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Evolution inclusions governed by subdifferentials in reflexive Banach spaces

Goro AKAGI* and Mitsuharu ÔTANI†

Abstract

The existence, uniqueness and regularity of strong solutions for Cauchy problem and periodic problem are studied for the evolution equation: $du(t)/dt + \partial\varphi(u(t)) \ni f(t)$, $t \in [0, T]$, where $\partial\varphi$ is the so-called subdifferential operator from a real Banach space V into its dual V^* . The study in the Hilbert space setting ($V = V^* = H$: Hilbert space) is already developed in detail so far. However, the study here is done in the V - V^* setting which is not yet fully pursued. Our method of proof relies on approximation arguments in a Hilbert space H . To assure this procedure, it is assumed that the embeddings $V \subset H \subset V^*$ are both dense and continuous.

1 Introduction

Let V and V^* be a real reflexive Banach space and its dual space respectively and let φ be a lower semicontinuous convex function from V into $]-\infty, +\infty]$ with $\varphi \not\equiv +\infty$. Then it is well known that the subdifferential $\partial\varphi$ (a generalization of Fréchet derivative) of φ becomes a maximal monotone operator from V into V^* (see Barbu [4]). The main purpose of this paper is to investigate the existence, uniqueness and regularity of the solution of the following evolution equation in V^* .

$$(E) \quad \frac{du}{dt}(t) + \partial\varphi(u(t)) \ni f(t), \quad t \in [0, T].$$

As for the case where V is a real Hilbert space H whose dual space is identified with H , H. Brézis [6] showed that Kōmura's theory [14] can be applied to (E) and moreover the subdifferential operator generates the nonlinear semigroup with the smoothing effect. Thereafter the generalizations of Kōmura-Brézis Theory have been developed by many people in various directions. Some of them are very successful in the application of semigroup theory to nonlinear partial differential equations such as the theory of time-dependent subdifferential operators ($\partial\varphi$ is replaced by $\partial\varphi^t$) [cf. Attouch-Bénilan-Damlamian-Picard [2], Kenmochi [11], Yamada [19] and Ôtani [18]] or the theory of non-monotone perturbations ($\partial\varphi$ is replaced by $\partial\varphi + B(\cdot)$) [cf. Attouch-Damlamian [3], Koi-Watanabe [13] and Ôtani [16], [17]].

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However, they are all done in the Hilbert space setting. As is well known, the theory of elliptic equations bears close relations with the theory of evolution equations, and in the theory of elliptic equations, the Fréchet derivative $d\phi$ of a C^1 -functional ϕ defined on V is usually regarded as the operator from V into V^* . For example, the statement of “Palais-Smale” condition and Mountain Pass lemma are formulated in the V - V^* setting, and this setting plays an essential role to show the well-known fact that the equation $-\Delta u(x) = |u(x)|^{q-2}u(x)$, $x \in \Omega$, $u|_{\partial\Omega} = 0$ admits a nontrivial positive solution if and only if $1 < q < 2^* = 2N/(N-2)$, provided that Ω is a bounded star-shaped domain.

From this point of view, it would be very important to investigate the solvability of (E) in the V - V^* setting. On the other hand, this kind of attempt was already fully developed in the book of J. L. Lions [15] for various types of evolution equations by using Faedo-Galerkin’s method. Our main tool here is the theory of nonlinear semigroup and our final goal is to present an abstract framework dealing with the evolution equation governed by subdifferential operator in the V - V^* setting. The treatment of (E) in this framework is not yet fully pursued. Brézis [7] discussed the existence of weak solutions of (E), and Kenmochi [11] studied the existence of strong solutions of (E) by employing the semi-discretisation. Our framework can assure the existence of strong solutions of (E) under weaker assumptions on f than those of [6], [7] and [11], which is very important when we aim at the perturbation problem for (E).

As will be exemplified in the applications, the advantage of this approach over Faedo-Galerkin’s method is that one can derive better regularity of solutions in a natural way.

This paper is composed of five sections. Section 2 contains some preliminaries which will be used later. In Section 3 we shall be concerned with Cauchy problem of (E). Section 4 deals with the periodic problem of (E). The last section is devoted to some applications of our abstract results to nonlinear heat equations governed by the so-called p-laplace operators in bounded and unbounded domains.

2 Preliminaries

Let V be a real reflexive Banach space and V^* be its dual space. We assume that there exists a real Hilbert space H identified with its dual such that

$$(2.1) \quad V \subset H \equiv H^* \subset V^*,$$

where $V \subset H$ and $H^* \subset V^*$ are both densely and continuously embedded. Hence ${}_V\langle u, v \rangle_V = (u, v)_H$ holds for every $u \in H$ and $v \in V$. For the sake of simplicity, we often denote ${}_V\langle \cdot, \cdot \rangle_V$ by $\langle \cdot, \cdot \rangle$.

Let $\Phi(V)$ be the set of all *proper lower semicontinuous convex functions* φ from V into $] -\infty, +\infty]$, where “*proper*” means that the *effective domain* $D(\varphi)$ of φ defined by $D(\varphi) = \{u \in V; \varphi(u) < +\infty\}$ is not empty. Define the subdifferential $\partial\varphi$ of φ by

$$\partial\varphi(u) = \{f \in V^*; \varphi(v) - \varphi(u) \geq \langle f, v - u \rangle \text{ for all } v \in D(\varphi)\}$$

with domain $D(\partial\varphi) = \{u \in V; \partial\varphi(u) \neq \emptyset\}$.

Then it is well known that $\partial\varphi$ becomes maximal monotone in $V \times V^*$ and has various nice properties (see [4], [6]). We recall two results of those which will be used later.

PROPOSITION 2.1 *Let $\varphi \in \Phi(H)$, and let $u \in W^{1,2}(0, T; H)$ be such that $u(t) \in D(\partial\varphi)$ for a.e. $t \in]0, T[$. Suppose that there exists $g \in L^2(0, T; H)$ such that $g(t) \in \partial\varphi(u(t))$ for a.e. $t \in]0, T[$. Then, the function $t \mapsto \varphi(u(t))$ is absolutely continuous on $[0, T]$ and the following equality holds:*

$$\frac{d}{dt}\varphi(u(t)) = \left(h(t), \frac{du}{dt}(t) \right)_H \quad \text{for a.e. } t \in]0, T[\text{ and for all } h(\cdot) \in \partial\varphi(u(\cdot)).$$

The following proposition plays an important role in the proof of Theorem 3.2.

PROPOSITION 2.2 *Let $\varphi \in \Phi(H)$. Then, for every $u_0 \in D(\varphi)$ and $f \in L^2(0, T; H)$, there exists a unique strong solution u of Cauchy problem of (E) with the initial data u_0 satisfying:*

$$\begin{aligned} u(t) &\in D(\partial\varphi) \text{ for a.e. } t \in]0, T[, \\ u &\in W^{1,2}(0, T; H), \quad u(+0) = u_0, \\ t &\mapsto \varphi(u(t)) \text{ is absolutely continuous on } [0, T]. \end{aligned}$$

Throughout the present paper, for every $i \in \mathbb{N}$, we denote by C_i positive constants which do not depend on the elements of the corresponding space or set.

Let p' be the Hölder conjugate of $p \in]1, +\infty[$, i.e., $1/p + 1/p' = 1$. For all $k > 0$, by Young's inequality, we have

$$(2.2) \quad ab \leq ka^p + \mathcal{M}_p(k)b^{p'} \quad \forall a \geq 0, \forall b \geq 0,$$

$$(2.3) \quad \mathcal{M}_p(k) = \{p'(pk)^{p'/p}\}^{-1}.$$

Furthermore, we always assume that $\varphi \geq 0$ and $0 \in D(\varphi)$ without any loss of generality. Actually from the fact that there exist $v^* \in V^*$ and $\mu \in \mathbb{R}$ such that

$$\varphi(u) \geq \langle v^*, u \rangle + \mu \quad \forall u \in V,$$

(see [4, Chap.II, Proposition 2.2]), we can choose the non-negative function $\tilde{\varphi}(u) := \varphi(u) - \langle v^*, u \rangle - \mu \geq 0$. Then $\tilde{\varphi}$ belongs to $\Phi(V)$ and

$$D(\tilde{\varphi}) = D(\varphi), \quad \partial\tilde{\varphi}(u) = \partial\varphi(u) - v^*.$$

For an arbitrary element $v_0 \in D(\varphi)$, put $\hat{\varphi}(u) := \tilde{\varphi}(u + v_0)$, then $\hat{\varphi}(u)$ belongs to $\Phi(V)$ and it follows that

$$D(\hat{\varphi}) = D(\tilde{\varphi}) - v_0 \ni 0, \quad \partial\tilde{\varphi}(u) = \partial\hat{\varphi}(u - v_0).$$

Hence put $\hat{u} = u - v_0$ and $\hat{f} = f + v^*$, then (E) is equivalent to the following evolution equation:

$$\frac{d\hat{u}}{dt}(t) + \partial\hat{\varphi}(\hat{u}(t)) \ni \hat{f}(t), \quad t \in [0, T].$$

3 Cauchy Problem

In this section, we study the existence of strong solutions of the following Cauchy problem (CP) when the initial data u_0 belongs to $\overline{D(\varphi)}^H$.

$$(CP) \quad \begin{cases} \frac{du}{dt}(t) + \partial\varphi(u(t)) \ni f(t) & \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

Here and henceforth, we are concerned with *strong solutions* of (CP) in the following sense.

