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Semistationary and stationary reflection

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Abstract

We study the relationship between the semistationary reflection principle and stationary reflection principles. We show that for all regular cardinals $\lambda \geq \omega_2$ the semistationary reflection principle in the space $[\lambda]^\omega$ implies that every stationary subset of $E^\lambda_\omega := \{\alpha \in \lambda \mid \mathrm{cf}(\alpha) = \omega\}$ reflects. We also show that for all cardinals $\lambda \geq \omega_3$ the semistationary reflection principle in $[\lambda]^\omega$ does not imply the stationary reflection principle in $[\lambda]^\omega$.

1 Introduction

In this paper we compare the semistationary reflection principle with stationary reflection principles. The notion of semistationary sets and the semistationary reflection principle were introduced by Shelah [10](Ch.XIII §1). They are closely related to the semiproperness of posets. We review this:

Notation 1.1. *For countable sets x and y, we write $x \sqsubseteq y$ if $x \subseteq y$ and $x \cap \omega_1 = y \cap \omega_1$.*

Definition 1.2 (Shelah [10] Ch.XIII, §1, 1.1.Def.)**.** *Let W be a set with $W \supseteq \omega_1$. A subset $S \subseteq [W]^\omega$ is called semistationary if the set $\{y \in [W]^\omega \mid (\exists x \in S)\ x \sqsubseteq y\}$ is stationary in $[W]^\omega$.*

Definition 1.3 (Shelah [10] Ch.XIII, §1, 1.5.Def.)**.** *For a cardinal $\lambda \geq \omega_2$, $\mathrm{SSR}([\lambda]^\omega)$, the semistationary reflection principle in $[\lambda]^\omega$, is the following:*

$\mathrm{SSR}([\lambda]^\omega) \equiv$ *For every semistationary $S \subseteq [\lambda]^\omega$, there exists $W \subseteq \lambda$ such that $|W| = \omega_1 \subseteq W$ and $S \cap [W]^\omega$ is semistationary in $[W]^\omega$.*

In [10](Ch.XIII, §1, 1.4.Claim) Shelah shows that a poset $\mathbb{P}$ is semiproper if and only if $\mathbb{P}$ preserves ω_1 and preserves semistationary subsets of $[W]^\omega$ for every W. He also shows that (†) holds if and only if $\mathrm{SSR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$. Here (†) is the principle, introduced in Foreman-Magidor-Shelah [3], that every poset preserving stationary subset of ω_1 is semiproper. This is known

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to have interesting consequences. Shelah [10](Ch.XII, §2) shows that if Namba forcing is semiproper then (a strong form of) Chang's conjecture holds. Hence (†) implies Chang's conjecture. Also, Foreman-Magidor-Shelah [3] shows that (†) implies precipitousness of the nonstationary ideal over ω_1.

In this paper we compare the semistationary reflection principle with the stationary reflection principles defined below. For a regular cardinal λ, E^λ_ω denotes the set $\{\alpha \in \lambda \mid \mathrm{cf}(\alpha) = \omega\}$.

Definition 1.4. *For a regular cardinal $\lambda \geq \omega_2$, let $\mathrm{SR}(\lambda)$ denote the following stationary reflection principle:*

$\mathrm{SR}(\lambda) \equiv$ *For every stationary $B \subseteq E^\lambda_\omega$, there exists $\gamma < \lambda$ such that $B \cap \gamma$ is stationary.*

For a cardinal $\lambda \geq \omega_2$, let $\mathrm{SR}([\lambda]^\omega)$ denote the stationary reflection principle in $[\lambda]^\omega$:

$\mathrm{SR}([\lambda]^\omega) \equiv$ *For every stationary $S \subseteq [\lambda]^\omega$, there exists $W \subseteq \lambda$ such that $|W| = \omega_1 \subseteq W$ and $S \cap [W]^\omega$ is stationary in $[W]^\omega$.*

It is easy to see that $\mathrm{SR}([\lambda]^\omega)$ implies $\mathrm{SSR}([\lambda]^\omega)$. (See Section 2.2.)

Our main results are as follows:

Theorem 1.5. *Let λ be a regular cardinal $\geq \omega_2$. Then $\mathrm{SSR}([\lambda]^\omega)$ implies $\mathrm{SR}(\lambda)$.*

Theorem 1.6. *If κ is a supercompact cardinal, then there exists a generic extension in which $\mathrm{SSR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$ but $\mathrm{SR}([\lambda]^\omega)$ does not hold for any $\lambda \geq \omega_3$.*

Foreman-Magidor-Shelah [3] shows that if $\mathrm{SR}([\lambda]^\omega)$ holds for every $\lambda \geq \omega_2$ then (†) holds. Theorem 1.6 claims that the converse is not true. Also, as we prove in Section 5, $\mathrm{SSR}([\omega_2]^\omega)$ implies $\mathrm{SR}([\omega_2]^\omega)$. Theorem 1.6 is optimal in this sense.

This paper is organized as follows: In Section 2 we discuss some preliminaries for this paper. In Section 3 we present a certain type of stationary subset of $[\lambda]^\omega$ which was first introduced by Shelah. This type of stationary set plays a central role in the proofs of both Theorems 1.5 and 1.6. In Section 4 we prove Theorem 1.5. In Section 5 we compare $\mathrm{SSR}([\lambda]^\omega)$ and $\mathrm{SR}([\lambda]^\omega)$. Among other things, we prove Theorem 1.6.

2 Preliminaries

2.1 Notations

We follow the notations of Jech [4]. Here we present those which may not be general.

For a regular cardinal γ and an inaccessible cardinal κ, let $\mathrm{Col}(\gamma, {<}\kappa)$ denote the Lévy collapse which forces κ to be γ^+.

For a regular cardinal γ and a limit ordinal $\delta > \gamma$, let E^δ_γ denote the set $\{\alpha \in \delta \mid \mathrm{cf}(\alpha) = \gamma\}$. Note that if $\mathrm{cf}(\delta) > \gamma$ then E^δ_γ is stationary in δ.

For a set x of ordinals let

$$\bar{\sup}\, x \ := \ \sup\{\alpha + 1 \mid \alpha \in x\} \ .$$

In this paper we use $\bar{\sup}$ rather than sup. We are mainly interested in sets of ordinals which do not have a greatest element. For such x, $\bar{\sup}\, x = \sup x$. The merit of using $\bar{\sup}$ is that $\bar{\sup}\, x$ is a limit ordinal if and only if x does not have a greatest element. This makes our definitions and arguments slightly simpler.

