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# **Principles of Biological Organization: Local-Global Negotiation based on “Material Cause”**

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Organization of living things is characterized by dynamical hierarchical structures inheriting discrepancy among levels. It can be expressed as a system consisting of two layers; the microscopic perspective (Extent) defined by a collection of elements and the macroscopic perspective (Intent) defined by the property as a whole, and the interplay between them. First we show that if the microscopic and macroscopic perspectives are consistent with each other (an ideal case), then the operation between the two layers can be expressed as a sheaf between a lattice and a quotient lattice, where a sheaf is a mathematical operation representing the gluing process. Second, we introduce an observer who cannot look out over the whole world, and this reveals discrepancy between the two layers. Third, we introduce a new mathematical construction, called skeleton, that is derived by the sheaf operation. The skeleton reduces the discrepancy between the micro- and macroscopic perspectives, and that reveals the perpetual transition between the perspectives. This process yields a basic framework of biological organizations. Finally, we argue that the skeleton mediating between the two levels is a particular expression for the material cause.

## **1. Introduction**

In this paper, we propose an abstract model for the organization of living systems to address the role of material cause. While the concept of self-organization is well accepted in a broad sense including non-living matters [1, 2], we believe that organizations of living matters should focus on two essential questions: (i) what is the dynamical hierarchy that exists in living things, and (ii) is it possible to abstract living things by ignoring the material nature? The two questions are intimately connected with each other.

The first question is always taken up whenever one thinks about a system which has the adaptability (or learning) to the open environments, the so-called heterarchy [3-5], such as cognitive, learning and evolutionary systems. These systems consist of at least two levels; one level is employed to estimate the environments and the other is employed to make the decision for the corresponding response. For instance, in brain activities, the level corresponding to higher level

activities, that directly related to consciousness, occurs in the frontal lobe, and the lower level activities occur in the primary visual and/or auditory areas. The most intriguing thing is the interplay between the top-down process and the bottom-up process [6], which can lead to a dynamical hierarchy which goes beyond the level of logical self-reference, since logical self-reference entailing a contradiction is based on the definite indication of the two layers (whole and part) [7,8]. In brain activity each layer cannot be closed and independently separated from the other layers. That is the reason why it does not entail a contradiction and is robust although it mimics logical self-reference.

Moreover dynamical hierarchy contributes to evolution even in a genomic population such that each layer is neither closed nor independently separated from the others. Adaptive mutation in bacteria shows that the environment estimation cannot be separated from the genesis of variants [9]. The complex structures of DNA-sequence and chromosomes are sensitive and so perpetually interact with their microscopic environments. It results in the anticipation by the genome society and the adaptive mutation [10]. Moreover, cooperative communications are also reported in a bacterial colony. Under the selective pressure on the colony, morphotype transition resulting from genetic changes appears. These transitions are beneficial to the colony but not directly to the individual cells [11,12]. This implies that there are two levels, the individual and the colony levels, and the interplay between them.

How can we describe the interplay between the two levels such as the individual and the population, or the local and the global? If one describes the interplay without contradiction, then the difference between the two levels with respect to the logical property is lost. On the contrary, if one sticks to the logical self-reference, it entails a contradiction although a real living system should be free from contradictions. We believe that both approaches are extreme due to the fact that both are based on consistent well defined mathematical expressions [5,7,8]. To resolve the problem, we introduce inconsistency in the mathematical expression, and show that (i) inconsistency invalidates the basis of self-reference and can avoid a contradiction, and (ii) inconsistent expression for an element makes it an agent or an internal observer [5,7,8,13,14]. For example, if a sensor of a cell is expressed as an inconsistent map, one cannot indicate the domain and the co-domain of the map so that it is impossible to determine whether a cell estimates environments or the cell is estimated. It then leads to the openness and adaptability of the cell. In order to overcome this difficulty, we introduce an “inconsistent mediator” between the two levels as a third component in order to describe the dynamical hierarchy or the interplay between the top-down and bottom-up processes.

The second question, “Is it possible to abstract living things by ignoring the material nature?” is usually forgotten in science. In the field of artificial life (AL), forgetting the material nature is rather purposeful [15] so that any mathematical model and computer program of AL lose their material nature. However, there are many endeavors on natural computing or biologically inspired computing (e.g., DNA computing, protein computing) [16-18]. The advantage of such

computing is the use of material nature as computing resources to make parallel process as massive as possible. It recover the material nature in theoretical biology. The notion of AL can be compared to that of artificial intelligence (AI) which goes into the impasse of encodingism. After the collapse of the concept of AI, robotics has been developed and it has been argued that consciousness cannot be separated from the body revealing community [19]. The advantage of thinking embodied mind [20] also treat the nature of the body as material. The question “What is the material nature in natural computing and robotics?” has not been raised explicitly, and still remains to be an open question. We cannot replace living things just by formal expressions without spelling out their material nature.

Aristotle classified causality into the efficient cause (e.g., carpenter for a house), the formal cause (e.g., the blue-print), the material cause (e.g., wood, stone, nails) and the final cause (e.g., to living in the house). Here we would like to focus on the material cause. Rosen argued that a dynamical approach contains the efficient cause as a map, the formal cause as a parameter related to the structural stability, the material cause as an initial and boundary condition. However the final cause is missing in science [21]. In contrast, we believe that the material cause is missing in science. The material cause is defined such that from which a *thing* comes into existence as from its parts, constitutes, substratum or materials. If a thing is regarded as a subsequent behavior in the framework of dynamical systems, then Rosen is right. However, if a thing is regarded as a real thing recognized by an observer in a real world, then the statement “from which a *thing* comes into existence” is not restricted to the framework of dynamical systems but rather interpreted as the interface between the framework and the outside real world. Here we use the material cause in the latter sense, and then re-define the formal and efficient cause in the framework of dynamical systems.

A dynamical system is a geometrical structure constituted by bundles of trajectories. Although one trajectory is an efficient cause and one point in a trajectory is a material cause in the sense of Rosen, a trajectory containing points is regarded as one unit and as an efficient cause. A whole geometry is regarded as a formal cause. The material cause is a mediator between the formal description and outside real world, that is employed to adjust the geometrical structure to a particular real phenomenon. Structural stability of a parameter controlling may be a candidate of such a mediator. It means that the mediator outside a dynamical system is embedded and re-implemented in the framework of dynamical system. It is here called embedding a mediator in a consistent manner. Such an embedding, however, falls into infinite regression. Once the mediator is embedded in the dynamical system, it loses the status of mediator, and that entails impossibility to implement the material cause. The material cause is at the edge of definite description consisting of formal and efficient causes that constitutes a dynamic triadic system. Therefore, the mediator as a material cause is described in an inconsistent manner.

Finally we can say that the essential property of biological organization is the dynamical hierarchy consisting of two levels that are different from each other with respect to the logical

property, and the inconsistent mediator as a third component corresponds to the material cause. In Section 2, two levels will be examined in terms of the parts and the whole, and of the Extent (collection of fragments) and the Intent (property as a whole). This distinction is consistent with the traditional mathematical expression for hierarchical systems, such as the lower level and the higher one generated by taking the limit (co-limit) of the lower one. This is typically expressed by a mathematical structure called “sheaf” [22], that we will introduce in Section 2. In Section 3 we will introduce the lattice and the quotient lattice [23], and we reconstruct the sheaf on a lattice in Section 4. Those sections yield the basic framework of our model for biological organization. In Section 5, we will introduce the interplay between the two levels and introduce an inconsistent mediator called “skeleton”. That provides a fundamental structure for biological organization. We will finally argue how skeleton corresponding to the material cause can contribute to the robustness and evolvability of a living system.

## 2. Parts and Whole, and Gluing

Several decades have passed since it was argued that a system as a whole is not just a collection of parts in system theory and cybernetics [24,25]. Although researchers of complex systems and artificial life introduce a system consisting of non-linear oscillators to address the collective phenomena and/or emergent behaviors, a system consisting of non-linear oscillators is defined just as a collection of oscillators, that is not a system as mentioned in section 1. It implies that the seed of emergent behavior is just hidden in the non-linear oscillators. On one hand, one accepts distinguishability in terms of trajectory. On the other hand, one has to accept indistinguishability at the global level where the collective phenomenon can be found as pointed out by Diebner [26]. In understanding emergence, two standards superimposed in an inconsistent manner are needed in order to reveal the notion of real emergence. Although such thinking may sound erroneous, we believe that it is inevitable. A question remains how we can describe the dynamical relationship between the local (with distinguishability) and the global (with indistinguishability). In other words, how can we define the parts and the whole?

The definition is reexamined such that a system as a whole is not just a collection of parts. A system is a formal concept that is expressed as the pair of Intent and Extent. While the Extent is a collection of objects, the Intent is expressed as a whole (e.g., [27]). In addition, the Extent is normally assumed to be equivalent to the Intent. For example, this can be illustrated by the concept of even numbers in the set theory. The Extent is the collection of 0, 2, 4, ... and the Intent is  $2n$  where  $n$  is a natural number. If an even number is replaced by “a system”, the definition of a system, however, implies that the Intent is not equivalent to the Extent. On one hand, the Intent has the indefiniteness inherited from the definition of natural numbers by  $n$  of  $2n$ . On the other hand, the

Extent has indefiniteness in the form of "...". It implies that the indefiniteness in the Intent (i.e., indefiniteness in  $n$ ) is different from those in the Extent (i.e., indefiniteness in "..."). We then have to accept discrepancy between the Intent and the Extent, which is true in the definition of a system.

In this sense, the contrast between the Extent and the Intent can be interpreted as the contrast between the individuals and the context. If the Intent (context) is consistent with the Extent (individuals), there is no dynamical negotiation so that the Intent is nothing but just representing the standard notion of boundary conditions. In the inconsistent case, Intent can then be regarded as a boundary condition where an internal observer sits and witnesses the outside.

First, we start from the concept of a system that is expressed as the pair of Intent and Extent, and we simultaneously accept the indefiniteness of a concept. It leads not only to the discrepancy between the Intent and the Extent, but also to the dynamical negotiation between them since the pair constitute the unity as a concept of the system. In terms of physics, the microscopic and macroscopic perspectives can correspond to the Extent and the Intent, respectively. One can imagine in a particular chemical substrate, the collection of molecules and molecular dynamics constitutes the Extent of the chemical substrate. If one idealizes a collection of huge number of molecules as the concentration and describes the dynamics with respect to the concentration, then the Intent of chemical substrate is represented by the reaction equations. Although the Intent and the Extent dynamically keep the concept of chemical substrate, researchers normally address only one of them. The dynamical process between the micro- and macroscopic perspectives is sometimes expressed in the form of Maxwell's demon [28] or vertical scheme in the context of biologically inspired computing [29]. It is, however, too hard to elucidate the process. Dynamical discrepancy and negotiation between the micro- (Extent) and macroscopic (Intent) views play an essential role in the science of consciousness since researchers have to explain how consciousness is generated from a collection of neurons [30]. The consciousness is regarded as the Intent of brain activities.

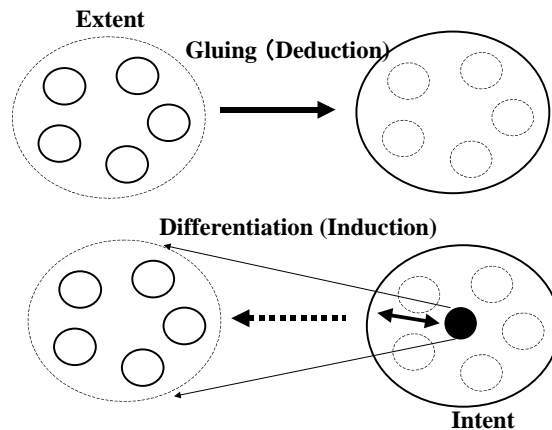


Fig. 1. Schematic diagram of gluing and differentiation. If a fragment is “glued” by a sheaf, fragments are regarded as parts in a whole system, and that implies Intent equivalent to Extent.

In order to elucidate the dynamical discrepancy and the negotiation between the Intent and the Extent, we use a particular mathematical structure called “sheaf” [22]. It is derived from the index set in the set theory [31]. Given an index set, a collection of elements is regarded as the Extent, and an index by which the elements are collected is regarded as the Intent. Once the index is determined, the Intent is consistent with the Extent, and the fragments are regarded as parts of the whole structure (Fig. 1). When this idea is expanded into the topological space, the sheaf can then be constructed. Now we introduce the definition of “sheaf”, and then gradually construct the dynamical negotiation between the Intent and the Extent. First, we define presheaf in the sense of category theory that is sketched in the Appendix.

### Definition 2-1 (Presheaf)

Let  $X$  be a topological space, and  $U, V \subseteq X$  be open sets in  $X$ .  $\mathfrak{I}$  is a category of  $X$ , that an object of  $\mathfrak{I}$  is an open set  $U$ , and an arrow  $V \rightarrow U$  is an inclusion such that  $V \subseteq U$ . Let **Sets** be a category of sets, whose objects are sets and arrows are maps. A presheaf is defined by a functor,  $F: \mathfrak{I}^{\text{op}} \rightarrow \mathbf{Sets}$ .

