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Semistationary and stationary reflection

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Abstract

We study the relationship between the semistationary reflection principle and stationary reflection principles. We show that for all regular cardinals $\lambda \geq \omega_2$ the semistationary reflection principle in the space $[\lambda]^\omega$ implies that every stationary subset of $E_\omega^\lambda := \{\alpha \in \lambda \mid \text{cf}(\alpha) = \omega\}$ reflects. We also show that for all cardinals $\lambda \geq \omega_3$ the semistationary reflection principle in $[\lambda]^\omega$ does not imply the stationary reflection principle in $[\lambda]^\omega$.

1 Introduction

In this paper we compare the semistationary reflection principle with stationary reflection principles. The notion of semistationary sets and the semistationary reflection principle were introduced by Shelah [10](Ch.XIII §1). They are closely related to the semiproperness of posets. We review this:

Notation 1.1. For countable sets x and y , we write $x \sqsubseteq y$ if $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$.

Definition 1.2 (Shelah [10] Ch.XIII, §1, 1.1.Def.). Let W be a set with $W \supseteq \omega_1$. A subset $S \subseteq [W]^\omega$ is called *semistationary* if the set $\{y \in [W]^\omega \mid (\exists x \in S) x \sqsubseteq y\}$ is stationary in $[W]^\omega$.

Definition 1.3 (Shelah [10] Ch.XIII, §1, 1.5.Def.). For a cardinal $\lambda \geq \omega_2$, $\text{SSR}([\lambda]^\omega)$, the *semistationary reflection principle in $[\lambda]^\omega$* , is the following:

$\text{SSR}([\lambda]^\omega) \equiv$ For every semistationary $S \subseteq [\lambda]^\omega$, there exists $W \subseteq \lambda$ such that $|W| = \omega_1 \subseteq W$ and $S \cap [W]^\omega$ is semistationary in $[W]^\omega$.

In [10](Ch.XIII, §1, 1.4.Claim) Shelah shows that a poset \mathbb{P} is semiproper if and only if \mathbb{P} preserves ω_1 and preserves semistationary subsets of $[W]^\omega$ for every W . He also shows that (\dagger) holds if and only if $\text{SSR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$. Here (\dagger) is the principle, introduced in Foreman-Magidor-Shelah [3], that every poset preserving stationary subset of ω_1 is semiproper. This is known

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to have interesting consequences. Shelah [10](Ch.XII, §2) shows that if Namba forcing is semiproper then (a strong form of) Chang's conjecture holds. Hence (\dagger) implies Chang's conjecture. Also, Foreman-Magidor-Shelah [3] shows that (\dagger) implies precipitousness of the nonstationary ideal over ω_1 .

In this paper we compare the semistationary reflection principle with the stationary reflection principles defined below. For a regular cardinal λ , E_ω^λ denotes the set $\{\alpha \in \lambda \mid \text{cf}(\alpha) = \omega\}$.

Definition 1.4. *For a regular cardinal $\lambda \geq \omega_2$, let $\text{SR}(\lambda)$ denote the following stationary reflection principle:*

$\text{SR}(\lambda) \equiv$ *For every stationary $B \subseteq E_\omega^\lambda$, there exists $\gamma < \lambda$ such that $B \cap \gamma$ is stationary.*

For a cardinal $\lambda \geq \omega_2$, let $\text{SR}([\lambda]^\omega)$ denote the stationary reflection principle in $[\lambda]^\omega$:

$\text{SR}([\lambda]^\omega) \equiv$ *For every stationary $S \subseteq [\lambda]^\omega$, there exists $W \subseteq \lambda$ such that $|W| = \omega_1 \subseteq W$ and $S \cap [W]^\omega$ is stationary in $[W]^\omega$.*

It is easy to see that $\text{SR}([\lambda]^\omega)$ implies $\text{SSR}([\lambda]^\omega)$. (See Section 2.2.)

Our main results are as follows:

Theorem 1.5. *Let λ be a regular cardinal $\geq \omega_2$. Then $\text{SSR}([\lambda]^\omega)$ implies $\text{SR}(\lambda)$.*

Theorem 1.6. *If κ is a supercompact cardinal, then there exists a generic extension in which $\text{SSR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$ but $\text{SR}([\lambda]^\omega)$ does not hold for any $\lambda \geq \omega_3$.*

Foreman-Magidor-Shelah [3] shows that if $\text{SR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$ then (\dagger) holds. Theorem 1.6 claims that the converse is not true. Also, as we prove in Section 5, $\text{SSR}([\omega_2]^\omega)$ implies $\text{SR}([\omega_2]^\omega)$. Theorem 1.6 is optimal in this sense.

This paper is organized as follows: In Section 2 we discuss some preliminaries for this paper. In Section 3 we present a certain type of stationary subset of $[\lambda]^\omega$ which was first introduced by Shelah. This type of stationary set plays a central role in the proofs of both Theorems 1.5 and 1.6. In Section 4 we prove Theorem 1.5. In Section 5 we compare $\text{SSR}([\lambda]^\omega)$ and $\text{SR}([\lambda]^\omega)$. Among other things, we prove Theorem 1.6.

2 Preliminaries

2.1 Notations

We follow the notations of Jech [4]. Here we present those which may not be general.

For a regular cardinal γ and an inaccessible cardinal κ , let $\text{Col}(\gamma, < \kappa)$ denote the Lévy collapse which forces κ to be γ^+ .

For a regular cardinal γ and a limit ordinal $\delta > \gamma$, let E_γ^δ denote the set $\{\alpha \in \delta \mid \text{cf}(\alpha) = \gamma\}$. Note that if $\text{cf}(\delta) > \gamma$ then E_γ^δ is stationary in δ .

For a set x of ordinals let

$$\text{s}\bar{\text{u}}\text{p } x := \sup\{\alpha + 1 \mid \alpha \in x\}.$$

In this paper we use $\text{s}\bar{\text{u}}\text{p}$ rather than \sup . We are mainly interested in sets of ordinals which do not have a greatest element. For such x , $\text{s}\bar{\text{u}}\text{p } x = \sup x$. The merit of using $\text{s}\bar{\text{u}}\text{p}$ is that $\text{s}\bar{\text{u}}\text{p } x$ is a limit ordinal if and only if x does not have a greatest element. This makes our definitions and arguments slightly simpler.