2.2 Basics on stationary sets and semistationary sets

For basics on the notion of club or stationary subsets of $\mathcal{P}_\kappa W$ consult Jech [4]. When $\kappa = \omega_1$, we prefer to use $[W]^\omega$ rather than $\mathcal{P}_{\omega_1} W$. A subset of $[W]^\omega$ is said to be club (stationary) if it is club (stationary) in $\mathcal{P}_{\omega_1} W$. This paper uses the following two facts without any reference:

Fact 2.1 (Kueker [6]). *Let κ be a regular uncountable cardinal, W be a set with $\kappa \subseteq W$, and let $C \subseteq \mathcal{P}_\kappa W$ be a club. Then there exists a function $f : [W]^{<\omega} \to W$ such that $\{x \in \mathcal{P}_\kappa W \mid f\,``[x]^{<\omega} \subseteq x \wedge x \cap \kappa \in \kappa\} \subseteq C$. If $\kappa = \omega_1$ then there exists a function $f : [W]^{<\omega} \to W$ such that $\{x \in \mathcal{P}_\kappa W \mid f\,``[x]^{<\omega} \subseteq x\} \subseteq C$.*

Fact 2.2 (Menas [9]). *Let κ be a regular uncountable cardinal, and let W and $\bar{W}$ be sets with $\kappa \subseteq W \subseteq \bar{W}$.*

(1) *If $C \subseteq \mathcal{P}_\kappa W$ is a club then the set $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in C\}$ is a club in $\mathcal{P}_\kappa \bar{W}$. Hence if $\bar{S} \subseteq \mathcal{P}_\kappa \bar{W}$ is stationary then the set $\{\bar{x} \cap W \mid \bar{x} \in \bar{S}\}$ is stationary in $\mathcal{P}_\kappa W$.*

(2) *If $\bar{C} \subseteq \mathcal{P}_\kappa \bar{W}$ is a club then the set $\{\bar{x} \cap W \mid \bar{x} \in \bar{C}\}$ contains a club in $\mathcal{P}_\kappa W$. Hence if $S \subseteq \mathcal{P}_\kappa W$ is stationary then the set $\{\bar{x} \in \mathcal{P}_\kappa \bar{W} \mid \bar{x} \cap W \in S\}$ is stationary in $\mathcal{P}_\kappa \bar{W}$.*

Basics on semistationary subsets of $[W]^\omega$ was studied in Shelah [10] (Ch.XIII, §1). The following lemma is an analogy of Fact 2.2 for semistationary sets. In the case of (2), a stronger result holds. Part (2) of the following lemma illustrates a unique property of semistationary sets:

Lemma 2.3. *Let W and $\bar{W}$ be sets with $\omega_1 \subseteq W \subseteq \bar{W}$.*

(1) *If $\bar{S} \subseteq [\bar{W}]^\omega$ is semistationary then the set $\{\bar{x} \cap W \mid x \in \bar{S}\}$ is semistationary.*

(2) *If $S \subseteq [W]^\omega$ is semistationary then S is also semistationary in $[\bar{W}]^\omega$.*

Proof. (1) is clear from Fact 2.2 (1). We prove (2).

Suppose that $S \subseteq [W]^\omega$ is semistationary. Let $T := \{y \in [W]^\omega \mid (\exists x \in S)\ x \sqsubseteq y\}$ and $\bar{T} := \{\bar{y} \in [\bar{W}]^\omega \mid (\exists x \in S)\ x \sqsubseteq \bar{y}\}$. Then T is stationary in $[W]^\omega$, and $\bar{T} = \{\bar{y} \in [\bar{W}]^\omega \mid \bar{y} \cap W \in T\}$. Hence $\bar{T}$ is stationary in $[\bar{W}]^\omega$. Therefore S is semistationary in $[\bar{W}]^\omega$. $\square$

2.3 Basics on reflection principles

In this paper we use the following reflection principles which are generalizations of $\mathrm{SR}([\lambda]^\omega)$ and $\mathrm{SSR}([\lambda]^\omega)$:

Definition 2.4. *For a cardinal $\lambda \geq \omega_2$ and a regular cardinal κ with $\omega_2 \leq \kappa \leq \lambda$, let $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ and $\mathrm{SR}([\lambda]^\omega, {<}\kappa)$ be the following reflection principles:*

$\mathrm{SR}([\lambda]^\omega, {<}\kappa) \equiv$ *For every stationary $S \subseteq [\lambda]^\omega$, there exists $W \in \mathcal{P}_\kappa\lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is stationary.*

$\mathrm{SSR}([\lambda]^\omega, {<}\kappa) \equiv$ *For every semistationary $S \subseteq [\lambda]^\omega$, there exists $W \in \mathcal{P}_\kappa\lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary.*

Here we review basics about the above reflection principles. First we observe that these are generalizations of $\mathrm{SR}([\lambda]^\omega)$ and $\mathrm{SSR}([\lambda]^\omega)$:

Lemma 2.5. *Let λ be a cardinal $\geq \omega_2$.*

(1) $\mathrm{SR}([\lambda]^\omega)$ *is equivalent to* $\mathrm{SR}([\lambda]^\omega, {<}\omega_2)$.

(2) $\mathrm{SSR}([\lambda]^\omega)$ *is equivalent to* $\mathrm{SSR}([\lambda]^\omega, {<}\omega_2)$.

Proof. (1). It suffices to show that $\mathrm{SR}([\lambda]^\omega)$ implies $\mathrm{SR}([\lambda]^\omega, < \omega_2)$. Before starting, take a surjection $\pi_\alpha : \omega_1 \to \alpha$ for each $\alpha < \omega_2$ and let $f : \omega_2 \times \omega_1 \to \omega_2$ be the function defined by $f(\alpha, \xi) = \pi_\alpha(\xi)$ for each $\langle \alpha, \xi \rangle \in \omega_2 \times \omega_1$.

Assume that $\mathrm{SR}([\lambda]^\omega)$ holds. To show that $\mathrm{SR}([\lambda]^\omega, < \omega_2)$ holds, take an arbitrary stationary $S \subseteq [\lambda]^\omega$. We may assume that every element of S is closed under f. By $\mathrm{SR}([\lambda]^\omega)$, we may choose a $W \subseteq \lambda$ such that $|W| = \omega_1 \subseteq W$ and $S \cap [W]^\omega$ is stationary. Note that W is closed under f because stationary many elements of $[W]^\omega$ are closed under f. Because $\omega_1 \subseteq W$, if $\alpha \in W \cap \omega_2$ then $\alpha \subseteq W$. Hence $W \cap \omega_2 \in \omega_2$. Therefore W witnesses that $\mathrm{SR}([\lambda]^\omega, {<}\omega_2)$ holds for S.

(2). It suffices to show that $\mathrm{SSR}([\lambda]^\omega)$ implies $\mathrm{SSR}([\lambda]^\omega, < \omega_2)$. Assume that $\mathrm{SSR}([\lambda]^\omega)$ holds. Take an arbitrary semistationary $S \subseteq [\lambda]^\omega$. Let $W \subseteq \lambda$ be a witness of $\mathrm{SSR}([\lambda]^\omega)$ for S and let $W' := W \cup \bar{\sup}(W \cap \omega_2)$. Then $\omega_1 \subseteq W' \cap \omega_2 \in \omega_2$. Moreover $S \cap [W]^\omega$ is semistationary in $[W']^\omega$ by Lemma 2.3. Hence $S \cap [W']^\omega$ is semistationary. Therefore W' witnesses $\mathrm{SSR}([\lambda]^\omega, {<}\omega_2)$ for S. □

Next we observe that $\mathrm{SR}([\lambda]^\omega, {<}\kappa)$ implies $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$.

Lemma 2.6. *Let λ be a cardinal $\geq \omega_2$ and κ be a regular cardinal with $\omega_2 \leq \kappa \leq \lambda$. Then $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ is equivalent to the following principle:*

$\mathrm{SSR}'([\lambda]^\omega, {<}\kappa) \equiv$ *For every stationary $S \subseteq [\lambda]^\omega$, there exists $W \in \mathcal{P}_\kappa\lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary.*

Therefore $\mathrm{SR}([\lambda]^\omega, {<}\kappa)$ *implies* $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$.