If an arrow  $V \rightarrow U$  in  $\mathfrak{I}$  is transformed to an arrow  $\rho_{U,V}: F(U) \rightarrow F(V)$  defined by the restriction map,  $F: \mathfrak{I}^{\text{op}} \rightarrow \mathbf{Sets}$  is a functor implies that it is a presheaf. Since  $W \subseteq V \subseteq U$  implies  $F(U) \rightarrow F(V) \rightarrow F(W)$ , in the typical example with  $F(U) = \{\text{continuous maps on } U\}$ , for  $\forall s \in F(U)$ , then we have  $\rho_{V,W} \rho_{U,V}(s) = \rho_{U,W}(s|_V) = s|_W = \rho_{U,W}(s)$ . It is also clear to see that given  $U \subseteq U$ , then  $\rho_{U,U} = \text{id}_{F(U)}$ . Hereafter, we denote  $F(U) \rightarrow F(V)$  as  $\rho_{U,V}$  for the presheaf.

A sheaf is a presheaf satisfying specific conditions, mono-presheaf and gluing conditions [22]. Two conditions are defined by the following.

### Definition 2-2 (Sheaf)

A presheaf  $F: \mathfrak{I}^{\text{op}} \rightarrow \mathbf{Sets}$  is a sheaf if and only if it satisfies (i) the mono-presheaf condition, and (ii) the gluing condition, where

- (i) Mono-presheaf condition:

Let  $A$  be an index set. Suppose that  $U$  is an open set of  $X$  and  $U = \bigcup_{\lambda \in A} U_{\lambda}$  is an open covering of  $U$ . For any  $\lambda \in A$ , if for any  $s, t \in F(U)$ , and  $\rho_{U, U_{\lambda}}(s) = \rho_{U, U_{\lambda}}(t)$ , then  $s = t$ .



(ii) Gluing condition:

Suppose that  $U$  is an open set of  $X$  and  $U = \cup_{\lambda \in A} U_\lambda$  is an open covering of  $U$ . Given a family  $(s_\lambda)_{\lambda \in A}$ , with  $\forall \lambda \in A, s_\lambda \in F(U_\lambda)$ , if  $\forall \lambda, \mu \in A$ , and  $\rho_{U_\lambda, U_\lambda \cap U_\mu}(s_\lambda) = \rho_{U_\mu, U_\lambda \cap U_\mu}(t_\mu)$ , then there exists  $s \in F(U)$  such that  $\forall \lambda \in A$ , and  $\rho_{U, U_\lambda}(s) = s_\lambda$ .

Due to the gluing condition, the parts are pasted up into the whole. Once the whole is constructed by the sheaf, any fragment can be expressed as a part of the whole. As a result, there is no discrepancy between the parts and the whole.

Here we sketch our strategy to describe a self-organizing system of living things by using the dynamical sheaf. We have introduced a sheaf as an operator indicating the distinction between the microscopic perspective (Extent) and the macroscopic one (Intent). We will show that it is impossible to obtain an operator (we here call reverse-sheaf) by which the Extent is reconstructed from the Intent without a super-observer. This reveals the inconsistency between the Intent and the Extent. The reverse sheaf has to be constructed “locally” and ad hoc, that cannot be uniquely determined. It implies that we can escape from the self-reference as a meta-physical trap.

We believe that self-reference is regarded as an evil thing as far as operations can be counted, such as “observation (1) of observation (2) of observation ... of observation (infinite)”. Here it is assumed that the observation is uniquely determined. But actually, this assumption is invalid since the observation is open to the outside (i.e., indicating that “observation” falls into, namely, the frame problem). Thus, the observation keeps changing perpetually, such as “the observation of the measurement of the thinking of ...”. As a result, one cannot see self-reference in the process since repetition cannot be seen. That is the reason why we argue that self-reference as an evil thing is just a meta-physical trap.

In our mathematical model, if the reverse-sheaf is uniquely determined, one can obtain a unique operational “measurement” by the composition of the sheaf and reverse-sheaf, and that falls into a meta-physical trap. In our framework, the reverse-sheaf cannot be determined, and indeed, the reverse-sheaf destroys the microscopic structure. The next question arises, namely, how one can repair the structure. We will show that the form derived from the sheaf, where the functionality of the sheaf is lost, can be employed to a special task of repair. The form losing its functionality of sheaf is called a skeleton. A skeleton plays as a mediator between the micro- and macroscopic perspectives.

To construct the dynamical gluing derived from the sheaf, we introduce the lattice since it is easy to construct the complex tools in the lattice. Before we define the sheaf as a map from the lattice to the quotient lattice, we introduce some notions on the quotient lattice.

### 3. Quotient Lattice

First we define a lattice [23]. A partially ordered set  $P$  is a set with a partially order  $\leq \subseteq P \times P$ , where for any  $a, b, c \in P$ , (i)  $a \leq a$ , (ii)  $a \leq b, b \leq a \Leftrightarrow a = b$ , (iii)  $a \leq b, b \leq c \Rightarrow a \leq c$ . Given a partially ordered set  $P$ , for a subset of  $S \subseteq P$ ,  $a$  is the upper bound of  $S$ , if, for any  $x \in S, x \leq a$ . The least upper bound is called join, and is represented by  $\vee S$ . Especially, if  $S$  is a two-elements set  $\{x, y\}$ , join is represented by  $x \vee y$ . Dually, the lower bound, meet is defined as the greatest lower bound denoted by  $x \wedge y$ . A lattice is a partially ordered set such that it is closed under the join and the meet for any two elements.

If a lattice  $L$  is distributive (i.e., for any  $a, b, c \in L, a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$  where  $L$  is finite), it is an open set lattice of a topological space. Since a quotient lattice derived from  $L$  is a collection of co-limits, we can construct the sheaf from the lattice to its quotient lattice. For this purpose, we first define congruence on a lattice.

#### Definition 3-1 (Congruence on a lattice)

Given a lattice  $L$ , let an equivalence relation on  $L$  be  $\theta = \{ \langle x, y \rangle \in L \times L \}$  such that, for  $\forall x, y, z \in L$ ,

- (i)  $\langle x, x \rangle \in \theta$ ,
- (ii)  $\langle x, y \rangle \in \theta \Leftrightarrow \langle y, x \rangle \in \theta$ ,
- (iii)  $\langle x, y \rangle \in \theta, \langle y, z \rangle \in \theta \Rightarrow \langle x, z \rangle \in \theta$ .

$\langle x, y \rangle \in \theta$  is expressed also as  $x \theta y$  or  $x \equiv y \pmod{\theta}$ . An equivalence relation  $\theta$  is a congruence on  $L$  if, for all  $a, b, c, d \in L$ ,

$$(a \equiv b \pmod{\theta} \text{ and } c \equiv d \pmod{\theta}) \Rightarrow (a \vee c \equiv b \vee d \pmod{\theta} \text{ and } a \wedge c \equiv b \wedge d \pmod{\theta}).$$

It is convenient to introduce the following proposition.

#### Proposition 3-2

An equivalence relation  $\theta$  on a lattice  $L$  is a congruence if and only if, for all  $a, b, c \in L$ ,

$$a \equiv b \pmod{\theta} \Rightarrow a \vee c \equiv b \vee c \pmod{\theta} \text{ and } a \wedge c \equiv b \wedge c \pmod{\theta}.$$

**Proof.** It is shown in [23], as lemma 6.6.

By using a congruence, a quotient lattice is well defined.

**Definition 3-3 (Quotient lattice)**

Let  $\theta$  be a congruence on a lattice  $L$ , and a set  $L/\theta$  is defined by

$$L/\theta = \{[a]_\theta \mid a \in L\} \text{ with } [a]_\theta = \{b \in L \mid b \equiv a \pmod{\theta}\}.$$

If meet ( $\wedge$ ) and join ( $\vee$ ) are defined by

$$[a]_\theta \wedge [b]_\theta := [a \wedge b]_\theta, \quad [a]_\theta \vee [b]_\theta := [a \vee b]_\theta$$

we call  $\langle L/\theta, \wedge, \vee \rangle$  the quotient lattice of  $L$  modulo  $\theta$ . If  $\theta$  is an equivalence relation but not a congruence, we call  $\langle L/\theta, \wedge, \vee \rangle$  the quasi-quotient lattice of  $L$  modulo  $\theta$ .

To show that the definition of quotient lattice is well defined, the following proposition is needed.

**Proposition 3-4**

An equivalent relation  $\theta$  on a lattice  $L$  is a congruence if and only if,

$$[a]_\theta = [c]_\theta, [b]_\theta = [d]_\theta \Rightarrow [a \wedge b]_\theta = [c \wedge d]_\theta, [a \vee b]_\theta = [c \vee d]_\theta$$

**Proof.** ( $\Rightarrow$ ) Assume that  $\theta$  is congruence.

$$\begin{aligned} [a]_\theta = [c]_\theta, [b]_\theta = [d]_\theta &\Leftrightarrow a \equiv c \pmod{\theta}, b \equiv d \pmod{\theta}, \\ &\Rightarrow a \wedge b \equiv c \wedge d \pmod{\theta}, a \vee b \equiv c \vee d \pmod{\theta} \quad (\text{congruence}) \\ &\Leftrightarrow [a \wedge b]_\theta = [c \wedge d]_\theta, [a \vee b]_\theta = [c \vee d]_\theta \end{aligned}$$

$$\begin{aligned} (\Leftarrow) a \equiv c \pmod{\theta}, b \equiv d \pmod{\theta} &\Leftrightarrow a \in [c]_\theta, b \in [d]_\theta \\ &\Leftrightarrow [a]_\theta = [c]_\theta, [b]_\theta = [d]_\theta \\ &\Rightarrow [a \wedge b]_\theta = [c \wedge d]_\theta, [a \vee b]_\theta = [c \vee d]_\theta \quad (\text{assumption}) \\ &\Leftrightarrow a \wedge b \equiv c \wedge d \pmod{\theta}, a \vee b \equiv c \vee d \pmod{\theta}. \end{aligned}$$

**Verification of Definition 3-3.** Since  $[a]_{\theta} \wedge [b]_{\theta} = [a \wedge b]_{\theta}$  and  $[a]_{\theta} \vee [b]_{\theta} = [a \vee b]_{\theta}$  we here show that they are independent of the elements chosen to represent the equivalence class. Assume  $c \in [a]_{\theta}$  and  $d \in [b]_{\theta}$ , it is equivalent to have  $[a]_{\theta} = [c]_{\theta}$ ,  $[b]_{\theta} = [d]_{\theta}$ . From propositions 3-4, we have  $[a \wedge b]_{\theta} = [c \wedge d]_{\theta}$ ,  $[a \vee b]_{\theta} = [c \vee d]_{\theta}$ . Therefore, meet and join in a quotient lattice is well defined.

If a quotient lattice is defined by using an equivalence relation derived from a subset of the lattice, it is easy to construct the sheaf from the lattice (topological space) to the quotient lattice (set). It is also easy to extend it to the dynamical sheaf. For this purpose, we introduce the equivalence relation derived from the ideal, where the ideal is a non-empty down-set that is closed under the join.

**Definition 3-5 (Equivalence relation derived from ideal)**

Let  $J$  be an ideal on a lattice  $L$ . The equivalence relation derived from  $J$  is defined by;

$$\theta_J := \{ \langle x, y \rangle \in L \times L \mid \exists z \in J (x \vee z = y \vee z) \}.$$

**Verification:** We show that  $\theta_J$  satisfies (i) reflective, (ii) symmetric, and (iii) transitive laws.

(i) reflective law: Since ideal  $J$  contains the least element 0 on  $L$ , for any  $x \in L$ , we have

$$x = x \vee 0 = x \vee 0.$$

It implies that  $\langle x, x \rangle$  satisfies  $\exists 0 \in J (x \vee 0 = x \vee 0)$ , and so  $x \equiv x \pmod{\theta_J}$ .

(ii) anti-symmetric law:

$$x \equiv y \pmod{\theta_J} \Leftrightarrow \exists z \in J (x \vee z = y \vee z) \Leftrightarrow \exists z \in J (y \vee z = x \vee z) \Leftrightarrow y \equiv x \pmod{\theta_J}.$$

(iii) transitive law:

$$x \equiv y \pmod{\theta_J}, y \equiv z \pmod{\theta_J} \Leftrightarrow \exists u \in J (x \vee u = y \vee u), \exists v \in J (y \vee v = z \vee v).$$

Since  $J$  is an ideal that is closed under join, from  $u \in J$  and  $v \in J$  we obtain  $u \vee v \in J$ . Now,

$$x \vee u = y \vee u, y \vee v = z \vee v \Rightarrow x \vee u \vee v = y \vee u \vee v, y \vee u \vee v = z \vee u \vee v \Rightarrow x \vee u \vee v = z \vee u \vee v.$$

It implies  $x \equiv z \pmod{\theta_J}$ .

An equivalence relation derived from an ideal has the following particular property.

**Proposition 3-6**

Let  $\theta_J$  be an equivalence relation on a lattice  $L$ , defined by definition 3-5, then  $J$  is a block of the corresponding partition of  $L$ .