2.2 Basics on stationary sets and semistationary sets

For basics on the notion of club or stationary subsets of $\mathcal{P}_\kappa W$ consult Jech [4]. When $\kappa = \omega_1$, we prefer to use $[W]^\omega$ rather than $\mathcal{P}_{\omega_1} W$. A subset of $[W]^\omega$ is said to be club (stationary) if it is club (stationary) in $\mathcal{P}_{\omega_1} W$. This paper uses the following two facts without any reference:

Fact 2.1 (Kueker [6]). *Let κ be a regular uncountable cardinal, W be a set with $\kappa \subseteq W$, and let $C \subseteq \mathcal{P}_\kappa W$ be a club. Then there exists a function $f : [W]^{<\omega} \rightarrow W$ such that $\{x \in \mathcal{P}_\kappa W \mid f''[x]^{<\omega} \subseteq x \wedge x \cap \kappa \in \kappa\} \subseteq C$. If $\kappa = \omega_1$ then there exists a function $f : [W]^{<\omega} \rightarrow W$ such that $\{x \in \mathcal{P}_\kappa W \mid f''[x]^{<\omega} \subseteq x\} \subseteq C$.*

Fact 2.2 (Menas [9]). *Let κ be a regular uncountable cardinal, and let W and \bar{W} be sets with $\kappa \subseteq W \subseteq \bar{W}$.*

- (1) *If $C \subseteq \mathcal{P}_\kappa W$ is a club then the set $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in C\}$ is a club in $\mathcal{P}_\kappa \bar{W}$. Hence if $\bar{S} \subseteq \mathcal{P}_\kappa \bar{W}$ is stationary then the set $\{\bar{x} \cap W \mid \bar{x} \in \bar{S}\}$ is stationary in $\mathcal{P}_\kappa W$.*
- (2) *If $\bar{C} \subseteq \mathcal{P}_\kappa \bar{W}$ is a club then the set $\{\bar{x} \cap W \mid \bar{x} \in \bar{C}\}$ contains a club in $\mathcal{P}_\kappa W$. Hence if $S \subseteq \mathcal{P}_\kappa W$ is stationary then the set $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in S\}$ is stationary in $\mathcal{P}_\kappa \bar{W}$.*

Basics on semistationary subsets of $[W]^\omega$ was studied in Shelah [10] (Ch.XIII, §1). The following lemma is an analogy of Fact 2.2 for semistationary sets. In the case of (2), a stronger result holds. Part (2) of the following lemma illustrates a unique property of semistationary sets:

Lemma 2.3. *Let W and \bar{W} be sets with $\omega_1 \subseteq W \subseteq \bar{W}$.*

- (1) *If $\bar{S} \subseteq [\bar{W}]^\omega$ is semistationary then the set $\{\bar{x} \cap W \mid \bar{x} \in \bar{S}\}$ is semistationary.*
- (2) *If $S \subseteq [W]^\omega$ is semistationary then S is also semistationary in $[\bar{W}]^\omega$.*

Proof. (1) is clear from Fact 2.2 (1). We prove (2).

Suppose that $S \subseteq [W]^\omega$ is semistationary. Let $T := \{y \in [W]^\omega \mid (\exists x \in S) x \sqsubseteq y\}$ and $\bar{T} := \{\bar{y} \in [\bar{W}]^\omega \mid (\exists x \in S) x \sqsubseteq \bar{y}\}$. Then T is stationary in $[W]^\omega$, and $\bar{T} = \{\bar{y} \in [\bar{W}]^\omega \mid \bar{y} \cap W \in T\}$. Hence \bar{T} is stationary in $[\bar{W}]^\omega$. Therefore S is semistationary in $[\bar{W}]^\omega$. \square

2.3 Basics on reflection principles

In this paper we use the following reflection principles which are generalizations of $\text{SR}([\lambda]^\omega)$ and $\text{SSR}([\lambda]^\omega)$:

Definition 2.4. For a cardinal $\lambda \geq \omega_2$ and a regular cardinal κ with $\omega_2 \leq \kappa \leq \lambda$, let $\text{SSR}([\lambda]^\omega, < \kappa)$ and $\text{SR}([\lambda]^\omega, < \kappa)$ be the following reflection principles:

$\text{SR}([\lambda]^\omega, < \kappa) \equiv$ For every stationary $S \subseteq [\lambda]^\omega$, there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is stationary.

$\text{SSR}([\lambda]^\omega, < \kappa) \equiv$ For every semistationary $S \subseteq [\lambda]^\omega$, there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary.

Here we review basics about the above reflection principles. First we observe that these are generalizations of $\text{SR}([\lambda]^\omega)$ and $\text{SSR}([\lambda]^\omega)$:

Lemma 2.5. Let λ be a cardinal $\geq \omega_2$.

- (1) $\text{SR}([\lambda]^\omega)$ is equivalent to $\text{SR}([\lambda]^\omega, < \omega_2)$.
- (2) $\text{SSR}([\lambda]^\omega)$ is equivalent to $\text{SSR}([\lambda]^\omega, < \omega_2)$.

Proof. (1). It suffices to show that $\text{SR}([\lambda]^\omega)$ implies $\text{SR}([\lambda]^\omega, < \omega_2)$. Before starting, take a surjection $\pi_\alpha : \omega_1 \rightarrow \alpha$ for each $\alpha < \omega_2$ and let $f : \omega_2 \times \omega_1 \rightarrow \omega_2$ be the function defined by $f(\alpha, \xi) = \pi_\alpha(\xi)$ for each $\langle \alpha, \xi \rangle \in \omega_2 \times \omega_1$.

Assume that $\text{SR}([\lambda]^\omega)$ holds. To show that $\text{SR}([\lambda]^\omega, < \omega_2)$ holds, take an arbitrary stationary $S \subseteq [\lambda]^\omega$. We may assume that every element of S is closed under f . By $\text{SR}([\lambda]^\omega)$, we may choose a $W \subseteq \lambda$ such that $|W| = \omega_1 \subseteq W$ and $S \cap [W]^\omega$ is stationary. Note that W is closed under f because stationary many elements of $[W]^\omega$ are closed under f . Because $\omega_1 \subseteq W$, if $\alpha \in W \cap \omega_2$ then $\alpha \subseteq W$. Hence $W \cap \omega_2 \in \omega_2$. Therefore W witnesses that $\text{SR}([\lambda]^\omega, < \omega_2)$ holds for S .

(2). It suffices to show that $\text{SSR}([\lambda]^\omega)$ implies $\text{SSR}([\lambda]^\omega, < \omega_2)$. Assume that $\text{SSR}([\lambda]^\omega)$ holds. Take an arbitrary semistationary $S \subseteq [\lambda]^\omega$. Let $W \subseteq \lambda$ be a witness of $\text{SSR}([\lambda]^\omega)$ for S and let $W' := W \cup \text{süp}(W \cap \omega_2)$. Then $\omega_1 \subseteq W' \cap \omega_2 \in \omega_2$. Moreover $S \cap [W]^\omega$ is semistationary in $[W']^\omega$ by Lemma 2.3. Hence $S \cap [W']^\omega$ is semistationary. Therefore W' witnesses $\text{SSR}([\lambda]^\omega, < \omega_2)$ for S . \square

Next we observe that $\text{SR}([\lambda]^\omega, < \kappa)$ implies $\text{SSR}([\lambda]^\omega, < \kappa)$.

