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On Potential Embedding and Versions of Martin's Axiom*

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1 Introduction

Let A and B be structures. For a condition $\mathcal{E}$ on p.o.-sets (e.g. ccc, proper, $<\kappa$-closed, etc.) let us say that A is $\mathcal{E}$-potentially embeddable into B if there exists a p.o.-set P with the property $\mathcal{E}$ such that $\Vdash_P$"A is embeddable into B". Similarly we shall say that A and B are $\mathcal{E}$-potentially isompophic if there exists a p.o.-set P with the property $\mathcal{E}$ such that $\Vdash_P$"$A \cong B$".

The notion of $(<\kappa, \infty)$-distributive-potentially isomorphism and $<\kappa$-closed-potentially isomorphism have been studied in M. Nadel and J. Stavi [7]. In [4] a characterization of $(<\kappa, \infty)$-distributive-potentially isomorphism to a free Boolean algebra is given under some set theoretic assumptions on κ.

In this note we shall consider the question if $\mathcal{E}$-potential embedding ($\mathcal{E}$-potential isomorphism) implies the real embedding (isomorphism).

The following example suggests that this question is by no means trivial for some instances of A and B even when we consider the ccc as the condition $\mathcal{E}$.

Example 1.1 *a) Let A be the subalgebra of the Boolean algebra $\wp(\omega_1)$ consisting of finite and co-finite subsets of ω_1. Assume that there exists a* ccc *Boolean algebra B which is not productively* ccc*. Let C be a* ccc *Boolean algebra such that $B \oplus C$ does not satisfy the* ccc*. By the* ccc *of B, A is not embeddable in B. But, since $\Vdash_{C^+}$"B does not satisfy the* ccc*", we obtain that $\Vdash_{C^+}$"A is embeddable into B".*
This situation can be also coded in structures in a language with only a binary relation symbol: Let B and C be as above. Let D and E be the structures defined by

$$D = (\omega_1, \{ (\alpha, \beta) \,:\, \alpha, \beta \in \omega_1,\, \alpha \neq \beta \}),$$

$$E = (B^+, \{ (a, b) \,:\, a, b \in B^+,\, a \cdot b = 0 \}).$$

Then no uncountable substructure of D is embeddable into E. But $\Vdash_{C^+}$"D is embeddable into E".
b) (CH) *Let A and B be mutually non-embeddable $\aleph_1$-dense subordering of $\mathbb{R}$. Let P be the standard p.o.-set forcing* MA + $2^{\aleph_0} = \aleph_2$. *Then, by J. Baumgartner* [1], *$\Vdash_P A \cong B$ holds.* ∎

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If B has some "good" properties we can deduce the real embedding (isomorphism) to B from the potential embedding (potential isomorphism). The following trivial lemma is such an example:

Lemma 1.2 *Let B be κ^+-universal for $\kappa = |A|$. If there exists a p.o.-set P such that $\Vdash_P$ "A is embeddable into B" then A is embeddable into B.* ∎

In section 2 we show that the ccc-potential isomorphism to a free Boolean algebra implies the real isomorphism (Proposition 2.4).

For a condition $\mathcal{E}$ on p.o.-sets and a cardinal κ let $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ ($\mathrm{PI}^{\mathcal{E}}_{\kappa}$) denote the following axiom:

> For every structures A, B such that $|A| = \kappa$, if A is $\mathcal{E}$-potentially embeddable into B (if A is $\mathcal{E}$-potentially isomorphic to B and $|A| = |B|$) then A is embeddable into (isomorphic to) B.

Similarly we shall also consider a class of axioms on potential partial embedding. Let $\mathrm{PPE}^{\mathcal{E}}_{\kappa}$ be the axiom saying:

> For every structures A, B in some relational language L, if $\Vdash_P$"there exists a substructure C of A of size κ which is embeddable into B" for some p.o.-set P with the property $\mathcal{E}$ then there exists a substructure C of A of size κ which is embeddable into B.

Here we call a language L relational when L contains no function symbols. Note that the definition of $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ and $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ would not be changed if the structures considered would have been restricted to be in relational languages.

In section 3 we show that the axioms $\mathrm{PE}^{\mathcal{E}}_{\kappa}$, $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ and $\mathrm{PPE}^{\mathcal{E}}_{\kappa}$ are closely related to the corresponding version of Martin's axiom. We show that Martin's axiom on the property $\mathcal{E}$ for κ dense sets is equivalent to $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ (Theorem 3.3). In particular $\mathrm{PE}^{\mathrm{ccc}}_{\kappa}$ is equivalent to Martin's axiom for κ dense sets.

In section 4 we give another characterization of versions of Martin's axiom which is related to the characterization of Martin's axiom given in S. Todorčević and B. Veličković [9].