**Proof.** We prove that  $J$  is a block of the corresponding partition of  $L$ , that is, if  $x \in J$ , then  $[x]_{\theta_J} = J$ . In assuming  $y \in J$ , we obtain  $x \vee y \in J$ , since  $x \in J$ . On the other hand,  $x \vee (x \vee y) = x \vee y = y \vee (x \vee y)$ . It implies that  $x \equiv y \pmod{\theta_J}$ , and so  $y \in [x]_{\theta_J}$ . Finally, we have  $J \subseteq [x]_{\theta_J}$ . Conversely, in assuming  $y \in [x]_{\theta_J}$ ,  $\exists z \in J$  ( $x \vee z = y \vee z$ ). Since  $J$  is an ideal  $x \in J$  and  $z \in J$ , we obtain  $x \vee z \in J$ . Therefore, we have  $y \vee z \in J$ . Moreover, since  $z \in J$ ,  $y \leq y \vee z \in J$  and  $J$  is an ideal with  $y \in J$ . It implies that  $[x]_{\theta_J} \subseteq J$ . Finally, we have  $[x]_{\theta_J} = J$ .

In our model of dynamical gluing, the following theorem plays an important role.

**Theorem 3-7**

Let  $\theta_J$  be an equivalence relation on a lattice  $L$ , defined by definition 3-5, then  $L$  is distributive if and only if  $\theta_J$  is a congruence on  $L$  for every ideal  $J$  of  $L$ .

**Proof.**  $\Rightarrow$  (i) Let  $L$  be distributive. In assuming that  $x \equiv y \pmod{\theta_J}$ ,  $\exists u \in J$  ( $x \vee u = y \vee u$ ). For any  $z \in L$ ,  $(x \vee u) \vee z = (y \vee u) \vee z$ , and then we have

$$(x \vee z) \vee u = (x \vee u) \vee z = (y \vee u) \vee z = (y \vee z) \vee u.$$

It means that  $x \vee z \equiv y \vee z \pmod{\theta_J}$ . In concerning about meet,

$$(x \wedge z) \vee u = (x \vee u) \wedge (z \vee u) = (y \vee u) \wedge (z \vee u) = (y \wedge z) \vee u.$$

It means that  $x \wedge z \equiv y \wedge z \pmod{\theta_J}$ . It verifies that  $\theta_J$  is a congruence from proposition 3-2.

$\Leftarrow$ ) To prove that  $L$  is distributive, first we prove that  $L$  is modular, that is,  $L$  does not contain  $N_5$  as a sublattice, and then prove that  $L$  does not contain  $M_3$  as a sublattice.

(i) First we prove that for  $\langle x, y \rangle \in L' \times L'$ ,  $\langle x, y \rangle \in \theta_J \Rightarrow \langle x, y \rangle \in \theta_S$  where  $S = J \cap L'$ , and  $L'$  is a sublattice of  $L$ . In assuming  $\langle x, y \rangle \in \theta_J$  such that  $\exists z \in J - S$  ( $(x \vee z) = (y \vee z)$ ). Let  $J = \downarrow p$ , with  $p \in L'$ , we

have  $p \vee z = p \in S$ , and  $x \vee z \vee p = y \vee z \vee p$ . It implies  $\langle x, y \rangle \in \theta_S$ .

(ii) We assume that  $L$  contains  $N_5$ . From (i), the partition in  $N_5$  coincides with that of  $L$ , and it is satisfied to estimate the equivalence class in  $N_5$ . In choosing an ideal  $J$ , as shown in top left diagram in Fig. 2, we obtain partitions derived from  $\theta_J$  as shown in the top right in Fig. 2. Since each loop indicates an equivalence class, we have

$$s_1 \equiv c \pmod{\theta_J}.$$

Although  $s_1 \vee a \equiv c \vee a \pmod{\theta_J}$  since  $s_1 \vee a = s_1 = c \vee a$ , we have  $s_1 \wedge a = a$  and  $c \wedge a = s_0$ . Since

$$s_1 \wedge a \equiv c \wedge a \pmod{\theta_J},$$

does not holds,  $\theta_J$  is not a congruence. That is a contradiction, and it implies that  $N_5$  is not contained in  $L$ . Thus, the lattice  $L$  is modular.

(iii) We assume that  $L$  contains  $M_3$ . From (i), what we have to do is just to estimate the equivalence classes in  $M_3$ . In choosing an ideal  $J'$ , as shown in the bottom left diagram of Fig. 2, we obtain the partitions derived from  $\theta_{J'}$  as shown in the bottom right of Fig. 2. Since each loop indicates an equivalence class, we have  $s_1 \equiv b \pmod{\theta_{J'}}$ . However, it does not imply  $s_1 \wedge c \equiv b \wedge c \pmod{\theta_{J'}}$  since  $s_1 \wedge c = c$  and  $b \wedge c = s_0$ . It means  $\theta_{J'}$  is not a congruence, and that is a contradiction. Therefore,  $L$  does not contain  $M_3$  as a sublattice.

All together from (ii) and (iii),  $L$  is distributive.

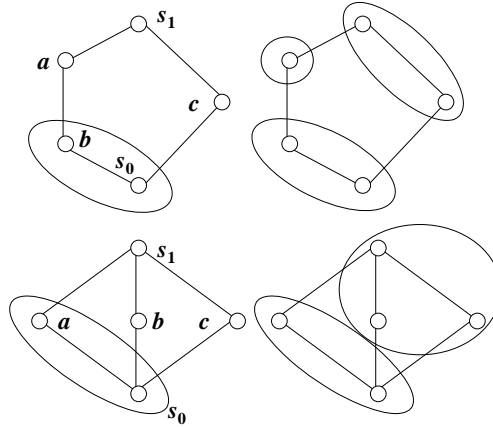


Fig. 2.  $N_5$  and an ideal  $J$  indicated on a diagram placing a loop around the elements (above left). Equivalence class derived from  $J$  (above right).  $M_3$  and an ideal  $J'$  (below left). Equivalence class derived from  $J'$  (below right).

A sheaf and a dynamical gluing on the lattice are defined by using the quotient and/or the

quasi-quotient lattice of  $L$  modulo  $\theta_J$ . In the next section, we will define the sheaf on the lattice, such as an operation from the lattice to the quotient lattice.

#### 4. Gluing on Lattice

To construct the dynamical gluing process in the lattice theory, we first define the sheaf on a lattice. If  $L$  is a finite distributive lattice, it can be regarded as an open set lattice of the topological space, then the map  $L \rightarrow L/\theta$  can be regarded as an operation from the topological space to the set. This leads to the sheaf construction. Before defining the sheaf on the lattice, we need to prove the following proposition.

##### Proposition 4-1

Let  $J$  be an ideal on a finite lattice  $L$ , and  $\theta_J$  is a congruence relation derived from the ideal  $J$ . We denote  $x' = \vee[x]_{\theta_J}$  for any  $x$  in  $L$ . Then, for any  $x, y, z \in L$ ,

- (i)  $x' \in [x]_{\theta_J}$
- (ii)  $x \geq y \Rightarrow x' \geq y'$
- (iii)  $x = y \vee z \Rightarrow x' = y' \vee z'$ .

**Proof.** (i) Since  $[x]_{\theta_J}$  is a finite set, we denote  $[x]_{\theta_J} = \{a_1, a_2, \dots, a_n\}$ . Also, we define  $a_1 \equiv x \pmod{\theta_J}$  and  $a_2 \equiv x \pmod{\theta_J}$  since  $a_1, a_2 \in [x]_{\theta_J}$ . From the definition of congruence, we have  $a_1 \vee a_2 \equiv x \vee x = x$ . Similarly, we have  $a_1 \vee a_2 \vee \dots \vee a_n = \vee[x]_{\theta_J} \equiv x$ , that is,  $x' \in \vee[x]_{\theta_J}$ .

(ii) From  $x' = \vee[x]_{\theta_J}$  and (i), we obtain  $x' \equiv x \pmod{\theta_J}$  and  $y' \equiv y \pmod{\theta_J}$ . By the definition of congruence, we obtain  $x' \vee y' \equiv x \vee y \pmod{\theta_J}$ . Also from the assumption, we have  $x \vee y = x \equiv x' \pmod{\theta_J}$ , that is,  $x' \vee y' \equiv x' \pmod{\theta_J}$ . This means

$$\exists j \in J (x' \vee y' \vee j = x' \vee j).$$

Since  $L$  is a finite lattice, the ideal  $J$  can be expressed as  $\downarrow p$  where  $p \in L$ . For any  $x \in L$ ,  $[x]_{\theta_J}$  is either  $J$  or not. If  $[x]_{\theta_J} = J$  with  $J = \downarrow x'$ , then  $j \leq x'$ . If  $[x]_{\theta_J} \neq J$ , we obtain  $j \leq p \leq x'$ . This leads to  $x' \vee j = x'$ , and then  $x' \vee y' = x' \vee y' \vee j = x' \vee j = x'$ , that is,  $x' \geq y'$ .

(iii) From the assumption, we have  $x' \equiv y' \vee z' \pmod{\theta_J}$ . Together with (ii),  $\exists j \in J (x' \vee j = y' \vee z' \vee j)$ ,  $x' \vee j = x'$  and  $y' \vee j = y'$ . Finally, we obtain  $y' \vee z' = y' \vee z' \vee j = x' \vee j = x'$ .

**Definition 4-2 (Lattice as a topological space and as a set)**

Let  $L$  be a distributive lattice, and we can regard  $C(L)$  as a category on the topological space whose object is an element of  $L$  such as  $x \in L$  and an arrow is partial order  $\geq$  defined on  $L$ . We call  $C(L)$  the lattice-top-space category. Let  $\theta_f$  be a congruence derived from the ideal  $J$  on  $L$ , and  $L/\theta_f$  be a quotient lattice of  $L$  modulo  $\theta_f$ . We can regard  $C(L/\theta_f)$  as a category on sets whose object is a pair of equivalence classes with supremum  $\langle [x]_{\theta_f}, x' \rangle$ , where  $x' = \vee [x]_{\theta_f}$ . The arrow is denoted by  $(-) \wedge y'$ :  $[x]_{\theta_f} \rightarrow [y]_{\theta_f}$  where  $y' = \vee [y]_{\theta_f}$ ,  $x \geq y$  in  $L$ . We call  $C(L/\theta_f)$  the lattice-set category. An object can also be expressed as  $[x]_{\theta_f}$  that is an abbreviation for  $\langle [x]_{\theta_f}, x' \rangle$ .

In addition, let  $\theta'$  be an equivalence class derived from an ideal  $J$ , and  $L/\theta'$  is a quasi-quotient lattice of  $L$  modulo  $\theta'$ , where  $[x]'_{\theta'} = \{x'\}$  and  $x' = \vee [x]_{\theta'}$ . This means that any other elements in the equivalence class are ignored. We call  $C'(L/\theta')$  the quasi-lattice-set category.

**Verification:** It is easy to see that a distributive lattice is a topological space, and  $C(L)$  is a category. In  $L/\theta_f$ , with any arrow  $(-) \wedge y'$ :  $[x]_{\theta_f} \rightarrow [y]_{\theta_f}$  for any  $a \in [x]_{\theta_f}$ , we have  $(-) \wedge y'(a) = a \wedge y'$ . Since  $\theta_f$  is a congruence and  $a \equiv x \pmod{\theta_f}$ , we obtain  $a \wedge y' \equiv x \wedge y' \pmod{\theta_f}$ . From proposition 4-1, we have  $y' \equiv y \pmod{\theta_f}$ , and thus  $x \wedge y' \equiv x \wedge y \pmod{\theta_f}$ . Because  $x \geq y$ ,  $a \wedge y' \equiv y \pmod{\theta_f}$ . This implies  $a \wedge y' \in [y]_{\theta_f}$  and therefore it is well defined. For any object  $[x]_{\theta_f}$ , there is an arrow  $(-) \wedge x'$ :  $[x]_{\theta_f} \rightarrow [x]_{\theta_f}$ . Also, for any  $a \in [x]_{\theta_f}$ ,  $(-) \wedge x'(a) = a \wedge x' = a$  since  $x' \geq a$ . This implies  $(-) \wedge x'$  is an identity. With respect to the composition, for  $x \geq y \geq z$  in  $L$ ,  $(-) \wedge y'$ :  $[x]_{\theta_f} \rightarrow [y]_{\theta_f}$  and  $(-) \wedge z'$ :  $[y]_{\theta_f} \rightarrow [z]_{\theta_f}$  can then be defined. For any  $a \in [x]_{\theta_f}$

$$((-) \wedge z')((-) \wedge y')(a) = (a \wedge y') \wedge z' = a \wedge (y' \wedge z') = a \wedge z' = (-) \wedge z'(a),$$

since  $y \geq z$  implies  $y' \geq z'$  from proposition 4-1. Therefore, the composition is well defined. Moreover, the associative law holds for  $[x]_{\theta_f} \rightarrow [y]_{\theta_f} \rightarrow [z]_{\theta_f} \rightarrow [w]_{\theta_f}$  since

$$\begin{aligned} ((a \wedge y') \wedge z') \wedge w' &= ((a \wedge (y' \wedge z'))) \wedge w' = (a \wedge z') \wedge w' = a \wedge w', \\ (a \wedge y') \wedge (z' \wedge w') &= (a \wedge y') \wedge w' = a \wedge (y' \wedge w') = a \wedge w'. \end{aligned}$$

In the quasi-lattice-set category, the only thing that we have to care about is, namely, for  $x \in [x]_{\theta'}$ ,  $(-) \wedge y(x) = x \wedge y = y \in [y]_{\theta'}$ . This implies that it is a category.

