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CONVERGENCE OF FUNCTIONALS AND ITS APPLICATIONS TO PARABOLIC EQUATIONS

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Asymptotic behavior of solutions of some parabolic equation associated with the p-Laplacian as $p \to +\infty$ is studied for the periodic problem as well as the initial-boundary value problem by pointing out the variational structure of the p-Laplacian, that is, $\partial\varphi_p(u) = -\Delta_p u$, where $\varphi_p : L^2(\Omega) \to [0,+\infty]$. To this end, the notion of Mosco convergence is employed and it is proved that φ_p converges to the indicator function over some closed convex set on $L^2(\Omega)$ in the sense of Mosco as $p \to +\infty$; moreover, an abstract theory relative to Mosco convergence and evolution equations governed by time-dependent subdifferentials is developed until the periodic problem falls within its scope. Further application of this approach to the limiting problem of porous-medium-type equations, such as $u_t = \Delta|u|^{m-2}u$ as $m \to +\infty$, is also given.

1. Introduction

The so-called p-Laplacian Δ_p given below could be regarded as a nonlinear differential operator generalizing the usual linear Laplacian:

$$\Delta_p u(x) := \nabla \cdot \left(|\nabla u(x)|^{p-2} \nabla u(x) \right), \quad 1 < p < +\infty. \tag{1.1}$$

This paper is motivated by the following naive question: what is the limit of Δ_p as $p \to +\infty$? This limiting problem was studied by several authors and their results were applied in various fields; for example, growing sandpile model [2], macroscopic models for type-II superconductors [1, 4, 13], and so on. In order to figure out the substantial features of this problem, we here recall the variational structure of p-Laplacian:

$$-\Delta_p u = dI_p(u), \quad I_p(u) := \frac{1}{p} \int_\Omega |\nabla u(x)|^p dx \quad \forall u \in W_0^{1,p}(\Omega), \tag{1.2}$$

where $dI_p(u)$ denotes the Fréchet derivative of the functional I_p at u, and we intend to investigate the limit of the functional I_p instead of Δ_p as $p \to +\infty$.

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However, it is easily expected that the limit of I_p may not belong to the class of Fréchet differentiable functionals. On the other hand, from the viewpoint of studies on evolution equations, it is convenient for applications to extend I_p on $L^2(\Omega)$ as follows:

$$\varphi_p(u) := \begin{cases} I_p(u) & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W_0^{1,p}(\Omega). \end{cases} \tag{1.3}$$

Then it is well known that φ_p is no longer Fréchet differentiable on $L^2(\Omega)$, but lower semicontinuous and convex on $L^2(\Omega)$; moreover, its subdifferential $\partial_{L^2(\Omega)}\varphi_p(u)$ coincides with $-\Delta_p u$ in the distribution sense.

Several authors also studied the asymptotic behavior of solutions for the following initial-boundary value problem as $p \to +\infty$:

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t) - \Delta_p u(x,t) = 0, \quad (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{gathered} \tag{1.4}$$

Here it is well known that (1.4) can be reduced to the following abstract Cauchy problem:

$$\begin{gathered} \frac{du}{dt}(t) + \partial_{L^2(\Omega)}\varphi_p(u(t)) = 0 \quad \text{in } L^2(\Omega),\ 0 < t < T, \\ u(0) = u_0. \end{gathered} \tag{1.5}$$

According to the previous studies, for example, [2, 4], every solution u_p of (1.5) converges to u as $p \to +\infty$ and the limit u gives a solution of the following Cauchy problem:

$$\begin{gathered} \frac{du}{dt}(t) + \partial_{L^2(\Omega)}\varphi_\infty(u(t)) \ni 0 \quad \text{in } L^2(\Omega),\ 0 < t < T, \\ u(0) = u_0, \end{gathered} \tag{1.6}$$

where φ_∞ is defined on $L^2(\Omega)$ by

$$\varphi_\infty(u) := \begin{cases} 0 & \text{if } u \in H_0^1(\Omega),\ |\nabla u|_{L^\infty(\Omega)} \le 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.7}$$

Hence one can easily expect that φ_p converges to φ_∞ as $p \to +\infty$ in a certain sense; however, it is not so obvious in what sense it is realized. In this paper, we prove that φ_p converges to φ_∞ on $L^2(\Omega)$ in the sense of Mosco as $p \to +\infty$; moreover, we discuss the asymptotic behavior of solutions for (1.4) as $p \to +\infty$ in a more general setting. These results will be shown in Section 3.1 whereas the definition of Mosco convergence will be given in Section 2. Moreover, our method can also be applied to the porous medium equation

$$\frac{\partial u}{\partial t}(x,t) - \Delta |u|^{m-2}u(x,t) = 0, \quad (x,t) \in \Omega \times (0,T). \tag{1.8}$$

In Section 3.2, we deal with the asymptotic behavior of solutions for (1.8) as $m \to +\infty$.

To formulate our results in an abstract form, we work in a more generalized setting, that is, we consider the following abstract evolution equations in a real Hilbert space H governed by time-dependent subdifferential operators $\partial_H \varphi_n^t$:

$$\frac{du_n}{dt}(t) + \partial_H \varphi_n^t(u_n(t)) \ni f_n(t) \quad \text{in } H,\ 0 < t < T,\ n \in \mathbb{N}, \tag{1.9}$$

where $f_n \in L^2(0,T;H)$, $f_n \to f$ strongly in $L^2(0,T;H)$, and φ_n^t is a time-dependent proper lower semicontinuous convex functional from H into $(-\infty, +\infty]$ such that $\varphi_n^t \to \varphi^t$ on H in the sense of Mosco as $n \to +\infty$. For the case where φ^t does not depend on t, that is, $\varphi^t = \varphi$, we can find related results in [3]. For the general case, we refer to Kenmochi [10].

However, all of the previous studies were done on the Cauchy problem for (1.9). As for the periodic problem for (1.9), there seems to be no attempt yet. The main objective here is to investigate the periodic problem as well as the Cauchy problem. The Cauchy problem has a unique solution, and the uniqueness of solution plays an essential role in deriving the convergence of u_n as $n \to +\infty$ in [3, 10]. On the other hand, in general, periodic solution is not unique. Hence the same procedure as in [3, 10] breaks down.

To cope with this difficulty, we introduce a remedy based on a compactness argument under a compactness assumption on the level set of $\{\varphi_n^t\}_{n\in\mathbb{N}}$. This result will be illustrated in the next section.

2. Evolution equations and Mosco convergence

This section deals with the following evolution equation $(\mathrm{E}(\varphi^t, f))$ in a Hilbert space H.

$(\mathrm{E}(\varphi^t, f))$

$$\frac{du}{dt}(t) + \partial_H \varphi^t(u(t)) \ni f(t) \quad \text{in } H,\ 0 < t < T, \tag{2.1}$$

where $f \in L^1(0,T;H)$ and $\partial_H \varphi^t$ is the subdifferential of a proper lower semicontinuous convex functional $\varphi^t : H \to (-\infty, +\infty]$ for every $t \in [0,T]$.

Throughout this paper, we denote by $\Psi(X)$ the set of all proper lower semicontinuous convex functionals ϕ from a Hilbert space X into $(-\infty, +\infty]$, where "proper" means that $\phi \not\equiv +\infty$. Moreover, the subdifferential $\partial_X \phi$ of $\phi \in \Psi(X)$ is defined as follows:

$$\partial_X \phi(u) := \{\xi \in X;\ \phi(v) - \phi(u) \geq (\xi, v-u)_X\ \forall v \in D(\phi)\}, \tag{2.2}$$

where $(\cdot,\cdot)_X$ denotes the inner product of X and $D(\phi)$ is the *effective domain* of ϕ given by

$$D(\phi) := \{u \in X;\ \phi(u) < +\infty\}. \tag{2.3}$$

Moreover, the domain $D(\partial_X \phi)$ of $\partial_X \phi$ is defined by

$$D(\partial_X \phi) := \{u \in D(\phi);\ \partial_X \phi(u) \neq \varnothing\}. \tag{2.4}$$

Now solutions of $(\mathrm{E}(\varphi^t, f))$ are defined as follows.

Definition 2.1. A function $u \in C([0,T];H)$ is said to be a strong solution of $(\mathrm{E}(\varphi^t, f))$ if the following are both satisfied:

(i) u is an H-valued absolutely continuous function on $[0,T]$;
(ii) $u(t) \in D(\partial_H \varphi^t)$ for a.e. $t \in (0,T)$ and there exists a section $g(t) \in \partial_H \varphi^t(u(t))$ such that

$$\frac{du}{dt}(t) + g(t) = f(t) \quad \text{in } H \text{ for a.e. } t \in (0,T). \tag{2.5}$$

Moreover, a function $u \in C([0,T];H)$ is said to be a weak solution of $(\mathrm{E}(\varphi^t, f))$ if there exist sequences $(f_n) \subset L^1(0,T;H)$ and $(u_n) \subset C([0,T];H)$ such that u_n is a strong solution of $(\mathrm{E}(\varphi^t, f_n))$, $f_n \to f$ strongly in $L^1(0,T;H)$, and $u_n \to u$ strongly in $C([0,T];H)$.

We next introduce a notion of the convergence of functionals.

Definition 2.2. Let X be a Hilbert space. Let (φ_n) be a sequence in $\Psi(X)$ and let $\varphi \in \Psi(X)$. Then $\varphi_n \to \varphi$ on X in the sense of Mosco as $n \to +\infty$ if the following conditions are all satisfied.

(1) For all $u \in D(\varphi)$, there exists a sequence (u_n) in X such that $u_n \to u$ strongly in X and $\varphi_n(u_n) \to \varphi(u)$.

(2) Let (u_n) be a sequence in X such that $u_n \to u$ weakly in X. Then $\liminf_{n\to+\infty} \varphi_n(u_n) \geq \varphi(u)$.

Remark 2.3. The second condition in Definition 2.2 is equivalent to the following.

(2)′ Let (u_k) be a sequence in X such that $u_k \to u$ weakly in X as $k \to +\infty$ and let (n_k) be a subsequence of (n). Then $\liminf_{k\to+\infty} \varphi_{n_k}(u_k) \geq \varphi(u)$.

Indeed, it is easily seen that (2) is derived immediately from (2)′. Hence it suffices to show that (2) implies (2)′. Suppose that (2) holds but (2)′ does not, that is, there exist a sequence (u_k) and a subsequence (n_k) of (n) such that

$$u_k \longrightarrow u \quad \text{weakly in } X, \qquad \liminf_{k\to+\infty} \varphi_{n_k}(u_k) < \varphi(u). \tag{2.6}$$

Now define the sequence $(\tilde{u}_n)$ as follows: $\tilde{u}_n = u_k$ if $n \in [n_k, n_{k+1})$ for each $k \in \mathbb{N}$. It then follows that $\tilde{u}_n \to u$ weakly in X as $n \to +\infty$. Moreover, (2.6) yields

$$\begin{aligned} \varphi(u) > \liminf_{k\to+\infty} \varphi_{n_k}(u_k) &= \lim_{K\to+\infty} \inf_{k\geq K} \varphi_{n_k}(u_k) \\ &\geq \lim_{K\to+\infty} \inf_{n\geq n_K} \varphi_n(\tilde{u}_n) = \liminf_{n\to+\infty} \varphi_n(\tilde{u}_n), \end{aligned} \tag{2.7}$$

which contradicts (2). Hence (2) implies (2)′.