Proof. It suffices to show that SSR$'$ implies SSR. Assume that $\mathrm{SSR}'([\lambda]^\omega, {<}\kappa)$ holds. Take an arbitrary semistationary $S \subseteq [\lambda]^\omega$. Then $T := \{y \in [\lambda]^\omega \mid (\exists x \in S)\ x \sqsubseteq y\}$ is stationary. Let $W \subseteq \lambda$ be a witness of $\mathrm{SSR}'([\lambda]^\omega, {<}\kappa)$ for T. Here note that

$$\{y \in [W]^\omega \mid (\exists x \in T \cap [W]^\omega)\ x \sqsubseteq y\} \ = \ \{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega)\ x \sqsubseteq y\}\ .$$

Hence $\{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega)\ x \sqsubseteq y\}$ is stationary and thus $S \cap [W]^\omega$ is semistationary. Therefore W witnesses $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ for S. □

The following is very easy:

Lemma 2.7. *Let λ and λ' be cardinals and let κ and κ' be regular cardinals such that $\omega_2 \leq \kappa \leq \kappa' \leq \lambda' \leq \lambda$.*

(1) $\mathrm{SR}([\lambda]^\omega, {<}\kappa)$ *implies* $\mathrm{SR}([\lambda']^\omega, {<}\kappa)$.

(2) $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ *implies* $\mathrm{SSR}([\lambda']^\omega, {<}\kappa')$.

Proof. (1) Assume that $\mathrm{SR}([\lambda]^\omega, {<}\kappa)$ holds. Take an arbitrary stationary $S' \subseteq [\lambda']^\omega$. Then $S := \{x \in [\lambda]^\omega \mid x \cap \lambda' \in S'\}$ is stationary. Hence there exists $W \in \mathcal{P}_\kappa\lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is stationary. Let $W' := W \cap \lambda'$. Then $W' \in \mathcal{P}_\kappa\lambda'$ and $\omega_1 \subseteq W' \cap \kappa \in \kappa$. Moreover $S' \cap [W']^\omega = \{x \cap W' \mid x \in S \cap [W]^\omega\}$. Thus $S' \cap [W']^\omega$ is stationary. Therefore W' witnesses $\mathrm{SSR}([\lambda']^\omega, {<}\kappa)$ for S'.

(2) Assume that $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ holds. Take an arbitrary semistationary $S' \subseteq [\lambda']^\omega$. Let S, W and W' be as in (1). Then, using Lemma 2.3, the same argument as (1) shows that $S' \cap [W']^\omega$ is semistationary. Let $W'' := W' \cup \bar{\sup}(W' \cap \kappa')$. Then $W'' \in \mathcal{P}_{\kappa'}\lambda'$ and $\omega_1 \subseteq W'' \cap \kappa' \in \kappa'$. Moreover $S' \cap [W']^\omega$ is semistationary in $[W'']^\omega$ by Lemma 2.3. Hence $S' \cap [W'']^\omega$ is semistationary. Therefore W'' witnesses $\mathrm{SSR}([\lambda']^\omega, {<}\kappa')$ for S'. □

We end this section with the following:

Lemma 2.8. *Let λ be a cardinal $\geq \omega_2$ and κ be a regular cardinal with $\omega_2 \leq \kappa \leq \lambda$.*

(1) (Feng-Jech [2]) *Assume that* $\mathrm{SR}([\lambda]^\omega, {<}\kappa)$ *holds. If $S \subseteq [\lambda]^\omega$ is stationary then the set $\{W \in \mathcal{P}_\kappa\lambda \mid S \cap [W]^\omega$ is stationary$\}$ is stationary in $\mathcal{P}_\kappa\lambda$.*

(2) *Assume that* $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ *holds. If $S \subseteq [\lambda]^\omega$ is semistationary then the set $\{W \in \mathcal{P}_\kappa\lambda \mid S \cap [W]^\omega$ is semistationary$\}$ is co-bounded, that is, there exists $W^* \in \mathcal{P}_\kappa\lambda$ such that $S \cap [W]^\omega$ is semistationary for every $W \in \mathcal{P}_\kappa\lambda$ with $W^* \supseteq W$.*

Proof. (2) is clear from Lemma 2.3 (2). We prove (1).

Take an arbitrary stationary $S \subseteq [\lambda]^\omega$ and an arbitrary function $f : [\lambda]^{<\omega} \to \lambda$. It suffices to find a $W \in \mathcal{P}_\kappa\lambda$ such that $W \cap \kappa \in \kappa$ and W is closed under f. Let S' be the set of all $x \in S$ closed under f. Then S' is stationary. Hence there exists $W \in \mathcal{P}_\kappa\lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S' \cap [W]^\omega$ is stationary. Note that W is closed under f because stationary many elements of $[W]^\omega$ are closed under f. Moreover $S \cap [W]^\omega$ is stationary because $S' \subseteq S$. Therefore W is a desired one. □

3 Sup depending stationary sets

Here we present a type of stationary set which plays a central role in the proofs of both Theorems 1.5 and 1.6:

Lemma 3.1 (The case when $n = 1$ is due to Shelah)**.** *Suppose that* $0 < n < \omega$ *and that* $\mu_0 < \mu_1 < \cdots < \mu_n$ *are regular uncountable cardinals. Moreover, suppose that* $A \subseteq E^{\mu_0}_\omega$ *is stationary and that, for each* m *with* $1 \leq m \leq n$, $\langle A^m_\alpha \mid \alpha \in \mu_{m-1} \rangle$ *is a sequence of stationary subsets of* $E^{\mu_m}_\omega$. *Let* S *be the set of all* $x \in \mathcal{P}_{\mu_0}\mu_n$ *such that*

(1) $x \cap \mu_0 \in A$,

(2) $\bar{\sup}(x \cap \mu_m) \in A^m_{\bar{\sup}(x \cap \mu_{m-1})}$ *for each* m *with* $1 \leq m \leq n$.

Then S *is stationary in* $\mathcal{P}_{\mu_0}\mu_n$.

This type of stationary set was considered by Shelah, and in Shelah-Shioya [12] and Shelah [11], such sets are used to obtain consequences of the stationary reflection principle. The proof of the above lemma for the case when $n = 1$ can be found in Shelah-Shioya [12]. Although there are no difficulties in generalization, we give the complete proof of Lemma 3.1.

We use the following game $\eth$:

Definition 3.2. *Suppose that* $0 < n < \omega$ *and that* $\mu_0 < \mu_1 < \cdots < \mu_n$ *are regular uncountable cardinals. For an* $\alpha \in \mu_0$ *and a function* $f : [\mu_n]^{<\omega} \to \mu_n$ *let* $\eth(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f)$ *be the following two players game of length* ω*:*

In the k*-th stage, first Player I plays a* $\langle \beta^m_k \mid 1 \leq m \leq n \rangle$ *and then Player II plays a* $\langle \gamma^m_k \mid 1 \leq m \leq n \rangle$ *so that* $\beta^m_k \leq \gamma^m_k < \mu_m$ *for each* m.