Using the axioms on potential partial embedding introduced above and their weaknings to be defined in section 5, we also give characterizations of some other known axioms related to Martin's axiom (Theorem 3.7 and Theorem 5.4).

As an application of the characterizations of versions of Martin's axiom by means of potential embedding, we prove in Theorem 5.6 that versions of Martin's axiom are equivalent to the corresponding axioms asserting merely the existence of a linked generic set in place of a generic filter (Theorem 5.6). This theorem generalizes an observation in M. G. Bell [2, pp150 − 151].

The axioms on potential embedding and some problems concerning with them came into my mind while I visited Prof. B. Węglorz and Prof. J. Cichoń at Wrocław University in November 1990. I would like to thank them for stimulating discussions.

Thanks are also due to Prof. S. Koppelberg and Prof. P. Vojtáš for some valuable comments.

2 Free Boolean algebras

A Boolean algebra B is said to be projective if B is a retract of a free Boolean algebra, i.e. if there are a free Boolean algebra F and homomorphisms $e : B \rightarrow F$, $f : F \rightarrow B$ such that $f \circ e = id_B$. For Boolean algebras$A$, B with $A \leq B$ (A is subalgebra of B), $A \leq_{\rm rc} B$ if for every $b \in B$ there exists a largest element a in A with $a \leq b$. For the proof of the following two theorems the reader may consult [6].

Theorem 2.1 (Ščepin) *A Boolean algebra B is projective if and noly if there exists a system $\mathcal{S}$ of subalgebras of B such that*

1) $2 \in \mathcal{S}$ *(where 2 is the two-element Boolean algebra),*

2) $S \in \mathcal{S}$ *implies* $S \leq_{\rm rc} B$*,*

3) *for every non-empty chain $\mathcal{C}$ in $\mathcal{S}$,* $\bigcup \mathcal{C} \in \mathcal{S}$*,*

4) *for every $S \in \mathcal{S}$ and $X \in [B]^{\leq\omega}$ there is $S' \in \mathcal{S}$ such that $S \cup X \subseteq S'$ and S' is countably generated over S.* ∎

Theorem 2.2 (Ščepin) *A Boolean algebra B is free if and only if B is projective and for any $X \in [B^+]^{<|B|}$ there exists $b \in B^+$ independent from X.* ∎

Proposition 2.3 *Let B be a Boolean algebra. If there exists a ccc p.o.-set P such that $\Vdash_P$ "B is projective" then B is projective.*

Proof By Theorem 2.1 there is a P-name $\dot{\mathcal{S}}$ such that $\Vdash_P$"$\dot{\mathcal{S}}$ satisfies 1) – 4) in Theorem 2.1". Let

$$\mathcal{S} = \{ A \leq B \ : \Vdash_P A \in \dot{\mathcal{S}} \}.$$

Then it is easy to show that $\mathcal{S}$ satisfies 1) – 4) in Theorem 2.1. ∎ (Proposition 2.3)

Proposition 2.4 *If a Boolean algebra B is* ccc*-potentially isomorphic to a free Boolean algebra then B is free.*

Proof By Theorem 2.2 and Proposition 2.3. ∎ (Proposition 2.4)

Problem 2.5 *Does* ccc*-potantial embedding into a free Boolean algebra imply the real embedding?*

While the above problem is still open, we have a complete charactarization of a similar situation of the potential embeddability in the power-set algebra over ω:

Proposition 2.6 ([3]) *Let B be a Boolean algebra. Then the followings are equivalent:*

1) *There exists a* ccc *p.o.-set P such that*

$$\Vdash_P \text{"}B \text{ is embeddable in } \wp(\omega)\text{"};$$

2) *There exists a* ccc *p.o.-set* P *such that*

$$\Vdash_P \text{“}B \textit{ has a finitely additive strictly positive measure”};$$

3) $\overbrace{B \oplus \cdots \oplus B}^{n \textit{ times}}$ *has the* ccc *for every* $n \in \omega$.

∎

Note that the condition 1) of Proposition 2.6 does not mean the ccc-potental embeddability of B into the power-set algebra $\wp(\omega)$ in our sense since the power-set algebra $\wp(\omega)$ in the generic extension is, in general, not equal to the power-set algebra $\wp(\omega)$ in the ground model.

3 Potential embedding and Martin's axiom

For a condition $\mathcal{E}$ on p.o.-sets and cardinal κ, let $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ be the following assertion:

> For any p.o.-set satisfying the condition $\mathcal{E}$ and for any family $\mathcal{D} = \{ D_\alpha : \alpha < \kappa \}$ of dense subsets of P, there exists a $\mathcal{D}$-generic filter over P.