In the following two subsections, we discuss the existence and uniqueness of solutions u_n for $(\mathrm{E}(\varphi_n^t, f_n))$ and the convergence of u_n as $n \to +\infty$ for the periodic problem as well as the Cauchy problem. To this end, we fix notations. From now on, we write $\{\varphi^t\}_{t\in[0,T]} \in \Psi(\alpha,\beta)$ for some functions $\alpha,\beta : [0,+\infty)\times[0,T] \to \mathbb{R}$ if the following hold true:

(i) $\varphi^t \in \Psi(H)$ for all $t \in [0,T]$;

(ii) there exists $\delta > 0$; for all $t_0 \in [0,T]$ and all $u_0 \in D(\varphi^{t_0})$, there exists a function u from $I_\delta(t_0) := [t_0-\delta, t_0+\delta]\cap[0,T]$ into H; for all $t \in I_\delta(t_0)$ and all $r \geq |u_0|_H$,

$$\begin{aligned} |u(t)-u_0|_H &\leq |\alpha(r,t)-\alpha(r,t_0)|\{|\varphi^{t_0}(u_0)|+1\}^{1/2}, \\ \varphi^t(u(t)) &\leq \varphi^{t_0}(u_0) + |\beta(r,t)-\beta(r,t_0)|\{|\varphi^{t_0}(u_0)|+1\}. \end{aligned} \tag{2.8}$$

Moreover, we say $\{\varphi^t\}_{t\in[0,T]} \in B(\alpha,\beta,C_0,\{M_r\}_{r\geq 0})$ for some functions $\alpha,\beta : [0,+\infty)\times[0,T] \to \mathbb{R}$ and constants C_0, $\{M_r\}_{r\geq 0}$ if the following are all satisfied.

(i) $\{\varphi^t\}_{t\in[0,T]} \in \Psi(\alpha,\beta)$.

(ii) $\varphi^t(u) \geq -C_0(|u|_H+1)$ for all $u \in H$ and all $t \in [0,T]$.

(iii) There exists a function $h : [0,T] \to H$ such that

$$\sup_{t\in[0,T]}\{|h(t)|_H + |\varphi^t(h(t))|\} + \left(\int_0^T \left|\frac{dh}{dt}(t)\right|_H^2 dt\right)^{1/2} \leq C_0. \tag{2.9}$$

(iv) For every $r \in [0,+\infty)$, it follows that

$$\int_0^T |\dot{\alpha}(r,t)|^2 dt + \int_0^T |\dot{\beta}(r,t)|\,dt \leq M_r, \tag{2.10}$$

where $\dot{\alpha}$ and $\dot{\beta}$ denote $\partial\alpha/\partial t$ and $\partial\beta/\partial t$, respectively.

Now let $\{\varphi^t\}_{t\in[0,T]} \in \Psi(\alpha,\beta)$ be such that $\alpha(r,\cdot) \in W^{1,2}(0,T)$ and $\beta(r,\cdot) \in W^{1,1}(0,T)$ for all $r \in [0,+\infty)$ and introduce the following functional Φ^S defined on $\mathcal{H}_S := L^2(0,S;H)$ for any $S \in (0,T]$:

$$\Phi^S(u) := \begin{cases} \displaystyle\int_0^S \varphi^t(u(t))\,dt & \text{if the function } t \longmapsto \varphi^t(u(t)) \in L^1(0,S), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.11}$$

Then we see that $\Phi^S \in \Psi(\mathcal{H}_S)$. Moreover, [9, Proposition 1.1] implies that for any $u, f \in \mathcal{H}_S$,

$$f \in \partial_{\mathcal{H}_S}\Phi^S(u) \Longleftrightarrow f(t) \in \partial_H\varphi^t(u(t)) \text{ for a.e. } t \in (0,S). \tag{2.12}$$

The following proposition plays an important role in investigating the convergence of strong solutions u_n for $(\mathrm{E}(\varphi_n^t, f_n))$ as $n \to +\infty$. For its proof, we refer to [10, Proposition 2.7.1].

Proposition 2.4 [10]. *For every $n \in \mathbb{N}$, let $\{\varphi_n^t\}_{t\in[0,T]} \in B(\alpha_n,\beta_n,C_0,\{M_r\}_{r\geq 0})$ and $\{\varphi^t\}_{t\in[0,T]} \in \Psi(\alpha,\beta)$ be such that $\alpha_n(r,\cdot),\alpha(r,\cdot) \in W^{1,2}(0,T)$ and $\beta_n(r,\cdot),\beta(r,\cdot) \in W^{1,1}(0,T)$ for every $r \in [0,+\infty)$. Suppose that $\varphi_n^t \to \varphi^t$ on H in the sense of Mosco for every $t \in [0,T]$ as $n \to +\infty$. Then for any $S \in (0,T]$, it follows that*

(1) *for each $u \in D(\Phi^S)$, there exists a sequence (u_n) in $\mathcal{H}_S$ such that $u_n \to u$ strongly in $\mathcal{H}_S$ and $\Phi_n^S(u_n) \to \Phi^S(u)$, where Φ_n^S is defined by (2.11) with φ^t replaced by φ_n^t;*

(2) *let (u_k) be a sequence in $\mathcal{H}_S$ such that (u_k) is bounded in $L^\infty(0,S;H)$ and $u_k(t) \to u(t)$ weakly in H for a.e. $t \in (0,S)$ as $k \to +\infty$ and let (n_k) be a subsequence of (n). Then $\liminf_{k\to+\infty} \Phi_{n_k}^S(u_k) \geq \Phi^S(u)$.*

Throughout the present paper, we denote by C or C_i $(i = 1,2,\dots)$ nonnegative constants which do not depend on the elements of the corresponding space or set.

2.1. Cauchy problem. In this subsection, we consider the following Cauchy problem $(\mathrm{CP}(\varphi^t,f,u_0))$ in a Hilbert space H.

$(\mathrm{CP}(\varphi^t,f,u_0))$

$$\frac{du}{dt}(t) + \partial_H\varphi^t(u(t)) \ni f(t) \quad \text{in } H,\ 0<t<T, \qquad u(0) = u_0, \tag{2.13}$$

where $\varphi^t \in \Psi(H)$ for all $t \in [0,T]$, $f \in L^1(0,T;H)$, and $u_0 \in H$.

We first give a definition of solutions for $(\mathrm{CP}(\varphi^t,f,u_0))$ as follows.

Definition 2.5. A function $u \in C([0,T];H)$ is said to be a strong (resp., weak) solution of $(\mathrm{CP}(\varphi^t,f,u_0))$ if u is a strong (resp., weak) solution of $(\mathrm{E}(\varphi^t,f))$ such that $u(t) \to u_0$ strongly in H as $t \to +0$.

As for the existence of solutions for $(\mathrm{CP}(\varphi^t,f,u_0))$, we here employ the following.

Theorem 2.6 [10]. *Let $\{\varphi^t\}_{t\in[0,T]} \in \Psi(\alpha,\beta)$ be such that $\alpha(r,\cdot) \in W^{1,2}(0,T)$ and $\beta(r,\cdot) \in W^{1,1}(0,T)$ for every $r \in [0,+\infty)$. Then for all $f \in L^1(0,T;H)$ and $u_0 \in \overline{D(\varphi^0)}^H$, $(\mathrm{CP}(\varphi^t,f,u_0))$ has a unique weak solution u such that the function $t \mapsto \varphi^t(u(t))$ is integrable on $(0,T)$. In particular, if $f \in L^2(0,T;H)$, then the weak solution u satisfies*

$$\sqrt{t}\frac{du}{dt} \in L^2(0,T;H), \qquad \sup_{t\in[0,T]} t\varphi^t(u(t)) < +\infty. \tag{2.14}$$

Moreover, if $f \in L^2(0,T;H)$ and $u_0 \in D(\varphi^0)$, then the unique weak solution u becomes a strong solution of $(\mathrm{CP}(\varphi^t,f,u_0))$ such that

$$\frac{du}{dt} \in L^2(0,T;H), \qquad \sup_{t\in[0,T]} \varphi^t(u(t)) < +\infty. \tag{2.15}$$

On account of Proposition 2.4, Kenmochi also proved the following result on the convergence of solutions u_n for $(\mathrm{CP}(\varphi_n^t,f_n,u_{0,n}))$ as $n \to +\infty$. Its proof can be found in [10, Theorem 2.7.1].

Theorem 2.7 [10]. *Under the same assumptions as in Proposition 2.4, let (f_n) and $(u_{0,n})$ be sequences in $L^2(0,T;H)$ and $\overline{D(\varphi_n^0)}^H$, respectively, such that $f_n \to f$ strongly in $L^2(0,T;H)$ and $u_{0,n} \to u_0 \in \overline{D(\varphi^0)}^H$ strongly in H. Then the unique weak solution u_n of $(\mathrm{CP}(\varphi_n^t, f_n, u_{0,n}))$ converges to u in the following sense:*

$$u_n \longrightarrow u \quad \text{strongly in } C([0,T];H) \tag{2.16}$$

and the limit u becomes the unique weak solution of $(\mathrm{CP}(\varphi^t, f, u_0))$. Moreover,

$$\int_0^T \varphi_n^t(u_n(t))\,dt \longrightarrow \int_0^T \varphi^t(u(t))\,dt. \tag{2.17}$$

In particular, if $\varphi_n^0(u_{0,n})$ is bounded for all $n \in \mathbb{N}$, then the limit u becomes a strong solution of $(\mathrm{CP}(\varphi^t, f, u_0))$.

2.2. Periodic problem. In this subsection, we consider the following periodic problem $(\mathrm{PP}(\varphi^t, f))$:

$(\mathrm{PP}(\varphi^t, f))$

$$\begin{gathered}\frac{du}{dt}(t) + \partial_H \varphi^t(u(t)) \ni f(t) \quad \text{in } H,\ 0 < t < T,\\ u(0) = u(T).\end{gathered} \tag{2.18}$$

We are concerned with strong solutions of $(\mathrm{PP}(\varphi^t, f))$ in the following sense.

Definition 2.8. A function $u \in C([0,T];H)$ is said to be a strong solution of $(\mathrm{PP}(\varphi^t, f))$ if u is a strong solution of $(\mathrm{E}(\varphi^t, f))$ such that $u(0) = u(T)$.