I	$\beta^1_0, \ldots, \beta^n_0$	$\beta^1_1, \ldots, \beta^n_1$	$\cdots\cdots$	$\beta^1_k, \ldots, \beta^n_k$	$\cdots\cdots$
II	$\gamma^1_0, \ldots, \gamma^n_0$	$\gamma^1_1, \ldots, \gamma^n_1$	$\cdots\cdots$	$\gamma^1_k, \ldots, \gamma^n_k$	$\cdots\cdots$

Player II wins if $\mathrm{cl}_f(\, \alpha \cup \{\gamma^m_k \mid 1 \leq m \leq n \wedge k \in \omega\}\,) \cap \mu_0 = \alpha$, *where* $\mathrm{cl}_f(x)$ *denotes the closure of* x *under* f. *Otherwise Player I wins.*

Note that $\eth(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f)$ is an open game for Player I. Hence it is determined. The following is a key lemma:

Lemma 3.3. *Suppose that* $0 < n < \omega$ *and that* $\mu_0 < \mu_1 < \cdots < \mu_n$ *are regular uncountable cardinals. Then for every function* $f : [\mu_n]^{<\omega} \to \mu_n$, *there are club many* $\alpha \in \mu_0$ *such that Player II has a winning strategy in* $\eth(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f)$.

Proof. Take an arbitrary function $f : [\mu_n]^{<\omega} \to \mu_n$ and let A be the set of all $\alpha \in \mu_0$ such that Player I has a winning strategy in $\eth(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f)$. It suffices to show that A is nonstationary.

Assume that A is stationary. For each $\alpha \in A$, take a winning strategy σ_α for Player I in $\eth(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha, f)$. Let θ be a sufficiently large regular cardinal.

Then we can take an elementary submodel M of $\langle \mathcal{H}_\theta, \in, \langle \sigma_\alpha \mid \alpha \in A \rangle \rangle$ such that $\alpha^* := M \cap \mu_0 \in A$.

By induction on k, we construct a sequence of moves $\langle \beta^1_k, \ldots, \beta^n_k, \gamma^1_k, \ldots, \gamma^n_k \mid k \in \omega \rangle$ in $\Game(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f)$ so that $\gamma^1_k, \ldots, \gamma^n_k \in M$ for each $k \in \omega$. Suppose that $k \in \omega$ and that $\langle \beta^1_l, \ldots, \beta^n_l \rangle$ and $\langle \gamma^1_l, \ldots, \gamma^n_l \rangle$ have been chosen for each $l < k$. Let

$$\langle \beta^1_k, \ldots, \beta^n_k \rangle \ := \ \sigma_{\alpha^*}(\, \langle \gamma^1_l, \ldots, \gamma^n_l \mid l < k \rangle \,)\ ,$$

and for each m with $1 \leq m \leq n$, let

$$\gamma^m_k \ := \ \bar{\sup}\{\pi_m \circ \sigma_\alpha\,(\, \langle \gamma^1_l, \ldots, \gamma^n_l \mid l < k \rangle \,) \mid \alpha \in A\}\ ,$$

where $\pi_m : \mu_1 \times \cdots \times \mu_n \to \mu_m$ is the m-th projection. Clearly $\beta^m_k \leq \gamma^m_k$ for each m. Note that $\gamma^m_k < \mu_m$ since μ_m is regular and $A \subseteq \mu_0 < \mu_m$. Note also that $\gamma^m_k \in M$ because $\langle \gamma^1_l, \ldots, \gamma^n_l \mid l < k \rangle \in M$. This completes the induction.

First note that $\langle \beta^1_k, \ldots, \beta^n_k, \gamma^1_k, \ldots, \gamma^n_k \mid k \in \omega \rangle$ is a sequence of moves in $\Game(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f)$ in which Player I has played according to winning strategy σ_{α^*}. Hence Player I wins. On the other hand, $\alpha^* \cup \{\gamma^m_k \mid 1 \leq m \leq n \wedge k \in \omega\} \subseteq M$, and M is closed under f. Thus $\mathrm{cl}_f(\, \alpha^* \cup \{\gamma^m_k \mid 1 \leq m \leq n \wedge k \in \omega\}\,) \cap \mu_0 \subseteq M \cap \mu_0 = \alpha^*$. Therefore $\mathrm{cl}_f(\, \alpha^* \cup \{\gamma^m_k \mid 1 \leq m \leq n \wedge k \in \omega\}\,) = \alpha^*$, so that Player II wins with this sequence of moves. This is a contradiction. □

Now we prove Lemma 3.1:

Proof of Lemma 3.1. We prove Lemma 3.1 by induction on n. Suppose that $n = 1$ or that $n > 1$ and that the lemma holds for $n - 1$. We prove the lemma for n. Take an arbitrary function $f : [\mu_n]^{<\omega} \to \mu_n$. It suffices to find $x^* \in S$ such that x^* is closed under f and $x^* \cap \mu_0 \in \mu_0$.

By Lemma 3.3, there exists $\alpha^* \in A$ such that Player II has a winning strategy σ^* in $\Game(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f)$. Let S' be the set of all $y \in \mathcal{P}_{\mu_1}\mu_n$ such that

(1) $y \cap \mu_1 \in A^1_{\alpha^*}$,

(2) $\bar{\sup}(y \cap \mu_m) \in A^m_{\bar{\sup}(y \cap \mu_{m-1})}$ for each m with $2 \leq m \leq n$.

Then S' is stationary in $\mathcal{P}_{\mu_1}\mu_n$. If $n = 1$ then this is clear. ((2) claims nothing.) If $n > 1$ then this follows from the lemma for $n - 1$.

Choose $y^* \in S'$ such that $\alpha^* \subseteq y^*$ and such that y^* is closed under σ^* and f. For each m with $1 \leq m \leq n$, take a cofinal sequence $\langle \beta^m_k \mid k \in \omega \rangle$ in $y^* \cap \mu_m$. Moreover let

$$\langle \gamma^1_k, \ldots, \gamma^n_k \rangle \ := \ \sigma^*(\, \langle \beta^1_l, \ldots, \beta^n_l \mid l < k \rangle \,)$$

for each $k \in \omega$. Note that $\langle \gamma^m_k \mid k \in \omega \rangle$ is a cofinal sequence in $y^* \cap \mu_m$ for each m with $1 \leq m \leq n$. This is because y^* is closed under σ^*. Finally let $x^* := \mathrm{cl}_f(\, \alpha^* \cup \{\gamma^m_k \mid 1 \leq m \leq n \wedge k \in \omega\}\,)$. We show that $x^* \in S$ and $x^* \cap \mu_0 \in \mu_0$.

First note that $x^* \subseteq y^*$ because $\alpha^* \cup \{\gamma^m_k \mid 1 \leq m \leq n \wedge k \in \omega\} \subseteq y^*$ and y^* is closed under f. Then, because $\langle \gamma^m_k \mid k \in \omega \rangle$ is cofinal in $y^* \cap \mu_m$, $\bar{\sup}(x^* \cap \mu_m) = \bar{\sup}(y^* \cap \mu_m)$ for each m with $1 \leq m \leq n$. Hence, by (2) above,

(i) $\bar{\sup}(x^* \cap \mu_m) \in A^m_{\bar{\sup}(x^* \cap \mu_{m-1})}$ for each m with $2 \le m \le n$.