Using this notation, the proper forcing axiom (PFA) and Martin's maximum (MM) can be denoted by $\mathrm{MA}^{\mathrm{proper}}_{\aleph_1}$ and $\mathrm{MA}^{\mathrm{stat.preserving}}_{\aleph_1}$ respectively. For these axioms see e.g. [5]. Martin's axiom for κ dense sets (i.e. $\mathrm{MA}^{\mathrm{ccc}}_{\kappa}$) is also denoted, as usual, by MA_κ.

Lemma 3.1 *For a condition $\mathcal{E}$ on p.o.-sets and an infinite cardinal κ, $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ implies $\mathrm{PE}^{\mathcal{E}}_{\kappa}$, $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ and $\mathrm{PPE}^{\mathcal{E}}_{\kappa}$.*

Proof Assume that $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ holds. We shall show that $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ and $\mathrm{PPE}^{\mathcal{E}}_{\kappa}$ hold. The proof of $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ can be done similarly.

Let A and B be structures of size κ. Suppose that there exists a p.o.-set P satisfying the condition $\mathcal{E}$ and a P-name $\dot{f}$ such that $\Vdash_P$ "$\dot{f}$ is an isomorphism from of A to B". For each $a \in A$ and $b \in B$ let

$$D_a = \{ p \in P : \text{there exists some } d \in B \text{ such that } p \Vdash_P \dot{f}(a) = d \},$$

$$D'_b = \{ p \in P : \text{there exists some } c \in A \text{ such that } p \Vdash_P \dot{f}(c) = b \}.$$

Clearly D_a and D'_b are dense subsets of P. Let $\mathcal{D} = \{ D_a : a \in A \} \cup \{ D'_b : b \in B \}$ and let G be a $\mathcal{D}$-generic filter over P. Then the mapping $f : A \to B$ defined by

$$f(a) = b \qquad \text{for some } b \in B \text{ such that there exists } p \in G \text{ with } p \Vdash_P \dot{f}(a) = b$$

is an isomorphism from A to B. This proves that $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ holds.

For $\mathrm{PPE}^{\mathcal{E}}_{\kappa}$ let A and B be structures in a relational language L. Suppose that there exists a p.o.-set P satisfying the condition $\mathcal{E}$ and P-names $\dot{C}$, $\dot{h}$ such that $\Vdash_P$ "$\dot{C}$ is a substructure of A of size κ and $\dot{h}$ is an embedding of $\dot{C}$ into B". Let $\dot{g}$ be a P-name of injective enumeration of $\dot{C}$ of length κ, i.e. $\Vdash_P$ "$\dot{g} : \kappa \to \dot{C}$ is 1-1 onto". For each $\alpha < \kappa$ let

$$D_\alpha = \{ p \in P : p \text{ decides } \dot{g}(\alpha) \text{ and } \dot{h} \circ \dot{g}(\alpha) \}.$$

Let $\mathcal{D} = \{ D_\alpha : \alpha \in \kappa \}$ and let G be a $\mathcal{D}$ generic filter over P. Let

$$C = \{ a \in A : \text{There exists } p \in G \text{ and } \alpha < \kappa \text{ such that } p \Vdash_P \dot{g}(\alpha) = a \}.$$

Then C is a substructure of A of size κ. Let $h : C \to B$ be defined by

$$h(a) = b \qquad \text{for } b \in B \text{ such that there is } p \in G \text{ and } \alpha < \kappa \text{ with } p \Vdash_P \dot{g}(\alpha) = a \wedge \dot{h} \circ \dot{g}(\alpha) = b.$$

It is easy to show that h is an embedding of C into B. ▮ (Lemma 3.1)

Lemma 3.2 $\mathrm{PI}^{\mathrm{ccc}}_{\aleph_1}$ *implies* $\neg$CH.

Proof Immediate from Example 1.1 *b*). ▮ (Lemma 3.2)

Similarly, using Example 1.1 *a*), we could show that $\mathrm{PE}^{\mathrm{ccc}}_{\aleph_1}$ or $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$ implies ¬CH. However we can actually prove much more general results. For a condition $\mathcal{E}$ on p.o.-sets, we shall say that $\mathcal{E}$ is a regular condition on p.o.-sets if, for any p.o.-set P and any dense subordering Q of P, P satisfies $\mathcal{E}$ whenever Q satisfies $\mathcal{E}$. Note that the conditions $\mathcal{E}$ on p.o.-sets used to define the usual versions of Martin's axiom of the form $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ (σ-centered, ccc, proper, stationary preserving, etc., see e.g. [5], [10]) are regular.