To state our results, define

$$\Psi_\pi(\alpha,\beta,C_0) := \left\{ \{\varphi^t\}_{t\in[0,T]} \in \Psi(\alpha,\beta);\ \begin{array}{l} |u|_H^2 \le C_0(\varphi^t(u)+1)\ \forall u \in D(\varphi^t),\ \forall t \in [0,T], \\ D(\varphi^T) \subset D(\varphi^0) \end{array} \right\} \tag{2.19}$$

for any positive constant C_0. Moreover, we write $\{\varphi^t\}_{t\in[0,T]} \in B_\pi(\alpha,\beta,C_0,\{M_r\}_{r\ge 0})$ if the following hold true.

(i) $\{\varphi^t\}_{t\in[0,T]} \in \Psi_\pi(\alpha,\beta,C_0)$.
(ii) There exists a function $h : [0,T] \to H$ such that (2.9) holds and $h(0) = h(T)$.
(iii) For every $r \in [0,+\infty)$, (2.10) holds.
(iv) $\varphi^0(u) \le \varphi^T(u)$ for all $u \in D(\varphi^T)$.

Then it is easily seen that $B_\pi(\alpha,\beta,C_0,\{M_r\}_{r\ge 0}) \subset B(\alpha,\beta,C_0,\{M_r\}_{r\ge 0})$.

The existence of strong solutions for $(\mathrm{PP}(\varphi^t, f))$ is assured by the following.

Theorem 2.9 [10]. *Let $\{\varphi^t\}_{t\in[0,T]} \in \Psi_\pi(\alpha,\beta,C_0)$ be such that $\alpha(r,\cdot) \in W^{1,2}(0,T)$, $\beta(r,\cdot) \in W^{1,1}(0,T)$ for all $r \in [0,+\infty)$. Then for all $f \in L^2(0,T;H)$, $(\mathrm{PP}(\varphi^t,f))$ has at least one strong solution u satisfying*

$$\frac{du}{dt} \in L^2(0,T;H), \qquad \sup_{t\in[0,T]} \varphi^t(u(t)) < +\infty. \tag{2.20}$$

In particular, if φ^t is strictly convex on H for a.e. $t \in (0,T)$, then every strong solution of $(\mathrm{PP}(\varphi^t,f))$ is unique.

We next focus on the convergence of strong solutions u_n for $(\mathrm{PP}(\varphi_n^t,f_n))$ when $\varphi_n^t \to \varphi^t$ on H in the sense of Mosco and $f_n \to f$ weakly in $L^2(0,T;H)$. However, any studies similar to Theorem 2.7 have not been done on the periodic problem $(\mathrm{PP}(\varphi_n^t,f_n))$ yet, which would be caused by a difficulty peculiar to the periodic problem. More precisely, by virtue of Theorems 2.6 and 2.7, for any $f \in L^2(0,T;H)$ and $u_0 \in \overline{D(\varphi^0)}^H$, every unique weak solution of $(\mathrm{CP}(\varphi^t,f,u_0))$ becomes the limit of unique weak solutions u_n for $(\mathrm{CP}(\varphi_n^t,f,u_0))$ as $n \to +\infty$. However, in general, periodic solutions could not be unique. Hence there could exist a strong solution u of $(\mathrm{PP}(\varphi^t,f))$ such that any strong solutions u_n of $(\mathrm{PP}(\varphi_n^t,f))$ never converge to u as $n \to +\infty$. In fact, we can give such a counter example (see Remark 3.10).

Thus because of the essential difference described above, the strong convergence of solutions u_n for $(\mathrm{PP}(\varphi_n^t,f_n))$ in $C([0,T];H)$ cannot be verified by the same manner as in the case of the Cauchy problem (see the proof of [10, Theorem 2.7.1]); so in order to cope with this difficulty, we introduce the following level set compactness assumption on $\{\varphi_n^t\}_{n\in\mathbb{N}}$.

(A1) For every $\lambda > 0$ and $t \in [0,T]$, any sequence (u_n) in H satisfying $\sup_{n\in\mathbb{N}}\{\varphi_n^t(u_n) + |u_n|_H\} \le \lambda$ is precompact in H.

Then our result can be stated as follows.

Theorem 2.10. *For every $n \in \mathbb{N}$, let $\{\varphi_n^t\}_{t\in[0,T]} \in B_\pi(\alpha_n,\beta_n,C_0,\{M_r\}_{r\ge 0})$ and let $\{\varphi^t\}_{t\in[0,T]} \in \Psi(\alpha,\beta)$ be such that $\alpha_n(r,\cdot),\alpha(r,\cdot) \in W^{1,2}(0,T)$ and $\beta_n(r,\cdot),\beta(r,\cdot) \in W^{1,1}(0,T)$ for every $r \in [0,+\infty)$. Suppose that $\varphi_n^t \to \varphi^t$ on H in the sense of Mosco as $n \to +\infty$ and that (A1) holds. Moreover, let (f_n) be a sequence in $L^2(0,T;H)$ such that $f_n \to f$ weakly in $L^2(0,T;H)$ and let (u_n) be a sequence of strong solutions for $(\mathrm{PP}(\varphi_n^t,f_n))$. Then there exists a subsequence (n_k) of (n) such that u_{n_k} converges to u in the following sense:*

$$u_{n_k} \longrightarrow u \quad \textit{strongly in } C([0,T];H),\ \textit{weakly in } W^{1,2}(0,T;H), \tag{2.21}$$

and the limit u becomes a strong solution of $(\mathrm{PP}(\varphi^t,f))$. Moreover,

$$\int_0^T \varphi_n^t(u_n(t))\,dt \longrightarrow \int_0^T \varphi^t(u(t))\,dt. \tag{2.22}$$

Remark 2.11. (1) In Theorem 2.10, the limit u possibly depends on the choice of the subsequence (n_k).

(2) By virtue of the assumptions on $\{\varphi_n^t\}_{t\in[0,T]}$ and $\{\varphi^t\}_{t\in[0,T]}$ in Theorem 2.10, we can verify that $\{\varphi^t\}_{t\in[0,T]} \in \Psi_\pi(\alpha,\beta,C_0)$. Indeed, let $u \in D(\varphi^T)$. Then we can take a sequence (u_n) in H such that $u_n \to u$ strongly in H and $\varphi_n^T(u_n) \to \varphi^T(u)$. Moreover, from the fact that $\varphi_n^0 \le \varphi_n^T$, it follows that

$$\varphi_n^0(u_n) \le \varphi_n^T(u_n) \longrightarrow \varphi^T(u). \tag{2.23}$$

Furthermore, since $\liminf_{n\to+\infty} \varphi_n^0(u_n) \ge \varphi^0(u)$, we have $\varphi^0(u) \le \varphi^T(u)$, which implies $D(\varphi^T) \subset D(\varphi^0)$. Similarly we can also deduce that $|u|_H^2 \le C_0(\varphi^t(u)+1)$ for all $u \in D(\varphi^t)$ and $t \in [0,T]$.

Proof of Theorem 2.10. Since $\{\varphi_n^t\}_{t\in[0,T]} \in B_\pi(\alpha_n,\beta_n,C_0,\{M_r\}_{r\ge 0})$ for all $n \in \mathbb{N}$, we can take a sequence (h_n) such that

$$h_n(0) = h_n(T), \tag{2.24}$$

$$\sup_{t\in[0,T]} \{ |h_n(t)|_H + |\varphi_n^t(h_n(t))| \} + \left(\int_0^T \left| \frac{dh_n}{dt}(t) \right|_H^2 dt \right)^{1/2} \le C_0. \tag{2.25}$$

Moreover, since $\{\varphi_n^t\}_{t\in[0,T]} \in \Psi_\pi(\alpha_n,\beta_n,C_0)$ for all $n \in \mathbb{N}$, we see that

$$|u|_H^2 \le C_0(\varphi_n^t(u)+1) \quad \forall u \in D(\varphi_n^t),\ \forall t \in [0,T],\ \forall n \in \mathbb{N}. \tag{2.26}$$

Now let u_n be a strong solution of (PP(φ_n^t, f_n)) for each $n \in \mathbb{N}$. Then multiplying the inclusion in (PP(φ_n^t, f_n)) by $u_n(t) - h_n(t)$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} |u_n(t) - h_n(t)|_H^2 + \varphi_n^t(u_n(t)) \\
&\quad \le \varphi_n^t(h_n(t)) + \left(f_n(t) - \frac{dh_n}{dt}(t), u_n(t) - h_n(t) \right)_H \\
&\quad \le C_0 + \left(|f_n(t)|_H + \left| \frac{dh_n}{dt}(t) \right|_H \right) |u_n(t) - h_n(t)|_H.
\end{aligned} \tag{2.27}$$

Now by (2.26) it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} |u_n(t) - h_n(t)|_H^2 + \alpha |u_n(t) - h_n(t)|_H^2 \\
&\quad \le C(|h_n(t)|_H^2 + 1) + \left(|f_n(t)|_H + \left| \frac{dh_n}{dt}(t) \right|_H \right) |u_n(t) - h_n(t)|_H
\end{aligned} \tag{2.28}$$

for some $\alpha > 0$. Hence by [11, Lemma 4.2], we get, by (2.25),

$$\sup_{t\in[0,T]} |u_n(t) - h_n(t)|_H \le C, \tag{2.29}$$

which implies

$$\sup_{t\in[0,T]} |u_n(t)|_H \le C_1. \tag{2.30}$$

Furthermore, integrating (2.27) over $(0,T)$, we have

$$\int_0^T \varphi_n^t(u_n(t))\,dt \le C_2. \tag{2.31}$$

Hence by (2.31) there exists $t_n \in (0,T)$ such that $\varphi_n^{t_n}(u_n(t_n)) \le C_2/T$.