DEFINITION 3.1 A function $u \in C([0, T]; V^*)$ is said to be a strong solution of (CP) on $[0, T]$, if the following conditions are satisfied:

- (i) $u(t)$ is a V^* -valued absolutely continuous function on $[0, T]$,
- (ii) $u(+0) = u_0$,
- (iii) $u(t) \in D(\partial\varphi)$ for a.e. $t \in]0, T[$
and there exists a function $g(t) \in \partial\varphi(u(t))$ satisfying:

$$(3.1) \quad \frac{du}{dt}(t) + g(t) = f(t) \text{ in } V^*, \quad \text{for a.e. } t \in]0, T[.$$

To assure the existence of strong solutions of (CP), we introduce the following coerciveness condition (A1) and boundedness condition (A2).

$$(A1) \quad |u|_V^p - C_1|u|_H^2 - C_2 \leq C_3\varphi(u) \quad \forall u \in D(\varphi), \quad 1 < p < +\infty,$$

$$(A2) \quad |g|_{V^*}^{p'} \leq \ell(|u|_H) \{\varphi(u) + 1\} \quad \forall [u, g] \in \partial\varphi,$$

where $\ell(\cdot)$ is a non-decreasing function on \mathbb{R} .

THEOREM 3.2 Let (A1) and (A2) be satisfied. Then, for every $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^{p'}(0, T; V^*)$, there exists a unique strong solution u of (CP) satisfying:

$$\begin{aligned} u &\in L^p(0, T; V) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*), \\ \text{the section } g(t) &\text{ of } \partial\varphi(u(t)) \text{ given in (3.1) belongs to } L^{p'}(0, T; V^*), \\ \varphi(u(\cdot)) &\in L^1(0, T). \end{aligned}$$

To prove this, we employ the following lemma (see e.g. Proposition 23.23 of [20]).

LEMMA 3.3 Let $u \in L^p(0, T; V)$ be such that $du/dt \in L^{p'}(0, T; V^*)$. Then $u \in C([0, T]; H)$ and the following holds true.

$$(3.2) \quad \frac{1}{2}|u(t)|_H^2 - \frac{1}{2}|u(s)|_H^2 = \int_s^t \left\langle \frac{du}{d\tau}(\tau), u(\tau) \right\rangle d\tau$$

for all $s, t \in [0, T]$ with $s < t$.

We now proceed to the proof of Theorem 3.2.

PROOF OF THEOREM 3.2

Uniqueness: Let u, v be strong solutions of (CP). Then $w(t) := u(t) - v(t)$ satisfies

$$(3.3) \quad \frac{dw}{dt}(t) + \partial\varphi(u(t)) - \partial\varphi(v(t)) \ni 0 \quad \text{for a.e. } t \in]0, T[.$$

Multiplying (3.3) by $w(t)$ and using the monotonicity of $\partial\varphi$ and Lemma 3.3, we have

$$(3.4) \quad \frac{1}{2}|w(t)|_H^2 - \frac{1}{2}|w(0)|_H^2 = \int_0^t \left\langle \frac{dw}{d\tau}(\tau), w(\tau) \right\rangle d\tau \leq 0 \quad \text{for all } t \in [0, T].$$

Now, the uniqueness follows from (3.4) at once.

Existence: The verification consists of three steps.

Step 1 (Approximation): First we consider the approximation problem in H for (CP). To this end, we introduce $\varphi_H: H \rightarrow [0, +\infty]$ by

$$\varphi_H(u) = \begin{cases} \varphi(u) & \text{if } u \in V, \\ +\infty & \text{if } u \in H \setminus V. \end{cases}$$

Then, it is clear that φ_H is proper and convex in H . We are going to show that φ_H is lower semicontinuous in H . Let $u_n \in H$ be such that $u_n \rightarrow u$ strongly in H as $n \rightarrow +\infty$. For the case where $\alpha := \liminf_{n \rightarrow +\infty} \varphi_H(u_n) \geq 0$ is finite, there exists a subsequence $u_{n'}$ of u_n such that $\varphi_H(u_{n'}) \rightarrow \alpha$ as $n' \rightarrow +\infty$. Then (A1) implies that $|u_{n'}|_V$ is bounded. Since V is reflexive, there exists a subsequence $u_{n''}$ of $u_{n'}$ such that $u_{n''} \rightarrow u$ weakly in V as $n'' \rightarrow +\infty$. Hence, by the lower semicontinuity and convexity of φ on V , we get

$$(3.5) \quad \varphi_H(u) \leq \liminf_{n'' \rightarrow +\infty} \varphi_H(u_{n''}) = \alpha.$$

For the case where $\alpha = +\infty$, it is obvious that $\varphi_H(u) \leq \alpha$ holds. Thus we find that $\varphi_H \in \Phi(H)$.

From the definition of φ_H , it follows immediately that

$$D(\varphi_H) = D(\varphi), \quad \partial\varphi_H \subset \partial\varphi.$$

Let $u_{0n} \in D(\varphi)$ and $f_n \in C^\infty([0, T]; H)$ be such that

$$\begin{aligned} u_{0n} &\rightarrow u_0 \text{ strongly in } H, \\ f_n &\rightarrow f \text{ strongly in } L^{p'}(0, T; V^*) \end{aligned}$$

as $n \rightarrow +\infty$ and consider the following Cauchy problem:

$$(CP)_n \quad \begin{cases} \frac{du_n}{dt}(t) + \partial\varphi_H(u_n(t)) \ni f_n(t) & \text{in } H, \quad 0 < t < T, \\ u_n(0) = u_{0n}. \end{cases}$$

The existence of the unique strong solution of $(CP)_n$ is assured by Proposition 2.2. In order to investigate the convergence of u_n , we need some a priori estimates.

Step 2 (A priori estimates): Multiplying $(CP)_n$ by $u_n(t)$, we have

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} |u_n(t)|_H^2 + \langle g_n(t), u_n(t) \rangle \leq |f_n(t)|_{V^*} |u_n(t)|_V$$

for a.e. $t \in]0, T[$,

where $g_n(t) = f_n(t) - du_n(t)/dt$ belongs to $\partial\varphi_H(u_n(t))$. By virtue of the fact that $0 \in D(\varphi)$, we get $\varphi(u_n(t)) \leq \langle g_n(t), u_n(t) \rangle + \varphi(0)$. Hence, by (A1), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |u_n(t)|_H^2 + \frac{1}{2} \varphi(u_n(t)) + C_4 |u_n(t)|_V^p \\ & \leq C_5 |u_n(t)|_H^2 + C_6 + |f_n(t)|_{V^*} |u_n(t)|_V \\ & \leq C_5 |u_n(t)|_H^2 + C_6 + \frac{C_4}{2} |u_n(t)|_V^p + \mathcal{M}_p \left(\frac{C_4}{2} \right) |f_n(t)|_{V^*}^{p'} \text{ for a.e. } t \in]0, T[, \end{aligned}$$

where $C_4 = 1/(2C_3)$, $C_5 = C_1/(2C_3)$, $C_6 = C_2/2C_3 + \varphi(0)$ and $\mathcal{M}_p(\cdot)$ is the function defined by (2.3). Then, using Gronwall's inequality, we deduce that

$$(3.7) \quad u_n \text{ is bounded in } C([0, T]; H),$$

$$(3.8) \quad u_n \text{ is bounded in } L^p(0, T; V),$$

$$(3.9) \quad \varphi(u_n(\cdot)) \text{ is bounded in } L^1(0, T).$$

Furthermore, it follows from (A2), (3.7) and (3.9) that

$$(3.10) \quad g_n \text{ is bounded in } L^{p'}(0, T; V^*),$$

and hence by virtue of the fact $du_n/dt = f_n - g_n$, we get

$$(3.11) \quad u_n \text{ is bounded in } W^{1,p'}(0, T; V^*).$$

Step 3 (Convergence of u_n): Multiplying $(CP)_n - (CP)_m$ by $u_n - u_m$ and using the monotonicity of $\partial\varphi_H$, we have

$$\frac{1}{2} \frac{d}{dt} |u_n(t) - u_m(t)|_H^2 \leq \langle f_n(t) - f_m(t), u_n(t) - u_m(t) \rangle \quad \text{for a.e. } t \in]0, T[$$

and therefore

$$(3.12) \quad \begin{aligned} |u_n(t) - u_m(t)|_H^2 & \leq |u_{0n} - u_{0m}|_H^2 \\ & + 2 \int_0^t |f_n(\tau) - f_m(\tau)|_{V^*} |u_n(\tau) - u_m(\tau)|_V d\tau. \end{aligned}$$

Since u_n is bounded in $L^p(0, T; V)$ and u_{0n} and f_n are convergent sequences in H and $L^{p'}(0, T; V^*)$ respectively, we find that u_n forms a Cauchy sequence in $C([0, T]; H)$.