Theorem 3.3 *For a regular condition $\mathcal{E}$ on p.o.-sets and an infinite cardinal κ, $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ is equivalent to $\mathrm{PE}^{\mathcal{E}}_{\kappa}$.*

Proof By Lemma 3.1 it is enough to show that $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ implies $\mathrm{MA}^{\mathcal{E}}_{\kappa}$. For $\kappa = \aleph_0$ this is clear since $\mathrm{MA}^{\mathcal{E}}_{\aleph_0}$ and $\mathrm{PE}^{\mathcal{E}}_{\aleph_0}$ already hold in ZFC. Let us assume that $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ does not hold for an uncountable κ. We shall show that $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ does not hold. Let P be a p.o.-set satisfying the condition $\mathcal{E}$ with a family $\mathcal{D} = \{ D_\alpha : \alpha < \kappa \}$ of dense subsets of P such that there exists no $\mathcal{D}$-generic filter over P.

Claim 3.3.1 *There exists a p.o.-set P' satisfying the condition $\mathcal{E}$ and a family $\mathcal{D}' = \{ D'_\alpha : \alpha < \kappa \}$ of dense subsets of P' such that*

a) *there exists no $\mathcal{D}'$-generic filter over P',*

b) *for every $\alpha < \kappa$ and $p \in D'_\alpha$ there exists $q \in P'$ such that $p \le q$ and*

$$q \in D'_\alpha \setminus \bigcup_{\beta \in \kappa \setminus \{\alpha\}} D'_\beta.$$

Proof of Claim 3.3.1 For every $p \in P$ let $T_p = \{ q_{p,\alpha} : \alpha < \kappa \}$ where we assume that $q_{p,\alpha} \notin P$ and $q_{p,\alpha} \neq q_{p',\alpha'}$ for $p, p' \in P$ and $\alpha, \alpha' \in \kappa$ such that $(p, \alpha) \neq (p', \alpha')$. Let

$$P' = P \dot{\cup} \dot{\bigcup_{p \in P}} T_p$$

and

$$\le^{P'} = \le^{P} \cup \{ (p', q_{p,\alpha}) : p, p' \in P,\ \alpha < \kappa,\ p' \le p \} \cup \{ (q_{p,\alpha}, q_{p,\beta}) : p \in P,\ \alpha \le \beta < \kappa \}.$$

Since P is dense in $P' = (P', \le^{P'})$, P' still satisfies the condition $\mathcal{E}$. For $\alpha < \kappa$ let

$$D'_\alpha = D_\alpha \cup \{ q_{p,\alpha} : p \in D_\alpha \}.$$

Then P' and $\mathcal{D}' = \{ D'_\alpha : \alpha < \kappa \}$ are as desired. ▮ (Claim 3.3.1)

By the claim above we may assume without loss of generality that for any $\alpha < \kappa$ and $p \in D_\alpha$ there exists $q \ge p$ such that

$$q \in D_\alpha \setminus \bigcup_{\beta \in \kappa \setminus \{\alpha\}} D_\beta.$$

Now let $A = (\kappa, \{\alpha\}, \kappa^n)_{\alpha < \kappa, n < \omega}$ and $B = (P, D_\alpha, C_n)_{\alpha < \kappa, n < \omega}$ where

$$C_n = \{ (p_1, \ldots, p_n) : p_1, \ldots, p_n \in P, \text{ there exists } q \in P \text{ such that } q \le p_1, \ldots, p_n \}.$$

Then the embeddability of A into B is equivalent to the exisistence of $\mathcal{D}$-generic filter over P. It follows that A is not embeddable into B. But we have: $\Vdash_P$"A is embeddable into B". Hence $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ does not hold. ▮ (Theorem 3.3)

From Theorem 3.3 and Lemma 3.1 it follows that $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ implies $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ for a regular condition $\mathcal{E}$ on p.o.-sets and every infinite cardinal κ.

Problem 3.4 *Is* $\mathrm{PI}^{\mathrm{ccc}}_{\aleph_1}$ *equivalent to* $\mathrm{PE}^{\mathrm{ccc}}_{\aleph_1}$*?*

Problem 3.5 *Is* $\mathrm{PI}^{\mathrm{ccc}}_{2^{\aleph_0}}$ *inconsistent?.*

Some structures constructed in [7] and [4] exemplify that $\mathrm{PI}^{\omega_1\text{ preserving}}_{\aleph_1}$ is inconsistent. These examples also show that, just as for MM, the axiom $\mathrm{PI}^{\text{stat.preserving}}_{\aleph_1}$ is maximally (possibly) consistent in the corresponding family of axioms.

Problem 3.6 *Is* $\mathrm{PI}^{\text{stat.preserving}}_{\aleph_1}$ *equivalent to* MM*?*

A subset Y of a p.o.-set P is said to be centered if for every $u \in [Y]^{<\omega}$ there exists $x \in P$ such that $x \leq y$ for all $y \in u$. A p.o.-set P has precaliber κ if for $X \subseteq P$ of size κ there exists $Y \subseteq X$ of size κ such that Y is centered.

