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Doubly nonlinear evolution equations governed by time-dependent subdifferentials in reflexive Banach spaces

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1 Introduction

Many authors proposed various types of doubly nonlinear evolution equations and tried to prove the existence of their solutions. In particular, some of these results are developed for applications to PDEs, which describes complex nonlinear phenomena, e.g., phase transition, dynamics of non-Newtonian fluid and so on.

In this paper, we study the existence of solutions of Cauchy problem for doubly nonlinear evolution equations governed by time-dependent subdifferential operator; more precisely, let V and V^* be a real reflexive Banach space and its dual space, and let H be a Hilbert space whose dual space H^* is identified itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

continuously and densely; let $\varphi^t : V \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function depending on a time variable t and let $\psi : H \to (-\infty, +\infty]$ be also proper lower semicontinuous and convex; then we consider

$$\text{(CP)} \quad \begin{cases} \dfrac{dv}{dt}(t) + \partial_V \varphi^t(u(t)) \ni f(t), \quad v(t) \in \partial_H \psi(u(t)), \quad 0 < t < T, \\ v(0) = v_0, \end{cases}$$

where $f : (0, T) \to V^*$ and $v_0 \in H$ are given.

V. Barbu [4] proved the existence of solutions of (CP) with φ^t replaced by φ independent of t for every initial data

$$v_0 \in \partial_H \psi(D(\partial_V \varphi)) := \{v \in \partial_H \psi(u);\ u \in D(\partial_H \psi) \cap D(\partial_V \varphi)\}.$$

He also applied his results on abstract evolution equation to the initial-boundary value problem for the doubly nonlinear parabolic equation of the form

$$\frac{\partial v}{\partial t}(x, t) - \Delta_p u(x, t) = f(x, t), \quad v(x, t) \in \alpha(u(x, t)), \quad (x, t) \in \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^N$ and f is a given function and Δ_p is the so-called p-Laplacian given by

$$\Delta_p u(x) \ := \ \nabla \cdot (|\nabla u(x)|^{p-2} \nabla u(x)), \quad 1 < p < +\infty,$$

for an arbitrary (possibly multi-valued) maximal monotone graph α in $\mathbb{R}^2$.

In this paper, we prove the existence of solutions for (CP) for each

$$v_0 \in R(\partial_H \psi) := \{ v \in \partial_H \psi(u); \ u \in D(\partial_H \psi) \} \tag{1}$$

by supposing a t-smoothness condition for φ^t in addition to the coercivity and the boundedness conditions for $\partial_V \varphi^t : V \to 2^{V^*}$ and an condition known as a sufficient one for the maximality of the sum of $\partial_H \psi$ and $\partial_H \varphi_H^t$ in H, where φ_H^t stands for an extension of φ^t onto H.

Our problem has two significant features; one is the fact that φ^t depends of time-variable t, and the other can be find in the assumption (1) on initial data. Compared with [4], we remark that $\partial_H \psi(D(\varphi)) \subset R(\partial_H \psi)$.

The method of our proof relies on a suitable approximation of initial data v_0 for the multi-valued operator $\partial_H \psi$ and approximate problems for (CP) by using resolvent and Yosida approximation of maximal monotone operator. Moreover, we also employ chain rule for time-dependent subdifferential operators developed in [11] and [1] to establish a priori estimates.

The following type of initial-boundary value problem falls within the scope of applications of our abstract framework.

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) - \mathrm{div}\ \mathbf{a}(x,t,\nabla u(x,t)) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ v(x,t) \in \alpha(u(x,t)), & (x,t) \in \partial\Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and $\mathbf{a} : \Omega \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times (0,T) \to \mathbb{R}$ are given, for an arbitrary (possibly multi-valued) maximal monotone graph α in $\mathbb{R}^2$.

In the next section, we arrange a couple of preliminary results, and in Section 3 is devoted to state our main result on existence of solutions for (CP). In Section 4, we construct an approximate sequences of initial data v_0, and by using the approximate sequence, we introduce approximate problems for (CP) and verify the existence of their solutions and establish a priori estimates for them. Moreover, in Section 5, we derive the convergences of those approximate solutions from the a priori estimates. Finally, in Section 6, we deal with (2) as a typical example of our abstract framework.