**Theorem 4-3 (Presheaf on a lattice)**

Let  $\theta_f$  be a congruence derived from the ideal  $J$  on a distributive lattice  $L$ . An operation



$F: C(L)^{\text{op}} \rightarrow C(L/\theta_f)$  defined as: for any object  $x$  in  $C(L)^{\text{op}}$ ,  $F(x) = \langle [x]_{\theta_f}, x' \rangle$  where  $x' = \vee [x]_{\theta_f}$ ; for an arrow,  $x \geq y$ ,  $\rho_{x,y} = (-) \wedge y'$ :  $[x]_{\theta} \rightarrow [y]_{\theta}$  is a presheaf.

**Proof.** From the definition of arrows in  $C(L/\theta)$ , it is clear to see that  $F$  preserves identity and composition.

#### Definition 4-4 (Semi-presheaf)

Let  $\theta'$  be an ideal-derived equivalence relation on a lattice  $L$ . The operation  $F: C(L)^{\text{op}} \rightarrow C'(L/\theta')$  is called a semi-presheaf.

Since we do not need the distributive law to show that  $F: C(L)^{\text{op}} \rightarrow C(L/\theta)$  is a presheaf, it is well-defined.

In  $C'(L/\theta')$ , we always choose the greatest element of the equivalence class as its representative, and ignore other elements. As a result,  $[x]_{\theta'} \wedge [y]_{\theta'} = [x \wedge y]_{\theta'}$  and  $[x]_{\theta'} \vee [y]_{\theta'} = [x \vee y]_{\theta'}$  are applied to  $C'(L/\theta')$ .

#### Lemma 4-5 (Sheaf conditions on a finite lattice)

If the topological space is given as a finite set, then the mono-presheaf condition is equivalent to the finite mono-presheaf condition, and the gluing condition is equivalent to the finite gluing condition and the mono-presheaf condition, where

(i) Finite mono-presheaf condition:

Given an open covering  $U = V_1 \cup V_2$ , for any  $s, t \in F(U)$ ,  $\rho_{U, V_1}(s) = \rho_{U, V_1}(t)$  and  $\rho_{U, V_2}(s) = \rho_{U, V_2}(t)$ , then  $s = t$ .

(ii) Finite gluing condition:

Given an open covering  $U = V_1 \cup V_2$ , suppose  $s_1 \in F(V_1)$ ,  $s_2 \in F(V_2)$  such that  $\rho_{U, V_1 \cap V_2}(s_1) = \rho_{U, V_1 \cap V_2}(s_2)$ , then there exists  $s \in F(U)$  such that  $\rho_{U, V_1}(s) = s_1$  and  $\rho_{U, V_2}(s) = s_2$ .

**Proof.** (i) Suppose the mono-presheaf condition is satisfied, it is clear to see that the finite mono-presheaf condition holds. Conversely, suppose the finite mono-presheaf condition holds, we can apply the mathematical induction in terms of the number of covering. Given  $U = V_1 \cup \dots \cup V_n \cup V_{n+1}$  and assume that for any  $s, t \in F(U)$ ,  $\rho_{U, V_i}(s) = \rho_{U, V_i}(t)$  with  $1 \leq i \leq n+1$ , and also  $V =$

$V_n \cup V_{n+1}$ . Since

$$\rho_{V, V_k}(\rho_{U, V}(s)) = \rho_{U, V_k}(s) = \rho_{U, V_k}(t) = \rho_{V, V_k}(\rho_{U, V}(t))$$

with  $k = n, n+1$ , we obtain  $\rho_{U, V}(s) = \rho_{U, V}(t)$  due to the finite mono-presheaf condition. By decreasing the number of open covering such that  $U = V_1 \cup \dots \cup V_{n-1} \cup V$ , we already obtain  $\rho_{U, V_i}(s) = \rho_{U, V_i}(t)$  with  $1 \leq i \leq n-1$ , and  $\rho_{U, V}(s) = \rho_{U, V}(t)$ . Thus from the assumption of induction, we have  $s = t$ .

(ii) Suppose the gluing condition holds, it is clear to see that the finite gluing condition also holds. Conversely, suppose the finite gluing condition holds, we again apply the mathematical induction in terms of the number of covering. Given  $U = V_1 \cup \dots \cup V_n \cup V_{n+1}$  and assume  $s_i \in F(V_i)$  with  $1 \leq i \leq n+1$  such that  $\rho_{V_i, V_i \cap V_j}(s_i) = \rho_{V_j, V_i \cap V_j}(s_j)$ . Let  $V = V_n \cup V_{n+1}$ , we then obtain  $\rho_{V_n, V_n \cap V_{n+1}}(s_n) = \rho_{V_{n+1}, V_n \cap V_{n+1}}(s_{n+1})$ . Thus, there exists  $s \in F(V)$  such that  $\rho_{V, V_n}(s) = s_n$  and  $\rho_{V, V_{n+1}}(s) = s_{n+1}$ . Let  $W = V_i \cap V = (V_i \cap V_n) \cup (V_i \cap V_{n+1})$ , then we have

$$\begin{aligned} \rho_{W, V_i \cap V_n}(\rho_{V_i, W}(s_i)) &= \rho_{V_i, V_i \cap V_n}(s_i) = \rho_{V_n, V_i \cap V_n}(s_n) = \rho_{V_n, V_i \cap V_n}(\rho_{V, V_n}(s)) \\ &= \rho_{V, V_i \cap V_n}(s) = \rho_{W, V_i \cap V_n}(\rho_{V, W}(s)). \end{aligned}$$

In a similar manner, we obtain  $\rho_{W, V_i \cap V_{n+1}}(\rho_{V_i, W}(s_i)) = \rho_{W, V_i \cap V_{n+1}}(\rho_{V, W}(s))$ . Due to the mono-presheaf condition, we have  $\rho_{V_i, W}(s_i) = \rho_{V, W}(s)$ , that is,

$$\rho_{V_i, V_i \cap V_n}(s_i) = \rho_{V, V_i \cap V_n}(s).$$

By decreasing the number of open covering such that  $U = V_1 \cup \dots \cup V_{n-1} \cup V$ , there exists  $p \in F(U)$  such that, for  $1 \leq i \leq n-1$ ,  $\rho_{U, V_i}(p) = s_i$  and  $\rho_{U, V}(p) = s$ . Thus, for  $k = n$  or  $n+1$ , we have  $\rho_{U, V_k}(p) = \rho_{V, V_k}(\rho_{U, V}(p)) = \rho_{V, V_k}(s) = s_k$ .

#### Theorem 4-6

Let  $\theta_J$  be an equivalence relation on the lattice  $L$  defined by definition 3-5, then  $\theta_J$  is a congruence for any  $J$ . The operation  $F: C(L)^{\text{op}} \rightarrow C(L/\theta_J)$  defined as: for any object  $x$  in  $C(L)^{\text{op}}$ ,  $F(x) = \langle [x]_{\theta_J}, x' \rangle$  where  $x' = \vee [x]_{\theta_J}$ ; for an arrow,  $x \geq y$ ,  $\rho_{x, y} = (-) \wedge y'$ :  $[x]_{\theta_J} \rightarrow [y]_{\theta_J}$ , is a sheaf.

**Proof.** (i) Mono-presheaf condition: Let  $x$  be a covering for  $y$  and  $z$  in  $L$ , that is,  $x = y \vee z$ . Due to the functor  $F$ ,  $x \geq y$  and  $x \geq z$  are mapped to arrows,  $\rho_{x, y} = (-) \wedge y'$ :  $[x]_{\theta_J} \rightarrow [y]_{\theta_J}$  and  $\rho_{x, z} = (-) \wedge z'$ :  $[x]_{\theta_J} \rightarrow [z]_{\theta_J}$ . The assumption of the mono-presheaf condition is given by: for  $a, b \in [x]_{\theta_J}$ ,  $a \wedge y' = b \wedge y'$  and  $a \wedge z' =$

$b \wedge z'$ . From proposition 4-1, we obtain  $x' = y' \vee z'$ . Thus,

$$a = a \wedge x' = a \wedge (y' \vee z') = (a \wedge y') \vee (a \wedge z') = (b \wedge y') \vee (b \wedge z') = b \wedge (y' \vee z') = b \wedge x' = b.$$

(ii) Gluing condition: Given  $(-) \wedge y': [x]_{\theta_I} \rightarrow [x \wedge y]_{\theta_I}$  and  $(-) \wedge x': [y]_{\theta_I} \rightarrow [x \wedge y]_{\theta_I}$ , if, for  $a \in [x]_{\theta_I}$  and  $b \in [y]_{\theta_I}$ ,  $a \wedge y' = b \wedge x'$ , then there exists  $c \in [x \vee y]_{\theta_I}$  such that  $c \wedge x' = a$  and  $c \wedge y' = b$ , with  $(-) \wedge x': [x \vee y]_{\theta_I} \rightarrow [x]_{\theta_I}$  and  $(-) \wedge y': [x \vee y]_{\theta_I} \rightarrow [y]_{\theta_I}$ . Since  $a \in [x]_{\theta_I}$ ,  $b \in [y]_{\theta_I}$  and  $\theta_I$  is a congruence, we can take  $a \vee b \in [x \vee y]_{\theta_I}$ . From  $a \in [x]_{\theta_I}$ ,  $a \leq x' \leq x' \vee y'$ , and then we have

$$\begin{aligned} (a \vee b) \wedge x' &= (a \wedge x') \vee (b \wedge x') = (a \wedge x') \vee (a \wedge y') = a \wedge (x' \vee y') = a, \\ (a \vee b) \wedge y' &= (a \wedge y') \vee (b \wedge y') = (b \wedge x') \vee (b \wedge y') = b \wedge (x' \vee y') = b. \end{aligned}$$

Fig. 3 shows an example of the sheaf. The equivalence class  $\theta_I$  derived from the ideal  $J$  is indicated in the diagram by placing a loop around the elements in the lattice  $L$ . The sheaf  $F$  is an operation from  $L$  to the quotient lattice  $L/\theta_I$ .

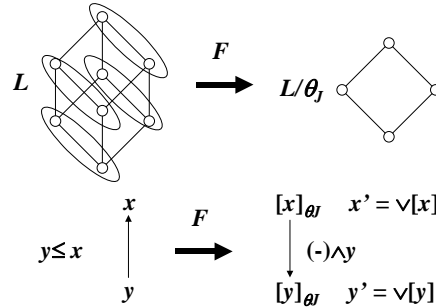


Fig. 3. An example of the sheaf from a lattice to a quotient lattice of  $L$  modulo  $\theta_I$ , where each equivalence class contains the elements.

#### Theorem 4-7 (Semi-sheaf)

Let  $\theta'$  be an equivalence relation derived from the ideal  $J$  on the lattice  $L$ . An operation  $G: C(L)^{\text{op}} \rightarrow C'(L/\theta')$  defined as: for any object  $x$  in  $C(L)^{\text{op}}$ ,  $G(x) = [x]'_{\theta'} = \{x'\}$ , where  $x' = \vee[x]_{\theta'}$ ; for an arrow,  $x \geq y$ ,  $\rho_{x,y} = (-) \wedge y': [x]'_{\theta'} \rightarrow [y]'_{\theta'}$  satisfies the sheaf condition. Since  $C(L)^{\text{op}}$  is not a topological space in the strict sense, we call  $G$  semi-sheaf.

**Proof.** (i) Mono-presheaf condition: Since  $[x]'_{\theta'} = \{x'\}$ , if  $a, b \in [x]'_{\theta'}$ ,  $a \wedge y = b \wedge y$ , then  $a = b = x$ .

(ii) Gluing condition: Since  $[x]'_{\theta} = \{x'\}$ , given  $(-)\wedge y': [x]'_{\theta} \rightarrow [x\wedge y]'_{\theta}$  and  $(-)\wedge x': [y]'_{\theta} \rightarrow [x\wedge y]'_{\theta}$ , the condition is trivially true such that for  $x' \in [x]'_{\theta}$  and  $y' \in [y]'_{\theta}$ ,  $x' \wedge y' = y' \wedge x'$ . Then there exists  $x' \vee y' \in [x \vee y]'_{\theta}$ , such that  $(x' \vee y') \wedge x' = x'$  and  $(x' \vee y') \wedge y' = y'$ .

By introducing the semi-presheaf and the semi-sheaf, we can also address the gluing process for the equivalence relation that is not a congruence. However, this implies a serious problem in terms of the relationship between the microscopic perspective as a lattice to the macroscopic perspective as a quotient lattice. As mentioned before, the equivalence relation derived from the ideal is always a congruence if the ideal is a subset of the distributive lattice so that the sheaf is well-defined. We, however, believe that the constraint of the distributive lattice is too hard to be valid in general gluing processes in order to expand the idea of sheaf to general lattices. This results in a semi-sheaf by which any element of the quotient lattice, such as  $[x]'_{\theta}$ , is obtained as a singleton set,  $\{x'\}$ . It also implies that the microscopic information is lost through the generalization procedure in the macroscopic perspective. Therefore, the next question arisen is whether the microscopic information can be reconstructed by a particular operation from the macroscopic perspective to the microscopic one.