Theorem 3.7 *a*) $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$ *is equivalent to* $\mathrm{MA}_{\aleph_1}$.
b) *Assume that every* ccc *p.o.-set is productively* ccc *(This is true e.g. under some weak version of* $\mathrm{MA}_{\aleph_1}$*.). Then for any cardinal* κ *of uncountable cofinality,* $\mathrm{PPE}^{\mathrm{ccc}}_{\kappa}$ *is equivalent to the assertion:*

($\mathcal{H}_\kappa$) *Every* ccc *p.o.-set has precaliber* κ*.*

Proof *a*) In [9] it is proved that $\mathrm{MA}_{\aleph_1}$ is equivalent to the assertion:

($\mathcal{H}$) Every ccc p.o.-set has precaliber $\aleph_1$.

So it is enough to show the equivalence of $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$ to this assertion. First let us assume $\mathcal{H}$. Let A and B be structures in some relational language L. Suppose that there exists a ccc p.o.-set P and P-names $\dot{C}$, $\dot{h}$ such that $\Vdash_P$"$\dot{C}$ is a substructure of A of size $\aleph_1$ and $\dot{h}$ is an embedding of $\dot{C}$ into B". Let $\dot{g}$ be a P-name of injective enumeration of $\dot{C}$ of length ω_1, i.e. $\Vdash_P$"$\dot{g} : \omega_1 \to \dot{C}$ is 1-1 onto". For each $\alpha \in \omega_1$ let $p_\alpha \in P$, $a_\alpha \in A$ and $b_\alpha \in B$ be such that $p_\alpha \Vdash_P \dot{g}(\alpha) = a_\alpha \wedge \dot{h} \circ \dot{g}(\alpha) = b_\alpha$. By the assumption there exists an uncountable $X \subseteq \omega_1$ such that $\{\, p_\alpha \,:\, \alpha \in X \,\}$ is centered. Let C be the substructure of A with the underlining set $\{\, a_\alpha \,:\, \alpha \in X \,\}$ and let $h : C \to B$ be defined by $h(a_\alpha) = b_\alpha$ for $\alpha \in X$. Then h is an embedding of C into B.

Now assume that there exists a ccc p.o.-set Q which does not have precaliber $\aleph_1$. So there exists an uncountable subset X of P which dones not have any uncountable centered subset. Let P be a ccc p.o.-set forcing $\mathrm{MA}_{\aleph_1}$. If $\Vdash_P$"Q does not have the ccc" then, as in Example 1.1 *a*), we can construct a counterexample to $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$. So let us assume $\Vdash_P$"Q satisfies the ccc". Then we have $\Vdash_P$"every uncountable subset of Q has an uncountable centered subset".

Let $A = (\aleph_1, \aleph_1, {\aleph_1}^n)_{n\in\omega}$ and $B = (Q, X, R_n)_{n\in\omega}$ where

$$R_n = \{\, (q_1, \ldots, q_n) \,:\, q_1, \ldots, q_n \in Q \text{ and there exists } r \in Q \text{ such that } r \leq q_1, \ldots, q_n \,\}.$$

Then any uncountable substructure of A is not embeddable into B but $\Vdash_P$"A is embeddable into B". This shows that $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1}$ does not hold.
b) can be proved similarly. ▮ (Theorem 3.7)

4 Forcing axioms for homogeneous covering of structures

Let $\mathcal{E}$ be a condition on p.o.-sets. A partition $[S]^{<\omega} = K_0 \cup K_1$ for a set S is said to be $\mathcal{E}$-destructible if there exist a p.o.-set P satisfying the condition $\mathcal{E}$ and a P-name $\dot{X}$ of a 0-homogeneous subset of S with respect to the partition (i.e. $\Vdash_P [\dot{X}]^{<\omega} \subseteq K_0$) such that for all $s \in S$ there exists $p \in P$ such that $p \Vdash_P s \in \dot{X}$. In S. Todorčević and B. Veličković [9] the following characterization of MA_κ is given:

Theorem 4.1 ([9], see also [8]) MA_κ *is equivalent to the following assertion.*

($\mathcal{L}_\kappa$) *Let S be a set of size $\leq \kappa$. Suppose that a partition $[S]^{<\omega} = K_0 \cup K_1$ is* ccc*-destructible. Then there exists a σ-covering $S = \bigcup_{i\in\omega} S_i$ of S such that each S_i is 0-homogeneous with respect to this partition (i.e. $[S_i]^{<\omega} \subseteq K_0$).*

∎

In this section we give a similar theorem in terms of potential embedding. Let A, B be structures in a language L which contains a unary relation symbol S. We say that S in B has an A-homogeneous σ-covering if there exist embeddings f_i of A into B for $i \in \omega$ such that $\mathsf{S}^B = \bigcup_{i\in\omega} f_i[\mathsf{S}^A]$.