Therefore, there exists $u \in C([0, T]; H)$ such that

$$(3.13) \quad u_n \rightarrow u \quad \text{strongly in } C([0, T]; H).$$

By virtue of (3.8), (3.10) and (3.11), we can extract a subsequence n' of n such that

$$(3.14) \quad u_{n'} \rightarrow u \quad \text{weakly in } L^p(0, T; V),$$

$$(3.15) \quad u_{n'} \rightarrow u \quad \text{weakly in } W^{1,p'}(0, T; V^*),$$

$$(3.16) \quad g_{n'} \rightarrow g \quad \text{weakly in } L^{p'}(0, T; V^*).$$

To complete the proof, it suffices to show that $g(t) \in \partial\varphi(u(t))$ for a.e. $t \in]0, T[$. Let $v \in D(\partial\varphi)$ and $h \in \partial\varphi(v)$ be fixed arbitrarily. Multiplying $(CP)_n$ by $u_n(t) - v$, we have

$$\frac{1}{2} \frac{d}{dt} |u_n(t) - v|_H^2 + \langle g_n(t), u_n(t) - v \rangle = \langle f_n(t), u_n(t) - v \rangle \quad \text{for a.e. } t \in]0, T[.$$

Integrating this equality over $]s, t[$ and using the monotonicity of $\partial\varphi$, we obtain

$$\frac{1}{2} (u_n(t) - u_n(s), u_n(t) + u_n(s) - 2v)_H \leq \int_s^t \langle -h + f_n(\tau), u_n(\tau) - v \rangle d\tau$$

for all $s, t \in [0, T]$ with $s < t$.

Hence, since $(u - w, w)_H \leq (u - w, u + w)_H/2$, we derive

$$\left(\frac{u_n(t) - u_n(s)}{t - s}, u_n(s) - v \right)_H \leq \frac{1}{t - s} \int_s^t \langle -h + f_n(\tau), u_n(\tau) - v \rangle d\tau$$

for all $s, t \in [0, T]$ with $s < t$.

Let $n \rightarrow +\infty$, then (3.13) and (3.14) give

$$\left(\frac{u(t) - u(s)}{t - s}, u(s) - v \right)_H \leq \frac{1}{t - s} \int_s^t \langle -h + f(\tau), u(\tau) - v \rangle d\tau$$

for all $s, t \in [0, T]$ with $s < t$.

Now, by letting $s \rightarrow t$, we deduce

$$\left\langle \frac{du}{dt}(t) - f(t) + h, u(t) - v \right\rangle \leq 0 \quad \text{for a.e. } t \in]0, T[.$$

Thus the arbitrariness of $[v, h] \in \partial\varphi$ as well as the maximal monotonicity of $\partial\varphi$ in $V \times V^*$ implies

$$g(t) = f(t) - \frac{du}{dt}(t) \in \partial\varphi(u(t)) \quad \text{for a.e. } t \in]0, T[.$$

This completes the proof. ■

Our another result is concerned with the regularity of solutions. More precisely, if we assume the higher regularity on f , e.g., $f \in W^{1,p'}(0, T; V^*)$, then the corresponding solutions enjoy the regularity higher than that of Theorem 3.2 in the following sense.

THEOREM 3.4 *Let (A1) and (A2) be satisfied. Then, for every $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^{p'}(0, T; V^*)$ with $t(df/dt) \in L^{p'}(0, T; V^*)$, the solution of (CP) satisfies:*

$$\begin{aligned} u &\in C([0, T]; V_w) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*), \\ u(t) &\in D(\varphi) \quad \forall t > 0, \quad \sup_{t \in [0, T]} t\varphi(u(t)) < +\infty, \\ t^{1/p'} \frac{du}{dt} &\in L^\infty(0, T; V^*), \quad t^{1/2} \frac{du}{dt} \in L^2(0, T; H). \end{aligned}$$

Moreover, if $u_0 \in D(\varphi)$ and $f \in W^{1,p'}(0, T; V^*)$, then

$$\begin{aligned} u &\in C([0, T]; V_w) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*), \\ u(t) &\in D(\varphi) \quad \forall t \geq 0, \quad \sup_{t \in [0, T]} \varphi(u(t)) < +\infty, \\ \frac{du}{dt} &\in L^2(0, T; H) \cap L^\infty(0, T; V^*). \end{aligned}$$

Here $C([0, T]; V_w)$ (resp. $C([0, T]; V_w)$) denotes the set of all V -valued weakly continuous functions on $]0, T[$ (resp. $[0, T]$).

PROOF OF THEOREM 3.4 Let u_{0n} be the same sequence as in the proof of Theorem 3.2, and let $f_n \in L^\infty(0, T; H) \cap C^\infty([0, T]; H)$ be such that $tf_n \in C^\infty([0, T]; H)$, $\lim_{t \rightarrow +0} tf_n(t) = 0$, $f_n \rightarrow f$ and $t(df_n/dt) \rightarrow t(df/dt)$ strongly in $L^{p'}(0, T; V^*)$. We here note that the fact that $f, t(df/dt) \in L^{p'}(0, T; V^*)$ enables us to take such a sequence f_n (see Remark 3.5). We again consider the appropriate equation (CP) $_n$. Multiply (CP) $_n$ by $t(du_n(t)/dt)$, then by Proposition 2.1, we get

$$\begin{aligned} (3.17) \quad & t \left| \frac{du_n}{dt}(t) \right|_H^2 + \frac{d}{dt} \left\{ t\varphi(u_n(t)) \right\} \\ &= \varphi(u_n(t)) + \frac{d}{dt} \left\{ t \langle f_n(t), u_n(t) \rangle \right\} \\ &\quad - \langle f_n(t), u_n(t) \rangle - t \left\langle \frac{df_n}{dt}(t), u_n(t) \right\rangle \text{ for a.e. } t \in]0, T[. \end{aligned}$$

Integrating both sides of (3.17) on $]0, t[$ and noting that $\lim_{t \rightarrow +0} tf_n(t) = 0$, we have

$$\begin{aligned} (3.18) \quad & \int_0^t \tau \left| \frac{du_n}{d\tau}(\tau) \right|_H^2 d\tau + t\varphi(u_n(t)) \\ &\leq \int_0^t \varphi(u_n(\tau)) d\tau + t|f_n(t)|_{V^*}|u_n(t)|_V \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^T |f_n(\tau)|_{V^*}^{p'} d\tau \right)^{1/p'} \left(\int_0^T |u_n(\tau)|_V^p d\tau \right)^{1/p} \\
& + \left(\int_0^T \left| \tau \frac{df_n}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau \right)^{1/p'} \left(\int_0^T |u_n(\tau)|_V^p d\tau \right)^{1/p}
\end{aligned}$$

for all $t \in [0, T]$. By virtue of (A1), we obtain

$$\begin{aligned}
(3.19) \quad t|f_n(t)|_{V^*}|u_n(t)|_V & \leq t\mathcal{M}_p\left(\frac{1}{2C_3}\right)|f_n(t)|_{V^*}^{p'} \\
& + T\left\{\frac{C_1}{2C_3}|u_n(t)|_H^2 + \frac{C_2}{2C_3}\right\} \\
& + \frac{t}{2}\varphi(u_n(t)).
\end{aligned}$$

Therefore, it follows from (3.18) and (3.19) that

$$\begin{aligned}
(3.20) \quad & \int_0^t \tau \left| \frac{du_n}{d\tau}(\tau) \right|_H^2 d\tau + \frac{1}{2}t\varphi(u_n(t)) \\
& \leq \int_0^T \varphi(u_n(\tau))d\tau + \mathcal{M}_p\left(\frac{1}{2C_3}\right) \sup_{t \in [0, T]} t|f_n(t)|_{V^*}^{p'} \\
& + \frac{C_1 T}{2C_3} \sup_{t \in [0, T]} |u_n(t)|_H^2 + \frac{C_2 T}{2C_3} \\
& + \left(\int_0^T |f_n(\tau)|_{V^*}^{p'} d\tau \right)^{1/p'} \left(\int_0^T |u_n(\tau)|_V^p d\tau \right)^{1/p} \\
& + \left(\int_0^T \left| \tau \frac{df_n}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau \right)^{1/p'} \left(\int_0^T |u_n(\tau)|_V^p d\tau \right)^{1/p}
\end{aligned}$$

for all $t \in [0, T]$. Here, by taking a sequence $\{t_m\}$ such that $|f_n(t_m)|_{V^*} \leq \|f_n\|_{L^\infty(0, T; V^*)}$ and $t_m \rightarrow 0$, we obtain