2 Preliminaries

We first recall the definition of subdifferential operators. Let $\Phi(X)$ be the set of all proper lower-semicontinuous convex functionals ϕ from a topological linear space X into

$(-\infty, +\infty]$, where "proper" means $\phi \not\equiv +\infty$. Then, the subdifferential $\partial_{X,X^*}\phi(u)$ of $\phi \in \Phi(X)$ at u is given by

$$\partial_{X,X^*}\phi(u) \ := \ \{\xi \in X^*; \phi(v) - \phi(u) \geq \langle \xi, v-u \rangle_X \quad \forall v \in D(\phi)\},$$

where $\langle \cdot, \cdot \rangle_X$ denotes the duality pairing between X and X^* and $D(\phi) := \{u \in X; \phi(u) < +\infty\}$. Hence we can define the subdifferential operator $\partial_{X,X^*}\phi : X \to 2^{X^*}; u \mapsto \partial_{X,X^*}\phi(u)$ with the domain $D(\partial_{X,X^*}\phi) := \{u \in D(\phi); \partial_{X,X^*}\phi(u) \neq \emptyset\}$. For simplicity of notation, we shall write $\partial_X\phi$ and $\langle \cdot, \cdot \rangle$ instead of $\partial_{X,X^*}\phi$ and $\langle \cdot, \cdot \rangle_X$, respectively, if no confusion can arise. It is well known that the graph of every subdifferential operator $\partial_X\phi$ becomes maximal monotone in $X \times X^*$.

In particular, if X is a Hilbert space H whose dual space is identified with itself, i.e., $H \equiv H^*$, then the subdifferential $\partial_H\phi(u)$ of $\phi \in \Phi(H)$ at u can be written by

$$\partial_H\phi(u) \ = \ \{\xi \in H; \phi(v) - \phi(u) \geq (\xi, v-u)_H \quad \forall v \in D(\phi)\},$$

since $\langle \cdot, \cdot \rangle_H$ coincides with the inner product $(\cdot, \cdot)_H$ of H; moreover, the graph of $\partial_H\phi$ is maximal monotone in $H \times H$.

Now we recall a couple of useful properties of the Legendre-Fenchel transform ϕ^* of $\phi \in \Phi(X)$ defined by

$$\phi^*(u) \ := \ \sup_{v \in X} \{\langle u, v \rangle - \phi(v)\} \quad \forall u \in X^*$$

in the following:

P:phi*:lsc (3) $\phi^* \in \Phi(X^*)$;

P:FM-id (4) $\phi^*(f) = \langle f, u \rangle - \phi(u) \quad \forall [u, f] \in \partial_X\phi$;

P:dphi* (5) $u \in \partial_{X^*}\phi^*(f) \quad \forall [u, f] \in \partial_X\phi$.

3 Main Result

Let V be a real reflexive Banach space and let V^* be its dual space. Moreover, let H be a real Hilbert space whose dual space H^* is identified with itself such that

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*$$

with continuous and densely defined canonical injections.

Now let $\varphi^t : V \to [0, +\infty]$ and $\psi : H \to [0, +\infty]$ be such that $\varphi^t \in \Phi(V)$ and $\psi \in \Phi(H)$ for every $t \in [0, T]$ and consider the Cauchy problem:

$$\text{(CP)} \quad \begin{cases} \dfrac{dv}{dt}(t) + \partial_V\varphi^t(u(t)) \ni f(t), \quad v(t) \in \partial_H\psi(u(t)), \quad 0 < t < T, \\ v(0) = v_0, \end{cases}$$

where $\partial_V\varphi^t$ and $\partial_H\psi$ denote subdifferential operators of φ^t and ψ, respectively, for every $t \in [0, T]$.

We are concerned with strong solutions of (CP) defined below.

:D:str-sol DEFINITION 3.1 *A pair of functions* $(u, v) : [0, T] \to V \times H$ *is said to be a strong solution of* (CP) *on* $[0, T]$ *if the following* (i)-(iii) *hold true*:

(i) v *is a* V^**-valued absolutely continuous function on* $[0, T]$;

(ii) $u(t) \in D(\partial_H \psi) \cap D(\partial_V \varphi^t)$ *for a.a.* $t \in (0, T)$, *and there exists a section* $g(t) \in \partial_V \varphi^t(u(t))$ *such that*

EQ (6) $$\frac{dv}{dt}(t) + g(t) = f(t) \quad \text{in } V^*, \quad v(t) \in \partial_H \psi(u(t)) \quad \text{for a.a. } t \in (0, T);$$

(iii) $v(t) \to v_0$ *strongly in* V^* *and weakly in* H *as* $t \to +0$.