Lemma 2.6. Let λ be a cardinal $\geq \omega_2$ and κ be a regular cardinal with $\omega_2 \leq \kappa \leq \lambda$. Then $\text{SSR}([\lambda]^\omega, < \kappa)$ is equivalent to the following principle:

$\text{SSR}'([\lambda]^\omega, < \kappa) \equiv$ For every stationary $S \subseteq [\lambda]^\omega$, there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary.

Therefore $\text{SR}([\lambda]^\omega, < \kappa)$ implies $\text{SSR}([\lambda]^\omega, < \kappa)$.

Proof. It suffices to show that SSR' implies SSR . Assume that $\text{SSR}'([\lambda]^\omega, < \kappa)$ holds. Take an arbitrary semistationary $S \subseteq [\lambda]^\omega$. Then $T := \{y \in [\lambda]^\omega \mid (\exists x \in S) x \sqsubseteq y\}$ is stationary. Let $W \subseteq \lambda$ be a witness of $\text{SSR}'([\lambda]^\omega, < \kappa)$ for T . Here note that

$$\{y \in [W]^\omega \mid (\exists x \in T \cap [W]^\omega) x \sqsubseteq y\} = \{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \sqsubseteq y\}.$$

Hence $\{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \sqsubseteq y\}$ is stationary and thus $S \cap [W]^\omega$ is semistationary. Therefore W witnesses $\text{SSR}([\lambda]^\omega, < \kappa)$ for S . \square

The following is very easy:

Lemma 2.7. *Let λ and λ' be cardinals and let κ and κ' be regular cardinals such that $\omega_2 \leq \kappa \leq \kappa' \leq \lambda' \leq \lambda$.*

- (1) $\text{SR}([\lambda]^\omega, < \kappa)$ implies $\text{SR}([\lambda']^\omega, < \kappa)$.
- (2) $\text{SSR}([\lambda]^\omega, < \kappa)$ implies $\text{SSR}([\lambda']^\omega, < \kappa')$.

Proof. (1) Assume that $\text{SR}([\lambda]^\omega, < \kappa)$ holds. Take an arbitrary stationary $S' \subseteq [\lambda']^\omega$. Then $S := \{x \in [\lambda]^\omega \mid x \cap \lambda' \in S'\}$ is stationary. Hence there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is stationary. Let $W' := W \cap \lambda'$. Then $W' \in \mathcal{P}_{\kappa'} \lambda'$ and $\omega_1 \subseteq W' \cap \kappa \in \kappa$. Moreover $S' \cap [W']^\omega = \{x \cap W' \mid x \in S \cap [W]^\omega\}$. Thus $S' \cap [W']^\omega$ is stationary. Therefore W' witnesses $\text{SR}([\lambda']^\omega, < \kappa)$ for S' .

(2) Assume that $\text{SSR}([\lambda]^\omega, < \kappa)$ holds. Take an arbitrary semistationary $S' \subseteq [\lambda']^\omega$. Let S, W and W' be as in (1). Then, using Lemma 2.3, the same argument as (1) shows that $S' \cap [W']^\omega$ is semistationary. Let $W'' := W' \cup \sup(W' \cap \kappa')$. Then $W'' \in \mathcal{P}_{\kappa'} \lambda'$ and $\omega_1 \subseteq W'' \cap \kappa' \in \kappa'$. Moreover $S' \cap [W']^\omega$ is semistationary in $[W'']^\omega$ by Lemma 2.3. Hence $S' \cap [W'']^\omega$ is semistationary. Therefore W'' witnesses $\text{SSR}([\lambda']^\omega, < \kappa')$ for S' . \square

We end this section with the following:

Lemma 2.8. *Let λ be a cardinal $\geq \omega_2$ and κ be a regular cardinal with $\omega_2 \leq \kappa \leq \lambda$.*

- (1) (Feng-Jech [2]) *Assume that $\text{SR}([\lambda]^\omega, < \kappa)$ holds. If $S \subseteq [\lambda]^\omega$ is stationary then the set $\{W \in \mathcal{P}_\kappa \lambda \mid S \cap [W]^\omega \text{ is stationary}\}$ is stationary in $\mathcal{P}_\kappa \lambda$.*
- (2) *Assume that $\text{SSR}([\lambda]^\omega, < \kappa)$ holds. If $S \subseteq [\lambda]^\omega$ is semistationary then the set $\{W \in \mathcal{P}_\kappa \lambda \mid S \cap [W]^\omega \text{ is semistationary}\}$ is co-bounded, that is, there exists $W^* \in \mathcal{P}_\kappa \lambda$ such that $S \cap [W]^\omega$ is semistationary for every $W \in \mathcal{P}_\kappa \lambda$ with $W^* \supseteq W$.*

Proof. (2) is clear from Lemma 2.3 (2). We prove (1).

Take an arbitrary stationary $S \subseteq [\lambda]^\omega$ and an arbitrary function $f : [\lambda]^{<\omega} \rightarrow \lambda$. It suffices to find a $W \in \mathcal{P}_\kappa \lambda$ such that $W \cap \kappa \in \kappa$ and W is closed under f . Let S' be the set of all $x \in S$ closed under f . Then S' is stationary. Hence there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S' \cap [W]^\omega$ is stationary. Note that W is closed under f because stationary many elements of $[W]^\omega$ are closed under f . Moreover $S \cap [W]^\omega$ is stationary because $S' \subseteq S$. Therefore W is a desired one. \square

3 Sup depending stationary sets

Here we present a type of stationary set which plays a central role in the proofs of both Theorems 1.5 and 1.6:

Lemma 3.1 (The case when $n = 1$ is due to Shelah). *Suppose that $0 < n < \omega$ and that $\mu_0 < \mu_1 < \dots < \mu_n$ are regular uncountable cardinals. Moreover, suppose that $A \subseteq E_{\omega}^{\mu_0}$ is stationary and that, for each m with $1 \leq m \leq n$, $\langle A_{\alpha}^m \mid \alpha \in \mu_{m-1} \rangle$ is a sequence of stationary subsets of $E_{\omega}^{\mu_m}$. Let S be the set of all $x \in \mathcal{P}_{\mu_0 \mu_n}$ such that*

- (1) $x \cap \mu_0 \in A$,
- (2) $\text{s\ddot{u}p}(x \cap \mu_m) \in A_{\text{s\ddot{u}p}(x \cap \mu_{m-1})}^m$ for each m with $1 \leq m \leq n$.

Then S is stationary in $\mathcal{P}_{\mu_0 \mu_n}$.

This type of stationary set was considered by Shelah, and in Shelah-Shioya [12] and Shelah [11], such sets are used to obtain consequences of the stationary reflection principle. The proof of the above lemma for the case when $n = 1$ can be found in Shelah-Shioya [12]. Although there are no difficulties in generalization, we give the complete proof of Lemma 3.1.

