{tikzcd}
\boldsymbol{E}_n \arrow[dr, "\boldsymbol{h}_n"] & \\
\boldsymbol{X} \arrow[u, "\boldsymbol{g}_n"] \arrow[r, "\boldsymbol{i}_n"'] & P_n(\boldsymbol{X})
\end{tikzcd}$$

where $\boldsymbol{E}_n = \boldsymbol{E}_{\boldsymbol{i}_n}$, the morphism $\boldsymbol{g}_n$ induces an isomorphism in pro-$\mathrm{H}(\mathsf{Top})_*$, and $\boldsymbol{h}_n$ is a strong pro-fibration. Let $\boldsymbol{F}_n$ be the fibre of $\boldsymbol{h}_n$. Then, by the naturality in Theorem 6.4, there is a map between pull-back diagrams:

$$\begin{tikzcd}
\boldsymbol{F}_n \arrow[rr, "\boldsymbol{k}_n"] \arrow[dr, "\boldsymbol{f}'_{n-1}"] \arrow[dd] & & \boldsymbol{E}_n \arrow[dr, "\gamma_{n-1}"] \arrow[dd, "\boldsymbol{h}_n"] & \\
& \boldsymbol{F}_{n-1} \arrow[rr, "\boldsymbol{k}_{n-1}"] \arrow[dd] & & \boldsymbol{E}_{n-1} \arrow[dd, "\boldsymbol{h}_{n-1}"] \\
* \arrow[rr] \arrow[dr] & & P_n(\boldsymbol{X}) \arrow[dr, "\boldsymbol{f}_{n-1}"] & \\
& * \arrow[rr] & & P_{n-1}(\boldsymbol{X})
\end{tikzcd}$$

We have the sequence of ANR objects:

$$\boldsymbol{E}_1 \xleftarrow{\boldsymbol{k}_1} \boldsymbol{F}_1 \xleftarrow{\boldsymbol{f}'_1} \boldsymbol{F}_2 \xleftarrow{\boldsymbol{f}'_2} \cdots \longleftarrow \boldsymbol{F}_{n-1} \xleftarrow{\boldsymbol{f}'_{n-1}} \boldsymbol{F}_n \longleftarrow \cdots$$

Let $\overline{P}_0(\boldsymbol{X}) = \boldsymbol{E}_1$, and using Remark 6.2, we inductively obtain ANR objects $\overline{P}_n(\boldsymbol{X})$ $(n = 1, 2, \ldots)$ which make the following diagram commute:

$$\begin{tikzcd}
\overline{P}_0(\boldsymbol{X}) = \boldsymbol{E}_1 & \boldsymbol{F}_1 \arrow[l, "\boldsymbol{k}_1"'] \arrow[d] & \boldsymbol{F}_2 \arrow[l, "\boldsymbol{f}'_1"'] \arrow[d] & \cdots \arrow[l, "\boldsymbol{f}'_2"'] & \boldsymbol{F}_n \arrow[l, "\boldsymbol{f}'_{n-1}"'] \arrow[d] & \cdots \arrow[l] \\
& \overline{P}_1(\boldsymbol{X}) \arrow[ul, "\overline{\boldsymbol{f}}_0"] & \overline{P}_2(\boldsymbol{X}) \arrow[l, "\overline{\boldsymbol{f}}_1"'] & \cdots \arrow[l, "\overline{\boldsymbol{f}}_2"'] & \overline{P}_n(\boldsymbol{X}) \arrow[l, "\overline{\boldsymbol{f}}_{n-1}"'] & \cdots \arrow[l]
\end{tikzcd}$$

where the vertical maps induce isomorphisms in pro-$\mathrm{H}(\mathsf{Top})_*$, and $\overline{\boldsymbol{f}}_n$ $(n = 0, 1, \cdots)$ are strong pro-fibrations. Thus we have properties (1) and (3). By the exact sequence of the strong pro-fibration $\boldsymbol{E}_n \to P_n(\boldsymbol{X})$, we have the second property (2).

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