$$\begin{aligned}
t|f_n(t)|_{V^*}^{p'} & = \int_{t_m}^t \frac{d}{d\tau} \left(\tau |f_n(\tau)|_{V^*}^{p'} \right) d\tau + t_m |f_n(t_m)|_{V^*}^{p'} \\
& = \int_{t_m}^t |f_n(\tau)|_{V^*}^{p'} d\tau \\
& + \int_{t_m}^t \tau \frac{p'}{2} \{ |f_n(\tau)|_{V^*}^2 \}^{(p'-2)/2} \frac{d}{d\tau} |f_n(\tau)|_{V^*}^2 d\tau + t_m |f_n(t_m)|_{V^*}^{p'} \\
& \leq \int_{t_m}^t |f_n(\tau)|_{V^*}^{p'} d\tau + p' \int_{t_m}^t |f_n(\tau)|_{V^*}^{p'-1} \left| \tau \frac{df_n}{d\tau}(\tau) \right|_{V^*} d\tau \\
& + t_m |f_n(t_m)|_{V^*}^{p'}
\end{aligned}$$

$$\begin{aligned}
 &\leq \int_{t_m}^t |f_n(\tau)|_{V^*}^{p'} d\tau \\
 &\quad + p' \left(\int_{t_m}^t |f_n(\tau)|_{V^*}^{p'} d\tau \right)^{(p'-1)/p'} \left(\int_{t_m}^t \left| \tau \frac{df_n}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau \right)^{1/p'} \\
 &\quad + t_m |f_n(t_m)|_{V^*}^{p'},
 \end{aligned}$$

since for all $h_n(\tau) \in F_V^{-1}(f_n(\tau))$,

$$\begin{aligned}
 \frac{d}{d\tau} |f_n(\tau)|_{V^*}^2 &= 2 \left\langle \frac{df_n}{d\tau}(\tau), h_n(\tau) \right\rangle \\
 &\leq 2 \left| \frac{df_n}{d\tau}(\tau) \right|_{V^*} |h_n(\tau)|_V = 2 \left| \frac{df_n}{d\tau}(\tau) \right|_{V^*} |f_n(\tau)|_{V^*},
 \end{aligned}$$

where F_V is the duality mapping from V into V^* (see [4, Chap.III, Lemma 1.2]). Then letting $t_m \rightarrow 0$, we find

$$\begin{aligned}
 (3.21) \quad t |f_n(t)|_{V^*}^{p'} &\leq \int_0^t |f_n(\tau)|_{V^*}^{p'} d\tau \\
 &\quad + p' \left(\int_0^t |f_n(\tau)|_{V^*}^{p'} d\tau \right)^{(p'-1)/p'} \left(\int_0^t \left| \tau \frac{df_n}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau \right)^{1/p'}.
 \end{aligned}$$

Moreover since $f_n \rightarrow f$ strongly in $L^{p'}(0, T; V^*)$, we can take a subsequence n' of n such that $f_{n'}(t) \rightarrow f(t)$ strongly in V^* for a.e. $t \in]0, T[$. Hence noting that $f \in W_{loc}^{1,p'}(]0, T[; V^*) \subset C(]0, T[; V^*)$ and letting $n' \rightarrow +\infty$ in (3.21) with n replaced by n' , we have

$$\begin{aligned}
 t |f(t)|_{V^*}^{p'} &\leq \int_0^t |f(\tau)|_{V^*}^{p'} d\tau \\
 &\quad + p' \left(\int_0^t |f(\tau)|_{V^*}^{p'} d\tau \right)^{(p'-1)/p'} \left(\int_0^t \left| \tau \frac{df}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau \right)^{1/p'},
 \end{aligned}$$

which implies $t |f(t)|_{V^*}^{p'} \in L^\infty(0, T)$ and $\lim_{t \rightarrow +0} t |f(t)|_{V^*}^{p'} = 0$. Furthermore by (3.20) and (3.21), it follows from (3.7), (3.8) and (3.9) that

$$(3.22) \quad t^{1/2} \frac{du}{dt} \in L^2(0, T; H), \quad \sup_{t \in [0, T]} t \varphi(u(t)) < +\infty.$$

Moreover (A1), (A2) and (3.22) yield:

$$\begin{aligned}
 t^{1/p} u &\in L^\infty(0, T; V) \cap C([0, T]; V_w), \\
 t^{1/p'} g &\in L^\infty(0, T; V^*), \\
 t^{1/p'} \frac{du}{dt} &\in L^\infty(0, T; V^*),
 \end{aligned}$$

where $g(t) = f(t) - du(t)/dt$ belongs to $\partial\varphi(u(t))$.

When u_0 belongs to $D(\varphi)$, we can take $u_{0n} = u_0$ for all $n \in \mathbb{N}$ and $f_n \rightarrow f$ strongly in $W^{1,p'}(0, T; V^*)$. In order to obtain a priori estimates, it suffices to multiply $(CP)_n$ by $du_n(t)/dt$. Then we have

$$\left| \frac{du_n}{dt}(t) \right|_H^2 + \frac{d}{dt} \varphi(u_n(t)) = \frac{d}{dt} \langle f_n(t), u_n(t) \rangle - \left\langle \frac{df_n}{dt}(t), u_n(t) \right\rangle$$

for a.e. $t \in]0, T[$. Integrating this over $]0, t[$, we get

$$\begin{aligned} & \int_0^t \left| \frac{du_n}{d\tau}(\tau) \right|_H^2 d\tau + \varphi(u_n(t)) - \varphi(u_0) \\ & \leq |f_n(t)|_{V^*} |u_n(t)|_V + |f_n(0)|_{V^*} |u_0|_V \\ & \quad + \left(\int_0^T \left| \frac{df_n}{d\tau}(\tau) \right|_{V^*}^{p'} d\tau \right)^{1/p'} \left(\int_0^T |u_n(\tau)|_V^p d\tau \right)^{1/p} \end{aligned}$$

for all $t \in [0, T]$. Repeating the same arguments as for (3.19) with the weight t replaced by 1 and noting that $W^{1,p'}(0, T; V^*)$ is embedded in $C([0, T]; V^*)$ continuously, we deduce

$$\begin{aligned} u & \in C([0, T]; V_w) \cap W^{1,2}(0, T; H) \cap W^{1,\infty}(0, T; V^*), \\ g & \in L^\infty(0, T; V^*), \\ \sup_{t \in [0, T]} \varphi(u(t)) & < +\infty, \end{aligned}$$

where $g(t) = f(t) - du(t)/dt$ belongs to $\partial\varphi(u(t))$. ■

REMARK 3.5 Since $f, t(df/dt) \in L^{p'}(0, T; V^*)$, there exists a sequence $f_n \in L^\infty(0, T; H) \cap C^\infty((0, T]; H)$ such that $tf_n \in C^\infty([0, T]; H)$, $\lim_{t \rightarrow +0} tf_n(t) = 0$, $f_n \rightarrow f$ and $t(df_n/dt) \rightarrow t(df/dt)$ strongly in $L^p(0, T; V^*)$. Indeed, from the fact that $f, t(df/dt) \in L^{p'}(0, T; V^*)$, it follows that $d(tf)/dt = f + t(df/dt) \in L^{p'}(0, T; V^*)$, which implies that $tf \in W^{1,p'}(0, T; V^*)$. Since $W^{1,p'}(0, T; V^*) \subset C([0, T]; V^*)$, tf can be regarded as a continuous function on $[0, T]$ with value in V^* . Then $\lim_{t \rightarrow +0} |tf(t)|_{V^*} = \alpha$ exists. If $\alpha \neq 0$, there exists a positive number δ such that $|f(t)|_{V^*} \geq |\alpha|/(2t)$ for all $t \in (0, \delta)$, which contradicts the fact that $f \in L^1(0, T; V^*)$. Thus we find that $tf(t)|_{t=+0} = 0$. Hence, in particular,

$$f(t) = \frac{1}{t} \int_0^t \frac{d}{d\tau} (\tau f(\tau)) d\tau.$$

Moreover, since H is densely and continuously embedded in V^* , we can choose a sequence $\rho_n \in C^\infty([0, T]; H)$ such that $\rho_n(0) = 0$ and $\rho_n \rightarrow tf$ strongly in $W^{1,p'}(0, T; V^*)$. Put $f_n(t) := (1/t)\rho_n(t)$, then

$$f_n \in C^\infty([0, T]; H) \quad \text{and} \quad f_n(t) = (1/t) \int_0^t \frac{d\rho_n}{d\tau}(\tau) d\tau.$$

Hence, from the fact that $\rho_n \in C^\infty([0, T]; H)$, it follows that

$$\sup_{t \in [0, T]} |f_n(t)|_{V^*} \leq \frac{1}{t} \cdot t \sup_{t \in [0, T]} \left| \frac{d\rho_n}{dt}(t) \right|_{V^*},$$

which implies $f_n \in L^\infty(0, T; V^*)$. Furthermore, recalling

$$f_n(t) - f(t) = \frac{1}{t} \int_0^t \frac{d}{d\tau} (\rho_n(\tau) - \tau f(\tau)) d\tau,$$

by Hardy's inequality, we find that

$$\|f_n - f\|_{L^{p'}(0, T; V^*)} \leq p \left\| \frac{d}{dt} (\rho_n - tf) \right\|_{L^{p'}(0, T; V^*)}.$$

Thus $f_n \rightarrow f$ strongly in $L^{p'}(0, T; V^*)$. Therefore, in view of $d\rho_n/dt = d(tf_n)/dt = f_n + t(df_n/dt) \rightarrow d(tf)/dt = f + t(df/dt)$ strongly in $L^{p'}(0, T; V^*)$, we also see that $t(df_n/dt) \rightarrow t(df/dt)$ strongly in $L^{p'}(0, T; V^*)$.