## 5. Dynamical Gluing and Skeleton

Dynamical gluing is defined as a pair of dynamics, from the parts (i.e., a collection of parts or Extent) to the whole (i.e., Intent), and one from the whole to the parts. We call the former the gluing process and the latter the differentiation process. Although these two dynamics are equivalent to each other in the sheaf, there is a discrepancy between the two dynamics. We first assume that the sheaf reveals an idealized case with respect to the relationship between the parts and the whole. As mentioned above, the equivalence relation derived from the ideal gives the sheaf. Thus we define two operations, the Intent and the Extent for the binary relation that can correspond to the two aspects of sheaf. In the idealized case, they are equivalent to each other.

### Definition 5-1 (Intent and Extent)

Let  $X$  be a set. Given a binary relation  $R \subseteq X \times X$ , a map  $E(R): X \times X \rightarrow 2$  derived from  $R$  is called the Extent of  $R$  if and only if for  $\langle x, y \rangle \in X \times X$ ,

$$E(R)(\langle x, y \rangle) = \begin{cases} 1 & \text{if } \langle x, y \rangle \in R ; \\ 0 & \text{if } \langle x, y \rangle \notin R. \end{cases}$$

Also, a map  $I(R):X \rightarrow 2^X$  such that, for  $y \in X$ ,  $I(R)(y) \in 2^X$  is defined by; for  $x \in X$   $I(R)(y)(x) = 1$  or  $0$ ; is called the Intent, if and only if,

$$E(R)(\langle x, y \rangle) = I(R)(y)(x).$$

Now we have defined the pair of Intent and Extent resulting from the equivalence relation. We recall the equivalence relation  $\theta_f$  derived from the ideal that is argued in section 3. The next theorem shows how to construct the map equivalent to the equivalence class of  $\theta_f$ .

**Theorem 5-2 (Reconstruction of a lattice from a quotient lattice)**

For the binary relation  $\theta_f$  derived from an ideal  $J \subseteq L$ , there exists a filter  $K \subseteq L$  such that

$$[x]_{\theta_f} = f^{-1}(x),$$

where for any  $x \in K$ ,

$$f^{-1}(x) := \downarrow x - \bigcup_{y \in K, y < x} \downarrow y.$$

**Proof.** Since  $J$  is an ideal, let  $J = \downarrow p$  with  $p \in L$ . A filter  $K$  is given by  $\uparrow p$ .

(i) We prove  $[x]_{\theta_f} \subseteq f^{-1}(x)$ . In taking  $y \in [x]_{\theta_f}$ ,  $y \equiv x \pmod{\theta_f}$ , that is,  $\exists z \in \downarrow p$  ( $x \vee z = y \vee z$ ). Therefore,  $z \leq p$ . Since  $x \in K = \uparrow p$ ,  $z \leq p \leq x$ . Also, since  $y \leq y \vee z = x \vee z = x$ ,  $y \in \downarrow x$ . Next we show that that  $u < x$ ,  $y \notin \downarrow u$ , for any  $u \in \uparrow p$ . Assume that  $y \in \downarrow u$  such that  $u < x$ , for any  $z \in \downarrow p = J$ ,  $z \leq p \leq u$ . From  $y \in \downarrow u$ , we have  $y \leq u$ . This implies that  $u$  is an upper bound of  $\{z, y\}$ ,  $z \vee y \leq u$ , therefore we have

$$z \vee y \leq u < x \leq z \vee x.$$

It implies that, for any  $z \in J$ ,  $z \vee y \neq z \vee x$ , and so  $y \notin [x]_{\theta_f}$ . That is a contradiction, and then we have  $y \notin \downarrow u$ . Since  $y \in \downarrow x$  and  $y \notin \downarrow u$  with  $u \in \uparrow p$  and  $u < x$ ,  $y \in \downarrow x - \bigcup_{u \in \uparrow p, u < x} \downarrow u = f^{-1}(x)$ .

(ii) We prove that  $f^{-1}(x) \subseteq [x]_{\theta_f}$ . We assume  $y \in f^{-1}(x)$ , then we have  $y \leq x$ . From  $x \in K = \uparrow p$ , we have  $p \leq x$ . Therefore,  $x$  is an upper bound of  $\{y, p\}$ , and  $y \vee p \leq x$ . If  $y \vee p \neq x$ ,  $y \notin f^{-1}(x)$  since  $f^{-1}(x)$  does not involve  $\downarrow(y \vee p)$ , and that is a contradiction. It implies  $y \vee p = x$ . On the other hand,  $x \vee p = x$  since  $p \leq x$ . As a result, we have

$$\exists p \in J = \downarrow p \text{ } (x \vee p = y \vee p).$$

This implies that  $y \in [x]_{\theta_I}$ .

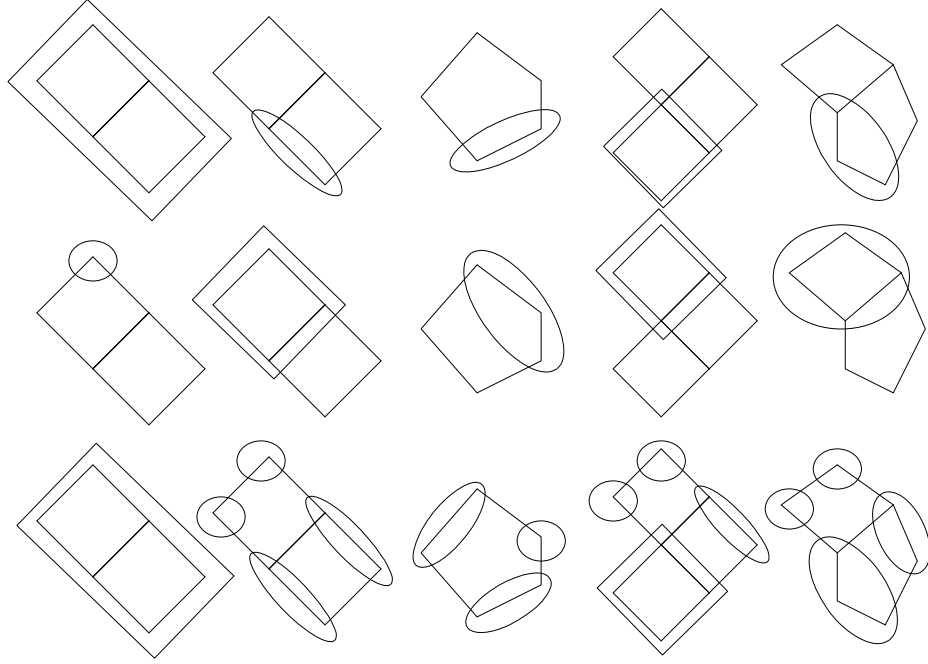


Fig. 4. Some example of  $[x]_{\theta_I} = f^{-1}(x)$ , where  $\theta_I$  is the binary relation derived from the ideal  $J \subseteq F$ ,  $J = \downarrow p$ . In the top row,  $J = \downarrow p$  is indicated on the diagram placing a loop around the elements. In the middle row, a loop indicates  $K = \uparrow p$ . In the bottom row  $[x]_{\theta_I} = f^{-1}(x)$  is indicated by placing a loop around the elements.

Fig. 4 shows some examples of  $[x]_{\theta_I} = f^{-1}(x)$  for various lattices. Given an ideal  $J = \downarrow p$  in a lattice (upper row), the corresponding filter  $K = \uparrow p$  is determined (middle row), and the partition  $[x]_{\theta_I} = f^{-1}(x)$  (bottom row) is also determined.

It is easy to construct the Extent of  $\theta_I$ . From the operation  $f^{-1}: K \rightarrow \wp(L)$  one can also construct the Intent of  $\theta_I$ . Strictly speaking, modifying  $\theta_I$  by  $\theta_I'$  can lead to its Intent and Extent.

### Theorem 5-2 (Intent and Extent in Reconstruction)

Let  $J = \downarrow p$  be an ideal on the lattice  $L$ ,  $K = \uparrow p$ ,  $\theta_I' = \{ \langle x, y \rangle \in L \times K \cap \theta_I \}$  and  $\theta_I = \{ \langle x, y \rangle \in L \times L \mid \exists z \in J (x \vee z = y \vee z) \}$ , and for any  $y \in K$ ,  $f^{-1}(y) = \downarrow y - \bigcup_{z \in K, z < y} \downarrow z$ . The map  $\langle f^{-1} \rangle: K \rightarrow 2^L$  such that, for any  $x \in L$ ,

$$\langle f^{-1} \rangle(y)(x) = \begin{cases} 1 & \text{if } x \in f^{-1}(y); \\ 0 & \text{if } x \notin f^{-1}(y). \end{cases}$$

is the Intent of  $\theta_I'$ , that is,

$$I(\theta_J') = \langle f^{-1} \rangle,$$

such that

$$\begin{array}{ccc} & L \times 2^L & \\ \text{id} \times \langle f^{-1} \rangle \uparrow & \searrow \text{ev} & \\ L \times K & \xrightarrow{E(\theta_J')} & 2 \end{array}$$

where the broken arrow indicates the unique map that commutes the diagram, and for any  $\langle x, g \rangle \in L \times 2^L$ ,  $\text{ev}(\langle x, g \rangle) = g(x)$ .

**Proof.** The Extent of  $\theta_J'$  is defined by  $E(\theta_J')(\langle x, y \rangle) = 1$  if  $\langle x, y \rangle \in \theta_J'$ ;  $E(\theta_J')(\langle x, y \rangle) = 0$  if  $\langle x, y \rangle \notin \theta_J'$ . Therefore, for any  $\langle x, y \rangle \in L \times K$ ,

$$\text{ev}(\text{id} \times \langle f^{-1} \rangle)(\langle x, y \rangle) = \text{ev}(\langle x, \langle f^{-1} \rangle(y) \rangle) = \langle f^{-1} \rangle(y)(x).$$

Since  $\langle f^{-1} \rangle(y)(x) = 1 \Leftrightarrow x \in f^{-1}(y) \Leftrightarrow x \in [y]_{\theta_J} \Leftrightarrow \langle x, y \rangle \in \theta_J' \Leftrightarrow I(\theta_J')(\langle x, y \rangle) = 1$ , the diagram is commutative. In assuming  $g: K \rightarrow 2^K$  also commutes the diagram, for any  $\langle x, y \rangle \in L \times K$ ,  $\langle f^{-1} \rangle(y)(x) = g(y)(x)$ . Then  $\langle f^{-1} \rangle(y) = g(y)$  for any  $y \in K$ , and implies  $\langle f^{-1} \rangle = g$ .

What is the relationship between the Extent and the Intent, or between  $\theta_J$  and  $f^{-1}$ ? We regard the sheaf (and semi-sheaf) as a gluing dynamics with respect to a particular function expressed as an equivalence relation derived from the ideal. Since the sheaf leads to the structured partition and any structure never holds without wholeness, it reveals the dynamics from the parts to the whole. Given  $\theta_J$ , we can define the sheaf. By contrast, we regard  $f^{-1}$  as a differentiation dynamics that is from the whole to the parts.

Notice that once the partitioned structure is made through the sheaf, each part forgets the structure. It implies that each  $[x]_{\theta_J}$  in  $L/\theta_J$  forgets the original lattice  $L$ . In this sense,  $f^{-1}$  can be regarded as a process reconstructing the original lattice. Since  $f^{-1}(x) = \downarrow x - \bigcup_{y \in K, y \prec x} \downarrow y$ , reconstruction by  $f^{-1}(x)$  is needed when  $\downarrow x$  is defined, and that requires the whole structure of the original lattice. From Theorem 5-2,  $f^{-1}(x) = [x]_{\theta_J}$ , and then  $\bigcup_{x \in K} f^{-1}(x) = L$ . This implies the paradoxical situation. Before reconstructing a lattice, each  $x$  has to know the lattice resulting from the reconstruction. In other words, differentiation needs the wholeness that cannot be definitely

known. Each  $x$  has to assume a quick-fix wholeness of  $L$  (Fig. 5).

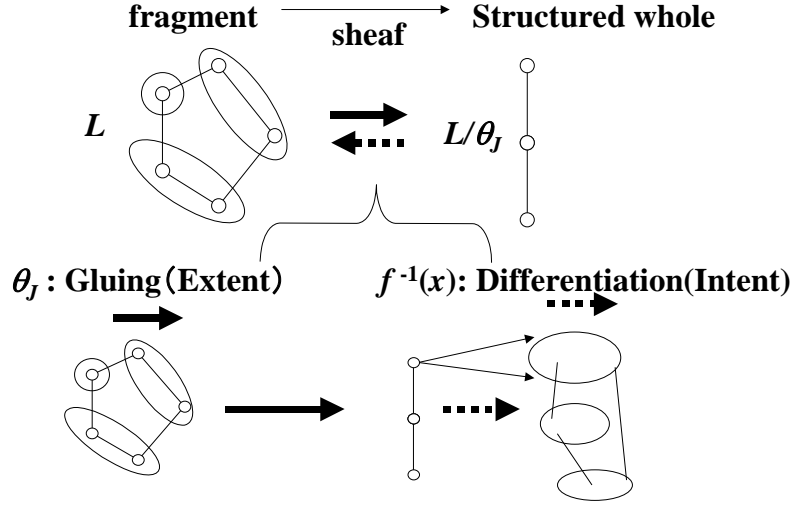


Fig. 5. Schematic diagram of the gluing and differentiation process derived from the sheaf. If the original lattice  $L$  is forgotten in the glued-up structure  $L/\theta_J$ , how can the differentiation process reconstruct  $L$ ? By contrast, if the information about  $L$  is not lost, we have  $[x]_{\theta_J} = f^{-1}(x)$  by Theorem 5-2.