Next multiplying the inclusion in $(\mathrm{PP}(\varphi_n^t, f_n))$ by $du_n(t)/dt$, we have

$$\begin{aligned} &\left|\frac{du_n}{dt}(t)\right|_H^2 + \left(g_n(t), \frac{du_n}{dt}(t)\right)_H \\ &\quad = \left(f_n(t), \frac{du_n}{dt}(t)\right)_H \le |f_n(t)|_H^2 + \frac{1}{4}\left|\frac{du_n}{dt}(t)\right|_H^2, \end{aligned} \tag{2.32}$$

where $g_n(t) := f_n(t) - du_n(t)/dt \in \partial_H \varphi_n^t(u_n(t))$. Hence put $r_0 = C_1$. Then by [12, Lemma 2.4], it follows from (2.30) that

$$\begin{aligned} &\left|\left(g_n(t), \frac{du_n}{dt}(t)\right)_H - \frac{d}{dt}\varphi_n^t(u_n(t))\right| \\ &\quad \le |\dot{\alpha}_n(r_0,t)|\,|g_n(t)|_H\{|\varphi_n^t(u_n(t))|+1\}^{1/2} + |\dot{\beta}_n(r_0,t)|\{|\varphi_n^t(u_n(t))|+1\} \end{aligned} \tag{2.33}$$

for a.e. $t \in (0,T)$. Thus

$$\begin{aligned} &\frac{3}{4}\left|\frac{du_n}{dt}(t)\right|_H^2 + \frac{d}{dt}\varphi_n^t(u_n(t)) \\ &\quad \le |f_n(t)|_H^2 + |\dot{\alpha}_n(r_0,t)|\left|f_n(t) - \frac{du_n}{dt}(t)\right|_H\{|\varphi_n^t(u_n(t))|+1\}^{1/2} \\ &\qquad + |\dot{\beta}_n(r_0,t)|\{|\varphi_n^t(u_n(t))|+1\} \\ &\quad \le \frac{5}{4}|f_n(t)|_H^2 + \frac{1}{4}\left|\frac{du_n}{dt}(t)\right|_H^2 \\ &\qquad + \left\{2|\dot{\alpha}_n(r_0,t)|^2 + |\dot{\beta}_n(r_0,t)|\right\}\{|\varphi_n^t(u_n(t))|+1\}. \end{aligned} \tag{2.34}$$

Integrating (2.34) over (t_n,t) and noting that (2.26) implies $|\varphi_n^t(u)| \le \varphi_n^t(u)+2$ for all $u \in D(\varphi_n^t)$, we observe

$$\begin{aligned} &\frac{1}{2}\int_{t_n}^t \left|\frac{du_n}{d\tau}(\tau)\right|_H^2 d\tau + \varphi_n^t(u_n(t)) \\ &\quad \le \varphi_n^{t_n}(u_n(t_n)) + \frac{5}{4}\int_0^T |f_n(\tau)|_H^2 d\tau \\ &\qquad + \int_{t_n}^t \left\{2|\dot{\alpha}_n(r_0,\tau)|^2 + |\dot{\beta}_n(r_0,\tau)|\right\}\{\varphi_n^\tau(u_n(\tau))+3\}\,d\tau \end{aligned} \tag{2.35}$$

for all $t \in [t_n, T]$. Thus from the fact that

$$\begin{gathered}\int_0^T |\dot{\alpha}_n(r_0,\tau)|^2 d\tau + \int_0^T |\dot{\beta}_n(r_0,\tau)| d\tau \le M_{r_0}, \\ \int_0^T |f_n(\tau)|_H^2 d\tau \le C, \qquad \varphi_n^{t_n}(u_n(t_n)) \le \frac{C_2}{T},\end{gathered} \tag{2.36}$$

by Gronwall's inequality, it follows that

$$\sup_{t\in[t_n,T]} \varphi_n^t(u_n(t)) \le C_3. \tag{2.37}$$

Hence since $u_n(0) = u_n(T)$ and $\varphi_n^0(u) \le \varphi_n^T(u)$ for all $u \in D(\varphi_n^T)$ and $n \in \mathbb{N}$, we find that $\varphi_n^0(u_n(0)) \le \varphi_n^T(u_n(T)) \le C_3$. Moreover, integrating (2.34) over $(0,t)$, we also get

$$\begin{aligned}\frac{1}{2}\int_0^t \left|\frac{du_n}{d\tau}(\tau)\right|_H^2 d\tau + \varphi_n^t(u_n(t)) \\ \le \varphi_n^0(u_n(0)) + \frac{5}{4}\int_0^T |f_n(\tau)|_H^2 d\tau \\ + \int_0^t \left\{2|\dot{\alpha}_n(r_0,\tau)|^2 + |\dot{\beta}_n(r_0,\tau)|\right\}\{\varphi_n^\tau(u_n(\tau)) + 3\} d\tau\end{aligned} \tag{2.38}$$

for all $t \in [0,T]$. Thus Gronwall's inequality implies

$$\sup_{t\in[0,T]} \varphi_n^t(u_n(t)) \le C. \tag{2.39}$$

Moreover, it follows from (2.26), (2.38), and (2.39) that

$$\int_0^T \left|\frac{du_n}{dt}(t)\right|_H^2 dt \le C. \tag{2.40}$$

Hence by (PP(φ_n^t, f_n)), (2.40) implies

$$\int_0^T |g_n(t)|_H^2 dt \le C. \tag{2.41}$$

By virtue of the above a priori estimates, we can take a subsequence (n_k) of (n) such that the following convergences hold true:

$$u_{n_k} \longrightarrow u \quad \text{weakly in } W^{1,2}(0,T;H), \tag{2.42}$$

$$g_{n_k} \longrightarrow g \quad \text{weakly in } L^2(0,T;H). \tag{2.43}$$

Moreover, by (A1), it follows from (2.30) and (2.39) that

$$(u_n(t)) \text{ is precompact in } H \quad \forall t \in [0,T]. \tag{2.44}$$

Furthermore, by (2.40), we can deduce that u_n is equicontinuous in $C([0,T];H)$ for all $n \in \mathbb{N}$. Thus Ascoli's theorem implies

$$u_{n_k} \longrightarrow u \quad \text{strongly in } C([0,T];H) \tag{2.45}$$

for a suitable subsequence (n_k) of (n). Hence since $u_n(0) = u_n(T)$ for all $n \in \mathbb{N}$, we have $u(0) = u(T)$.

In the rest of this proof, we write n simply for n_k. Now define Φ^T and Φ_n^T as in (2.11) with obvious replacements and let $v \in D(\Phi^T)$ be fixed. Then by Proposition 2.4 we can take a sequence (v_n) in $\mathcal{H}_T := L^2(0,T;H)$ such that

$$v_n \longrightarrow v \quad \text{strongly in } \mathcal{H}_T, \quad \Phi_n^T(v_n) \longrightarrow \Phi^T(v). \tag{2.46}$$

Now since $g_n \in \partial_{\mathcal{H}_T} \Phi_n^T(u_n)$, we have

$$\begin{aligned} \int_0^T \left(f_n(t) - \frac{du_n}{dt}(t), u_n(t) - v_n(t) \right)_H dt \\ = \int_0^T (g_n(t), u_n(t) - v_n(t))_H dt \\ \geq \Phi_n^T(u_n) - \Phi_n^T(v_n). \end{aligned} \tag{2.47}$$

Moreover, by Proposition 2.4, it follows from (2.45) that

$$\liminf_{n \to +\infty} \Phi_n^T(u_n) \geq \Phi^T(u). \tag{2.48}$$

Thus passing to the limit $n \to +\infty$ in (2.47), by (2.42) and (2.45), we find

$$\int_0^T \left(f(t) - \frac{du}{dt}(t), u(t) - v(t) \right)_H dt \geq \Phi^T(u) - \Phi^T(v), \tag{2.49}$$

which together with the arbitrariness of $v \in D(\Phi^T)$ implies $u \in D(\partial_{\mathcal{H}_T} \Phi^T)$ and $g = f - du/dt \in \partial_{\mathcal{H}_T} \Phi^T(u)$. Hence by [9, Proposition 1.1], we deduce that $g(t) \in \partial_H \varphi^t(u(t))$ for a.e. $t \in (0,T)$. Therefore u is a strong solution of (PP(φ^t, f)).

Finally we prove (2.22). Since $u \in D(\Phi^T)$, by Proposition 2.4, we can take a sequence (w_n) in $\mathcal{H}_T$ such that $w_n \to u$ strongly in $\mathcal{H}_T$ and $\Phi_n^T(w_n) \to \Phi^T(u)$. Hence we get, by (2.40) and (2.45),

$$\begin{aligned} &\Phi_n^T(u_n) - \Phi_n^T(w_n) \\ &\quad \leq \int_0^T \left(f_n(t) - \frac{du_n}{dt}(t), u_n(t) - w_n(t) \right)_H dt \\ &\quad \leq C \left\{ \int_0^T |u_n(t) - u(t)|_H^2 dt + \int_0^T |w_n(t) - u(t)|_H^2 dt \right\}^{1/2} \longrightarrow 0, \end{aligned} \tag{2.50}$$

which implies

$$\limsup_{n\to+\infty} \Phi_n^T(u_n) \le \Phi^T(u). \tag{2.51}$$

Therefore combining (2.48) and (2.51), we can derive (2.22). □

3. Applications to parabolic equations

The first application is concerned with quasilinear parabolic equations associated with the p-Laplacian such as

$$\frac{\partial u_p}{\partial t}(x,t) - \Delta_p u_p(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{3.1}$$

where Ω denotes a domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The asymptotic behavior of solutions u_p to the initial-boundary value problem for (3.1) as $p \to +\infty$ has already been studied by several authors (see, e.g., [2, 4]). We here generalize (3.1) as follows:

$$\frac{\partial u_p}{\partial t}(x,t) - \Delta_p^\gamma u_p(x,t) = f_p(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{(P)$_p$}$$

where $f_p \to f$ strongly in $L^2(0,T;L^2(\Omega))$ as $p \to +\infty$ and Δ_p^γ is defined by

$$\Delta_p^\gamma u(x) := \nabla \cdot \left\{ \left(\frac{1}{\gamma(x,t)} \right)^p |\nabla u(x)|^{p-2} \nabla u(x) \right\} \tag{3.2}$$

for some function $\gamma : \Omega \times (0,T) \to \mathbb{R}$. This generalization is motivated by some macroscopic model for type-II superconductors (see [1]).

In Section 3.1, we discuss the existence and uniqueness of solutions u_p for ((P)$_p$) and the asymptotic behavior of u_p as $p \to +\infty$ for the periodic problem as well as the initial-boundary value problem. Solutions of ((P)$_p$) are defined as follows.

Definition 3.1. A function $u \in C([0,T];L^2(\Omega))$ is said to be a strong solution of ((P)$_p$) if the following are both satisfied:

(i) $u(\cdot,t)$ is an $L^2(\Omega)$-valued absolutely continuous function on $[0,T]$;

(ii) $u(\cdot,t) \in W_0^{1,p}(\Omega)$ for a.e. $t \in (0,T)$ and

$$\begin{aligned}\int_\Omega \frac{\partial u}{\partial t}(x,t)\phi(x)dx + \int_\Omega \left(\frac{1}{\gamma(x,t)}\right)^p |\nabla u|^{p-2} \nabla u(x,t) \cdot \nabla \phi(x) dx \\ = \int_\Omega f_p(x,t)\phi(x)dx\end{aligned} \tag{3.3}$$

for all $\phi \in W_0^{1,p}(\Omega)$ and a.e. $t \in (0,T)$.

Moreover, a function $u \in C([0,T];L^2(\Omega))$ is said to be a weak solution of ((P)$_p$) if there exist sequences $(f_{p,n}) \subset L^1(0,T;L^2(\Omega))$ and $(u_n) \subset C([0,T];L^2(\Omega))$ such that u_n is a strong solution of ((P)$_p$), $f_{p,n} \to f_p$ strongly in $L^1(0,T;L^2(\Omega))$, and $u_n \to u$ strongly in $C([0,T];L^2(\Omega))$ as $n \to +\infty$.