Moreover $x^* \cap \mu_0 = \alpha^*$. This is because σ^* is a winning strategy for Player II in $\Game(\langle \mu_0, \mu_1, \ldots, \mu_n \rangle, \alpha^*, f)$. Also recall that $\bar{\sup}(x^* \cap \mu_1) = \bar{\sup}(y^* \cap \mu_1) \in A^1_{\alpha^*}$. Thus

(ii) $\bar{\sup}(x^* \cap \mu_0) \in A$,

(iii) $\bar{\sup}(x^* \cap \mu_1) \in A^1_{\bar{\sup}(x^* \cap \mu_1)}$.

Now it follows from (i), (ii) and (iii) that $x^* \in S$ and $x^* \cap \mu_0 \in \mu_0$.

This completes the proof. □

4 SSR($[\lambda]^\omega$) and SR(λ)

In this section, we prove Theorem 1.5. In fact, we prove the following more general theorem:

Theorem 4.1. *Let λ and κ be regular cardinals such that $\omega_2 \le \kappa \le \lambda$. Then* $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ *implies* $\mathrm{SR}(\lambda)$.

By Lemma 2.5 and 2.7, Theorem 1.5 follows from Theorem 4.1. Theorem 4.1 can be easily obtained from Lemma 3.1 and the following lemma:

Lemma 4.2. *Let λ be a cardinal and κ be a regular cardinal such that $\omega_2 \le \kappa \le \lambda$. Assume that $S \subseteq [\lambda]^\omega$ and that there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary. Let $W^* \in \mathcal{P}_\kappa \lambda$ be such that*

(1) *$\omega_1 \subseteq W^* \cap \kappa \in \kappa$ and $S \cap [W^*]^\omega$ is semistationary,*

(2) *for every $W \in \mathcal{P}_\kappa \lambda$, if $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary then $\bar{\sup} W^* \le \bar{\sup} W$.*

Then

$$S_0 := \{y \in [W^*]^\omega \mid (\exists x \in S \cap [W^*]^\omega)\ x \sqsubseteq y \wedge \bar{\sup} x = \bar{\sup} y\}$$

is stationary in $[W^*]^\omega$.

Proof. Assume that S_0 is not stationary. Then $S_1 := \{y \in [W^*]^\omega \mid (\exists x \in S \cap [W^*]^\omega)\ x \sqsubseteq y \wedge \bar{\sup} x < \bar{\sup} y\}$ is stationary. For each $y \in S_1$, choose $x_y \in S$ with $x_y \sqsubseteq y$ and $\bar{\sup} x_y < \bar{\sup} y$ and choose $\alpha_y \in y$ with $\bar{\sup} x_y \le \alpha_y$. Then there exists $\alpha' \in W^*$ such that $S' := \{y \in S_1 \mid \alpha_y = \alpha'\}$ is stationary. Let $W' := W^* \cap \alpha'$. Clearly $W' \in \mathcal{P}_\kappa \lambda$ and $\omega_1 \subseteq W' \cap \kappa \in \kappa$. Moreover $\bar{\sup} W' < \bar{\sup} W^*$. Hence if we show that $S \cap [W']^\omega$ is semistationary then this contradicts property (2) of W^*.

First note that if $y \in S'$ then $x_y \in [W']^\omega$ and $x_y \sqsubseteq y \cap W'$. Thus

$$\{y \cap W' \mid y \in S'\} \subseteq \{z \in [W']^\omega \mid (\exists x \in S \cap [W']^\omega)\ x \sqsubseteq z\}.$$

Now the left side is stationary in $[W']^\omega$ because S' is stationary in $[W^*]^\omega$. Therefore the right side is stationary, that is, $S \cap [W']^\omega$ is semistationary.

This completes the proof. □

Proof of Theorem 4.1. Assume that $\mathrm{SSR}([\lambda]^\omega, <\kappa)$ holds. Take an arbitrary stationary $B \subseteq E^\lambda_\omega$. We show that B reflects.

Take a pairwise disjoint sequence $\langle B_\alpha \mid \alpha \in \omega_1 \rangle$ of stationary subsets of B. Let S be the set of all $x \in [\lambda]^\omega$ such that $x \cap \omega_1 \in \omega_1$ and $\bar{\sup} x \in B_{x \cap \omega_1}$. Then S is stationary by Lemma 3.1. By Lemma 4.2, there exists $W \in \mathcal{P}_\kappa \lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S_0 := \{y \in [W]^\omega \mid (\exists x \in S \cap [W]^\omega)\ x \sqsubseteq y \wedge \bar{\sup} x = \bar{\sup} y\}$ is stationary in $[W]^\omega$. Here note that $S_0 \subseteq S$. Hence $S \cap [W]^\omega$ is stationary. We claim that $\mathrm{cf}(\bar{\sup} W) > \omega$.

Clearly $\bar{\sup} W$ is a limit ordinal. Assume that $\mathrm{cf}(\bar{\sup} W) = \omega$. Then $C := \{y \in [W]^\omega \mid \bar{\sup} y = \bar{\sup} W\}$ is club in $[W]^\omega$. But if $y_1, y_2 \in S \cap C$ then $\bar{\sup}(y_1 \cap \omega_1) = \bar{\sup}(y_2 \cap \omega_1)$ by the construction of S. Hence $|\{y \cap \omega_1 \mid y \in S_0 \cap C\}| \leq 1$. This contradicts $\omega_1 \subseteq W$ and $S \cap [W]^\omega$ is stationary.

Now $\mathrm{cf}(\bar{\sup} W) > \omega$ and $S \cap [W]^\omega$ is stationary. Hence $\{\bar{\sup} y \mid y \in S \cap [W]^\omega\}$ is stationary in $\bar{\sup} W$. Recall that $\bar{\sup} y \in B$ for every $y \in S$. Therefore $B \cap \bar{\sup} W$ is stationary. This completes the proof. □

5 $\mathrm{SSR}([\lambda]^\omega)$ and $\mathrm{SR}([\lambda]^\omega)$

As we mentioned in Section 1, first we prove that $\mathrm{SSR}([\omega_2]^\omega)$ implies $\mathrm{SR}([\omega_2]^\omega)$ and thus that they are equivalent. This is essentially proved in Todorčević [13]. After that, we prove Theorem 1.6.

Theorem 5.1. $\mathrm{SSR}([\omega_2]^\omega)$ *implies* $\mathrm{SR}([\omega_2]^\omega)$.

Proof. Assume that $\mathrm{SSR}([\omega_2]^\omega)$ holds. To show that $\mathrm{SR}([\omega_2]^\omega)$ holds, take an arbitrary stationary $S \subseteq [\omega_2]^\omega$. Fix a bijection $\pi_\alpha : \omega_1 \to \alpha$ for each $\alpha \in [\omega_1, \omega_2)$. We may assume that if $x \in S$ then $\omega_1 < \bar{\sup} x$ and x is closed under $\pi_\alpha, \pi_\alpha^{-1}$ for each $\alpha \in x \setminus \omega_1$.

By Lemma 2.5 let α^* be the least $\alpha \in \omega_2$ such that $S \cap [\alpha]^\omega$ is semistationary. Then let S_0 be the set of all $y \in [\alpha^*]^\omega$ such that

(1) for some $x \in S \cap [\alpha^*]^\omega$, $x \sqsubseteq y$ and $\bar{\sup} x = \bar{\sup} y$,

(2) y is closed under $\pi_\alpha, \pi_\alpha^{-1}$ for each $\alpha \in y \setminus \omega_1$.