REMARK 3.6

- (1) In Theorem 3.4, it is not necessary to assume that (A2) is satisfied. Indeed, (3. 20) already assures the a priori bound for $t^{1/2}(du_n/dt)$ in $L^2(0, T; H)$. Hence the a priori bound for $t^{1/2}g_n := t^{1/2}(f_n - du_n/dt)$ in $L^2(0, T; V^*)$ follows, since

$$\|t^{1/2}f_n\|_{L^2(0, T; V^*)} \leq \|tf_n\|_{L^\infty(0, T; V^*)}^{1/2} \|f_n\|_{L^1(0, T; V^*)}^{1/2}.$$

In this case, however, we can not conclude that $t^{1/p'}(du/dt) \in L^\infty(0, T; V^*)$.

- (2) Lemma 3.3 assures that the solution u in Theorem 3.2 enjoys the following property:

$$t \mapsto |u(t)|_H^2 \text{ is absolutely continuous on } [0, T].$$

- (3) Let u^i ($i = 1, 2$) be the solutions of (CP) with f and u_0 replaced by f^i and u_0^i . As in the proof of Theorem 3.2, we can define $u_n^i(t)$ as the solutions of approximate equations (CP) $_n$ with f_n and u_{0n} replaced by appropriate f_n^i and u_{0n}^i . Then, by the same verification for (3. 12), we get

$$\begin{aligned} |u_n^1(t) - u_n^2(t)|_H^2 &\leq |u_{0n}^1 - u_{0n}^2|_H^2 \\ &\quad + 2 \int_0^t |f_n^1(\tau) - f_n^2(\tau)|_{V^*} |u_n^1(\tau) - u_n^2(\tau)|_V d\tau. \end{aligned}$$

Since $u_n^i \rightarrow u^i$ strongly in $C([0, T]; H)$ and $|u_n^i(t)|_V$ is bounded in $L^p(0, T)$, by letting $n \rightarrow +\infty$, we deduce

$$\begin{aligned} (3. 23) \quad |u^1(t) - u^2(t)|_H^2 &\leq |u_0^1 - u_0^2|_H^2 \\ &\quad + C_7 \left(\int_0^t |f^1(\tau) - f^2(\tau)|_{V^*}^{p'} d\tau \right)^{1/p'}, \end{aligned}$$

where C_7 depends on $\sum_{i=1}^2 \left(|u_0^i|_H^2 + \int_0^T |f^i(t)|_{V^*}^{p'} dt \right)$ and T , i.e., the continuous dependence of solutions on the initial data and external forces.

- (4) For any $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^{p'}(0, T; V^*)$ with $t(df/dt) \in L^{p'}(0, T; V^*)$, the strong solution u of (CP) satisfies that $\varphi(u(\cdot)) \in C([0, T])$. To see this, putting $\varphi^t(v) := \varphi(v) - \langle f(t), v \rangle$ for all $v \in V$, we find that $\varphi^t \in \Phi(V)$ and $D(\varphi^t) = D(\varphi)$ for all $t \in [0, T]$. Moreover we define φ_H^t as in the proof of Theorem 3.2. Then $\varphi_H^t \in \Phi(H)$ for all $t \in [0, T]$ and φ_H^t satisfies the smoothness condition $(A\varphi^t)$ with $\alpha_1 = \alpha_2 = 0$ introduced in [18] for all $t \in [\delta, T]$ with an arbitrary positive number δ . Furthermore it is clear that $\partial\varphi^t(v) = \partial\varphi(v) - f(t)$. Theorem 3.4 implies

$$\partial\varphi^t(u(t)) = \partial\varphi(u(t)) - f(t) \ni -\frac{du}{dt}(t), \quad t^{1/2} \frac{du}{dt} \in L^2(0, T; H),$$

hence $-du(t)/dt \in \partial\varphi_H^t(u(t))$ for a.e. $t \in]0, T[$. Then, Proposition 2.6 of [18] assures that $t \mapsto \varphi^t(u(t)) = \varphi(u(t)) - \langle f(t), u(t) \rangle$ is absolutely continuous on $]0, T[$. Here we note that $t \mapsto \langle f(t), u(t) \rangle$ is continuous on $]0, T[$. Actually from the fact that $f \in C([0, T]; V^*)$ and $u \in C([0, T]; V_w)$, it follows that for all $s, t \in]0, T[$,

$$\begin{aligned} \langle f(t), u(t) \rangle - \langle f(s), u(s) \rangle &= \langle f(t) - f(s), u(t) \rangle + \langle f(s), u(t) - u(s) \rangle \\ &\rightarrow 0 \end{aligned}$$

as $s \rightarrow t$. Therefore we deduce that $\varphi(u(\cdot)) \in C([0, T])$. Moreover, when $u_0 \in D(\varphi)$ and $f \in W^{1,p'}(0, T; V^*)$, we conclude that $\varphi(u(\cdot)) \in C([0, T])$.

- (5) Assume the following (A3):

- (A3) There exists a continuous mapping $\mathcal{N}(\cdot, \cdot) : [0, +\infty[\times [0, +\infty[\rightarrow [0, +\infty[$ such that $|\cdot|_\varphi := \mathcal{N}(\varphi(\cdot), |\cdot|_H)$ gives a norm of V which is equivalent to $|\cdot|_V$ and $(V, |\cdot|_\varphi)$ is uniformly convex.

Then for any $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^{p'}(0, T; V^*)$ with $t(df/dt) \in L^{p'}(0, T; V^*)$, the strong solution u of (CP) belongs to $C([0, T]; V)$. In fact, from remarks (2) and (4) above, we know that $t \mapsto \varphi(u(t))$ and $t \mapsto |u(t)|_H$ are continuous on $]0, T[$. Hence $t \mapsto |u(t)|_\varphi = \mathcal{N}(\varphi(u(t)), |u(t)|_H)$ is continuous on $]0, T[$. Moreover since $u \in C([0, T], V_w)$, we find that $u(s) \rightarrow u(t)$ weakly in $(V, |\cdot|_\varphi)$ as $s \rightarrow t$. To see this, let $t_n, t \in]0, T[$ be such that $t_n \rightarrow t$ as $n \rightarrow +\infty$ and $u(t_n) \rightarrow u(t)$ weakly in $(V, |\cdot|_V)$ as $n \rightarrow +\infty$. Since $|u(t_n)|_V$ is bounded, by (A3), $|u(t_n)|_\varphi$ is bounded. Then we can extract a subsequence n' of n such that $u(t_{n'}) \rightarrow \chi$ weakly in $(V, |\cdot|_\varphi)$ as $n' \rightarrow +\infty$. Now, since $(V, |\cdot|_V)$ is embedded in H continuously, (A3) implies that $(V, |\cdot|_\varphi)$ is embedded in H continuously. Hence we can extract a subsequence n'' of n' such that $u(t_{n''}) \rightarrow \chi$ weakly in H as $n'' \rightarrow +\infty$ and $\chi = u(t)$. Since this argument does not depend on the choice of subsequences, we finally deduce that $u(t_n) \rightarrow u(t)$ weakly in $(V, |\cdot|_\varphi)$ as $n \rightarrow +\infty$. Therefore it

follows from the uniformly convexity of $(V, |\cdot|_\varphi)$ that $u(s) \rightarrow u(t)$ strongly in $(V, |\cdot|_\varphi)$ as $s \rightarrow t$. Since $|\cdot|_\varphi$ is equivalent to $|\cdot|_V$, we conclude that $u \in C([0, T]; V)$. Especially when $u_0 \in D(\varphi)$ and $f \in W^{1,p'}(0, T; V^*)$, it holds that $u \in C([0, T]; V)$.