The second application is for the porous medium equation

$$\frac{\partial u_m}{\partial t}(x,t) - \Delta |u_m|^{m-2} u_m(x,t) = 0, \quad (x,t) \in \Omega \times (0,T). \tag{3.4}$$

Bénilan and Crandall [6] studied the asymptotic behavior of solutions u_n to the initial-boundary value problem for

$$\frac{\partial u_n}{\partial t}(x,t) - \Delta \varphi_n(u_n(x,t)) = 0, \quad (x,t) \in \mathbb{R}^N \times (0,T), \tag{3.5}$$

where φ_n is a maximal monotone function from $\mathbb{R}$ into itself, when $\varphi_n \to \varphi$ in a proper sense as $n \to +\infty$. Moreover, their results cover (3.4) for the case where $\Omega = \mathbb{R}^N$.

In Section 3.2, we deal with the following.

$((\mathrm{PM})_m)$

$$\frac{\partial u_m}{\partial t}(x,t) - \Delta \beta_m(x,t,u_m(x,t)) = f_m(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{3.6}$$

where

$$\beta_m(x,t,r) := \frac{|r|^{m-2} r}{\gamma(x,t)^m} \quad \forall (x,t) \in \Omega \times (0,T),\ \forall r \in \mathbb{R}. \tag{3.7}$$

We then define solutions for $((\mathrm{PM})_m)$ in the following sense.

Definition 3.2. A function $u \in C([0,T];H^{-1}(\Omega))$ is said to be a strong solution of $((\mathrm{PM})_m)$ if the following are both satisfied:

(i) $u(\cdot,t)$ is an $H^{-1}(\Omega)$-valued absolutely continuous function on $[0,T]$;
(ii) $u(\cdot,t) \in L^m(\Omega)$, $\beta_m(\cdot,t,u(\cdot,t)) \in H_0^1(\Omega)$, and

$$\left\langle \frac{\partial u}{\partial t}(\cdot,t), \phi \right\rangle_{H_0^1(\Omega)} + \int_\Omega \nabla \beta_m(x,t,u(x,t)) \cdot \nabla \phi(x)dx = \langle f_m(\cdot,t), \phi \rangle_{H_0^1(\Omega)} \tag{3.8}$$

for all $\phi \in H_0^1(\Omega)$ and a.e. $t \in (0,T)$.

Moreover, a function $u \in C([0,T];H^{-1}(\Omega))$ is said to be a weak solution of $((\mathrm{PM})_m)$ if there exist sequences $(f_{m,n}) \subset L^1(0,T;H^{-1}(\Omega))$ and $(u_n) \subset C([0,T];H^{-1}(\Omega))$ such that u_n is a strong solution of $((\mathrm{PM})_m)$, $f_{m,n} \to f_m$ strongly in $L^1(0,T;H^{-1}(\Omega))$, and $u_n \to u$ strongly in $C([0,T];H^{-1}(\Omega))$ as $n \to +\infty$.

We also investigate the existence and uniqueness of solutions u_m for $((\mathrm{PM})_m)$ and the asymptotic behavior of u_m as $m \to +\infty$ for the periodic problem as well as the initial-boundary value problem.

3.1. Asymptotic behavior of solutions for parabolic equations associated with p-Laplacian as $p \to +\infty$

Problem 3.3. Find a unique solution u_p of the initial-boundary value problem for $((\mathrm{P})_p)$ with the boundary condition $u_p(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, and the initial condition $u_p(x,0) = u_{0,p}(x)$, $x \in \Omega$, which is denoted by $(\mathrm{IBVP1})_p$, and investigate the asymptotic behavior of u_p as $p \to +\infty$.

To this end, we introduce the following hypotheses:

(H1) Ω is a bounded domain in $\mathbb{R}^N$,

$$\begin{aligned} &\gamma(x,t) = \pi(x)\phi(t), \quad \pi \in L^\infty(\Omega),\ \phi \in W^{1,2}(0,T), \\ &\gamma(x,t) \geq \delta_0 > 0 \quad \text{for a.e. } x \in \Omega \text{ and all } t \in [0,T]. \end{aligned} \tag{3.9}$$

Moreover, set $H := L^2(\Omega)$ and define $\varphi_p^t : H \to [0,+\infty]$ as follows:

$$\varphi_p^t(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega \left(\frac{|\nabla u(x)|}{\gamma(x,t)}\right)^p dx & \text{if } u \in W_0^{1,p}(\Omega),\ \dfrac{\nabla u}{\gamma(\cdot,t)} \in \left(L^p(\Omega)\right)^N, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.10}$$

Then (H1) implies

$$\varphi_p^t \in \Psi(H), \quad D(\varphi_p^t) = W_0^{1,p}(\Omega) \quad \forall t \in [0,T]. \tag{3.11}$$

Moreover, $\partial_H \varphi_p^t(u)$ coincides with $-\Delta_p^\gamma u$ with the homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ in the distribution sense. Hence by Definition 3.1, $(\mathrm{IBVP1})_p$ is equivalent to $(\mathrm{CP}(\varphi_p^t, f_p, u_{0,p}))$.

As for the existence of a unique solution to $(\mathrm{CP}(\varphi_p^t, f, u_0))$, our result is stated as follows.

Theorem 3.4. *Suppose that (H1) is satisfied and let $p \in (1,+\infty)$. Then for all $f \in L^1(0,T; L^2(\Omega))$ and $u_0 \in L^2(\Omega)$, $(\mathrm{CP}(\varphi_p^t, f, u_0))$ has a unique weak solution u_p. In particular, if $f \in L^2(0,T;L^2(\Omega))$ and $u_0 \in W_0^{1,p}(\Omega)$, the weak solution u_p becomes a strong solution of $(\mathrm{CP}(\varphi_p^t, f, u_0))$.*

Proof of Theorem 3.4. We first claim that $\{\varphi_p^t\}_{t\in[0,T]} \in \Psi(\alpha_1, 0)$ for some function $\alpha_1 : [0,+\infty) \times [0,T] \to \mathbb{R}$. Indeed, let $t_0 \in [0,T]$ and $u_0 \in D(\varphi_p^{t_0})$ be fixed and define the function $u : [0,T] \to H$ as follows:

$$u(t) := \frac{\phi(t)}{\phi(t_0)} u_0 \in D(\varphi_p^t) \quad \forall t \in [0,T]. \tag{3.12}$$

Since (H1) says $\gamma(x,t) = \pi(x)\phi(t)$, it follows that

$$\nabla u(t) = \frac{\phi(t)}{\phi(t_0)} \nabla u_0 = \frac{\gamma(x,t)}{\gamma(x,t_0)} \nabla u_0 \quad \forall t \in [0,T], \tag{3.13}$$

which implies

$$\varphi_p^t(u(t)) = \frac{1}{p}\int_\Omega \left(\frac{|\nabla u(x,t)|}{\gamma(x,t)}\right)^p dx = \frac{1}{p}\int_\Omega \left(\frac{|\nabla u_0(x)|}{\gamma(x,t_0)}\right)^p dx = \varphi_p^{t_0}(u_0). \tag{3.14}$$

Moreover, we see

$$\begin{aligned}|u(t)-u_0|_H &= \left|\frac{\phi(t)}{\phi(t_0)} - 1\right| |u_0|_H \\ &\le \frac{1}{\delta_0}|\pi|_{L^\infty(\Omega)}|\phi(t)-\phi(t_0)|\,|u_0|_H \\ &\le |\alpha_1(r,t)-\alpha_1(r,t_0)| \quad \forall r \ge |u_0|_H,\end{aligned} \tag{3.15}$$

where α_1 is given by

$$\alpha_1(r,t) = \frac{r}{\delta_0}|\pi|_{L^\infty(\Omega)}\phi(t) \in W^{1,2}(0,T). \tag{3.16}$$

Therefore we conclude that $\{\varphi_p^t\}_{t\in[0,T]} \in \Psi(\alpha_1, 0)$ for every $p \in (1,+\infty)$. Then applying Theorem 2.6 to $(\mathrm{CP}(\varphi_p^t, f, u_0))$, we can derive the desired result. □

Now we are going to describe our result on the asymptotic behavior of u_p as $p \to +\infty$.

Theorem 3.5. *Suppose that (H1) is satisfied and define*

$$K^t := \{u \in H_0^1(\Omega);\ |\nabla u(x)| \le \gamma(x,t) \text{ for a.e. } x \in \Omega\}. \tag{3.17}$$

Let (p_n) be a sequence in $(1,+\infty)$ such that $p_n \to +\infty$ as $n \to +\infty$. Moreover, let $f_n, f \in L^2(0,T;L^2(\Omega))$, $u_{0,n} \in L^2(\Omega)$, and $u_0 \in K^0$ be such that

$$f_n \longrightarrow f \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \tag{3.18}$$

$$u_{0,n} \longrightarrow u_0 \quad \text{strongly in } L^2(\Omega). \tag{3.19}$$

Then the unique weak solution u_n of $(\mathrm{CP}(\varphi_{p_n}^t, f_n, u_{0,n}))$ converges to u as $n \to +\infty$ in the following sense:

$$u_n \longrightarrow u \quad \text{strongly in } C([0,T];L^2(\Omega)). \tag{3.20}$$

Moreover, the limit u is a unique weak solution of $(\mathrm{CP}(\varphi_\infty^t, f, u_0))$, where φ_∞^t is defined by

$$\varphi_\infty^t(u) := \begin{cases} 0 & \text{if } u \in K^t, \\ +\infty & \text{if } u \in L^2(\Omega)\setminus K^t. \end{cases} \tag{3.21}$$

In particular, if $(1/p_n)\int_\Omega |\nabla u_{0,n}(x)|^{p_n} dx$ is bounded as $n \to +\infty$, then the limit u becomes a strong solution of $(\mathrm{CP}(\varphi_\infty^t, f, u_0))$.

Proof of Theorem 3.5. On account of Theorem 2.7, it suffices to show that

$$\{\varphi_{p_n}^t\}_{t\in[0,T]} \in B(\alpha_1,0,C_0,\{M_r\}_{r\geq 0}) \quad \text{for some constants } C_0,\ \{M_r\}_{r\geq 0}, \tag{3.22}$$

$$\{\varphi_\infty^t\}_{t\in[0,T]} \in \Psi(\alpha_1,0), \tag{3.23}$$

$$\varphi_{p_n}^t \longrightarrow \varphi_\infty^t \quad \text{on } H \text{ in the sense of Mosco as } p_n \longrightarrow +\infty. \tag{3.24}$$

We first prove (3.22). We have already seen that $\{\varphi_p^t\}_{t\in[0,T]} \in \Psi(\alpha_1,0)$ for all $p \in (1,+\infty)$. Moreover, it is obvious that $\varphi_p^t \geq 0$ and that $h \equiv 0$ satisfies

$$\frac{dh}{dt}(t) = 0, \quad \varphi_p^t(h(t)) = 0 \quad \forall t \in [0,T],\ \forall p \in (1,+\infty). \tag{3.25}$$

Hence we can take $C_0 = 0$. Furthermore, we see

$$\int_0^T |\dot{\alpha}_1(r,t)|^2 dt = \left(\frac{r}{\delta_0}|\pi|_{L^\infty(\Omega)}\right)^2 \int_0^T |\dot{\phi}(t)|^2 dt =: M_r, \tag{3.26}$$

where we note that M_r is independent of p. Therefore (3.22) holds.