By Lemma 4.2, S_0 is stationary in $[\alpha^*]^\omega$. For each $y \in S_0$, choose an $x_y \in S \cap [\alpha^*]^\omega$ witnessing (1).

Note that if $y \in S_0$ then

$$y \cap \alpha \;=\; \pi_\alpha``(y \cap \omega_1) \;=\; \pi_\alpha``(x_y \cap \omega_1) \;=\; x_y \cap \alpha$$

for each $\alpha \in x_y \setminus \omega_1$. Then, because $\bar{\sup} y = \bar{\sup} x_y$, $y = x_y$ for every $y \in S_0$. Hence $S_0 \subseteq S \cap [\alpha^*]^\omega$. Therefore $S \cap [\alpha^*]^\omega$ is stationary. This completes the proof. □

Now we turn our attention to Theorem 1.6. It is well-known that if a λ-supercompact cardinal is Lévy collapsed to ω_2 then $\mathrm{SR}([\lambda]^\omega)$ holds. It was shown by Shelah [10] that collapsing a λ-strongly compact cardinal suffices to obtain a model of $\mathrm{SSR}([\lambda]^\omega)$. First we review this:

Lemma 5.2 (Shelah [10] Ch.XIII, §1, 1.6.Claim, 1.10.Claim)**.** *Suppose that* κ *is a* λ*-strongly compact cardinal, where* λ *is a cardinal* $\geq \kappa$*. Then* $\mathrm{SSR}([\lambda]^\omega, {<}\kappa)$ *holds. Moreover if* γ *is a regular uncountable cardinal* $< \kappa$ *then* $\Vdash_{\mathrm{Col}(\gamma,<\kappa)}$ *"*$\mathrm{SR}([\lambda]^\omega, {<}\kappa)$ *".*

Proof. Both statements can be proved by similar arguments, but the latter is slightly harder than the former. We will prove only the latter.

We discuss some preliminaries in V. Take a fine ultrafilter U over $\mathcal{P}_\kappa\lambda$. Let M be the transitive collapse of $\mathrm{Ult}(V, U)$, and let $j : V \to M$ be the ultrapower map. Moreover, let $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ be a function such that $f(W) = W \cup \overline{\sup}(W \cap \kappa)$ for each $W \in \mathcal{P}_\kappa\lambda$ and let $W^* := [f]_U \in M$. Then $j``\lambda \subseteq W^*$, and in M, $W^* \in \mathcal{P}_{j(\kappa)}j(\lambda)$ and $\omega_1 \subseteq W^* \cap j(\kappa) \in j(\kappa)$.

Suppose that γ is a regular uncountable cardinal $< \kappa$ and that G is a $\mathrm{Col}(\gamma, {<}\kappa)$-generic filter over V. In $V[G]$, take an arbitrary stationary $S \subseteq [\lambda]^\omega$. We must show that, in $V[G]$, there exists $W \in \mathcal{P}_\kappa\lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary.

Let $\bar{G}$ be a $\mathrm{Col}(\gamma, {<}j(\kappa))$-generic filter over V with $\bar{G} \cap \mathrm{Col}(\gamma, {<}\kappa) = G$. We work in $V[\bar{G}]$. Define a map $\bar{j} : V[G] \to M[\bar{G}]$ by $\bar{j}(\dot{a}_G) = j(\dot{a})_{\bar{G}}$ for each $\mathrm{Col}(\gamma, {<}\kappa)$-name $\dot{a} \in V$. Then $\bar{j}$ is well-defined, and $\bar{j} : V[G] \to M[\bar{G}]$ is an elementary embedding which extends j. For simplicity of notation, we let j denote $\bar{j}$.

Note that S remains stationary because $V[\bar{G}]$ is a γ-closed forcing extension of $V[G]$. Hence $\{j``x \mid x \in S\}$ is stationary in $[j``\lambda]^\omega$. Moreover, for each $x \in S$, $j``x = j(x)$ because $x$ is countable in $V[G]$. Thus $\{j``x \mid x \in S\} \subseteq j(S)$, and therefore $j(S) \cap [j``\lambda]^\omega$ is stationary. Then $j(S) \cap [W^*]^\omega$ is semistationary by Lemma 2.3 (2). This is also true in $M[\bar{G}]$. Hence W^* witnesses that the following holds in $M[\bar{G}]$:

> There exists $W \in \mathcal{P}_{j(\kappa)}j(\lambda)$ such that $\omega_1 \subseteq W \cap j(\kappa) \in j(\kappa)$ and $j(S) \cap [W]^\omega$ is semistationary.

Therefore, by the elementarity of j, it holds in $V[G]$ that there exists $W \in \mathcal{P}_\kappa\lambda$ such that $\omega_1 \subseteq W \cap \kappa \in \kappa$ and $S \cap [W]^\omega$ is semistationary. This completes the proof. □

We will prove that collapsing a λ-strongly compact cardinal does not suffice to obtain a model of $\mathrm{SR}([\lambda]^\omega)$. The core of Theorem 1.6 is the following theorem. As we see later, Theorem 1.6 will be obtained by further Lévy collapsing κ to ω_2.

Theorem 5.3. *If* κ *is a supercompact cardinal, then there exists a generic extension in which* κ *is a strongly compact cardinal and* $\mathrm{SR}([\kappa^+]^\omega, {<}\kappa)$ *does not hold.*

First we prove Theorem 5.3. Krueger [7] constructed a model in which κ is strongly compact but $S(\kappa, \kappa^+) := \{x \in \mathcal{P}_\kappa\kappa^+ \mid \mathrm{o.t.}(x) = (x \cap \kappa)^+\}$ is not stationary. (Note that $S(\kappa, \kappa^+)$ is stationary if κ is κ^+-supercompact.) We show that $\mathrm{SR}([\kappa^+]^\omega, {<}\kappa)$ does not hold in this model.

We start with a review of Krueger's model. Krueger's model was obtained from a model with κ supercompact by two step forcing extension. The first step forces a partial square principle at κ with preserving supercompactness of κ. This type of partial square principle was first introduced by Baumgertner in his unpublished note, and Apter-Cummings [1] showed that it can hold at a supercompact cardinal. In fact, the first step of Krueger's construction is due to Apter-Cummings [1]. The second step is the iteration of Prikry forcing below κ which was developed by Magidor [8]. We summarize basic properties of these forcings below.

Definition 5.4. *For an uncountable cardinal κ and an $E \subseteq \mathrm{Lim}(\kappa^+)$, let $\square_\kappa^E$ be the following principle:*

$\square_\kappa^E \equiv$ *There exists a sequence $\langle c_\beta \mid \beta \in E \rangle$ such that*

(1) *c_β is a club in β.*
(2) *if $\mathrm{cf}(\beta) < \kappa$ then $\mathrm{o.t.}(c_\beta) < \kappa$,*
(3) *if $\beta' \in \mathrm{Lim}(c_\beta)$ then $c_{\beta'} = c_\beta \cap \beta'$,*

for each $\beta, \beta' \in E$.