We use the following game \mathfrak{D} :

Definition 3.2. *Suppose that $0 < n < \omega$ and that $\mu_0 < \mu_1 < \dots < \mu_n$ are regular uncountable cardinals. For an $\alpha \in \mu_0$ and a function $f : [\mu_n]^{<\omega} \rightarrow \mu_n$ let $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$ be the following two players game of length ω :*

In the k -th stage, first Player I plays a $\langle \beta_k^m \mid 1 \leq m \leq n \rangle$ and then Player II plays a $\langle \gamma_k^m \mid 1 \leq m \leq n \rangle$ so that $\beta_k^m \leq \gamma_k^m < \mu_m$ for each m .

I	$\beta_0^1, \dots, \beta_0^n$	$\beta_1^1, \dots, \beta_1^n$	$\beta_k^1, \dots, \beta_k^n$
II	$\gamma_0^1, \dots, \gamma_0^n$	$\gamma_1^1, \dots, \gamma_1^n$	$\gamma_k^1, \dots, \gamma_k^n$

Player II wins if $\text{cl}_f(\alpha \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\}) \cap \mu_0 = \alpha$, where $\text{cl}_f(x)$ denotes the closure of x under f . Otherwise Player I wins.

Note that $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$ is an open game for Player I. Hence it is determined. The following is a key lemma:

Lemma 3.3. *Suppose that $0 < n < \omega$ and that $\mu_0 < \mu_1 < \dots < \mu_n$ are regular uncountable cardinals. Then for every function $f : [\mu_n]^{<\omega} \rightarrow \mu_n$, there are club many $\alpha \in \mu_0$ such that Player II has a winning strategy in $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$.*

Proof. Take an arbitrary function $f : [\mu_n]^{<\omega} \rightarrow \mu_n$ and let A be the set of all $\alpha \in \mu_0$ such that Player I has a winning strategy in $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$. It suffices to show that A is nonstationary.

Assume that A is stationary. For each $\alpha \in A$, take a winning strategy σ_{α} for Player I in $\mathfrak{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha, f)$. Let θ be a sufficiently large regular cardinal.

Then we can take an elementary submodel M of $\langle \mathcal{H}_\theta, \in, \langle \sigma_\alpha \mid \alpha \in A \rangle \rangle$ such that $\alpha^* := M \cap \mu_0 \in A$.

By induction on k , we construct a sequence of moves $\langle \beta_k^1, \dots, \beta_k^n, \gamma_k^1, \dots, \gamma_k^n \mid k \in \omega \rangle$ in $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$ so that $\gamma_k^1, \dots, \gamma_k^n \in M$ for each $k \in \omega$. Suppose that $k \in \omega$ and that $\langle \beta_l^1, \dots, \beta_l^n \rangle$ and $\langle \gamma_l^1, \dots, \gamma_l^n \rangle$ have been chosen for each $l < k$. Let

$$\langle \beta_k^1, \dots, \beta_k^n \rangle := \sigma_{\alpha^*}(\langle \gamma_l^1, \dots, \gamma_l^n \mid l < k \rangle),$$

and for each m with $1 \leq m \leq n$, let

$$\gamma_k^m := \sup\{\pi_m \circ \sigma_\alpha(\langle \gamma_l^1, \dots, \gamma_l^n \mid l < k \rangle) \mid \alpha \in A\},$$

where $\pi_m : \mu_1 \times \dots \times \mu_n \rightarrow \mu_m$ is the m -th projection. Clearly $\beta_k^m \leq \gamma_k^m$ for each m . Note that $\gamma_k^m < \mu_m$ since μ_m is regular and $A \subseteq \mu_0 < \mu_m$. Note also that $\gamma_k^m \in M$ because $\langle \gamma_l^1, \dots, \gamma_l^n \mid l < k \rangle \in M$. This completes the induction.

First note that $\langle \beta_k^1, \dots, \beta_k^n, \gamma_k^1, \dots, \gamma_k^n \mid k \in \omega \rangle$ is a sequence of moves in $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$ in which Player I has played according to winning strategy σ_{α^*} . Hence Player I wins. On the other hand, $\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\} \subseteq M$, and M is closed under f . Thus $\text{cl}_f(\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\}) \cap \mu_0 \subseteq M \cap \mu_0 = \alpha^*$. Therefore $\text{cl}_f(\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\}) = \alpha^*$, so that Player II wins with this sequence of moves. This is a contradiction. \square

Now we prove Lemma 3.1:

Proof of Lemma 3.1. We prove Lemma 3.1 by induction on n . Suppose that $n = 1$ or that $n > 1$ and that the lemma holds for $n - 1$. We prove the lemma for n . Take an arbitrary function $f : [\mu_n]^{<\omega} \rightarrow \mu_n$. It suffices to find $x^* \in S$ such that x^* is closed under f and $x^* \cap \mu_0 \in \mu_0$.

By Lemma 3.3, there exists $\alpha^* \in A$ such that Player II has a winning strategy σ^* in $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$. Let S' be the set of all $y \in \mathcal{P}_{\mu_1 \mu_n}$ such that

- (1) $y \cap \mu_1 \in A_{\alpha^*}^1$,
- (2) $\sup(y \cap \mu_m) \in A_{\sup(y \cap \mu_{m-1})}^m$ for each m with $2 \leq m \leq n$.

Then S' is stationary in $\mathcal{P}_{\mu_1 \mu_n}$. If $n = 1$ then this is clear. ((2) claims nothing.) If $n > 1$ then this follows from the lemma for $n - 1$.

Choose $y^* \in S'$ such that $\alpha^* \subseteq y^*$ and such that y^* is closed under σ^* and f . For each m with $1 \leq m \leq n$, take a cofinal sequence $\langle \beta_k^m \mid k \in \omega \rangle$ in $y^* \cap \mu_m$. Moreover let

$$\langle \gamma_k^1, \dots, \gamma_k^n \rangle := \sigma^*(\langle \beta_l^1, \dots, \beta_l^n \mid l < k \rangle)$$

for each $k \in \omega$. Note that $\langle \gamma_k^m \mid k \in \omega \rangle$ is a cofinal sequence in $y^* \cap \mu_m$ for each m with $1 \leq m \leq n$. This is because y^* is closed under σ^* . Finally let $x^* := \text{cl}_f(\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\})$. We show that $x^* \in S$ and $x^* \cap \mu_0 \in \mu_0$.

First note that $x^* \subseteq y^*$ because $\alpha^* \cup \{\gamma_k^m \mid 1 \leq m \leq n \wedge k \in \omega\} \subseteq y^*$ and y^* is closed under f . Then, because $\langle \gamma_k^m \mid k \in \omega \rangle$ is cofinal in $y^* \cap \mu_m$, $\sup(x^* \cap \mu_m) = \sup(y^* \cap \mu_m)$ for each m with $1 \leq m \leq n$. Hence, by (2) above,

- (i) $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_m) \in A_{\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_{m-1})}^m$ for each m with $2 \leq m \leq n$.