In much the same way as in the proof of Theorem 3.4, we can derive (3.23). Indeed, $u(t)$ appearing in (3.12) satisfies

$$|\nabla u(x,t)| = \left|\frac{\gamma(x,t)}{\gamma(x,t_0)}\nabla u_0(x)\right| \leq \gamma(x,t) \quad \text{for a.e. } x \in \Omega \text{ and all } t \in [0,T] \tag{3.27}$$

for each $u_0 \in K^{t_0}$ and $t_0 \in [0,T]$. Hence we deduce that $\varphi_\infty^t(u(t)) = \varphi_\infty^{t_0}(u_0) = 0$ for all $t \in [0,T]$, which together with (3.15) implies $\{\varphi_\infty^t\}_{t\in[0,T]} \in \Psi(\alpha_1,0)$.

Finally (3.24) is derived from the following lemma.

Lemma 3.6. *For each $t \in [0,T]$, it follows that*

$$\varphi_{p_n}^t \longrightarrow \varphi_\infty^t \quad \text{on } H \text{ in the sense of Mosco as } p_n \longrightarrow +\infty. \tag{3.28}$$

Proof of Lemma 3.6. Let $t \in [0,T]$ be fixed. We first claim that

$$\begin{gathered} \forall u \in D(\varphi_\infty^t), \quad \exists (u_n) \subset H; \\ u_n \longrightarrow u \quad \text{strongly in } H, \quad \varphi_{p_n}^t(u_n) \longrightarrow \varphi_\infty^t(u). \end{gathered} \tag{3.29}$$

Indeed, let $u \in D(\varphi_\infty^t) = K^t$ and put $u_n := u$ for all $n \in \mathbb{N}$. Then since $K^t \subset W_0^{1,p_n}(\Omega)$ for all $n \in \mathbb{N}$, it follows immediately that

$$\begin{aligned} 0 \leq \varphi_{p_n}^t(u_n) &= \frac{1}{p_n}\int_\Omega \left(\frac{|\nabla u(x)|}{\gamma(x,t)}\right)^{p_n} dx \\ &\leq \frac{1}{p_n}|\Omega| \longrightarrow 0 = \varphi_\infty^t(u) \quad \text{as } p_n \longrightarrow +\infty. \end{aligned} \tag{3.30}$$

Hence we deduce that (3.29) holds true.

We next show that

$$\begin{gathered}\forall (u_n) \subset H \quad \text{satisfying } u_n \longrightarrow u \text{ weakly in } H,\\ \liminf_{n\to+\infty} \varphi^t_{p_n}(u_n) \geq \varphi^t_\infty(u).\end{gathered} \tag{3.31}$$

For the case where $u \in D(\varphi^t_\infty) = K^t$, it is obvious that $\liminf_{n\to+\infty} \varphi^t_{p_n}(u_n) \geq 0 = \varphi^t_\infty(u)$. For the case where $u \notin K^t$, we give a proof by contradiction. Suppose that

$$\exists (u_n) \subset H; \quad u_n \longrightarrow u \quad \text{weakly in } H, \quad \liminf_{n\to+\infty} \varphi^t_{p_n}(u_n) < \varphi^t_\infty(u) = +\infty. \tag{3.32}$$

Then we can take a subsequence (n') of (n) such that

$$\varphi^t_{p_{n'}}(u_{n'}) \leq C \quad \forall n', \tag{3.33}$$

which implies

$$\begin{aligned}&\left\{\int_\Omega \left(\frac{|\nabla u_{n'}(x)|}{\gamma(x,t)}\right)^{p_{n'}} dx\right\}^{1/p_{n'}}\\ &\quad \leq \left\{p_{n'}\varphi^t_{p_{n'}}(u_{n'})\right\}^{1/p_{n'}} \leq \left(p_{n'}C\right)^{1/p_{n'}} \longrightarrow 1 \quad \text{as } n' \longrightarrow +\infty.\end{aligned} \tag{3.34}$$

For simplicity of notation, we write p and u_p for $p_{n'}$ and $u_{n'}$, respectively. Hence by (H1) we have

$$\left(\int_\Omega |\nabla u_p(x)|^p dx\right)^{1/p} \leq C, \tag{3.35}$$

which yields

$$\begin{aligned}\left(\int_\Omega |\nabla u_p(x)|^q dx\right)^{1/q} &\leq \left(\int_\Omega |\nabla u_p(x)|^p dx\right)^{1/p} |\Omega|^{(p-q)/(pq)}\\ &\leq C(|\Omega|+1)^{1/q} \quad \forall q \in [1,p].\end{aligned} \tag{3.36}$$

Thus for each $q \in (1,+\infty)$, we can take a subsequence (p_q) of (p) such that

$$\nabla u_{p_q} \longrightarrow \nabla u \quad \text{weakly in } \left(L^q(\Omega)\right)^N. \tag{3.37}$$

Here we also observe that $u \in H^1_0(\Omega)$. In the rest of this proof, we drop q in p_q. Moreover, by (H1), we can also derive

$$\frac{\nabla u_p}{\gamma(\cdot,t)} \longrightarrow \frac{\nabla u}{\gamma(\cdot,t)} \quad \text{weakly in } \left(L^q(\Omega)\right)^N. \tag{3.38}$$

Hence it follows from (3.34) and (3.38) that

$$\begin{aligned}\left\{\int_\Omega\left(\frac{|\nabla u(x)|}{\gamma(x,t)}\right)^q dx\right\}^{1/q} &\le \liminf_{p\to+\infty}\left\{\int_\Omega\left(\frac{|\nabla u_p(x)|}{\gamma(x,t)}\right)^q dx\right\}^{1/q}\\ &\le \liminf_{p\to+\infty}\left\{\int_\Omega\left(\frac{|\nabla u_p(x)|}{\gamma(x,t)}\right)^p dx\right\}^{1/p}|\Omega|^{(p-q)/(pq)}\\ &\le \lim_{p\to+\infty}(pC)^{1/p}|\Omega|^{(p-q)/(pq)}\\ &= |\Omega|^{1/q}.\end{aligned}\tag{3.39}$$

Therefore passing to the limit $q\to+\infty$, we deduce that

$$\left|\frac{\nabla u}{\gamma(\cdot,t)}\right|_{L^\infty(\Omega)}\le 1,\tag{3.40}$$

which contradicts the fact that $u\notin K^t$. Hence (3.31) holds true. □

Then by (3.22), (3.23), and (3.24), Theorem 2.7 assures the desired conclusion. □

Problem 3.7. Find a solution of the periodic problem for ($(\mathrm{P})_p$) with the boundary condition $u(x,t)=0$, $(x,t)\in\partial\Omega\times(0,T)$, and the periodic condition $u(x,0)=u(x,T)$, $x\in\Omega$, which is denoted by $(\mathrm{PP1})_p$. Moreover, investigate the asymptotic behavior of u_p as $p\to+\infty$.

Just as in Problem 3.3, $(\mathrm{PP1})_p$ can be also reduced to $(\mathrm{PP}(\varphi_p^t,f_p))$. Then as for the existence of a solution to $(\mathrm{PP}(\varphi_p^t,f))$, we have the following.

THEOREM 3.8. *Suppose that (H1) is satisfied and let $p\in[2,+\infty)$. Then for all $f\in L^2(0,T;L^2(\Omega))$, $(\mathrm{PP}(\varphi_p^t,f))$ has a unique strong solution u_p.*

Proof of Theorem 3.8. We claim that $\{\varphi_p^t\}_{t\in[0,T]}\in\Psi_\pi(\alpha_1,0,C_0)$ for some positive constant C_0. Since $H_0^1(\Omega)$ is continuously embedded in H, we see that

$$\begin{aligned}|u|_H^2 &\le C|\nabla u|_H^2\\ &\le C|\gamma|^2_{L^\infty(Q)}\int_\Omega\left(\frac{|\nabla u(x)|}{\gamma(x,t)}\right)^2dx\\ &\le C|\gamma|^2_{L^\infty(Q)}\left\{\frac{2}{p}\int_\Omega\left(\frac{|\nabla u(x)|}{\gamma(x,t)}\right)^p dx+\frac{p-2}{p}|\Omega|\right\}\\ &\le 2C|\gamma|^2_{L^\infty(Q)}\{\varphi_p^t(u)+|\Omega|\}\quad \forall u\in D(\varphi_p^t),\ \forall p\ge 2,\end{aligned}\tag{3.41}$$

where $Q:=\Omega\times[0,T]$. Hence since we have already known that $D(\varphi_p^t)=W_0^{1,p}(\Omega)$ for all $t\in[0,T]$ and $\{\varphi_p^t\}_{t\in[0,T]}\in\Psi(\alpha_1,0)$ for every $p\in(1,+\infty)$, we deduce that $\{\varphi_p^t\}_{t\in[0,T]}\in\Psi_\pi(\alpha_1,0,C_0)$ for some positive constant C_0 independent of p. Therefore Theorem 2.9 assures the existence of a strong solution u_p for $(\mathrm{PP}(\varphi_p^t,f))$. Moreover, since φ_p^t is strictly convex on H, the periodic solution u_p is unique. □

As for the asymptotic behavior of u_p as $p \to +\infty$, we have the following.

Theorem 3.9. *Suppose that (H1) is satisfied and that $\gamma(x,0) \geq \gamma(x,T)$ for a.e. $x \in \Omega$. Let (p_n) be a sequence in $[2,+\infty)$ such that $p_n \to +\infty$ as $n \to +\infty$ and let $f_n, f \in L^2(0,T;L^2(\Omega))$ be such that*

$$f_n \longrightarrow f \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{3.42}$$

Then a subsequence (n_k) of (n) can be taken such that the unique strong solution u_{n_k} of $(\mathrm{PP}(\varphi^t_{p_{n_k}}, f_{n_k}))$ converges to u as $k \to +\infty$ in the following sense:

$$u_{n_k} \longrightarrow u \quad \text{strongly in } C([0,T];L^2(\Omega)),\ \text{weakly in } W^{1,2}(0,T;L^2(\Omega)). \tag{3.43}$$

Moreover, the limit u is a strong solution of $(\mathrm{PP}(\varphi^t_\infty, f))$.

Proof of Theorem 3.9. We first claim that $\{\varphi^t_{p_n}\}_{t\in[0,T]} \in B_\pi(\alpha_1, 0, C_0, \{M_r\}_{r\geq 0})$ for some constants $\{M_r\}_{r\geq 0}$ independent of n. Indeed, we have already seen that $\{\varphi^t_p\}_{t\in[0,T]} \in \Psi_\pi(\alpha_1,0,C_0)$, where C_0 is a positive constant independent of p. Moreover, since $\gamma(x,0) \geq \gamma(x,T)$ for a.e. $x \in \Omega$, it is obvious that

$$\varphi^0_p(u) \leq \varphi^T_p(u) \quad \forall u \in D(\varphi^T_p),\ \forall p \in (1,+\infty). \tag{3.44}$$

The rest of the proof for this claim can be derived as in the proof of Theorem 3.5.