We call a sequence $\langle c_\beta \mid \beta \in E \rangle$ satisfying (1)-(3) above a $\square_\kappa^E$-sequence.

The proof of the following fact can be also found in Krueger [7]:

Fact 5.5 (Apter-Cummings [1])**.** *Assume that κ is a supercompact cardinal. Then there exists a poset $\mathbb{P}$ with the following properties:*

(1) *$\mathbb{P}$ preserves supercompactness of κ.*
(2) *$\Vdash_{\mathbb{P}}$ "$\square_\kappa^E$ holds for*

$$E = \mathrm{Lim}(\kappa^+) \setminus \bigcup \{E^{\kappa^+}_{\alpha^+} \mid \alpha \text{ is a measurable cardinal} < \kappa\} \text{"}.$$

Fact 5.6 (Magidor [8])**.** *Assume that κ is a supercompact cardinal. Then there exists a poset $\mathbb{Q}$ with the following properties:*

(1) *$\mathbb{Q}$ has the κ^+-c.c.*
(2) *$\Vdash_{\mathbb{Q}}$ "κ is strongly compact".*
(3) *For every measurable cardinal $\alpha < \kappa$, $\Vdash_{\mathbb{Q}}$ "$\mathrm{cf}(\alpha) = \omega$".*
(4) *For every measurable cardinal $\alpha < \kappa$, $\mathbb{Q}$ can be factored as $\mathbb{Q}_{\leq\alpha} * \dot{\mathbb{Q}}_{>\alpha}$ so that $\mathbb{Q}_{\leq\alpha}$ has the α^+-c.c. and $\Vdash_{\mathbb{Q}_{\leq\alpha}}$ "$\dot{\mathbb{Q}}_{>\alpha}$ does not add subsets of α^+".*

Before starting the proof of Theorem 5.3, we give a technical lemma:

Lemma 5.7. *Suppose that κ and θ are regular cardinals with $\omega_2 \leq \kappa < \theta$. Let M be an elementary submodel of $\langle \mathcal{H}_\theta, \in \rangle$ such that $M \cap \kappa \in \kappa$ and such that both $\mathrm{cf}(M \cap \kappa)$ and $\mathrm{cf}(\bar{\sup}(M \cap \kappa^+))$ are uncountable. Then $M \cap \kappa^+$ is ω-closed, that is, $\bar{\sup}\, b \in M$ for every countable $b \subseteq M \cap \kappa^+$.*

Proof. We prove this by contradiction. Assume that $b \subseteq M \cap \kappa^+$ is countable and that $\bar{\sup} b \notin M \cap \kappa^+$. Then b does not have a greatest element. Also, b is bounded in $M \cap \kappa^+$ because $\mathrm{cf}(\bar{\sup}(M \cap \kappa^+))$ is uncountable. Let β^* be the least element of $(M \cap \kappa^+) \setminus \bar{\sup} b$. Note that β^* is a limit ordinal and $\bar{\sup}(M \cap \beta^*) = \bar{\sup} b < \beta^*$. Take an increasing continuous cofinal map $\sigma : \mathrm{cf}(\beta^*) \to \beta^*$.

First assume that $\mathrm{cf}(\beta^*) < \kappa$. Then $\mathrm{cf}(\beta^*) \subseteq M$ because $\mathrm{cf}(\beta^*) \in M \cap \kappa \in \kappa$. Hence $\mathrm{ran}\, \sigma \subseteq M$ and thus $M \cap \beta^*$ is cofinal in β^*. This contradicts $\bar{\sup}(M \cap \beta^*) < \beta^*$.

Next assume that $\mathrm{cf}(\beta^*) = \kappa$. Then it is easy to see that $\bar{\sup}(M \cap \beta^*) = \bar{\sup}(\sigma``(M \cap \kappa))$. Hence $\mathrm{cf}(\bar{\sup}(M \cap \beta^*))$ is uncountable because $\mathrm{cf}(M \cap \kappa)$ is uncountable. But this contradicts $\bar{\sup}(M \cap \beta^*) = \bar{\sup} b$ and b is countable.

This completes the proof. $\square$

Now we prove Theorem 5.3:

Proof of Theorem 5.3. Assume that κ is supercompact in V. Let V_0 be a forcing extension of V by the poset $\mathbb{P}$ of Fact 5.5, and let V_1 be a forcing extension of V_0 by the poset $\mathbb{Q}$ of Fact 5.6. It suffices to show that $\mathrm{SR}([\kappa^+]^\omega, {<}\kappa)$ does not hold in V_1. Before starting, we summarize properties of V_0 and V_1.

In V_0, let $E := \mathrm{Lim}(\kappa^+) \setminus \bigcup\{E^{\kappa^+}_{\alpha^+} \mid \alpha \text{ is a measurable cardinal } < \kappa\}$. Then $\square^E_\kappa$ holds in V_0. Let $\langle c_\gamma \mid \gamma \in E \rangle$ be a $\square^E_\kappa$-sequence. Note that, in V_0, there are unboundedly many measurable cardinals below κ.

V_1 is a κ^+-c.c. forcing extension of V_0. Moreover, in V_1, the following hold:

(*1) If $\alpha < \kappa$ is a measurable cardinal in V_0 the $\mathrm{cf}(\alpha) = \omega$.

(*2) Suppose that $\alpha < \kappa$ is measurable in V_0. Let $\gamma := (\alpha^+)^{V_0}$. Then $(E^\gamma_\alpha)^{V_0}$is a stationary subset of E^γ_ω.

(*3) $E^{\kappa^+}_\omega \subseteq E$.

(*2) and (*3) hold by Fact 5.6 (4).

Now we show that $\mathrm{SR}([\kappa^+]^\omega, {<}\kappa)$ does not hold in V_1. We work in V_1. We show that there exists a stationary $S \subseteq [\kappa^+]^\omega$ such that $\{W \in \mathcal{P}_\kappa \kappa^+ \mid S \cap [W]^\omega$ is stationary$\}$ is nonstationary in $\mathcal{P}_\kappa \kappa^+$. By Lemma 2.8 this suffices. S will be constructed using Lemma 3.1.

First take a pairwise disjoint partition $\langle A_\xi \mid \xi < \omega_1 \rangle$ of E^κ_ω into stationary sets. Next take an injection $\sigma : E^\kappa_\omega \to \kappa$ such that, for every $\alpha \in \kappa$, $\sigma(\alpha) > \alpha$ and $\sigma(\alpha)$ is measurable in V_0. Then let $B_\alpha := (E^{\kappa^+}_{\sigma(\alpha)})^{V_0}$ for each $\alpha \in E^\kappa_\omega$. Note the following:

- $\langle B_\alpha \mid \alpha \in E^\kappa_\omega \rangle$ is a pairwise disjoint sequence of stationary subsets of $E^{\kappa^+}_\omega$.
- For each $\alpha \in E^\kappa_\omega$ and each $\beta \in B_\alpha$, $\mathrm{cf}^{V_0}(\beta) > \alpha$.

Now let S be the set of all $x \in [\kappa^+]^\omega$ such that

(1) $\bar{\sup}(x \cap \kappa) \in A_{\bar{\sup}(x \cap \omega_1)}$,

(2) $\bar{\sup}(x) \in B_{\bar{\sup}(x \cap \kappa)}$.