As shown in Fig. 5, each  $x$  has to construct a subspace (indicated by an ellipse) of the original lattice in the differentiation process represented by the broken arrow. Since each  $x$  has no knowledge about the original lattice in constructing the subspace which is independently separated from each other, it cannot be expected that the resulting subspaces constitute a unified structure such as the lattice. Thus, making a subspace independently separated from each other can lead to the collapse of the lattice. Then, how can one overcome this problem?

We here define a tool called the skeleton that can move freely between the parts and the whole. Even if a collection of  $f^{-1}(x)$  can lead to the collapse of the lattice, the skeleton can recover the lattice. A skeleton is expressed as the wrecked function.

### Definition 5-3 (Skeleton)

Given a map  $f(x)$ ,  $f(?)$  is a skeleton, where  $?$  indicates an indefinite symbol, and

$$f'(x) = f(x) + f(?),$$



where any symbol can be substituted to ?. If some  $m$  is substituted for ?,

$$f(m) \approx f(m) + f(?).$$

The symbol  $\approx$  indicates an equivalence relation on the product of the set of all symbols, and includes  $=$ . If an expression  $f(m)$  is well defined and has a value, the transition is always stopped and  $f(?)$  is ignored. If  $f(m)$  is not defined, the transition proceeds by using  $\approx$ . If  $f(m)$  has no value (such a situation is indicated by  $\#f(m)$ ), the symbol % representing “no value” is computed, and

$$f(m) \approx \#f(m) + f(?) \approx \% + f(?).$$

The condition of no value is defined depending on the definition of  $f(x)$ . The % satisfies a particular structure toward which no value is verified, and can be distinguished from each other depending on the structure. It is denoted by  $\%_1, \%_2, \%_3, \dots \%_n$ . The equivalence relation  $\approx$  also satisfies

$$f(?) + f(?) \approx f(?),$$

where the symbol  $+$  just represents the concatenation of expressions, and if  $f(x)$  is a set,

$$f(?) + f(?) \approx f(?) \cup f(?).$$

We also define that for an element  $x$ ,

$$\{x\} \approx x.$$

Other conditions specialized to a concrete form of  $f(x)$  are also defined if  $f(x)$  is defined.

From the definition,  $f(?)$  always exists, and it always computes for concrete symbol substituted for ?. To define a skeleton for  $f^{-1}(x)$ , we introduce the following operation.

**Definition 5-4 (Point-shift operation)**

Let  $L$  be a lattice, and  $K$  and  $S$  be subsets of  $L$ . The point-shift operation,  $(-)^+$  is defined as

$$(S)^+_K = \{z \in L \mid z \leq y, (\exists x \in S)(y < x, \exists y \in K)\}.$$

It is clear to see that  $\bigcup_{y \in K, y < x} \downarrow y = (\downarrow x)^+_K$ , and then  $f^{-1}(x) = \downarrow x - (\downarrow x)^+_K$ .

Now we define the skeleton of  $f^{-1}(x) = \downarrow x - (\downarrow x)^+_K$ . Note that  $x$  is an element of the lattice. By introducing the skeleton, what we want is to assimilate an element with a subset of the lattice. In this sense, we introduce the symbol “?” that is not only an element but also a set. As for a set  $S$ ,

$$S = S' \cap S''$$

makes sense, although for an element,  $x = x' \cap x''$  does not make sense. Imagine  $S = \{x\}$ , a singleton set can be expressed as the conjunction of some sets, such that  $\{x\} = \{x, y\} \cap \{y, z\}$ . If one assimilates  $x$  with  $\{x\}$ , the expression  $S = S' \cap S''$  bridges an element with a set.

From the definition of  $f^{-1}(x) = \downarrow x - (\downarrow x)^+_K$ ,  $x$  in this expression must be an element of the partially ordered set. Thus,  $\downarrow x$  makes sense even though  $\downarrow S$  does not. Note that  $\downarrow x$  is a set when  $x$  is an element. If one assimilates  $x$  with  $S$ , and represents it by  $m$ , one can obtain

$$\downarrow m = m.$$

It is also a particular expression bridging an element with a set.

Note that  $f^{-1}(x) = \downarrow x - (\downarrow x)^+_K$  is well defined if and only if the whole structure  $L$  is given, since  $\downarrow x$  and  $K$  are subsets in  $L$ , that is,

$$L = \bigcup_{x \in K} (\downarrow x - (\downarrow x)^+_K).$$

It implies that, although each element  $x$  needs the whole structure  $L$  to compute  $f^{-1}(x)$ ,  $L$  results from the collection of  $f^{-1}(x)$  for all elements. Although it is a contradictory situation, each  $x$  needs  $L$ . One of the hopeful ways is to make a quick-fix  $L(x)$  for an element  $x$ . In other words, making a decision to construct  $L(x)$  is destined to be decision with reservation. The next question might arise how one can express the notion of “reservation”. The skeleton is nothing but an expression for “reservation”.

#### **Definition 5-5 (Skeleton in $f^{-1}(x)$ )**

Let  $L$  be a lattice that can supply all elements used. Let  $f^{-1}(x) = \downarrow x - (\downarrow x)^+_K$  that is defined in the lattice  $L(x)$ . A skeleton of  $f^{-1}(x)$  is,

$$f^{-1}(?) = \downarrow ? - (\downarrow ?)^+,$$

that is obtained by dropping  $K$ . It implies that  $L(?) \approx f^{-1}(?)$ . With respect to the equivalence relation  $\approx$ , it satisfies

$$\begin{aligned}\downarrow? &\approx ?, \\ ? &\approx ?' \cap ?'',\end{aligned}$$

where  $?$  is the concatenation of  $?'$  and  $''$ . Since  $\downarrow? \approx \downarrow(?)' \cap ?'' \approx \downarrow(\downarrow?' \cap \downarrow?'') \approx \downarrow?' \cap \downarrow?''$ , the symbol  $?$  satisfies

$$\downarrow(?)' \cap ?'' \approx \downarrow?' \cap \downarrow?''.$$

Either a set or an element can be substituted into  $?$  in  $\downarrow?$ , then either the set or the element can be recognized as the concatenation of the symbols by  $\downarrow?$ , that is, for the set  $\{a, b, c, \dots\}$ , we have  $\downarrow(abc\dots)$ .

The condition of existence of a value is defined as: For any symbol  $m$ , a set or an element,

$$\exists x \in L (\downarrow m \approx \downarrow x).$$

**Verification:** All we have to show is  $L(?) \approx f^{-1}(?)$ . Since, for  $y \in K$ ,  $L(y) = \bigcup_{x \in K} (\downarrow x - (\downarrow x)^+_K)$ .

$$L(?) = \bigcup (\downarrow? - (\downarrow?)^+) \approx (\downarrow? - (\downarrow?)^+) = f^{-1}(?).$$

The condition  $\downarrow? \approx ?$  reveals the assimilation of a set with an element. The condition  $? \approx ?' \cap ?''$  is derived from a set such that  $S = S' \cap S''$ . From the definition, if an element  $a$  is substituted into  $\downarrow?$ ,  $\downarrow a$  is obtained. Since  $\downarrow x \approx \downarrow a$  where  $x = a$ , it has a value. If a set  $\{a, b, c, \dots\}$  is substituted into  $?$ ,  $\downarrow(abc\dots)$  is obtained, and

$$\downarrow(abc\dots) \approx \downarrow(a \cap b \cap c \cap \dots) \approx \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots,$$

is obtained since  $\downarrow(abc\dots)$  and  $\downarrow(a \cap b \cap c \cap \dots)$  are not well defined, then  $\approx$  is applied to expressions. It implies that the expression such as  $\downarrow a \cap \downarrow b \cap \downarrow c \cap \dots$ , is well defined and can be reached by using  $\approx$ . If there exists  $\exists x \in L$  such that

$$\downarrow x = \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots,$$

an expression  $\downarrow(abc\dots)$  has a value, and then the transition is stopped, and  $\downarrow(abc\dots) \approx x$ . If there is

no  $x$  such that  $\downarrow x = \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots$ ,

$$\downarrow(abc\dots) \approx \%.$$

Since the symbol  $\%$  is indefinite, it also satisfies

$$\% \approx \downarrow\%.$$

If the symbol  $\%$  is regarded as an element,  $\downarrow\%$  is a set, and then for any  $y$  in  $\downarrow\%$ ,  $y < a$ ,  $y < b$ ,  $y < c, \dots$ , since  $\downarrow\%$  is assimilated with  $\downarrow a \cap \downarrow b \cap \downarrow c \cap \dots$ , and  $y < \%$ .

Although the condition, such as  $? \approx \downarrow?$ , is similar to the definition of the ordinal number, they are essentially different from each other. While the relation  $\approx$  is an equivalence relation, it is applied to an expression only if the expression is not well defined. Thus,  $\downarrow(a \cap b \cap c \cap \dots) \approx \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots$ , but  $\downarrow a \cap \downarrow b \cap \downarrow c \cap \dots \approx \downarrow(a \cap b \cap c \cap \dots)$  does not hold.

#### Lemma 5-6

Given  $f^{-1}(?) = \downarrow? - (\downarrow?)^+$ , for a partially ordered set  $S$ ,  $S \cup f^{-1}(?)$  is closed under the meet.

**Proof:** Any subset,  $M \subseteq S$ , can be substituted into  $?$  in  $f^{-1}(?)$ . Let  $M$  be  $\{a, b, c, \dots, d\}$ , and then we obtain

$$\begin{aligned} f^{-1}(S) &= f^{-1}(abc\dots d) = \downarrow(abc\dots d) - (\downarrow(abc\dots d))^+ \\ &\approx \downarrow(a \cap b \cap c \cap \dots \cap d) - (\downarrow(a \cap b \cap c \cap \dots \cap d))^+ \\ &\approx \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots \cap \downarrow d - (\downarrow a \cap \downarrow b \cap \downarrow c \cap \dots \cap \downarrow d)^+. \end{aligned}$$

If there exists  $\downarrow x = \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots \cap \downarrow d$ ,  $f^{-1}(S)$  is equivalent to

$$\approx \downarrow x - (\downarrow x)^+ \approx \downarrow x - (\downarrow x - \{x\}) = \{x\} \approx x.$$

Since  $\downarrow x = \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots \cap \downarrow d$ ,

$$y \in \downarrow x \Leftrightarrow y \in \downarrow a \text{ and } y \in \downarrow b \text{ and } y \in \downarrow c \dots \text{ and } y \in \downarrow d.$$

Thus,  $y \leq a$ ,  $y \leq b$ ,  $y \leq c, \dots$ ,  $y \leq d$ . It implies that  $y$  is the lower bound of  $\{a, b, c, \dots, d\}$ . Since  $y \leq x$ ,  $x$  is the greatest lower bound of  $\{a, b, c, \dots, d\}$ , that is  $x = \wedge \{a, b, c, \dots, d\}$ .

If there exists no  $x$  such that  $\downarrow x = \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots \cap \downarrow d$ , we obtain

$$f^{-1}(S) \approx \% \approx \downarrow \%.$$

If we obtain  $\downarrow \%$ , it implies that  $\downarrow \% = \downarrow a \cap \downarrow b \cap \downarrow c \cap \dots \cap \downarrow d$ , and then  $\% = \wedge \{a, b, c, \dots, d\}$ . Finally, if  $S$  is not closed under the meet due to a skeleton, new element appears and then  $S \cup \{\%_1, \%_2, \dots\}$  is closed under the meet.

### Lemma 5-7

Given a lattice  $L$  and  $f^{-1}(?) = \downarrow ? - (\downarrow ?)^+$ , if  $x \approx \downarrow x$  is allowed, a skeleton can indicate an ideal.

**Proof.** Let  $S$  be a subset of  $L$ . From lemma 5-6,  $f^{-1}(S) \approx x$  or  $\%$ , and then  $x \approx \downarrow x$  or  $\% \approx \downarrow \%$ , those are ideals.

**Example:** Given a subset of the set lattice  $\{\{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}$ , and ' $f^{-1}(x) = \downarrow x - (\downarrow x)^+_K$  +  $\downarrow ? - (\downarrow ?)^+$ ', all subsets of  $\{\{a\}, \{b\}, \{a, b, c\}, \{a, b, d\}\}$  are substituted to  $?$ , that is,

$$\begin{aligned} f^{-1}(\{\{a\}, \{a, b, c\}\}) &\approx f^{-1}(\{a\}\{a, b, c\}) \approx \downarrow (\{a\}\{a, b, c\}) - (\downarrow (\{a\}\{a, b, c\}))^+ \\ &\approx \downarrow (\{a\} \cap \{a, b, c\}) - (\downarrow (\{a\} \cap \{a, b, c\}))^+ \\ &\approx \downarrow \{a\} \cap \downarrow \{a, b, c\} - (\downarrow \{a\} \cap \downarrow \{a, b, c\})^+ \approx \downarrow \{a\} - (\downarrow \{a\})^+ \approx a. \end{aligned}$$

$$\begin{aligned} f^{-1}(\{\{a, b, d\}, \{a, b, c\}\}) &\approx f^{-1}(\{a, b, d\}\{a, b, c\}) \\ &\approx \downarrow (\{a, b, d\} \cap \{a, b, c\}) - (\downarrow (\{a, b, d\} \cap \{a, b, c\}))^+ \\ &\approx \downarrow \{a, b, d\} \cap \downarrow \{a, b, c\} - (\downarrow \{a, b, d\} \cap \downarrow \{a, b, c\})^+ \\ &\approx \downarrow \% - (\downarrow \%)^+ \approx \%, \end{aligned}$$

where  $\%$  is equivalent to  $\{a, b\}$ , and the term  $+\downarrow ? - (\downarrow ?)^+$  is omitted.