Before describing our main result, we introduce the following assumptions for some $p \in (1, +\infty)$ and $r_0 \geq 0$.

($A\varphi^t$) There exist functions $a \in W^{1,p}(0, T)$, $b \in W^{1,1}(0, T)$ and a constant $\delta > 0$ such that for every $t_0 \in [0, T]$ and $x_0 \in D(\varphi^{t_0})$, we can take a function $x : I_\delta(t_0) := [t_0 - \delta, t_0 + \delta] \cap [0, T] \to V$ satisfying:
$$\begin{cases} |x(t) - x_0|_V & \leq \; |a(t) - a(t_0)| \left\{\varphi^{t_0}(x_0) + r_0\right\}^{1/p}, \\ \quad \varphi^t(x(t)) & \leq \; \varphi^{t_0}(x_0) + |b(t) - b(t_0)| \left\{\varphi^{t_0}(x_0) + r_0\right\} \quad \forall t \in I_\delta(t_0). \end{cases}$$

(A1) There exists a constant C_1 such that
$|u|_V^p \leq C_1\{\varphi^t(u) + r_0\} \quad \forall u \in D(\varphi^t), \ \forall t \in [0, T].$

(A2) There exists a constant C_2 such that
$|\xi|_{V^*}^{p'} \leq C_2\{\varphi^t(u) + r_0\} \quad \forall [u, \xi] \in \partial_V \varphi^t, \ \forall t \in [0, T].$

(A3) There exists a constant C_3 such that
$\varphi^t(J_\varepsilon u) \leq \varphi^t(u) + \varepsilon C_3 \quad \forall \varepsilon > 0, \ \forall u \in D(\varphi^t), \ \forall t \in [0, T],$
where J_ε and $\partial_H \psi_\varepsilon$ denote the resolvent and the Yosida approximation of $\partial_H \psi$, respectively, with the identity I of H.

(A4) V is compactly embedded in H.

Our main result reads:

T:abst-ex THEOREM 3.2 *Let* $p \in (1, +\infty)$ *and* $r_0 \geq 0$ *be fixed and assume that* ($A\varphi^t$), (A1), (A2), (A3) *and* (A4) *are all satisfied. Then for all* $f \in L^{p'}(0, T; V^*) \cap L^2(0, T; H)$ *and* $v_0 \in H$ *satisfying that* $t(df/dt) \in L^{p'}(0, T; V^*)$ *and*

$$v_0 \in R(\partial_H \psi) := \{v \in \partial_H \psi(u); \ u \in D(\partial_H \psi)\},$$

(CP) *admits a strong solution* (u, v) *on* $[0, T]$ *such that*

$$u \in L^p(0, T; V) \cap L^\infty_{loc}((0, T]; V), \quad v \in C_w([0, T]; H) \cap W^{1,p'}(0, T; V^*).$$

M:R:psi>0 REMARK 3.3 Our result is valid also for $\varphi^t : V \to (-\infty, +\infty]$ and $\psi : H \to (-\infty, +\infty]$. By (A1), we can always assume that $\varphi^t \geq 0$ without any loss of generality. On the other hand, since there exist $\xi \in H$ and $C_0 \in \mathbb{R}$ such that

$$\psi(u) \;\; \geq \;\; -(\xi, u)_H - C_0 \quad \forall u \in H$$

(see Proposition 2.1 of [3]), we can define the non-negative function $\tilde{\psi}(u) := \psi(u) + (\xi, u)_H + C_0 \geq 0$. It then follows that $D(\tilde{\psi}) = D(\psi)$, $D(\partial_H \tilde{\psi}) = D(\partial_H \psi)$, $\partial_H \tilde{\psi}(u) = \partial_H \psi(u) + \xi$ for all $u \in D(\partial_H \psi)$. Therefore, in order to prove the existence of solutions for (CP), it suffices to do so for (CP) with ψ replaced by $\tilde{\psi}$. Indeed, let (u, w) be a strong solution of (CP) with ψ and v_0 replaced by $\tilde{\psi}$ and $v_0 + \xi$, respectively, on $[0, T]$, and put $v(t) := w(t) + \xi$. Then, observing that

$$
\begin{aligned}
&v(t) = w(t) - \xi \in \partial_H \psi(u(t)), \quad \text{for a.a. } t \in (0, T), \\
&dv(t)/dt = dw(t)/dt \in f(t) - \partial_V \varphi^t(u(t)), \quad \text{for a.a. } t \in (0, T), \\
&v(t) = w(t) - \xi \to v_0 \ \text{ strongly in } V^* \ \text{ as } t \to +0,
\end{aligned}
$$

we can deduce that (u, v) becomes a strong solution of (CP) on [0,T].