Then S is stationary by Lemma 3.1. We show that $\{W \in \mathcal{P}_\kappa \kappa^+ \mid S \cap [W]^\omega$ is stationary$\}$ is nonstationary. Take a sufficiently large regular cardinal θ and let Ω be the set of all $M \in \mathcal{P}_\kappa \mathcal{H}_\theta$ such that $M \cap \kappa \in \kappa$ and $M \prec \langle \mathcal{H}_\theta, \in, \mathcal{H}_\theta^{V_0}, \kappa, \langle c_\gamma \mid \gamma \in E \rangle \rangle$. It suffices to show that if $M \in \Omega$ then $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. The proof of this splits into two cases:

Case 1: $\mathrm{cf}(M \cap \kappa) = \omega$ or $\mathrm{cf}(\bar{\sup}(M \cap \kappa^+)) = \omega$.

First suppose that $\mathrm{cf}(M \cap \kappa) = \omega$. Then $C := \{x \in [M \cap \kappa^+]^\omega \mid \bar{\sup}(x \cap \kappa) = M \cap \kappa\}$ is club in $[M \cap \kappa^+]^\omega$. Note that if $x, y \in S \cap C$ then $\bar{\sup}(x \cap \omega_1) = \bar{\sup}(y \cap \omega_1)$ by (1) of the construction of S. Thus $|\{\bar{\sup}(x \cap \omega_1) \mid x \in S \cap C\}| \leq 1$. But $\omega_1 \subseteq M \cap \kappa^+$. Hence $S \cap C$ is nonstationary in $[M \cap \kappa^+]^\omega$. Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary.

The case when $\mathrm{cf}(\bar{\sup}(M \cap \kappa^+)) = \omega$ is similar. Suppose that $\mathrm{cf}(\bar{\sup}(M \cap \kappa^+)) = \omega$. Then $C := \{x \in [M \cap \kappa^+]^\omega \mid \bar{\sup}(x) = \bar{\sup}(M \cap \kappa^+)\}$ is club in $[M \cap \kappa^+]^\omega$. If $x, y \in S \cap C$ then $\bar{\sup}(x \cap \kappa) = \bar{\sup}(y \cap \kappa)$ by (2) of the construction of S and thus $\bar{\sup}(x \cap \omega_1) = \bar{\sup}(y \cap \omega_1)$ by (1). Hence $|\{x \cap \omega_1 \mid x \in S \cap C\}| \leq 1$. Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. ■(Case 1)

Case 2: Both $\mathrm{cf}(M \cap \kappa)$ and $\mathrm{cf}(\bar{\sup}(M \cap \kappa^+))$ are uncountable.

First we claim the following:

Claim . $\mathrm{cf}^{V_0}(\bar{\sup}(M \cap \kappa^+)) \leq M \cap \kappa$.

$\vdash$ Let $\delta := \bar{\sup}(M \cap \kappa^+)$.

First suppose that $\delta \in E$. Note that $M \cap \kappa^+$ is ω-closed by Lemma 5.7. Hence, by (*3), $\mathrm{Lim}(c_\delta) \cap M \cap E$ is unbounded in δ. Moreover if $\beta \in \mathrm{Lim}(c_\delta) \cap M \cap E$ then $\mathrm{o.t.}(c_\delta \cap \beta) = \mathrm{o.t.}(c_\beta) \in M \cap \kappa$. Thus $\mathrm{o.t.}(c_\delta) \leq M \cap \kappa$. But $\mathrm{cf}^{V_0}(\delta) \leq \mathrm{o.t.}(c_\delta)$ because $c_\delta \in V_0$. Therefore $\mathrm{cf}^{V_0}(\delta) \leq M \cap \kappa$.

Next suppose that $\delta \notin E$. Then there exists $\alpha < \kappa$ such that, in V_0, α is a measurable cardinal and $\mathrm{cf}(\delta) = \alpha^+$. Then, by (*2), $(E^\delta_\alpha)^{V_0}$ is a stationary subset of E^δ_ω. On the other hand, $M \cap \kappa^+$ is ω-closed unbounded in δ. Thus there exists $\beta \in M \cap \kappa^+$ such that $\mathrm{cf}^{V_0}(\beta) = \alpha$. Then, by the elementarity of M, $(\alpha^+)^{V_0} \in M \cap \kappa$. Therefore $\mathrm{cf}^{V_0}(\delta) = (\alpha^+)^{V_0} < M \cap \kappa$.

This completes the proof of the claim. $\dashv$

Take an increasing continuous sequence $\langle \beta_\gamma \mid \gamma < \mathrm{cf}^{V_0}(\bar{\sup}(M \cap \kappa^+)) \rangle \in V_0$ which is cofinal in $\bar{\sup}(M \cap \kappa^+)$. Let C be the set of all $x \in [M \cap \kappa^+]^\omega$ such that, for some limit $\gamma < \mathrm{cf}^{V_0}(\bar{\sup}(M \cap \kappa^+))$, $\bar{\sup}\, x = \beta_\gamma$ and $\gamma \leq \bar{\sup}(x \cap \kappa)$. Then C is club in $[M \cap \kappa^+]^\omega$ by the claim above. Moreover if $x \in C$ and $\bar{\sup}\, x = \beta_\gamma$ then

$$\mathrm{cf}^{V_0}(\bar{\sup}\, x) \; = \; \mathrm{cf}^{V_0}(\beta_\gamma) \; \leq \; \gamma \; \leq \; \bar{\sup}(x \cap \kappa)$$

That is, $\mathrm{cf}^{V_0}(\bar{\sup}\, x) \leq \bar{\sup}(x \cap \kappa)$ for every $x \in C$. On the other hand, if $x \in S$ then $\mathrm{cf}^{V_0}(\bar{\sup}\, x) > \bar{\sup}(x \cap \kappa)$ by (2) of the construction of S. Thus $S \cap C = \emptyset$. Therefore $S \cap [M \cap \kappa^+]^\omega$ is nonstationary. ■(Case 2)

This completes the proof of Theorem 5.3. □

Now we turn our attention to Theorem 1.6. As we mentioned before, the forcing of Theorem 1.6 followed by Lévy collapsing κ to ω_2 gives Theorem 1.6.

Let V_0, V_1 and E be as in the proof of Theorem 1.6. Let V_2 be an extension of V_1 by $\mathrm{Col}(\omega_1, <\kappa)$. Then, by Lemma 5.2, $\mathrm{SSR}([\lambda]^\omega)$ holds in V_2 for every $\lambda \geq \omega_2$. So, by Lemma 2.5 (1) and 2.7 (1), it suffices to show that $\mathrm{SR}([\omega_3]^\omega, < \omega_2)$ does not hold in V_2.

Here note that V_2 is a κ^+-c.c. forcing extension of V_0 and that (*1), (*2) and (*3) all hold in V_2. (*2) holds because $\mathrm{Col}(\omega_1, <\kappa)$ preserves stationary subsets of E^γ_ω. (*3) holds because $\mathrm{Col}(\omega_1, <\kappa)$ preserves ordinals having uncountable cofinalities. Hence the same argument shows that $\mathrm{SR}([\kappa^+]^\omega, <\kappa)$ does not hold in V_2. But $\kappa = \omega_2$ in V_2. Therefore $\mathrm{SR}([\omega_3]^\omega, < \omega_2)$ does not hold in V_2.

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