Moreover $x^* \cap \mu_0 = \alpha^*$. This is because σ^* is a winning strategy for Player II in $\mathcal{D}(\langle \mu_0, \mu_1, \dots, \mu_n \rangle, \alpha^*, f)$. Also recall that $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_1) = \text{s}\bar{\text{u}}\text{p}(y^* \cap \mu_1) \in A_{\alpha^*}^1$. Thus

- (ii) $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_0) \in A$,
 (iii) $\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_1) \in A_{\text{s}\bar{\text{u}}\text{p}(x^* \cap \mu_1)}^1$.

Now it follows from (i), (ii) and (iii) that $x^* \in S$ and $x^* \cap \mu_0 \in \mu_0$.

This completes the proof. \square

4 $\text{SSR}([\lambda]^\omega)$ and $\text{SR}(\lambda)$

In this section, we prove Theorem 1.5. In fact, we prove the following more general theorem:

Theorem 4.1. *Let λ and κ be regular cardinals such that $\omega_2 \leq \kappa \leq \lambda$. Then $\text{SSR}([\lambda]^\omega, < \kappa)$ implies $\text{SR}(\lambda)$.*

By Lemma 2.5 and 2.7, Theorem 1.5 follows from Theorem 4.1. Theorem 4.1 can be easily obtained from Lemma 3.1 and the following lemma:

Lemma 4.2. *Let λ be a cardinal and κ be a regular cardinal such that $\omega_2 \leq \kappa \leq \lambda$. Assume that $S \subseteq [\lambda]^\omega$ and that there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary. Let $W^* \in \mathcal{P}_\kappa \lambda$ be such that*

- (1) $\omega_1 \subseteq W^* \cap \kappa \in \kappa$ and $S \cap [W^*]^\omega$ is semistationary,
- (2) for every $W \in \mathcal{P}_\kappa \lambda$, if $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary then $\text{s}\bar{\text{u}}\text{p} W^* \leq \text{s}\bar{\text{u}}\text{p} W$.

Then

$$S_0 := \{y \in [W^*]^\omega \mid (\exists x \in S \cap [W^*]^\omega) x \sqsubseteq y \wedge \text{s}\bar{\text{u}}\text{p} x = \text{s}\bar{\text{u}}\text{p} y\}$$

is stationary in $[W^*]^\omega$.

Proof. Assume that S_0 is not stationary. Then $S_1 := \{y \in [W^*]^\omega \mid (\exists x \in S \cap [W^*]^\omega) x \sqsubseteq y \wedge \text{s}\bar{\text{u}}\text{p} x < \text{s}\bar{\text{u}}\text{p} y\}$ is stationary. For each $y \in S_1$, choose $x_y \in S$ with $x_y \sqsubseteq y$ and $\text{s}\bar{\text{u}}\text{p} x_y < \text{s}\bar{\text{u}}\text{p} y$ and choose $\alpha_y \in y$ with $\text{s}\bar{\text{u}}\text{p} x_y \leq \alpha_y$. Then there exists $\alpha' \in W^*$ such that $S' := \{y \in S_1 \mid \alpha_y = \alpha'\}$ is stationary. Let $W' := W^* \cap \alpha'$. Clearly $W' \in \mathcal{P}_\kappa \lambda$ and $\omega_1 \subseteq W' \cap \kappa \in \kappa$. Moreover $\text{s}\bar{\text{u}}\text{p} W' < \text{s}\bar{\text{u}}\text{p} W^*$. Hence if we show that $S \cap [W']^\omega$ is semistationary then this contradicts property (2) of W^* .

First note that if $y \in S'$ then $x_y \in [W']^\omega$ and $x_y \sqsubseteq y \cap W'$. Thus

$$\{y \cap W' \mid y \in S'\} \subseteq \{z \in [W']^\omega \mid (\exists x \in S \cap [W']^\omega) x \sqsubseteq z\}.$$

Now the left side is stationary in $[W']^\omega$ because S' is stationary in $[W^*]^\omega$. Therefore the right side is stationary, that is, $S \cap [W']^\omega$ is semistationary.

This completes the proof. \square

Proof of Theorem 4.1. Assume that $\text{SSR}([\lambda]^\omega, < \kappa)$ holds. Take an arbitrary stationary $B \subseteq E_\omega^\lambda$. We show that B reflects.

Take a pairwise disjoint sequence $\langle B_\alpha \mid \alpha \in \omega_1 \rangle$ of stationary subsets of B . Let S be the set of all $x \in [\lambda]^\omega$ such that $x \cap \omega_1 \in \omega_1$ and $\text{süp } x \in B_{x \cap \omega_1}$. Then S is stationary by Lemma 3.1. By Lemma 4.2, there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S_0 := \{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega) x \sqsubseteq y \wedge \text{süp } x = \text{süp } y\}$ is stationary in $[W]^\omega$. Here note that $S_0 \subseteq S$. Hence $S \cap [W]^\omega$ is stationary. We claim that $\text{cf}(\text{süp } W) > \omega$.

Clearly $\text{süp } W$ is a limit ordinal. Assume that $\text{cf}(\text{süp } W) = \omega$. Then $C := \{y \in [W]^\omega \mid \text{süp } y = \text{süp } W\}$ is club in $[W]^\omega$. But if $y_1, y_2 \in S \cap C$ then $\text{süp}(y_1 \cap \omega_1) = \text{süp}(y_2 \cap \omega_1)$ by the construction of S . Hence $|\{y \cap \omega_1 \mid y \in S_0 \cap C\}| \leq 1$. This contradicts $\omega_1 \subseteq W$ and $S \cap [W]^\omega$ is stationary.

Now $\text{cf}(\text{süp } W) > \omega$ and $S \cap [W]^\omega$ is stationary. Hence $\{\text{süp } y \mid y \in S \cap [W]^\omega\}$ is stationary in $\text{süp } W$. Recall that $\text{süp } y \in B$ for every $y \in S$. Therefore $B \cap \text{süp } W$ is stationary. This completes the proof. \square

5 $\text{SSR}([\lambda]^\omega)$ and $\text{SR}([\lambda]^\omega)$

As we mentioned in Section 1, first we prove that $\text{SSR}([\omega_2]^\omega)$ implies $\text{SR}([\omega_2]^\omega)$ and thus that they are equivalent. This is essentially proved in Todorćević [13]. After that, we prove Theorem 1.6.

Theorem 5.1. $\text{SSR}([\omega_2]^\omega)$ implies $\text{SR}([\omega_2]^\omega)$.

Proof. Assume that $\text{SSR}([\omega_2]^\omega)$ holds. To show that $\text{SR}([\omega_2]^\omega)$ holds, take an arbitrary stationary $S \subseteq [\omega_2]^\omega$. Fix a bijection $\pi_\alpha : \omega_1 \rightarrow \alpha$ for each $\alpha \in [\omega_1, \omega_2)$. We may assume that if $x \in S$ then $\omega_1 < \text{süp } x$ and x is closed under $\pi_\alpha, \pi_\alpha^{-1}$ for each $\alpha \in x \setminus \omega_1$.