REMARK 3.7 We can weaken the sufficient condition (A1) in Theorem 3.2. More precisely, assume that (A2) and the following $(A1)_q$ with $q \in [0, 2p]$ hold.

$$(A1)_q \quad |u|_V^p - C_1|u|_H^q - C_2 \leq C_3\varphi(u) \quad \forall u \in D(\varphi).$$

Then for any $f \in L^r(0, T; V^*)$ with $r = \max(p', 2p/(2p-q))$ (especially if $q = 2p$, then $r := +\infty$) and $u_0 \in \overline{D(\varphi)}^H$, we can assure the existence and the uniqueness of the strong solution of (CP). For this proof, as in the proof of Theorem 3.2, we can consider the approximation problem $(CP)_n$, since the lower semicontinuity of φ_H in H is also assured by $(A1)_q$. Moreover we need a slight modification of a priori estimates for the solution of $(CP)_n$. Recalling (3. 6), we get

$$\frac{1}{2} \frac{d}{dt} |u_n(t)|_H^2 + \varphi(u_n(t)) \leq \varphi(0) + |f_n(t)|_{V^*} |u_n(t)|_V \quad \text{for a.e. } t \in]0, T[.$$

By $(A1)_q$ with $q \in [0, 2p]$, it follows from Young's inequality that for the case where $q \in [0, 2p[$:

$$\begin{aligned} |f_n(t)|_{V^*} |u_n(t)|_V &\leq |f_n(t)|_{V^*} \{C_1|u_n(t)|_H^q + C_2 + C_3\varphi(u_n(t))\}^{1/p} \\ &\leq |f_n(t)|_{V^*} \left\{ C_1^{1/p} |u_n(t)|_H^{q/p} + C_2^{1/p} + C_3^{1/p} (\varphi(u_n(t)))^{1/p} \right\} \\ &\leq C_8 \left\{ |f_n(t)|_{V^*}^{2p/(2p-q)} + |f_n(t)|_{V^*}^{p'} + 1 \right\} + |u_n(t)|_H^2 \\ &\quad + \frac{1}{2} \varphi(u_n(t)), \end{aligned}$$

for the case where $q = 2p$:

$$\begin{aligned} |f_n(t)|_{V^*} |u_n(t)|_V &\leq C_1^{1/p} \left(\sup_{\tau \in [0, T]} |f_n(\tau)|_{V^*} \right) |u_n(t)|_H^2 + C_8 \left\{ |f_n(t)|_{V^*}^{p'} + 1 \right\} \\ &\quad + \frac{1}{2} \varphi(u_n(t)). \end{aligned}$$

Hence we can obtain a priori estimates (3. 7) and (3. 9), which together with $(A1)_q$ imply (3. 8). Furthermore the convergence of u_n can be verified by the same arguments as in the proof of Theorem 3.2. Thus we have

THEOREM 3.8 Let $(A1)_q$ with $q \in [0, 2p]$ and (A2) be satisfied. Then, for every $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^r(0, T; V^*)$ with $r = \max(p', 2p/(2p-q))$ (especially if $q = 2p$, then $r := +\infty$), there exists a unique strong solution u of (CP) satisfying:

$$u \in L^p(0, T; V) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*),$$

the section $g(t)$ of $\partial\varphi(u(t))$ given in (3. 1) belongs to $L^{p'}(0, T; V^*)$,

$$\varphi(u(\cdot)) \in L^1(0, T).$$

In Theorem 3.4, we can replace (A1) by $(A1)_q$ with $q \in [0, 2p]$ when f satisfies the all conditions required in Theorem 3.4 and that $f \in L^r(0, T; V^*)$. Furthermore since the arguments in Remark 3.6 are independent of the assumption (A1), the same conclusion can be derived for the case where $f \in L^r(0, T; V^*)$, and $(A1)_q$ with $q \in [0, 2p]$ holds instead of (A1).

REMARK 3.9 Just as in Theorem 3.2, we can assure the existence and uniqueness of the strong solution of the following Cauchy problem with Lipschitz perturbation:

$$(CP)' \begin{cases} \frac{du}{dt}(t) + \partial\varphi(u(t)) \ni f(t) - Bu(t) & \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where B is a Lipschitz continuous operator from H into H , that is, there exists $L \geq 0$ such that $|Bu - Bv|_H \leq L|u - v|_H$ for all $u, v \in H$. The strong solution of $(CP)'$ is defined by Definition 3.1 with (CP) and $f(t)$ replaced by $(CP)'$ and $f(t) - Bu(t)$ respectively. To verify the existence part, let us consider the following approximation problem:

$$\begin{cases} \frac{du_n}{dt}(t) + \partial\varphi_H(u_n(t)) + Bu_n(t) \ni f_n(t) & \text{in } H, \quad 0 < t < T, \\ u(0) = u_{0n} \end{cases}$$

and the existence of its strong solution u_n is assured by [6, Proposition 3.12]. Then, using the fact that $|Bu|_H \leq C_9 + L|u|_H$ for all $u \in H$, we can obtain a priori estimates for u_n by the same arguments as in the proof of Theorem 3.2 with obvious modifications. Moreover, when $u_n \rightarrow u$ strongly in $C([0, T]; H)$ as $n \rightarrow +\infty$, it follows from the Lipschitz continuity of B that $Bu_n \rightarrow Bu$ strongly in $C([0, T]; H)$ as $n \rightarrow +\infty$. From these facts, we can handle with the convergence as in the proof of Theorem 3.2. Thus the following result holds.

THEOREM 3.10 *Let (A1) and (A2) be satisfied. Then, for every $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^{p'}(0, T; V^*)$, there exists a unique strong solution u of $(CP)'$ satisfying:*

$$\begin{aligned} u &\in L^p(0, T; V) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*), \\ \text{the section } g(t) \text{ of } \partial\varphi(u(t)) &\text{ given in (3. 1) belongs to } L^{p'}(0, T; V^*), \\ \varphi(u(\cdot)) &\in L^1(0, T), \quad Bu \in C([0, T]; H). \end{aligned}$$

Moreover when $f \in L^{p'}(0, T; V^*)$ satisfies $t(df/dt) \in L^{p'}(0, T; V^*)$, just as in the proof of Lemma 3.4, we obtain the following theorem.

THEOREM 3.11 *Let (A1) and (A2) be satisfied. Then, for every $u_0 \in \overline{D(\varphi)}^H$ and $f \in L^{p'}(0, T; V^*)$ with $t(df/dt) \in L^{p'}(0, T; V^*)$, the solution u of $(CP)'$ satisfies:*

$$u \in C([0, T]; V_w) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*),$$

$$u(t) \in D(\varphi) \quad \forall t > 0, \quad \sup_{t \in [0, T]} t\varphi(u(t)) < +\infty,$$

$$t^{1/p'} \frac{du}{dt} \in L^\infty(0, T; V^*), \quad t^{1/2} \frac{du}{dt} \in L^2(0, T; H), \quad t^{1/2} \frac{d}{dt} Bu \in L^2(0, T; H).$$

Moreover, if $u_0 \in D(\varphi)$ and $f \in W^{1,p'}(0, T; V^*)$, then

$$u \in C([0, T]; V_w) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*),$$

$$u(t) \in D(\varphi) \quad \forall t \geq 0, \quad \sup_{t \in [0, T]} \varphi(u(t)) < +\infty,$$

$$\frac{du}{dt} \in L^2(0, T; H) \cap L^\infty(0, T; V^*), \quad \frac{d}{dt} Bu \in L^2(0, T; H).$$

4 Periodic Problem

In this section, we study the following periodic problem (PP).

$$(PP) \quad \begin{cases} \frac{du}{dt}(t) + \partial\varphi(u(t)) \ni f(t) & \text{in } V^*, \quad 0 < t < T, \\ u(0) = u(T). \end{cases}$$

We begin with the definition of *strong solutions* of (PP).

DEFINITION 4.1 A function $u \in C([0, T]; V^*)$ is said to be a strong solution of (PP) on $[0, T]$, if the following conditions are satisfied:

- (i) $u(t)$ is a V^* -valued absolutely continuous function on $[0, T]$,
- (ii) $u(0) = u(T)$,
- (iii) $u(t) \in D(\partial\varphi)$ for a.e. $t \in]0, T[$
and there exists a function $g(t) \in \partial\varphi(u(t))$ satisfying (3. 1).

Then our result is stated as follows:

THEOREM 4.2 Let (A1) with $C_1 = 0$ and (A2) be fulfilled. Then, for every $f \in L^{p'}(0, T; V^*)$, (PP) has at least one strong solution u satisfying:

$$u \in L^p(0, T; V) \cap C([0, T]; H) \cap W^{1,p'}(0, T; V^*),$$

$$g \text{ in (3. 1) belongs to } L^{p'}(0, T; V^*),$$

$$\varphi(u(\cdot)) \in L^1(0, T).$$

REMARK 4.3 By using the fact that there exists a constant C such that $|u|_H \leq C|u|_V$ and applying Young's inequality, we can easily see that $(A1)_q$ with $q \in [0, p[$ implies (A1) with $C_1 = 0$, i.e.,

$$|u|_V^p - C_2 \leq C_3 \varphi(u) \quad \forall u \in D(\varphi).$$

First, we prepare the following lemma for later use.