### Definition 5-8 (Differentiation)

Let  $L$  be a lattice,  $K \subseteq L$  be a sublattice. For  $x \in K$ ,

$$\begin{aligned} g^{-1}(x) &= (\downarrow x - (\downarrow x)^+_K)_{L(x)} = (\downarrow x)_{L(x)} - ((\downarrow x)^+_K)_{L(x)}, \\ (\downarrow x)_{L(x)} &= \{z \in L(x) \mid z \leq x\}, \end{aligned}$$

$$((M)^+_K)_{L(x)} = \{z \in L(x) \mid z \leq y, (\exists x \in M)(y < x, y \in K)\},$$

where  $L(x)$  is a sublattice of  $L$  in which  $K$  is a filter. The operation  $(\downarrow x)_{L(x)}$  indicates  $\downarrow x$  in  $L(x)$ . We call ' $g^{-1}$ ' the differentiation process. We also call

$$\bigcup_{x \in K} g^{-1}(x)' = \bigcup_{x \in K} (g^{-1}(x) + g^{-1}(?)) = \bigcup_{x \in K} g^{-1}(x) + \downarrow ? - (\downarrow ?)^+$$

the result of the differentiation, where any element and subset of  $\bigcup_{x \in K} g^{-1}(x)$  can be substituted to ? in a skeleton.

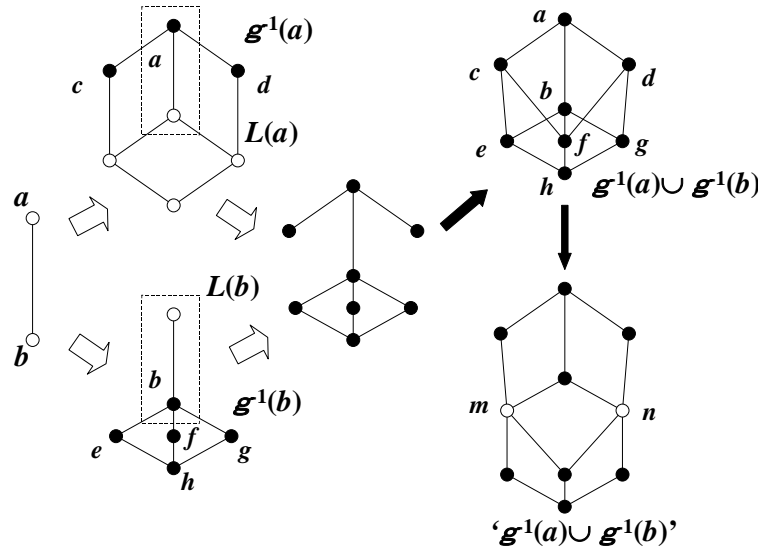


Fig. 6. An example of the differentiation process. The non-lattice  $g^{-1}(a) \cup g^{-1}(b)$  is regenerated by ' $g^{-1}(a) \cup g^{-1}(b)$ ' due to the skeleton. In  $L(a)$  and  $L(b)$  broken loop represents  $K = \uparrow p$ . See text for detailed discussion.

Fig. 6 shows an example of the differentiation process, ' $g^{-1}(x)$ '. First we assume that the set  $X = \{x, y, z, u\}$  and its power set  $\wp(X)$  are given. The set lattice is defined as  $L = \langle \wp(X), \subseteq \rangle$ . Also assume that through structuring (gluing), the lattice  $\subseteq L$  shown at left hand is obtained, where  $a = \{x, y, z, u\}$  and  $b = \{x, y, z\}$ . The proliferation ' $g^{-1}$ ' is applied to  $a$  and  $b$ , and;

$$g^{-1}(a)' = (\downarrow a - (\downarrow a)^+_K)_{L(a)} + \downarrow ? - (\downarrow ?)^+ \text{ and } g^{-1}(b)' = (\downarrow b - (\downarrow b)^+_K)_{L(b)} + \downarrow ? - (\downarrow ?)^+,$$

where  $L(a)$  and  $L(b)$  are determined as shown in Fig. 6. In  $L(a)$ ,  $c = \{x, y, u\}$  and  $d = \{y, z, u\}$ . In  $L(b)$ ,  $e = \{x\}$ ,  $f = \{y\}$ ,  $g = \{z\}$  and  $h = \emptyset$ . It results in a set with the skeleton.

$$g^{-1}(a) \cup g^{-1}(b)' = \{\{x, y, z, u\}, \{x, y, z\}, \{x, y, u\}, \{y, z, u\}, \{x\}, \{y\}, \{z\}, \emptyset\}$$

$$+\downarrow? - (\downarrow?)^+,$$

that is not a lattice since  $\{\{x, y, z\}, \{x, y, u\}\}$  has no least upper bound. Due to the skeleton, any subset of  $g^{-1}(a) \cup g^{-1}(b)$  are substituted to  $?$ , and that leads to;

$$\begin{aligned} & \downarrow(\{\{x, y, z\}, \{x, y, u\}\}) - (\downarrow\{\{x, y, z\}, \{x, y, u\}\})^+ + \downarrow? - (\downarrow?)^+ \\ & \approx \downarrow\{x, y, z\} \cap \downarrow\{x, y, u\} - (\downarrow\{x, y, z\} \cap \downarrow\{x, y, u\})^+ + \downarrow? - (\downarrow?)^+ \\ & \approx \#_m + \downarrow? - (\downarrow?)^+, \end{aligned}$$

since  $\downarrow\{x, y, z\} \cap \downarrow\{x, y, u\}$  has no value. New element  $\#_m$  that is equivalent to  $\{x, y\}$  is generated. Similarly,  $\#_n$  that is equivalent to  $\{y, z\}$  is obtained. It results in the lattice ' $g^{-1}(a) \cup g^{-1}(b)$ ' as shown in Fig.6.

#### Definition 5-9 (Organization as a pair of gluing and differentiation)

Let  $L$  be a finite lattice,  $L \subseteq L$  be a sublattice and  $p \in L$ ,  $J = \downarrow p$ . The gluing is defined by  $F: C(L)^{\text{op}} \rightarrow C(L/\theta_J)$ , where  $\theta_J = \{ \langle x, y \rangle \in L \times L \mid \exists z \in J (x \vee z = y \vee z) \}$ . Let  $K = L/\theta_J$ , the corresponding differentiation is defined by ' $g^{-1}(x)$ '. The organization on  $L$  is defined as the pair  $\langle F, 'g^{-1}' \rangle$ . We call  $\langle L, \bigcup_{x \in K} 'g^{-1}(x)' \rangle$  the transition of organization. After the differentiation,  $x \approx \downarrow x$  is allowed, and that indicates an ideal.

It is clear to see that  $F(L)$  is a lattice. In the differentiation of each  $x$  in  $K$ , there exists a lattice  $L(x)$  in which  $K$  is a filter. Although  $\bigcup_{x \in K} g^{-1}(x)$  is not a lattice,  $\bigcup_{x \in K} 'g^{-1}(x)'$  is a lattice due to the skeleton, and that can be verified by the following theorem.

#### Theorem 5-10

Let  $L$  be a finite lattice. In any organization on  $L \subseteq L$ , the result of the differentiation, that is expressed as  $\bigcup_{x \in K} 'g^{-1}(x)'$ , is a lattice.

**Proof:** Even if  $\bigcup_{x \in K} g^{-1}(x)$  is not closed under the meet,  $\bigcup_{x \in K} 'g^{-1}(x)'$  is closed under the meet from lemma 5-6. For any  $x$ ,  $L(x)$  including  $K$  as a filter contains the greatest element of  $L$ , and thus  $\bigcup_{x \in K} g^{-1}(x)$  has the greatest element. For any subset  $S$  in  $\bigcup_{x \in K} 'g^{-1}(x)'$ , there exists an upper bound due to  $\bigcup_{x \in K} 'g^{-1}(x)'$ . Let the set of the upper bound be  $S''$ , there exists a greatest lower bound for  $S''$  that is the least upper bound for  $S$ . Thus,  $\bigcup_{x \in K} 'g^{-1}(x)'$  is also closed under the join.

Fig. 7 shows a typical example of the organization consisting of the gluing and

differentiation. First, given a lattice and an ideal, it leads to a partition of the equivalence class derived from the ideal. By the gluing process  $F:C(L)^{\text{op}} \rightarrow C(L/\theta_f)$ , we obtain the quotient lattice that corresponds to the structured wholeness mentioned before. After that, the differentiation process  $\bigcup_{x \in K} g^{-1}(x)$  proceeds, that is, each  $x$  in  $K = L/\theta_f$  determines its own wholeness as  $L(x)$  and makes  $g^{-1}(x)$  in  $L(x)$ . At first, it just results in a collection of  $g^{-1}(x)$  that is  $\bigcup_{x \in K} g^{-1}(x)$  and is just a partially ordered set. When it is accompanied with the skeleton,  $f^{-1}(?) = \downarrow? - (\downarrow?)^+$ . Even if  $\bigcup_{x \in K} g^{-1}(x)$  is not a lattice and is just a partially ordered set, it can be replaced by a particular lattice due to the skeleton. After the lattice is obtained, the skeleton still remains, and  $x \approx \downarrow x$  is allowed after a while, and then the skeleton can indicate the ideal. As a result, the gluing process can proceed again. These operations are iterated, so that the transition  $\langle L, \bigcup_{x \in K} g^{-1}(x) \rangle$  is obtained. For example, from the lattice shown in Fig. 7A to the other one shown in Fig. 7F.

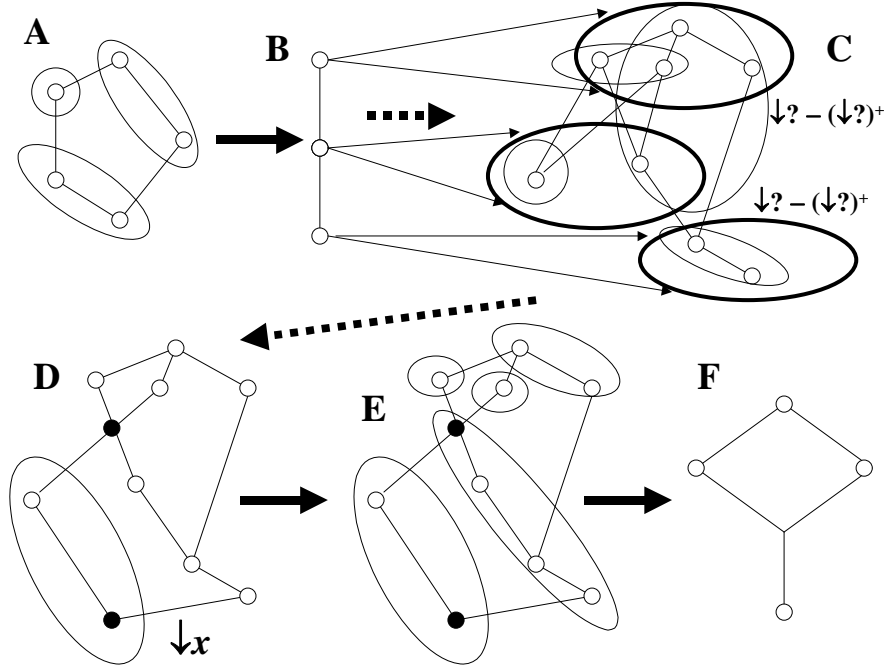


Fig. 7. An example of the organization consisting of the gluing and differentiation. Given an ideal in a lattice, the equivalence class derived from the ideal is obtained (A) and then the quotient lattice is obtained by gluing (B). By differentiation  $\bigcup_{x \in K} g^{-1}(x)$  is obtained, although it cannot be a lattice (C). Each  $g^{-1}(x)$  is indicated by a thick loop. There are skeletons in  $F:C(L)^{\text{op}} \rightarrow C(L/\theta_f)$ , and subsets of  $\bigcup_{x \in K} g^{-1}(x)$  indicated by thin loops can be applied to a skeleton. Due to the skeletons, a new lattice is obtained by adding new elements indicated by black dots (D), and at this stage the skeleton can represent the ideal indicated by the loop. The ideal leads to the equivalence class (E), and results in the new quotient lattice (F).