By Lemma 2.5 let α^* be the least $\alpha \in \omega_2$ such that $S \cap [\alpha]^\omega$ is semistationary. Then let S_0 be the set of all $y \in [\alpha^*]^\omega$ such that

- (1) for some $x \in S \cap [\alpha^*]^\omega$, $x \sqsubseteq y$ and $\text{süp } x = \text{süp } y$,
- (2) y is closed under $\pi_\alpha, \pi_\alpha^{-1}$ for each $\alpha \in y \setminus \omega_1$.

By Lemma 4.2, S_0 is stationary in $[\alpha^*]^\omega$. For each $y \in S_0$, choose an $x_y \in S \cap [\alpha^*]^\omega$ witnessing (1).

Note that if $y \in S_0$ then

$$y \cap \alpha = \pi_\alpha''(y \cap \omega_1) = \pi_\alpha''(x_y \cap \omega_1) = x_y \cap \alpha$$

for each $\alpha \in x_y \setminus \omega_1$. Then, because $\text{süp } y = \text{süp } x_y$, $y = x_y$ for every $y \in S_0$. Hence $S_0 \subseteq S \cap [\alpha^*]^\omega$. Therefore $S \cap [\alpha^*]^\omega$ is stationary. This completes the proof. \square

Now we turn our attention to Theorem 1.6. It is well-known that if a λ -supercompact cardinal is Lévy collapsed to ω_2 then $\text{SR}([\lambda]^\omega)$ holds. It was shown by Shelah [10] that collapsing a λ -strongly compact cardinal suffices to obtain a model of $\text{SSR}([\lambda]^\omega)$. First we review this:

Lemma 5.2 (Shelah [10] Ch.XIII, §1, 1.6.Claim, 1.10.Claim). *Suppose that κ is a λ -strongly compact cardinal, where λ is a cardinal $\geq \kappa$. Then $\text{SSR}([\lambda]^\omega, < \kappa)$ holds. Moreover if γ is a regular uncountable cardinal $< \kappa$ then $\Vdash_{\text{Col}(\gamma, < \kappa)} \text{"SR}([\lambda]^\omega, < \kappa)\text{"}$.*

Proof. Both statements can be proved by similar arguments, but the latter is slightly harder than the former. We will prove only the latter.

We discuss some preliminaries in V . Take a fine ultrafilter U over $\mathcal{P}_\kappa \lambda$. Let M be the transitive collapse of $\text{Ult}(V, U)$, and let $j : V \rightarrow M$ be the ultrapower map. Moreover, let $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$ be a function such that $f(W) = W \cup \text{süp}(W \cap \kappa)$ for each $W \in \mathcal{P}_\kappa \lambda$ and let $W^* := [f]_U \in M$. Then $j''\lambda \subseteq W^*$, and in M , $W^* \in \mathcal{P}_{j(\kappa)} j(\lambda)$ and $\omega_1 \subseteq W^* \cap j(\kappa) \in j(\kappa)$.

Suppose that γ is a regular uncountable cardinal $< \kappa$ and that G is a $\text{Col}(\gamma, < \kappa)$ -generic filter over V . In $V[G]$, take an arbitrary stationary $S \subseteq [\lambda]^\omega$. We must show that, in $V[G]$, there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary.

Let \bar{G} be a $\text{Col}(\gamma, < j(\kappa))$ -generic filter over V with $\bar{G} \cap \text{Col}(\gamma, < \kappa) = G$. We work in $V[\bar{G}]$. Define a map $\bar{j} : V[\bar{G}] \rightarrow M[\bar{G}]$ by $\bar{j}(\dot{a}_{\bar{G}}) = j(\dot{a})_{\bar{G}}$ for each $\text{Col}(\gamma, < \kappa)$ -name $\dot{a} \in V$. Then \bar{j} is well-defined, and $\bar{j} : V[\bar{G}] \rightarrow M[\bar{G}]$ is an elementary embedding which extends j . For simplicity of notation, we let j denote \bar{j} .

Note that S remains stationary because $V[\bar{G}]$ is a γ -closed forcing extension of $V[G]$. Hence $\{j''x \mid x \in S\}$ is stationary in $[j''\lambda]^\omega$. Moreover, for each $x \in S$, $j''x = j(x)$ because x is countable in $V[G]$. Thus $\{j''x \mid x \in S\} \subseteq j(S)$, and therefore $j(S) \cap [j''\lambda]^\omega$ is stationary. Then $j(S) \cap [W^*]^\omega$ is semistationary by Lemma 2.3 (2). This is also true in $M[\bar{G}]$. Hence W^* witnesses that the following holds in $M[\bar{G}]$:

There exists $W \in \mathcal{P}_{j(\kappa)} j(\lambda)$ such that $\omega_1 \subseteq W \cap j(\kappa) \in j(\kappa)$ and $j(S) \cap [W]^\omega$ is semistationary.

Therefore, by the elementarity of j , it holds in $V[G]$ that there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary. This completes the proof. \square

We will prove that collapsing a λ -strongly compact cardinal does not suffice to obtain a model of $\text{SR}([\lambda]^\omega)$. The core of Theorem 1.6 is the following theorem. As we see later, Theorem 1.6 will be obtained by further Lévy collapsing κ to ω_2 .

Theorem 5.3. *If κ is a supercompact cardinal, then there exists a generic extension in which κ is a strongly compact cardinal and $\text{SR}([\kappa^+]^\omega, < \kappa)$ does not hold.*

First we prove Theorem 5.3. Krueger [7] constructed a model in which κ is strongly compact but $S(\kappa, \kappa^+) := \{x \in \mathcal{P}_\kappa \kappa^+ \mid \text{o.t.}(x) = (x \cap \kappa)^+\}$ is not stationary. (Note that $S(\kappa, \kappa^+)$ is stationary if κ is κ^+ -supercompact.) We show that $\text{SR}([\kappa^+]^\omega, < \kappa)$ does not hold in this model.

We start with a review of Krueger's model. Krueger's model was obtained from a model with κ supercompact by two step forcing extension. The first step forces a partial square principle at κ with preserving supercompactness of κ . This type of partial square principle was first introduced by Baumgartner in his unpublished note, and Apter-Cummings [1] showed that it can hold at a supercompact cardinal. In fact, the first step of Krueger's construction is due to Apter-Cummings [1]. The second step is the iteration of Prikry forcing below κ which was developed by Magidor [8]. We summarize basic properties of these forcings below.

Definition 5.4. *For an uncountable cardinal κ and an $E \subseteq \text{Lim}(\kappa^+)$, let \square_κ^E be the following principle:*

$\square_\kappa^E \equiv$ *There exists a sequence $\langle c_\beta \mid \beta \in E \rangle$ such that*

- (1) *c_β is a club in β .*
- (2) *if $\text{cf}(\beta) < \kappa$ then $\text{o.t.}(c_\beta) < \kappa$,*
- (3) *if $\beta' \in \text{Lim}(c_\beta)$ then $c_{\beta'} = c_\beta \cap \beta'$,*

for each $\beta, \beta' \in E$.