LEMMA 4.4 *Let $f(\cdot) \in L^1(0, T)$ and let $j(\cdot)$ be a nonnegative absolutely continuous function on $[0, T]$ such that*

$$(4.1) \quad \frac{d}{dt}j(t) + \alpha j^{p-1}(t) \leq K|f(t)| \quad \text{for a.e. } t \in]0, T[,$$

where $\alpha > 0$, $K > 0$ and $p > 1$. Suppose that $j(0) \leq r$ and $\|f\|_{1,T} \leq r^{p-1}$ ($r > 0$), where

$$\|f\|_{1,T} := \begin{cases} \sup_{t \in [1, T]} \int_{t-1}^t |f(\tau)| d\tau & \text{if } 1 \leq T, \\ \int_0^T |f(\tau)| d\tau & \text{if } 0 < T < 1. \end{cases}$$

Then there exists a monotone non-decreasing function $M_{\alpha, K, p}(\cdot)$ depending on α, K, p such that

$$j(t) \leq M_{\alpha, K, p}(r)r \quad \text{for all } t \in [0, T].$$

PROOF OF LEMMA 4.4 The verification for the case where $2 \leq p$ can be done as in the proof of Lemma 4.3 of [16] with obvious modifications. As for the case where $1 < p < 2$, we can modify the arguments in the proofs of Lemma 4.3 of [16], Lemma 4.6 of [17] and Lemma 3.4 of [10]. We put

$$\phi_a(t) := \left(|j^{2-p}(a) - \alpha(2-p)(t-a)|^+ \right)^{1/(2-p)} + K \int_a^t |f(s)| ds, \quad a \in [0, T],$$

where $|m|^+ := \max\{m, 0\}$. Then $\phi_a(t)$ satisfies

$$(4.2) \quad \begin{cases} \frac{d\phi_a}{dt}(t) + \alpha\phi_a^{p-1}(t) \geq K|f(t)| & \text{for a.e. } t \in]a, T[, \\ \phi_a(a) = j(a). \end{cases}$$

Consequently, we find

$$j(t) \leq \phi_a(t) \quad \text{for all } t \in [a, T].$$

Now, we claim that $M_{\alpha, K, p}$ can be taken as follows:

$$(4.3) \quad M_{\alpha, K, p}(r) := 2(1 + Kr^{p-2}) + \left(\frac{2K}{\alpha(2-p)} \right)^{1/(p-1)}.$$

Suppose that there exists $t_2 \in [0, T]$ such that $j(t_2) > M_{\alpha, K, p}(r)r$. Since $j(0) \leq r \leq M_{\alpha, K, p}(r)r$ and $t \mapsto j(t)$ is continuous, there exists $t_1 \in [0, t_2[$ such that $j(t_1) = M_{\alpha, K, p}(r)r$ and $j(t) < M_{\alpha, K, p}(r)r$ for all $t \in [0, t_1[$.

Then, by putting $\tau_0 := (M_{\alpha, K, p}(r)r)^{2-p}/\alpha(2-p)$, we deduce

$$(4.4) \quad \begin{aligned} M_{\alpha, K, p}(r)r &= j(t_1) \leq \phi_0(t_1) \leq j(0) + K \int_0^{t_1} |f(t)| dt \\ &\leq r + K(1 + \tau_0)r^{p-1} \quad \text{if } t_1 \in [0, \tau_0], \end{aligned}$$

and

$$(4.5) \quad \begin{aligned} M_{\alpha,K,p}(r)r &= j(t_1) \leq \phi_{t_1-\tau_0}(t_1) \\ &\leq K(1+\tau_0)r^{p-1} \quad \text{if } t_1 \in]\tau_0, T], \end{aligned}$$

since $|j^{2-p}(t_1-\tau_0)-\alpha(2-p)\{t_1-(t_1-\tau_0)\}|^+ \leq |(M_{\alpha,K,p}(r)r)^{2-p}-\alpha(2-p)\tau_0|^+ = 0$. However, simple calculations show that (4.4) or (4.5) contradicts (4.3), the definition of $M_{\alpha,K,p}$. Indeed, it follows from (4.4) or (4.5) that

$$\begin{aligned} M_{\alpha,K,p}(r) &\leq 1 + Kr^{p-2} + K\tau_0 r^{p-2} \\ &\leq \frac{1}{2}M_{\alpha,K,p}(r) - \frac{1}{2}\left(\frac{2K}{\alpha(2-p)}\right)^{1/(p-1)} + K\frac{(M_{\alpha,K,p}(r)r)^{2-p}}{\alpha(2-p)}r^{p-2} \\ &\leq \frac{1}{2}M_{\alpha,K,p}(r) + K\frac{(M_{\alpha,K,p}(r))^{2-p}}{\alpha(2-p)}. \end{aligned}$$

Hence we can verify

$$M_{\alpha,K,p}(r) \leq \left(\frac{2K}{\alpha(2-p)}\right)^{1/(p-1)},$$

which contradicts (4.3). Therefore we deduce

$$\sup_{t \in [0, T]} j(t) \leq M_{\alpha,K,p}(r)r. \blacksquare$$

PROOF OF THEOREM 4.2 By Theorem 3.2, for every $u_0 \in \overline{D(\varphi)}^H$, there exists a unique strong solution u of (CP). For a fixed $f \in L^p(0, T; V^*)$, we consider the mapping S_f from $\overline{D(\varphi)}^H$ into $\overline{D(\varphi)}^H$ such that

$$S_f(u_0) = u(T).$$

In order to show the existence of the periodic solution, it suffices to show that S_f has a fixed point. To this end, we use the following fixed point theorem.

LEMMA 4.5 (*Browder-Petryshyn* [9]) *Let X be a uniformly convex Banach space, let C be a closed convex subset of X and let S be a non-expansive map from C into C . Then S has a fixed point if and only if there exists $x \in C$ such that $S^n x$ is bounded for all $n \in \mathbb{N}$.*

We are going to apply this lemma with $S = S_f$. So, for all $n \in \mathbb{N}$, we consider the following Cauchy problem:

$$(4.6) \quad \begin{cases} \frac{du_n}{dt}(t) + \partial\varphi(u_n(t)) \ni f(t) & \text{in } V^*, \quad 0 < t < T, \\ u_n(0) = (S_f)^{n-1}u_0, \end{cases}$$

where $(S_f)^0 u_0 := u_0 \in \overline{D(\varphi)}^H$. Then, we see that $(S_f)^n u_0 = u_n(T)$.

For every $n \in \mathbb{N}$, let $F(t) := f(t - nT)$ and $U(t) := u_n(t - nT)$ for all $t \in [nT, (n+1)T[$, then (4. 6) implies

$$(4. 7) \quad \begin{cases} \frac{dU}{dt}(t) + \partial\varphi(U(t)) \ni F(t) & \text{in } V^*, \quad 0 < t < +\infty, \\ U(0) = u_0. \end{cases}$$

Since $u_n(T) = U((n+1)T)$, we have only to show that $U(nT)$ is bounded in H . By much the same verification as for (3. 6), it follows from (A1) with $C_1 = 0$ and (4. 7) that there exist positive constants C_{10}, C_{11} such that

$$(4. 8) \quad \frac{d}{dt}|U(t)|_H^2 + C_{10}|U(t)|_H^p \leq C_{11} \left(|F(t)|_{V^*}^{p'} + 1 \right) \quad \text{for a.e. } t \in]0, +\infty[.$$

In view of

$$\left\| |F(\cdot)|_{V^*}^{p'} + 1 \right\|_{1, nT} \leq \left(1 + \frac{1}{T} \right) \|f\|_{L^{p'}(0, T; V^*)}^{p'} + 1 \quad \text{for all } n \in \mathbb{N},$$

applying Lemma 4.4 to (4. 8), we deduce

$$\sup_{n \in \mathbb{N}} |U(nT)|_H < +\infty.$$

In order to see that S_f is non-expansive, we have only to recall estimate (3. 23) with $f^1 = f^2$. Thus Lemma 4.5 completes the proof. ■

We have the following uniqueness theorem.

THEOREM 4.6 *Let all assumptions in Theorem 4.2 and the following (A4) be satisfied.*

$$(A4) \quad \varphi \text{ is strictly convex.}$$

Then, for every $f \in L^{p'}(0, T; V^)$, the strong solution of (PP) is unique.*

PROOF OF THEOREM 4.6 Let u and v be any periodic solutions of (E), then by virtue of (3. 4), we obtain

$$\frac{1}{2}|u(T) - v(T)|_H^2 - \frac{1}{2}|u(0) - v(0)|_H^2 + \int_0^T \langle g(\tau) - h(\tau), u(\tau) - v(\tau) \rangle d\tau = 0,$$

where $g(t) \in \partial\varphi(u(t))$ and $h(t) \in \partial\varphi(v(t))$ for a.e. $t \in]0, T[$. From the periodicity of the solutions, it follows that

$$(4. 9) \quad \int_0^T \langle g(\tau) - h(\tau), u(\tau) - v(\tau) \rangle d\tau = 0.$$

Since the strict convexity of φ implies the strict monotonicity of $\partial\varphi$, whence follows $u = v$. ■

REMARK 4.7 We can obtain regularity results for (PP) similar to those in Theorem 3.4 and Remark 3.6 for (CP).