Following the above terminology of S. Todorčević and B. Veličković, let us say that S in B is $\mathcal{E}$-destructible by A if there exist a p.o.-set P with the property $\mathcal{E}$ and a P-name $\dot{f}$ of embedding of A into B such that for every $b \in \mathsf{S}^B$ there is a $p \in P$ such that $p \Vdash_P$ "$b \in \dot{f}[\mathsf{S}^A]$".

Let $\mathrm{HC}^{\mathcal{E}}_\kappa$ denote the following assertion:

> Let A and B be structures in a language L which contains a unary relation symbol S such that $|\, A \,|$, $|\, \mathsf{S}^B \,| \leq \kappa$. If S in B is $\mathcal{E}$-destructible by A then S in B has an A-homogeneous σ-covering.

The following lemma shows that $\mathrm{HC}^{\mathrm{ccc}}_\kappa$ is a generalization of $\mathcal{L}_\kappa$.

Lemma 4.2 *Let S be any infinite set and let $[S]^{<\omega} = K_0 \cup K_1$ be a partition. Then there exist structures A, B in a language L which contains a unary relation symbol S such that*
a) $|\, A \,| = |\, B \,| = |\, S \,|$,
b) $[S]^{<\omega} = K_0 \cup K_1$ is $\mathcal{E}$-destructible if and only if S in B is $\mathcal{E}$-destructible and
c) S has a σ-covering $\{ S_i : i \in \omega \}$ of 0-homogeneous sets with respect to the partition $[S]^{<\omega} = K_0 \cup K_1$ such that $|\, S_i \,| = |\, S \,|$ for all $i \in \omega$ if and only if S in B has an A-homogeneous σ-covering.

Proof Let

$$A = (\, [S]^{<\omega}, [S]^1, [S]^{<\omega}, g_n \,)_{n\in\omega},$$

$$B = (\, [S]^{<\omega}, [S]^1, K_0, g_n \,)_{n\in\omega}.$$

where $[S]^1$ is supposed to be the interpretation of S in both the structures and g_n is an n-place function defined by

$$g_n(a_0, \ldots, a_{n-1}) = \begin{cases} \{\, s_0, \ldots, s_{n-1} \,\}; & \text{if } a_i \text{ is the singleton } \{s_i\} \text{ for } i < n \\ \emptyset; & \text{otherwise.} \end{cases}$$

∎ (Lemma 4.2)

Proposition 4.3 *Let $\mathcal{E}$ be a regular condition on p.o.-sets and κ an infinite cardinal. Suppose that $\mathrm{MA}_\kappa^{\mathcal{E}}$ implies that, for any p.o.-set P satisfying the condition $\mathcal{E}$, the finite support product of ω copies of P satisfies the condition $\mathcal{E}$. Then $\mathrm{HC}_\kappa^{\mathcal{E}}$ is equivalent to $\mathrm{MA}_\kappa^{\mathcal{E}}$*

Proof First assume $\mathrm{MA}_\kappa^{\mathcal{E}}$. Let A, B be structures in a language L which contains a unary relation symbol S such that $|A|$, $|\mathsf{S}^B| \leq \kappa$ and S in B is $\mathcal{E}$-destructible by A. Let P and $\dot{f}$ be as in the definition of $\mathcal{E}$-destructibility. Let

$$Q = \{ f \in {}^\omega P \,:\, \{ i \in \omega \,:\, f(i) \neq 1_P \} \text{ is finite} \}$$

with the ordering

$$f \leq g \text{ if } f(i) \leq g(i) \text{ for all } i \in \omega.$$

By the assumption, Q satisfies the condition $\mathcal{E}$.

For each $b \in \mathsf{S}^B$ let

$$D_b = \{ f \in Q \,:\, f(i) \Vdash_P b \in \dot{f}[\mathsf{S}^A] \text{ for some } i \in \omega \}.$$

Then D_b is dense in Q. For $a \in A$ and $i \in \omega$ let

$$E_{a,i} = \{ f \in Q \,:\, f(i) \Vdash_P \dot{f}(a) = b \text{ for some } b \in B \}.$$

Then $E_{a,i}$ is dense in Q. Let

$$\mathcal{D} = \{ D_b \,:\, b \in \mathsf{S}^B \} \cup \{ E_{a,i} \,:\, a \in A,\, i \in \omega \}.$$

By $\mathrm{MA}_\kappa^{\mathcal{E}}$ there is a $\mathcal{D}$-generic filter G over Q. For $i \in \omega$ let $f_i : A \to B$ be defined by $f_i(a) = b$ for $a \in A$ where $b \in B$ is such that $f(i) \Vdash_P \dot{f}(a) = b$ for some $f \in Q$. Then each f_i is well-defined embedding of A into B and $\bigcup_{i\in\omega} f_i[\mathsf{S}^A] = \mathsf{S}^B$.