We call a sequence $\langle c_\beta \mid \beta \in E \rangle$ satisfying (1)-(3) above a \square_κ^E -sequence.

The proof of the following fact can be also found in Krueger [7]:

Fact 5.5 (Apter-Cummings [1]). *Assume that κ is a supercompact cardinal. Then there exists a poset \mathbb{P} with the following properties:*

- (1) *\mathbb{P} preserves supercompactness of κ .*
- (2) *$\Vdash_{\mathbb{P}}$ “ \square_κ^E holds for*

$$E = \text{Lim}(\kappa^+) \setminus \bigcup \{E_{\alpha^+}^\kappa \mid \alpha \text{ is a measurable cardinal } < \kappa\}.”.$$

Fact 5.6 (Magidor [8]). *Assume that κ is a supercompact cardinal. Then there exists a poset \mathbb{Q} with the following properties:*

- (1) *\mathbb{Q} has the κ^+ -c.c.*
- (2) *$\Vdash_{\mathbb{Q}}$ “ κ is strongly compact”.*
- (3) *For every measurable cardinal $\alpha < \kappa$, $\Vdash_{\mathbb{Q}}$ “ $\text{cf}(\alpha) = \omega$ ”.*
- (4) *For every measurable cardinal $\alpha < \kappa$, \mathbb{Q} can be factored as $\mathbb{Q}_{\leq \alpha} * \dot{\mathbb{Q}}_{> \alpha}$ so that $\mathbb{Q}_{\leq \alpha}$ has the α^+ -c.c. and $\Vdash_{\mathbb{Q}_{\leq \alpha}}$ “ $\dot{\mathbb{Q}}_{> \alpha}$ does not add subsets of α^+ ”.*

Before starting the proof of Theorem 5.3, we give a technical lemma:

Lemma 5.7. *Suppose that κ and θ are regular cardinals with $\omega_2 \leq \kappa < \theta$. Let M be an elementary submodel of $\langle \mathcal{H}_\theta, \in \rangle$ such that $M \cap \kappa \in \kappa$ and such that both $\text{cf}(M \cap \kappa)$ and $\text{cf}(\text{süp}(M \cap \kappa^+))$ are uncountable. Then $M \cap \kappa^+$ is ω -closed, that is, $\text{süp } b \in M$ for every countable $b \subseteq M \cap \kappa^+$.*

Proof. We prove this by contradiction. Assume that $b \subseteq M \cap \kappa^+$ is countable and that $\sup b \notin M \cap \kappa^+$. Then b does not have a greatest element. Also, b is bounded in $M \cap \kappa^+$ because $\text{cf}(\sup(M \cap \kappa^+))$ is uncountable. Let β^* be the least element of $(M \cap \kappa^+) \setminus \sup b$. Note that β^* is a limit ordinal and $\sup(M \cap \beta^*) = \sup b < \beta^*$. Take an increasing continuous cofinal map $\sigma : \text{cf}(\beta^*) \rightarrow \beta^*$.

First assume that $\text{cf}(\beta^*) < \kappa$. Then $\text{cf}(\beta^*) \subseteq M$ because $\text{cf}(\beta^*) \in M \cap \kappa \in \kappa$. Hence $\text{ran } \sigma \subseteq M$ and thus $M \cap \beta^*$ is cofinal in β^* . This contradicts $\sup(M \cap \beta^*) < \beta^*$.

Next assume that $\text{cf}(\beta^*) = \kappa$. Then it is easy to see that $\sup(M \cap \beta^*) = \sup(\sigma''(M \cap \kappa))$. Hence $\text{cf}(\sup(M \cap \beta^*))$ is uncountable because $\text{cf}(M \cap \kappa)$ is uncountable. But this contradicts $\sup(M \cap \beta^*) = \sup b$ and b is countable.

This completes the proof. \square

Now we prove Theorem 5.3:

Proof of Theorem 5.3. Assume that κ is supercompact in V . Let V_0 be a forcing extension of V by the poset \mathbb{P} of Fact 5.5, and let V_1 be a forcing extension of V_0 by the poset \mathbb{Q} of Fact 5.6. It suffices to show that $\text{SR}([\kappa^+]^\omega, < \kappa)$ does not hold in V_1 . Before starting, we summarize properties of V_0 and V_1 .

In V_0 , let $E := \text{Lim}(\kappa^+) \setminus \bigcup \{E_{\alpha^+}^\kappa \mid \alpha \text{ is a measurable cardinal } < \kappa\}$. Then \square_κ^E holds in V_0 . Let $\langle c_\gamma \mid \gamma \in E \rangle$ be a \square_κ^E -sequence. Note that, in V_0 , there are unboundedly many measurable cardinals below κ .

V_1 is a κ^+ -c.c. forcing extension of V_0 . Moreover, in V_1 , the following hold:

(*1) If $\alpha < \kappa$ is a measurable cardinal in V_0 the $\text{cf}(\alpha) = \omega$.

(*2) Suppose that $\alpha < \kappa$ is measurable in V_0 . Let $\gamma := (\alpha^+)^{V_0}$. Then $(E_\alpha^\gamma)^{V_0}$ is a stationary subset of E_ω^γ .

(*3) $E_\omega^{\kappa^+} \subseteq E$.

(*2) and (*3) hold by Fact 5.6 (4).

Now we show that $\text{SR}([\kappa^+]^\omega, < \kappa)$ does not hold in V_1 . We work in V_1 . We show that there exists a stationary $S \subseteq [\kappa^+]^\omega$ such that $\{W \in \mathcal{P}_\kappa \kappa^+ \mid S \cap [W]^\omega \text{ is stationary}\}$ is nonstationary in $\mathcal{P}_\kappa \kappa^+$. By Lemma 2.8 this suffices. S will be constructed using Lemma 3.1.

First take a pairwise disjoint partition $\langle A_\xi \mid \xi < \omega_1 \rangle$ of E_ω^κ into stationary sets. Next take an injection $\sigma : E_\omega^\kappa \rightarrow \kappa$ such that, for every $\alpha \in \kappa$, $\sigma(\alpha) > \alpha$ and $\sigma(\alpha)$ is measurable in V_0 . Then let $B_\alpha := (E_{\sigma(\alpha)}^\kappa)^{V_0}$ for each $\alpha \in E_\omega^\kappa$. Note the following:

- $\langle B_\alpha \mid \alpha \in E_\omega^\kappa \rangle$ is a pairwise disjoint sequence of stationary subsets of $E_\omega^{\kappa^+}$.
- For each $\alpha \in E_\omega^\kappa$ and each $\beta \in B_\alpha$, $\text{cf}^{V_0}(\beta) > \alpha$.

Now let S be the set of all $x \in [\kappa^+]^\omega$ such that

- (1) $\sup(x \cap \kappa) \in A_{\sup(x \cap \omega_1)}$,
- (2) $\sup(x) \in B_{\sup(x \cap \kappa)}$.