Since a sheaf is an operation by which the co-limit is taken, it can be regarded as an operation from the microscopic description (collection of particles) to the macroscopic one. In the lattice description, a sheaf is an operation by which the equivalence class is taken, that is, making the structured wholeness. Once the parts (the elements) are glued up, a particle is regarded as a part. In this sense, the whole is consistent with the parts. We pay attention to the inverse operation from the structured whole to the collection of elements, that is, the differentiation  $g^{-1}(x)$ . If the inverse process is based on the complete knowledge on the whole structure, it can reconstruct the original lattice, otherwise, it collapses. That is a problem with respect to the endo-perspective in which an observer cannot see the whole world [5,7,13,14]. Discrepancy between the top-down and the bottom-up process can result from the endo-perspective. Now the problem is rephrased by the following: If the endo-observer has just a limited partial perspective and has no ability to look at the outside, the system (the lattice) collapses. However, a real living system does not collapse. It implies that the endo-observer has the ability to look at the outside through a particular interface. In our organization scheme, the endo-observer appears as an element in  $K = L/\theta_j$ . The endo-observer is able to look at the outside or to negotiate between the subspaces resulting from the skeleton. That is a particular expression for the material cause.

The idea of skeleton appears only when the outside of the formal framework is addressed. Recall the pair of Intent and Extent in the even number case. Analogously, we can say that the process is an induction in which the Intent  $2n$  is obtained from the sequence 0,2,4, ... Similarly the process is a deduction in which a concrete number, such as 4, is obtained from  $2n$ . If it is sufficient to describe the concept only by using the pair of Intent and Extent, there is no room to think about the outside of the pair. In this sense, the induction and deduction can be compared to the sheaf and the reverse-sheaf such as  $f^{-1}(x)$ . The case that we concentrate on, however, is an inconsistent case between the Intent and the Extent, and that leads to the reference of the outside of the formal framework. Referring to the outside is explicitly expressed as a skeleton. It is nothing but just a process of abduction.

## 6. Discussion and Conclusion

In our framework, there is an intrinsic discrepancy between the microscopic and macroscopic perspectives. A special mediator between them called skeleton is introduced. The essential point is that a skeleton is implicitly embedded in the explicit framework of the pair of micro- and macroscopic perspectives since the function of skeleton appears by depriving the original functionality for  $f^{-1}(x)$ . We here discuss about the material cause and the skeleton. Our mathematical model with the skeleton is a model for the hierarchical structure with the material cause.

Imagine two levels in making a house. At the upper level, one can design the house. Only in that level, we can examine the relative location of the rooms by assuming that all one can do is just examining. At the lower level, we make each room, such as a toilet, a kitchen, by some tools. We assume that each room is made independently that is separated from each other. Since there are differences between the two levels with respect to the work (examining and making), the two levels can be regarded as a metaphoric model for the micro- and macroscopic perspectives. Note that a problem exists as follows: Each room is made by itself in the lower level. Thus, when all rooms are examined at the upper level, one might find that the location of the toilet is not adequate such that the door of the toilet is facing the garden. One then has to relocate the toilet door. However, all the tools one has is only specialized to make a toilet, or is specialized in the lower level. So it is formally impossible to repair. Our answer to resolve the problem is the “material cause” of the tool. Note that a big wrench for water pipe is specialized to make a toilet and cannot be used for any other purpose from a first thought. However, a big wrench is a heavy object made of iron and can also be used as a hammer. If one regards the heavy iron object that we call the wrench as a hammer, then one can break the wall and fix the location of the door.

The material cause has latent functions hidden in the tool specialized to a particular purpose. We here abstract the concept of material to construct a formal model with the material cause by embedding a mediator in an inconsistent manner (i.e., a formal model implemented in a material framework). We first recall the idea of material as a particular constraint. For example, the aim of the artificial life is to be free from the constraint of material such that any life on the earth is based on carbon in order to concentrate only on the formal cause. The function of material as a constraint is regarded as an operation indicating the distinction of the reality from the possibility. Carbon distinguishes real lives on earth from possible ideal lives. For example, a ball-point pen, that is a tool for writing distinguishes real pattern drawn by the pen from the possible ideal patterns. Although it is possible to imagine a black  $1\text{m}^2$  pattern on a paper, but it would not be real if it is drawn solely by a ball-point pen.

The latent function of a tool contradicts the function such as indicating the distinction of the reality from the possibility. In the case of the ball-point pen, one can find the latent functionality of the pen such as the hardness on the point when inks are lost. By utilizing the hardness on the point, one can write down something by scratching a sheet of paper. It implies that even if the ink is lost one can still write down more. This is inconsistent with the distinction between the reality and the possibility mentioned before. A wrench for water pipe has also dual functions, where the explicit function may contradict the latent implicit functions. The explicit function indicates the distinction of the real special function from all possible functions by a general tool, while the implicit function invalidates that distinction. We here generalize that idea and abstract the role of material as a dual function; the former explicit one indicates the distinction between the reality and the possibility, and

the latter implicit one invalidates the former distinction. A dual function is defined by the inconsistency between the explicit and implicit functions.

In our framework, the material cause corresponds to the skeleton that is the wrecked map. By assimilating domains with co-domains of a map, the notion of the map is destroyed, and that leads to the skeleton. The skeleton  $f^1(?)$  and the reverse sheaf  $f^1(x)$  are the two sides of the same coin, and they correspond to the implicit latent function and the explicit function of a “mathematical tool”, respectively. The function of  $f^1(x)$  is to indicate the distinction between the microscopic and macroscopic perspectives, on one hand, and the function of skeleton  $f^1(?)$  is to invalidate such a distinction, on the other hand. Actually, an element of the macroscopic perspective is an element of the lattice since any equivalent set is a singleton set. Therefore, the aim of  $f^1(x)$  is to distinguishably assign an element of the lattice in the domain by assigning a subset of the lattice in the co-domain. In  $f^1(x)$  one has to distinguish an element from a subset, and in  $f^1(?)$  one has to assimilate an element with a subset by introducing  $\approx$ . That is the reason why a skeleton has double functions in which one contradicts the other, and it is well-defined in terms of the material cause.

Moreover, a skeleton carries the material cause. In real living systems the material cause contributes to the local-global negotiation. Cytoskeleton has a function to maintain an individual cell as a three dimensional structure (intra-cellular level). At the inter-cellular level cytoskeleton reveals a special radiated structure and can contribute to the bondage of cells. Why is it possible? Because a cytoskeleton consists of microtubules, the arrangements of microtubules can be changed depending on its own micro-environment. It is almost impossible to spell out all functions of microtubules in advance. We have to accept the microtubules as a material carrying latent functions.

The relationship between the lattice (microscopic perspective) and the quotient lattice (macroscopic perspective) can be deciphered in terms of molecular dynamics. As mentioned in Section 2, the contrast between the microscopic and macroscopic perspectives can be illustrated by the contrast between the molecular dynamics approach and the dynamical system approach with respect to the notion of concentration. Note that the concentration cannot be obtained only by summing up a huge numerous molecules. The function of individual molecules depends on the micro-environment surrounding the molecule or the “context of molecules”. Even if the number of molecules are the same, the function of a society of molecules is different from each other. We have to pay attention to the difference in quality, and that constitutes the essential discrepancy between the molecular dynamics and the dynamical system with respect to the concentration. There are some mathematical approaches in which the microscopic perspective of biochemical substrates is expressed as a lattice and the macroscopic one is expressed as a quotient lattice [34]. We will apply our idea of dynamical negotiation to the robust biochemical cycle with skeleton according to that framework.

Starting from the definition that a system is not only a collection of parts, we focus on the

biological organization in which two hierarchical levels interact with each other. The two hierarchical levels can correspond to the Intent and the Extent or correspond to the macro- and the microscopic levels, respectively. Since the macroscopic description has the property of a whole and the microscopic one is a collection of elements, the macroscopic description is regarded as a limit (co-limit) of microscopic one no matter whether they contain a structure.

There are several researches on hierarchical systems in terms of the category theory. The operation taking a co-limit called the co-limit functor raises the upper layer, and it yields the hierarchical layers which are different from each other with respect to the logical property [32,33]. These models are, however, too static to understand the biological organization. In our dynamical sheaf scheme, we argue that a sheaf is an ideal and a static case in which the Intent is equivalent to the Extent. This is expressed as a one-to-one relationship between  $\theta_j$  and  $f^1(x)$ . It corresponds exactly to the Cartesian closed category and the Cartesian cut from which an object is independently separated from an observer.

On the other hand, the endo-perspective is in contrast to the Cartesian cut, and it cannot be assumed that an observer cannot look out to the whole world, but this does not mean that an observer's perspective is restricted. An observer always attempts to witness the outside or the rest of his own perspective. The dynamical interfacing between the inside and outside is expressed as a dynamical negotiation between the Intent and the Extent, or the macro- and microscopic perspectives in our dynamical gluing model. On one hand, there is a discrepancy between the Intent and the Extent since an observer cannot look out to the whole world. On the other hand, an observer can empirically observe a phenomenon as one unity or one concept, and that leads to the negotiation.

The negotiation between the Intent and the Extent can be revealed only through the skeleton. Although there are two operations, one is from the lattice (Extent; a collection of elements) to the quotient lattice (Intent; a whole as an equivalence class) and the another is the inverse operation. These two operations contribute to the interaction but not to the negotiation. Here the interaction is defined by the well-defined operations between the Intent and the Extent, and never contains inconsistency. The consistency is kept as long as an element of the lattice cannot be assimilated with a subset of the lattice, while the structure of the lattice is broken without the eye by which the whole structure can be overlooked. The question arises how a lattice can be reconstructed, and our answer is the skeleton. A skeleton introduces an inconsistency by breaking this forbidden clause by the mixing an element with a set. It contributes to reconstruct the lattice. Strictly speaking, the inconsistency contributes to the implementation of a skeleton that can lead to the negotiation different from the interaction. Negotiation is revealed due to the help of the inconsistency outside the formal description. As mentioned before, it is nothing but just the material cause. Since the description and/or expression results from the interaction between the subject and the world from

any perspective, any material property can be expressed as a contradiction, a paradox, a discrepancy and/or an impossibility, and also as the negotiator for the subject and the world. The material cause is expressed at the edge of the formal expression. It negotiates between the formal perspective and the outside with the material nature.

Although the significance of the body and/or the embodied mind has been focused in robotics, there still exists one quest: What does the body or the material remain? It would be too optimistic if one thinks that an embodied mind can be implemented by a particular program (artificial intelligence) encapsulated by a particular material (artificial body). Although the encapsulation may implement real interactions between a formal system (program) and a natural system (body), there are only little negotiation between them. If one has to pay attention to the negotiation between the formal and natural system, one has to implement the outside of the formal system and the inconsistency. Actually, the formal system consisting of data and a program should be redesigned and reconstructed such that the propagating data, that perpetually breaks the program and the skeleton-like structure, repairs and reconstructs the program. It also reveals the perpetually propagating modification of a program through destruction. The outside material nature can then be implemented and re-entered in the formal system and that is the essence of the notion of skeleton.

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## Appendix: A functor in the category theory

In the text, we define the presheaf as a functor from a category of the topological space to a category of the set. Here we define the category and show what a functor is.

A category is an abstract system of the pair of objects and arrows. An object is represented by the capital symbol such as  $A, B, C, \dots$ . An arrow is a directed edge from an object to another, and is represented by the non-capital such as  $f, g, h, \dots$ . An arrow from  $A$  to  $B$  is represented by  $f: A \rightarrow B$ , where  $\text{dom}f = A$  and  $\text{codom}f = B$ . A pair of objects and arrows is a category if it satisfies the following axioms.

- (i) Composition of arrows is an arrow, namely, given  $f: A \rightarrow B$  and  $g: B \rightarrow C$ ,  $gf: A \rightarrow C$  is an arrow.
- (ii) Arrows satisfy associative law, namely, given  $f: A \rightarrow B$ ,  $g: B \rightarrow C$  and  $h: C \rightarrow D$ ,  $hgf = h(gf) = (hg)f$ .
- (iii) Any object  $A$  has an identity arrow  $\text{id}_A: A \rightarrow A$  such that for  $f: X \rightarrow A$ ,  $f = f \text{id}_A$ , and for  $g: A \rightarrow Y$ ,  $g = \text{id}_A g$ .

A category of sets is a category in which an object is a set and an arrow is a map. In a category of the topological space, an object is an open set and an arrow is an inclusion relation. For example, since the inclusion is a partial order, it satisfies the reflective and transitive laws, that leads to (ii) and (iii).

Given categories  $\mathbf{C}$  and  $\mathbf{D}$ , a functor  $F: \mathbf{C} \Rightarrow \mathbf{D}$  is defined as: An object  $C$  in  $\mathbf{C}$  is transformed to  $FC$  in  $\mathbf{D}$ , and an arrow  $f: C \rightarrow C'$  is transformed to  $Ff: FC \rightarrow FC'$ , with the following two conditions satisfied:

- (i) For any identity arrow  $\text{id}_C: C \rightarrow C$  in  $\mathbf{C}$ ,  $F \text{id}_C = \text{id}_{FC}: FC \rightarrow FC$  in  $\mathbf{D}$ .
- (ii) For arrows,  $f: C \rightarrow C'$  and  $g: C' \rightarrow C''$  in  $\mathbf{C}$ ,  $Fgf = Fg(Ff): FC \rightarrow FC''$ .

A presheaf is just defined as a functor from a category of the topological space to a category of the set.