Assume now that $\mathrm{MA}_\kappa^{\mathcal{E}}$ does not hold. Then as in the proof of Theorem 3.3 there are p.o-set P of size κ satisfying the condition $\mathcal{E}$ and a family $\mathcal{D} = \{ D_\alpha \,:\, \alpha < \kappa \}$ of dense subsets of P such that there exists no $\mathcal{D}$-generic filter over P and for every $\alpha < \kappa$ and $p \in D_\alpha$ there exists $q \geq p$ such that

$$q \in D_\alpha \setminus \bigcup_{\beta \in \kappa \setminus \{\alpha\}} D_\beta.$$

Now let X be any set of size $\leq \kappa$ disjoint from P. Let $A = (\kappa \,\dot{\cup}\, X, X, \{\alpha\}, \kappa^n)_{\alpha<\kappa, n<\omega}$ and $B = (P \,\dot{\cup}\, X, X, D_\alpha, C_n)_{\alpha<\kappa, n<\omega}$ where X is supposed to be the interpretation of S in both the structures and C_n, $n \in \omega$ are defined as in the proof of Theorem 3.3. Then S in B is $\mathcal{E}$-destructible by A but there is no A-homogeneous σ-partition of S in B. ▮ (Proposition 4.3)

5 Some other axioms

In this section we shall consider the following weaknings of the axioms defined in the introduction: For $n \geq 1$, $n \in \omega$, a language L is said to be n-ary if L contains only $\leq (n-1)$-ary function symbols and $\leq n$-ary relation symbols. For a condition $\mathcal{E}$ on p.o.-sets and a cardinal κ let $\mathrm{PE}^{\mathcal{E}}_{\kappa,n}$ ($\mathrm{PI}^{\mathcal{E}}_{\kappa,n}$) be the following axiom:

> For every structures A, B in some n-ary language L such that $|A| = \kappa$, if A is $\mathcal{E}$-potentially embeddable into B (if A is $\mathcal{E}$-potentially isomorphic to B and $|A| = |B|$) then A is embeddable into (isomorphic to) B.

Similarly let $\mathrm{PPE}^{\mathcal{E}}_{\kappa,n}$ be the axiom saying:

> For every structures A, B in some n-ary relational language L, if $\Vdash_P$ "there exists a substructure C of A of size κ which is embeddable into B" for some p.o.-set P with the property $\mathcal{E}$ then there exists a substructure C of A of size κ which is embeddable into B.

$\mathrm{MA}^{\mathcal{E}}_{\kappa,n}$ is the axiom:

> For every p.o.-set P satisfying the condition $\mathcal{E}$, if D_α is a dense subset of P for $\alpha < \kappa$, then there exists an n-linked subset G of P such that $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \kappa$.

Lemma 5.1 *a)* $\mathrm{PE}^{\mathrm{ccc}}_{\kappa,1}$, $\mathrm{PI}^{\mathrm{ccc}}_{\kappa,1}$ *and* $\mathrm{PPE}^{\mathrm{ccc}}_{\kappa,1}$ *hold in* ZFC *for any cardinal* κ.
b) $\mathrm{PE}^{\omega_1 \text{ preserving}}_{\aleph_1,1}$, $\mathrm{PI}^{\omega_1 \text{ preserving}}_{\aleph_1,1}$ *and* $\mathrm{PPE}^{\omega_1 \text{ preserving}}_{\aleph_1,1}$ *hold in* ZFC.

Proof *a)* Immedeate from the fact that a generic extension with a ccc p.o.-set preserves every cardinals. *b)* Easy. ▮ (Lemma 5.1)

For $n \geq 2$, every thing proved in the previous section can be rewritten to the corresponding assertions for n-indexed axioms:

Lemma 5.2 *For a condition $\mathcal{E}$ on p.o.-sets ,an infinite cardinal κ, and $n \geq 2$, $\mathrm{MA}^{\mathcal{E}}_{\kappa,n}$ implies $\mathrm{PE}^{\mathcal{E}}_{\kappa,n}$, $\mathrm{PI}^{\mathcal{E}}_{\kappa,n}$ and $\mathrm{PPE}^{\mathcal{E}}_{\kappa,n}$.*

Proof Similar to Lemma 3.1. ▮ (Lemma 5.2)