We next prove that $\{\varphi^t_{p_n}\}_{n\in\mathbb{N}}$ satisfies (A1). Let $\lambda > 0$ and $t \in [0,T]$ be fixed and let (u_n) be a sequence in H such that

$$\varphi^t_{p_n}(u_n) + |u_n|_H \leq \lambda \quad \forall n \in \mathbb{N}. \tag{3.45}$$

For every $p_n \geq 2$, we get

$$\begin{aligned}\left(\int_\Omega |\nabla u_n(x)|^2 dx\right)^{1/2} &\leq |\gamma(\cdot,t)|_{L^\infty(\Omega)} \left\{\int_\Omega \left(\frac{|\nabla u_n(x)|}{\gamma(x,t)}\right)^{p_n} dx\right\}^{1/p_n} |\Omega|^{(p_n-2)/(2p_n)} \\ &\leq |\gamma(\cdot,t)|_{L^\infty(\Omega)} (p_n\lambda)^{1/p_n} |\Omega|^{(p_n-2)/(2p_n)} \leq C,\end{aligned} \tag{3.46}$$

where C is a constant independent of n. Hence since $H^1_0(\Omega)$ is compactly embedded in H, we deduce that (u_n) becomes precompact in H, which implies (A1) with φ^t_n replaced by $\varphi^t_{p_n}$.

Moreover, Lemma 3.6 says

$$\varphi^t_{p_n} \longrightarrow \varphi^t_\infty \quad \text{on } H \text{ in the sense of Mosco}. \tag{3.47}$$

Hence by Theorem 2.10 we can take a subsequence (n_k) of (n) such that the unique strong solution u_{n_k} of $(\mathrm{PP}(\varphi^t_{p_{n_k}}, f_{n_k}))$ satisfies

$$u_{n_k} \longrightarrow u \quad \text{strongly in } C([0,T];H), \text{ weakly in } W^{1,2}(0,T;H); \tag{3.48}$$

moreover, u becomes a strong solution of $(\mathrm{PP}(\varphi^t_\infty, f))$. □

Remark 3.10. As mentioned in Theorem 3.8, $(\mathrm{PP}(\varphi^t_p, f))$ has a unique strong solution. On the other hand, $(\mathrm{PP}(\varphi^t_\infty, f))$ may have multiple strong solutions. Indeed, let $t_0 \in [0,T]$ be a minimizer of ϕ, that is, $0 < \phi(t_0) \leq \phi(t)$ for all $t \in [0,T]$. Then we have $K^{t_0} \subset K^t$ for all $t \in [0,T]$. Hence for every $u_0 \in K^{t_0}$, $\partial_H \varphi^t_\infty(u_0) \ni 0$ for all $t \in [0,T]$ and $u \equiv u_0$ becomes a strong solution for $(\mathrm{PP}(\varphi^t_\infty, 0))$. Therefore since K^{t_0} has infinitely many elements, $(\mathrm{PP}(\varphi^t_\infty, 0))$ admits infinitely many strong solutions.

Furthermore, since $u_p \equiv 0$ is a unique strong solution of $(\mathrm{PP}(\varphi^t_p, 0))$ for all $p \in (1, +\infty)$, u_p never converges to any strong solution u of $(\mathrm{PP}(\varphi^t_\infty, 0))$ except $u \equiv 0$ as $p \to +\infty$.

3.2. Asymptotic behavior of solutions for porous medium equation as $m \to +\infty$

Problem 3.11. Find a unique solution u_m of the initial-boundary value problem for $((\mathrm{PM})_m)$ with the boundary condition $u(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, and the initial condition $u(x,0) = u_0(x)$, $x \in \Omega$, which is denoted by $(\mathrm{IBVP2})_m$, and investigate the asymptotic behavior of u_m as $m \to +\infty$.

We will make the following assumptions:

(H2) Ω is a bounded domain in $\mathbb{R}^N$,

$$\begin{gathered} \gamma(x,t) \in W^{1,2}(0,T;L^\infty(\Omega)), \\ \gamma(x,t) \geq \delta_0 > 0 \quad \text{for a.e. } x \in \Omega \text{ and all } t \in [0,T]. \end{gathered} \tag{3.49}$$

Set $H := H^{-1}(\Omega)$ and define $\psi^t_m : H \to [0, +\infty]$ as follows:

$$\psi^t_m(u) := \begin{cases} \dfrac{1}{m} \displaystyle\int_\Omega \left(\frac{|u(x)|}{\gamma(x,t)} \right)^m dx & \text{if } \dfrac{u}{\gamma(\cdot,t)} \in L^m(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{3.50}$$

Moreover, we denote ψ^t_m with $\gamma \equiv 1$ simply by ψ_m. Then by (H2), we see that

$$D(\psi^t_m) = L^m(\Omega) \quad \forall t \in [0,T]. \tag{3.51}$$

Now define

$$(u,v)_H := \left\langle u, (-\Delta_D)^{-1} v \right\rangle_{H^1_0(\Omega)} \quad \forall u, v \in H, \tag{3.52}$$

where Δ_D denotes the Laplacian with the homogeneous Dirichlet boundary condition and $\langle \cdot, \cdot \rangle_{H^1_0(\Omega)}$ denotes the natural duality pairing between $H^1_0(\Omega)$ and H.

We now observe that $H_0^1(\Omega) \subset L^{m'}(\Omega)$ for every $m \in [2N/(N+2), +\infty)$. Hence let $t \in [0,T]$ be fixed and let $[u,f] \in \partial_H \psi_m^t$. We then get, for all $v \in L^m(\Omega)$,

$$\begin{aligned}\psi_m\left(\frac{u}{\gamma(\cdot,t)}\right) - \psi_m(v) &= \psi_m^t(u) - \psi_m^t(\gamma(\cdot,t)v)\\ &\leq (f, u - \gamma(\cdot,t)v)_H\\ &= \left\langle u - \gamma(\cdot,t)v, (-\Delta_D)^{-1} f\right\rangle_{H_0^1(\Omega)}\\ &= \int_\Omega \gamma(x,t)(-\Delta_D)^{-1} f(x)\left(\frac{u(x)}{\gamma(x,t)} - v(x)\right)dx.\end{aligned} \tag{3.53}$$

Therefore since ψ_m is Fréchet differentiable on $L^m(\Omega)$ and its derivative at $u/\gamma(\cdot,t)$ coincides with $|u/\gamma(\cdot,t)|^{m-2}u/\gamma(\cdot,t)$, we can verify

$$f = \partial_H \psi_m^t(u) \Longleftrightarrow (-\Delta_D)^{-1} f(x) = \beta_m(x,t,u(x)) \text{ for a.e. } x \in \Omega \tag{3.54}$$

for every $m \in [2N/(N+2), +\infty)$, where we use the maximality of $\partial_H \psi_m^t$. Hence (IBVP2)$_m$ is reduced to $(\mathrm{CP}(\psi_m^t, f_m, u_{0,m}))$.

The existence of solutions is assured by Theorem 2.6.

Theorem 3.12. *Suppose that (H2) is satisfied and let $m \in [2, +\infty)$. Then for all $f \in L^1(0, T; H^{-1}(\Omega))$ and $u_0 \in H^{-1}(\Omega)$, $(\mathrm{CP}(\psi_m^t, f, u_0))$ has a unique weak solution u_m. In particular, if $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^m(\Omega)$, the unique weak solution u_m becomes a strong solution of $(\mathrm{CP}(\psi_m^t, f, u_0))$.*

Proof of Theorem 3.12. By Theorem 2.6, it suffices to verify that $\{\psi_m^t\}_{t\in[0,T]} \in \Psi(\alpha_2, 0)$ for some function $\alpha_2 : [0,+\infty) \times [0,T] \to \mathbb{R}$. Let $t_0 \in [0,T]$ and let $u_0 \in D(\psi_m^{t_0})$ be fixed. Define

$$u(t) := \frac{\gamma(\cdot,t)}{\gamma(\cdot,t_0)} u_0 \in L^m(\Omega) \quad \forall t \in [0,T]. \tag{3.55}$$

Then we find that

$$|u(x,t)| = \gamma(x,t)\frac{|u_0(x)|}{\gamma(x,t_0)} \quad \text{for a.e. } x \in \Omega, \tag{3.56}$$

which implies $\psi_m^t(u(t)) = \psi_m^{t_0}(u_0)$ for all $t \in [0,T]$. Furthermore, for any $\phi \in H_0^1(\Omega)$, we see that

$$\begin{aligned}\langle u(t) - u_0, \phi\rangle_{H_0^1(\Omega)} &= \int_\Omega \{\gamma(x,t) - \gamma(x,t_0)\}\frac{u_0(x)}{\gamma(x,t_0)}\phi(x)dx\\ &\leq |\gamma(\cdot,t) - \gamma(\cdot,t_0)|_{L^\infty(\Omega)}\left|\frac{u_0}{\gamma(\cdot,t_0)}\right|_{L^m(\Omega)}|\phi|_{L^{m'}(\Omega)}.\end{aligned} \tag{3.57}$$

Here since $m \geq 2$, it follows that

$$|\phi|_{L^{m'}(\Omega)} \leq |\phi|_{L^2(\Omega)}|\Omega|^{(2-m')/2} \leq C|\phi|_{H_0^1(\Omega)}(|\Omega| + 1), \tag{3.58}$$

where C is independent of m. Thus

$$\begin{aligned} |u(t)-u_0|_H &\le C(|\Omega|+1)\,|\gamma(\cdot,t)-\gamma(\cdot,t_0)|_{L^\infty(\Omega)}\{m\psi_m^{t_0}(u_0)\}^{1/m} \\ &\le Ce^{1/e}(|\Omega|+1)\,|\gamma(\cdot,t)-\gamma(\cdot,t_0)|_{L^\infty(\Omega)}\{\psi_m^{t_0}(u_0)+1\}^{1/2}. \end{aligned} \tag{3.59}$$

Hence it follows that

$$\{\psi_m^t\}_{t\in[0,T]} \in \Psi(\alpha_2,0), \tag{3.60}$$

where α_2 is given by

$$\alpha_2(r,t) = Ce^{1/e}(|\Omega|+1)\int_0^t \left|\frac{\partial\gamma}{\partial\tau}(\cdot,\tau)\right|_{L^\infty(\Omega)} d\tau \in W^{1,2}(0,T). \tag{3.61}$$

□

As for the asymptotic behavior of u_m as $m \to +\infty$, our result is stated as follows.