Then S is stationary by Lemma 3.1. We show that $\{W \in \mathcal{P}_{\kappa\kappa^+} \mid S \cap [W]^\omega \text{ is stationary}\}$ is nonstationary. Take a sufficiently large regular cardinal θ and let Ω be the set of all $M \in \mathcal{P}_{\kappa}\mathcal{H}_\theta$ such that $M \cap \kappa \in \kappa$ and $M \prec \langle \mathcal{H}_\theta, \in, \mathcal{H}_\theta^{V_0}, \kappa, \langle c_\gamma \mid \gamma \in E \rangle \rangle$. It suffices to show that if $M \in \Omega$ then $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. The proof of this splits into two cases:

Case 1: $\text{cf}(M \cap \kappa) = \omega$ or $\text{cf}(\text{süp}(M \cap \kappa^+)) = \omega$.

First suppose that $\text{cf}(M \cap \kappa) = \omega$. Then $C := \{x \in [M \cap \kappa^+]^\omega \mid \text{süp}(x \cap \kappa) = M \cap \kappa\}$ is club in $[M \cap \kappa^+]^\omega$. Note that if $x, y \in S \cap C$ then $\text{süp}(x \cap \omega_1) = \text{süp}(y \cap \omega_1)$ by (1) of the construction of S . Thus $|\{\text{süp}(x \cap \omega_1) \mid x \in S \cap C\}| \leq 1$. But $\omega_1 \subseteq M \cap \kappa^+$. Hence $S \cap C$ is nonstationary in $[M \cap \kappa^+]^\omega$. Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary.

The case when $\text{cf}(\text{süp}(M \cap \kappa^+)) = \omega$ is similar. Suppose that $\text{cf}(\text{süp}(M \cap \kappa^+)) = \omega$. Then $C := \{x \in [M \cap \kappa^+]^\omega \mid \text{süp}(x) = \text{süp}(M \cap \kappa^+)\}$ is club in $[M \cap \kappa^+]^\omega$. If $x, y \in S \cap C$ then $\text{süp}(x \cap \kappa) = \text{süp}(y \cap \kappa)$ by (2) of the construction of S and thus $\text{süp}(x \cap \omega_1) = \text{süp}(y \cap \omega_1)$ by (1). Hence $|\{x \cap \omega_1 \mid x \in S \cap C\}| \leq 1$. Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. \blacksquare (Case 1)

Case 2: Both $\text{cf}(M \cap \kappa)$ and $\text{cf}(\text{süp}(M \cap \kappa^+))$ are uncountable.

First we claim the following:

Claim . $\text{cf}^{V_0}(\text{süp}(M \cap \kappa^+)) \leq M \cap \kappa$.

\vdash Let $\delta := \text{süp}(M \cap \kappa^+)$.

First suppose that $\delta \in E$. Note that $M \cap \kappa^+$ is ω -closed by Lemma 5.7. Hence, by (*3), $\text{Lim}(c_\delta) \cap M \cap E$ is unbounded in δ . Moreover if $\beta \in \text{Lim}(c_\delta) \cap M \cap E$ then $\text{o.t.}(c_\delta \cap \beta) = \text{o.t.}(c_\beta) \in M \cap \kappa$. Thus $\text{o.t.}(c_\delta) \leq M \cap \kappa$. But $\text{cf}^{V_0}(\delta) \leq \text{o.t.}(c_\delta)$ because $c_\delta \in V_0$. Therefore $\text{cf}^{V_0}(\delta) \leq M \cap \kappa$.

Next suppose that $\delta \notin E$. Then there exists $\alpha < \kappa$ such that, in V_0 , α is a measurable cardinal and $\text{cf}(\delta) = \alpha^+$. Then, by (*2), $(E_\alpha^\delta)^{V_0}$ is a stationary subset of E_ω^δ . On the other hand, $M \cap \kappa^+$ is ω -closed unbounded in δ . Thus there exists $\beta \in M \cap \kappa^+$ such that $\text{cf}^{V_0}(\beta) = \alpha$. Then, by the elementarity of M , $(\alpha^+)^{V_0} \in M \cap \kappa$. Therefore $\text{cf}^{V_0}(\delta) = (\alpha^+)^{V_0} < M \cap \kappa$.

This completes the proof of the claim. \dashv

Take an increasing continuous sequence $\langle \beta_\gamma \mid \gamma < \text{cf}^{V_0}(\text{süp}(M \cap \kappa^+)) \rangle \in V_0$ which is cofinal in $\text{süp}(M \cap \kappa^+)$. Let C be the set of all $x \in [M \cap \kappa^+]^\omega$ such that, for some limit $\gamma < \text{cf}^{V_0}(\text{süp}(M \cap \kappa^+))$, $\text{süp} x = \beta_\gamma$ and $\gamma \leq \text{süp}(x \cap \kappa)$. Then C is club in $[M \cap \kappa^+]^\omega$ by the claim above. Moreover if $x \in C$ and $\text{süp} x = \beta_\gamma$ then

$$\text{cf}^{V_0}(\text{süp} x) = \text{cf}^{V_0}(\beta_\gamma) \leq \gamma \leq \text{süp}(x \cap \kappa)$$

That is, $\text{cf}^{V_0}(\text{süp} x) \leq \text{süp}(x \cap \kappa)$ for every $x \in C$. On the other hand, if $x \in S$ then $\text{cf}^{V_0}(\text{süp} x) > \text{süp}(x \cap \kappa)$ by (2) of the construction of S . Thus $S \cap C = \emptyset$. Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. \blacksquare (Case 2)

This completes the proof of Theorem 5.3. \square

Now we turn our attention to Theorem 1.6. As we mentioned before, the forcing of Theorem 1.6 followed by Lévy collapsing κ to ω_2 gives Theorem 1.6.

Let V_0 , V_1 and E be as in the proof of Theorem 1.6. Let V_2 be an extension of V_1 by $\text{Col}(\omega_1, < \kappa)$. Then, by Lemma 5.2, $\text{SSR}([\lambda]^\omega)$ holds in V_2 for every $\lambda \geq \omega_2$. So, by Lemma 2.5 (1) and 2.7 (1), it suffices to show that $\text{SR}([\omega_3]^\omega, < \omega_2)$ does not hold in V_2 .

Here note that V_2 is a κ^+ -c.c. forcing extension of V_0 and that (*1), (*2) and (*3) all hold in V_2 . (*2) holds because $\text{Col}(\omega_1, < \kappa)$ preserves stationary subsets of E_ω^γ . (*3) holds because $\text{Col}(\omega_1, < \kappa)$ preserves ordinals having uncountable cofinalities. Hence the same argument shows that $\text{SR}([\kappa^+]^\omega, < \kappa)$ does not hold in V_2 . But $\kappa = \omega_2$ in V_2 . Therefore $\text{SR}([\omega_3]^\omega, < \omega_2)$ does not hold in V_2 .

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