Lemma 5.3 *For a regular condition $\mathcal{E}$ on p.o.-sets, an infinite cardinal κ and $n \geq 2$, $\mathrm{MA}^{\mathcal{E}}_{\kappa,n}$ is equivalent to $\mathrm{PE}^{\mathcal{E}}_{\kappa,n}$.*

Proof Similar to Theorem 3.3. ▮ (Lemma 5.3)

Theorem 5.4 *a) For $n \geq 2$, $\mathrm{PPE}^{\mathrm{ccc}}_{\aleph_1,n}$ is equivalent to the assertion:*

> ($\mathcal{K}_n$) *For every* ccc *p.o.-set P, every uncountable subset X of P has an uncountable subset Y which is n-linked.*

b) Assume that every ccc *p.o.-set is productively ccc. Then for any cardinal κ of uncountable cofinality, $\mathrm{PPE}^{\mathrm{ccc}}_{\kappa,n}$ is equivalent to the assertion:*

($\mathcal{K}_{\kappa,n}$) *For every* ccc *p.o.-set P, every subset X of P of size κ has a subset Y of size κ such that Y is n-linked.*

Proof Similar to Theorem 3.7. ▮ (Theorem 5.4)

Lemma 5.5 *Let $\mathcal{E}$ be any condition on p.o.-sets and κ a cardinal. Then a)* $\mathrm{PE}^{\mathcal{E}}_{\kappa}$ *is equivalent to* $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ *and b)* $\mathrm{PI}^{\mathcal{E}}_{\kappa}$ *is equivalent to* $\mathrm{PI}^{\mathcal{E}}_{\kappa,2}$.

Proof *a*): We shall prove that $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ imples $\mathrm{PE}^{\mathcal{E}}_{\kappa}$. Assume that $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ holds. For a structure $A = (A, g_i, R_j)_{i\in I, j\in J}$ where g_i is k_i-ary function on A for $i \in I$ and R_j is k_j-ary relation on A for $j \in J$, let $\tilde{A}$ be the structure defined by

$$\tilde{A} = (A^{<\omega}, \emptyset, \tilde{g}_i, p_l, A^1, \tilde{R}_j)_{i\in I, j\in J, l\in\omega}$$

where

$$\tilde{g}_i((a_0,\ldots,a_{k-1})) = \begin{cases} (g_i(a_0,\ldots,a_{k-1})), & \text{if } k = k_i,\\ \emptyset, & \text{otherwise,}\end{cases}$$

for $i \in I$,

$$p_l((a_0,\ldots,a_{k-1})) = \begin{cases} (a_l), & \text{if } l < k,\\ \emptyset, & \text{otherwise,}\end{cases}$$

for $l \in \omega$ and

$$\tilde{R}_j = \{\, ((a_0,\ldots,a_{k_j-1})) \,:\, (a_0,\ldots,a_{k_j-1}) \in A^{<\omega} \text{ and } R_j(a_0,\ldots,a_{k_j-1}) \,\}$$

for $j \in J$. $\tilde{A}$ is a structure in a binary language. For any structures A and B we have that A is embeddable into B if and only if $\tilde{A}$ is embeddable into $\tilde{B}$. Now if $\Vdash_P$"A is embeddable into B" then $\Vdash_P$"$\tilde{A}$ is embeddable into $\tilde{B}$". By $\mathrm{PE}^{\mathcal{E}}_{\kappa,2}$ it follows that $\tilde{A}$ is embeddable into $\tilde{B}$. Hence A is embeddable into B.
b): is proved similarly. ▮ (Lemma 5.5)

For the case that $\mathcal{E}$ permits the Löwenheim-Skolem argument (e.g. σ-centered, ccc etc.) the following result was already proved in [2].

Theorem 5.6 *For a regular condition $\mathcal{E}$ on p.o.-sets and an infinite cardinal κ,* $\mathrm{MA}^{\mathcal{E}}_{\kappa}$ *is equivalent to* $\mathrm{MA}^{\mathcal{E}}_{\kappa,2}$.

Proof By Theorem 3.3, Lemma 5.3 and Lemma 5.5. ▮ (Theorem 5.6)

Let ϕ be the empty condition on p.o.-sets.

Problem 5.7 *Is* $\mathrm{PPE}^{\phi}_{\aleph_1,2}$ *consistent?*

This problem is also connected with the problem of consistency of the axiom RFA considered in [9] and [8], since $\mathrm{PPE}^{\phi}_{\aleph_1,n}$ implies RFA^n where RFA is equal to RFA^2. In [9] it is proved that RFA^n is inconsistent for every $n \geq 3$. From this we obtain:

Proposition 5.8 $\mathrm{PPE}^{\phi}_{\aleph_1,n}$ *is inconsistent for all $n \geq 3$. In particular* $\mathrm{PPE}^{\phi}_{\aleph_1}$ *is inconsistent.*

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