THEOREM 3.13. *Suppose that (H2) is satisfied and define*

$$\kappa^t := \{u \in L^2(\Omega);\, |u(x)| \le \gamma(x,t) \text{ for a.e. } x \in \Omega\}. \tag{3.62}$$

Let (m_n) be a sequence in $[2,+\infty)$ such that $m_n \to +\infty$ as $n \to +\infty$. Moreover, let $f_n, f \in L^2(0,T;H^{-1}(\Omega))$, $u_{0,n} \in H^{-1}(\Omega)$, and $u_0 \in \kappa^0$ be such that

$$f_n \longrightarrow f \quad \text{strongly in } L^2(0,T;H^{-1}(\Omega)), \tag{3.63}$$

$$u_{0,n} \longrightarrow u_0 \quad \text{strongly in } H^{-1}(\Omega). \tag{3.64}$$

Then the unique weak solution u_n of $(\mathrm{CP}(\psi_{m_n}^t, f_n, u_{0,n}))$ converges to u as $n \to +\infty$ in the following sense:

$$u_n \longrightarrow u \quad \text{strongly in } C([0,T];H^{-1}(\Omega)). \tag{3.65}$$

Moreover, the limit u is a unique weak solution of $(\mathrm{CP}(\psi_\infty^t, f, u_0))$, where

$$\psi_\infty^t(u) := \begin{cases} 0 & \text{if } u \in \kappa^t, \\ +\infty & \text{if } u \in H^{-1}(\Omega)\setminus\kappa^t. \end{cases} \tag{3.66}$$

In particular, if $(1/m_n)\int_\Omega |u_{0,n}(x)|^{m_n}dx$ is bounded as $n \to +\infty$, then the limit u becomes a strong solution of $(\mathrm{CP}(\psi_\infty^t, f, u_0))$.

As in Lemma 3.6, we have the following.

LEMMA 3.14. *For each $t \in [0,T]$, it follows that*

$$\psi_{m_n}^t \longrightarrow \psi_\infty^t \quad \text{on } H \text{ in the sense of Mosco as } m_n \longrightarrow +\infty. \tag{3.67}$$

Proof of Lemma 3.14. Let $t \in [0,T]$ be fixed. Then as in the proof of Lemma 3.6, we can easily derive that

$$\begin{gathered} \forall u \in D(\psi_\infty^t), \quad \exists (u_n) \subset H; \\ u_n \longrightarrow u \quad \text{strongly in } H, \quad \psi_{m_n}^t(u_n) \longrightarrow \psi_\infty^t(u). \end{gathered} \tag{3.68}$$

Our next claim is

$$\begin{gathered} \forall (u_n) \subset H \quad \text{satisfying } u_n \longrightarrow u \quad \text{weakly in } H, \\ \liminf_{n\to+\infty} \psi_{m_n}^t(u_n) \geq \psi_\infty^t(u). \end{gathered} \tag{3.69}$$

For the case where $u \in D(\psi_\infty^t) = \kappa^t$, (3.69) follows immediately; for the case where $u \notin \kappa^t$, to obtain a contradiction, suppose that

$$\exists (u_n) \subset H; \quad u_n \longrightarrow u \quad \text{weakly in } H, \quad \liminf_{n\to+\infty} \psi_{m_n}^t(u_n) < \psi_\infty^t(u) = +\infty. \tag{3.70}$$

Then we can extract a subsequence (n') of (n) such that

$$\psi_{m_{n'}}^t(u_{n'}) \leq C \quad \forall n'. \tag{3.71}$$

We write m and u_m simply for $m_{n'}$ and $u_{n'}$, respectively. Thus we have

$$\left\{\int_\Omega \left(\frac{|u_m(x)|}{\gamma(x,t)}\right)^m dx\right\}^{1/m} \leq \left\{m\psi_m^t(u_m)\right\}^{1/m} \leq (mC)^{1/m} \longrightarrow 1 \tag{3.72}$$

as $m \to +\infty$. Then in much the same way as in the proof of Lemma 3.6, we obtain

$$\left|\frac{u}{\gamma(\cdot,t)}\right|_{L^\infty(\Omega)} \leq 1, \tag{3.73}$$

which contradicts the fact that $u \notin \kappa^t$. Therefore (3.69) follows. □

Proof of Theorem 3.13. Just as in the proof of Theorem 3.5, we can derive $\{\psi_m^t\}_{t\in[0,T]} \in B(\alpha_2, 0, C_0, \{M_r\}_{r\geq 0})$ for some constants $C_0 > 0$, $\{M_r\}_{r\geq 0}$ independent of n. Moreover, recall (3.57). Then for all $t_0 \in [0,T]$ and $u_0 \in \kappa^{t_0}$, we have

$$\langle u(t) - u_0, \phi\rangle_{H_0^1(\Omega)} \leq |\gamma(\cdot,t) - \gamma(\cdot,t_0)|_{L^\infty(\Omega)} |\Omega|^{1/m} |\phi|_{L^{m'}(\Omega)}, \tag{3.74}$$

which together with (3.58) implies

$$|u(t) - u_0|_H \leq C(|\Omega| + 1)^2 |\gamma(\cdot,t) - \gamma(\cdot,t_0)|_{L^\infty(\Omega)}. \tag{3.75}$$

Furthermore, we can easily check $\psi_\infty^t(u(t)) = \psi_\infty^{t_0}(u_0) = 0$. Hence $\{\psi_\infty^t\}_{t\in[0,T]} \in \Psi(\tilde{\alpha}_2, 0)$ with $\tilde{\alpha}_2(\cdot) := (|\Omega| + 1)\alpha_2(\cdot)$. Therefore by Theorem 2.7 and Lemma 3.14, we complete the proof. □

Remark 3.15. In [6], the authors rewrite (3.5) in the form

$$\frac{du_n}{dt}(t) + A_n u_n(t) = 0 \quad \text{in } L^1(\Omega), \tag{3.76}$$

where A_n is an m-accretive operator in $L^1(\Omega)$, and employ an abstract theory of evolution equations governed by m-accretive operators in a general Banach space developed in [5, 7, 8]. Particularly the strong convergence of (u_n) in $C([0,T];L^1(\Omega))$ is derived from the convergence of the resolvent of A_n, which means

$$(I + A_n)^{-1} u \longrightarrow (I + A)^{-1} u \quad \text{strongly in } L^1(\Omega)\ \forall u \in L^1(\Omega) \tag{3.77}$$

as $n \to +\infty$ for some m-accretive operator A. We here note that the following conditions are equivalent to each other:

(i) $\psi_m \to \psi_\infty$ on $H := H^{-1}(\Omega)$ in the sense of Mosco as $m \to +\infty$;
(ii) $(I + \partial_H \psi_m)^{-1} u \to (I + \partial_H \psi_\infty)^{-1} u$ strongly in H for all $u \in H$.

Hence our approach could also be regarded as an $H^{-1}(\Omega)$-framework version of [6].

Problem 3.16. Find a solution of the periodic problem for $((\mathrm{PM})_m)$ with the boundary condition $u(x,t) = 0$, $(x,t) \in \partial\Omega \times (0,T)$, and the periodic condition $u(x,0) = u(x,T)$, $x \in \Omega$, which is denoted by $(\mathrm{PP2})_m$. Moreover, investigate the asymptotic behavior of u_m as $m \to +\infty$.

Just as in Problem 3.11, $(\mathrm{PP2})_m$ is equivalent to $(\mathrm{PP}(\psi_m^t, f_m))$. Concerning the existence of strong solutions, we have the following.

THEOREM 3.17. *Suppose that (H2) is satisfied and let $m \in [2,+\infty)$. Then for all $f \in L^2(0,T;H^{-1}(\Omega))$, $(\mathrm{PP}(\psi_m^t, f))$ has the unique strong solution u_m.*

Proof of Theorem 3.17. We observe

$$\begin{aligned} |u|^2_{L^2(\Omega)} &\le |\gamma|^2_{L^\infty(Q)} \left\{ \frac{2}{m} \int_\Omega \left(\frac{|u(x)|}{\gamma(x,t)} \right)^m dx + \frac{m-2}{m} |\Omega| \right\} \\ &= 2|\gamma|^2_{L^\infty(Q)} \left\{ \psi_m^t(u) + |\Omega| \right\} \quad \forall u \in L^m(\Omega),\ \forall m \ge 2, \end{aligned} \tag{3.78}$$

where $Q := \Omega \times [0,T]$. Hence since $L^2(\Omega)$ is continuously embedded in H, we can take a positive number C_0 independent of m such that

$$|u|_H^2 \le C_0 \left(\psi_m^t(u) + 1 \right) \quad \forall u \in D(\psi_m^t),\ \forall t \in [0,T]. \tag{3.79}$$

Therefore on account of (3.51) and (3.60), we have

$$\{\psi_m^t\}_{t \in [0,T]} \in \Psi_\pi(\alpha_2, 0, C_0). \tag{3.80}$$

Thus Theorem 2.9 ensures the existence of a strong solution u_m of $(\mathrm{PP}(\psi_m^t, f))$. Moreover, since ψ_m^t is strictly convex on H, every periodic solution is unique. □

As for the convergence of u_m as $m \to +\infty$, our result is the following.

Theorem 3.18. *Suppose that (H2) is satisfied and that $\gamma(x,0) \geq \gamma(x,T)$ for a.e. $x \in \Omega$. Let (m_n) be a sequence in $[2,+\infty)$ such that $m_n \to +\infty$ as $n \to +\infty$. Moreover, let $f_n, f \in L^2(0,T;H^{-1}(\Omega))$ be such that*

$$f_n \longrightarrow f \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{3.81}$$

Then a subsequence (n_k) of (n) can be taken such that the unique strong solution u_{n_k} of $(\mathrm{PP}(\psi^t_{m_{n_k}}, f_{n_k}))$ converges to u as $k \to +\infty$ in the following sense:

$$u_{n_k} \longrightarrow u \quad \text{strongly in } C([0,T];H^{-1}(\Omega)),\ \text{weakly in } W^{1,2}(0,T;H^{-1}(\Omega)). \tag{3.82}$$

Moreover, the limit u is a strong solution of $(\mathrm{PP}(\psi^t_\infty, f))$.

Proof of Theorem 3.18. We claim that any sequence (u_n) in H satisfying

$$\sup_{n\in\mathbb{N}} \{\psi^t_{m_n}(u_n) + |u_n|_H\} \leq \lambda \tag{3.83}$$

is precompact in H for every $\lambda > 0$ and $t \in [0,T]$. For every $m_n \geq 2$, we get

$$\begin{aligned}\left(\int_\Omega |u_n(x)|^2 dx\right)^{1/2} &\leq \left(\int_\Omega |u_n(x)|^{m_n} dx\right)^{1/m_n} |\Omega|^{(m_n-2)/(2m_n)} \\ &\leq |\gamma(\cdot,t)|_{L^\infty(\Omega)} (m_n\lambda)^{1/m_n} |\Omega|^{(m_n-2)/(2m_n)} \leq C,\end{aligned} \tag{3.84}$$

where C denotes a constant independent of n. Then since $L^2(\Omega)$ is compactly embedded in H, it follows immediately that (u_n) becomes precompact in H. Moreover, since $\gamma(x,0) \geq \gamma(x,T)$ for a.e. $x \in \Omega$, we can easily see that $\psi^T_m(u) \geq \psi^0_m(u)$ for all $u \in D(\psi^T_m)$ and $n \in \mathbb{N}$. Then the rest of proof can be derived just as in the proof of Theorem 3.9. □

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