5 Application

In this section, we give a typical example to which the preceding theory can be applied.

Let Ω be a domain in \mathbb{R}^N with smooth boundary $\partial\Omega$ and $p \in]1, +\infty[$.

$$(NHE) \begin{cases} \frac{\partial u}{\partial t}(x, t) - \Delta_p u(x, t) = f(x, t), & (x, t) \in \Omega \times]0, T[, \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times]0, T[, \end{cases}$$

where Δ_p denotes the so-called p -Laplacian given by

$$\Delta_p u := \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad 1 < p < +\infty.$$

In [15], by using Faedo-Galerkin's method, J. L. Lions studied the existence of weak solutions of the initial-boundary value problem and periodic problem for (NHE) when Ω is a bounded domain. Our abstract framework can be applied not only for the bounded domain case but also for the unbounded domain case. Moreover our abstract treatment provides us more minute informations on the regularity of solutions in a natural way.

Let $X_p := \left\{ u \in L^2(\Omega); \nabla u \in (L^p(\Omega))^N \right\}$ be with the norm

$$|u|_{X_p} := \left\{ |u|_{L^2(\Omega)}^p + |\nabla u|_{L^p(\Omega)}^p \right\}^{1/p} \quad \text{for all } u \in X_p,$$

where $|u|_{L^2(\Omega)} = \left\{ \int_{\Omega} |u(x)|^2 dx \right\}^{1/2}$ and $|\nabla u|_{L^p(\Omega)} = \left\{ \int_{\Omega} |\nabla u(x)|^p dx \right\}^{1/p}$. Moreover let $V_p := \overline{C_0^\infty(\Omega)}^{X_p}$ with $|\cdot|_{V_p} := |\cdot|_{X_p}$. Then we find that V_p is a uniformly convex Banach space, since V_p is a closed subspace of X_p which is a uniformly convex Banach space (see [1, 1.21, 1.22]). Moreover from the definition of V_p , it is easily obtained that V_p is embedded in $L^2(\Omega)$ with continuous injection. Furthermore we can verify that V_p is dense in $L^2(\Omega)$. Indeed, from the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, it is obvious that $L^2(\Omega) \supset \overline{V_p}^{L^2(\Omega)} \supset \overline{C_0^\infty(\Omega)}^{L^2(\Omega)} = L^2(\Omega)$. Hence we deduce that $\overline{V_p}^{L^2(\Omega)} = L^2(\Omega)$.

Now putting $V = V_p$ and $H = L^2(\Omega)$ with $|\cdot|_V := |\cdot|_{V_p}$ and $|\cdot|_H := |\cdot|_{L^2(\Omega)}$, we find that (2. 1) holds. We now define a function ϕ_p on V by setting

$$\phi_p(u) := \frac{1}{p} \int_{\Omega} |\nabla u(x)|^p dx \quad \text{for all } u \in V.$$

Then, it follows that $D(\phi_p) = V$ and $\partial\phi_p(u) = -\Delta_p u$ (see [4, p53, Example 1]). Hence, (NHE) can be reduced to (E) with $\partial\varphi$ replaced by $\partial\phi_p$.

5.1 Bounded Domain Case

First we consider the case where Ω is a bounded domain.

For the case where $2N/(N+2) \leq p < +\infty$, by Sobolev's embedding theorem and Poincaré's inequality together with the boundedness of Ω , X_p and V_p coincide with $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively and we can reset $|u|_V := |\nabla u|_{L^p(\Omega)}$ for all $u \in V$.

Hence we find that $\phi_p(u) = (1/p)|u|_V^p$, which implies (A1) with $C_1 = C_2 = 0$, (A3) and (A4). Moreover, since $\sup\{\int_{\Omega} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla w(x) dx; w \in V \text{ with } |\nabla w|_{L^p(\Omega)} = 1\} \leq |\nabla u|_{L^p(\Omega)}^{p-1}$, we easily get

$$|\partial \phi_p(u)|_{V^*}^{p'} \leq p \phi_p(u),$$

which implies (A2). Therefore we can apply Theorems 3.2, 3.4, Remark 3.6 and Theorem 4.2 to the initial-boundary value problem and periodic problem for (NHE).

As for the case where $1 < p < 2N/(N+2)$, X_p coincides with $W^{1,p}(\Omega) \cap L^2(\Omega)$, since by Sobolev's embedding theorem and the boundedness of Ω , $|\cdot|_{X_p}$ is equivalent to $|\cdot|_{W^{1,p}(\Omega)} + |\cdot|_{L^2(\Omega)}$. Moreover we can verify that the closure of $C_0^\infty(\Omega)$ in X_p coincides with $W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Indeed, it is clear that the closure of $C_0^\infty(\Omega)$ in X_p is a subset of $W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Moreover, by the slight modification of the proof of Proposition IX.18 in [5], we can show that all elements in $W_0^{1,p}(\Omega) \cap L^2(\Omega)$ belong to the closure of $C_0^\infty(\Omega)$ in X_p . Therefore $V_p = W_0^{1,p}(\Omega) \cap L^2(\Omega)$. Furthermore V^* is homeomorphic to $W^{-1,p'}(\Omega) + L^2(\Omega)$.

By the definition of $|\cdot|_V$, $|\cdot|_H$ and ϕ_p , we have

$$\begin{aligned} |u|_V^p &= |\nabla u|_{L^p(\Omega)}^p + |u|_H^p \\ &\leq p \phi_p(u) + |u|_H^2 + \mathcal{M}_{2/p}(1), \end{aligned}$$

which implies (A1). Moreover (A2) and (A3) hold. Therefore we can apply Theorem 3.2, 3.4 and Remark 3.6 to the initial-boundary value problem for (NHE).

5.2 Unbounded Domain Case

As for the case where Ω is an unbounded domain, from the definition of ϕ_p ,

$$\phi_p(u) = \frac{1}{p} |\nabla u|_{L^p(\Omega)}^p = \frac{1}{p} \{|u|_V^p - |u|_H^p\},$$

which implies (A1)_p. Moreover (A2) and (A3) hold. Hence Theorem 3.8 can be applicable to the initial-boundary value problem for (NHE).

5.3 Results

Here note the following coincidence:

$$V_p = \begin{cases} W_0^{1,p}(\Omega) \cap L^2(\Omega) & \text{if } 1 < p < \frac{2N}{N+2}, \Omega \text{ is bounded,} \\ W_0^{1,p}(\Omega) & \text{if } \frac{2N}{N+2} \leq p < +\infty, \Omega \text{ is bounded.} \end{cases}$$

Summarizing observations above, we obtain the following results for Cauchy problem.

THEOREM 5.1 (*Initial-boundary value problem: bounded domain case*) *When $1 < p < +\infty$ and Ω is a bounded domain, for every $u_0 \in L^2(\Omega)$ and $f \in L^{p'}(0, T; V_p^*)$, there exists a unique solution u of the initial-boundary value problem for (NHE) with the initial data u_0 such that*

$$(5.1) \quad u \in L^p(0, T; V_p) \cap C([0, T]; L^2(\Omega)) \cap W^{1,p'}(0, T; V_p^*).$$

Moreover if $f, t(df/dt) \in L^{p'}(0, T; V_p^)$, then the following (5.2) holds for any $\delta > 0$.*

$$(5.2) \quad u \in C([\delta, T]; V_p) \cap W^{1,2}(\delta, T; L^2(\Omega)) \cap W^{1,\infty}(\delta, T; V_p^*).$$

Furthermore if $u_0 \in V_p$ and $f \in W^{1,p'}(0, T; V_p^)$, then (5.2) with $\delta = 0$ is satisfied.*

THEOREM 5.2 (*Initial-boundary value problem: unbounded domain case*) *When $1 < p < +\infty$ and Ω is an unbounded domain, for every $u_0 \in L^2(\Omega)$ and $f \in L^q(0, T; V_p^*)$ with $q = \max(p', 2)$, there exists a unique solution u of the initial-boundary value problem for (NHE) with the initial data u_0 such that (5.1) holds. Furthermore if $f, t(df/dt) \in L^{p'}(0, T; V_p^*)$, (resp. $u_0 \in V_p$ and $f \in W^{1,p'}(0, T; V_p^*)$), then (5.2) for any $\delta > 0$, (resp. (5.2) with $\delta = 0$), holds.*

As for the periodic problem, our result is stated as follows.

THEOREM 5.3 (*Periodic problem: bounded domain case*) *When $2N/(N+2) \leq p < +\infty$ and Ω is a bounded domain, for every $f \in L^{p'}(0, T; W_0^{-1,p'}(\Omega))$, there exists a unique solution u of the periodic problem for (NHE) such that (5.1) holds. Furthermore if $f \in W^{1,p'}(0, T; W^{-1,p'}(\Omega))$, then (5.2) with $\delta = 0$ holds.*

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