In order to construct a strong solution of (CP), we approximate initial data v_0 in an appropriate way (see Section 4) and construct solutions of the following approximate problems of (CP) (see Section 5):

$$
(\mathrm{CP})_\varepsilon \quad \begin{cases} \sqrt{\varepsilon} \dfrac{du_\varepsilon}{dt}(t) + \dfrac{d}{dt} \partial_H \psi_\varepsilon(u_\varepsilon(t)) + g_\varepsilon(t) = f_\varepsilon(t), \ g_\varepsilon(t) \in \partial_H \varphi^t_H(u_\varepsilon(t)), \ 0 < t < T, \\ \sqrt{\varepsilon} u_\varepsilon(+0) + \partial_H \psi_\varepsilon(u_\varepsilon(+0)) = \sqrt{\varepsilon} u_{0,\varepsilon} + \partial_H \psi_\varepsilon(u_{0,\varepsilon}), \end{cases}
$$

where $\partial_H \psi_\varepsilon$ stands for the Yosida approximation of $\partial_H \psi$ and φ^t_H denotes the extension of φ^t on H given by

$$
\varphi^t_H(u) \ := \ \begin{cases} \varphi^t(u) & \text{if } \ u \in V, \\ +\infty & \text{otherwise} \end{cases}
$$

and f_ε denotes a smooth approximation of f such as $f_\varepsilon \in C^1([0, T]; V)$ satisfying $f_\varepsilon \to f$ strongly in $L^{p'}(0, T; V^*) \cap L^2(0, T; H)$ and $df_\varepsilon/dt \to df/dt$ strongly in $L^{p'}(0, T; V^*)$ as $\varepsilon \to +0$. Further, by establishing a priori estimates for $(u_\varepsilon, \partial_H \psi_\varepsilon(u_\varepsilon(\cdot))$, we shall get a strong solution (u, v) of (CP) as a limit of $(u_\varepsilon, \partial_H \psi_\varepsilon(u_\varepsilon(\cdot))$ as $\varepsilon \to +0$ (see Section 6).

4 Approximation of Initial Data

In this section, we propose an approximation of initial data v_0 suitable for the construction of solutions for $(\mathrm{CP})_\varepsilon$.

Since $v_0 \in R(\partial_H \psi)$, there exists $u_0 \in D(\partial_H \psi)$ such that $v_0 \in \partial_H \psi(u_0)$. Since $\partial_H \psi_\varepsilon$ is maximal monotone in H, we can obtain a unique solution $u_{0,\varepsilon} \in H$ of the equation

$$
\sqrt{\varepsilon} u_{0,\varepsilon} + \partial_H \psi_\varepsilon(u_{0,\varepsilon}) = \sqrt{\varepsilon} u_0 + v_0. \tag{7}
$$

Moreover, the following lemma provides information on the convergence of $u_{0,\varepsilon}$ as $\varepsilon \to +0$.

L:u0e **Lemma 4.1** *Let $u_{0,\varepsilon} \in H$ be a unique solution of (7). Then it follows that*

$$
\begin{aligned}
&u_{0,\varepsilon} \to u_0 \quad \text{strongly in } H,\\
&J_\varepsilon u_{0,\varepsilon} \to u_0 \quad \text{strongly in } H,\\
&\partial_H \psi_\varepsilon(u_{0,\varepsilon}) \to v_0 \quad \text{strongly in } H,\\
&\psi^*(\partial_H \psi_\varepsilon(u_{0,\varepsilon})) \to \psi^*(v_0),\\
&\psi(J_\varepsilon u_{0,\varepsilon}) \to \psi(u_0),
\end{aligned}
$$

where J_ε and $\partial_H \psi_\varepsilon$ denote the resolvent and the Yosida approximation of $\partial_H \psi$, respectively, as $\varepsilon \to +0$.

Proof of Lemma 4.1 Multiplying both sides of (7) by $u_{0,\varepsilon} - u_0$, we get

$$\sqrt{\varepsilon}|u_{0,\varepsilon} - u_0|_H^2 + (\partial_H \psi_\varepsilon(u_{0,\varepsilon}) - v_0, u_{0,\varepsilon} - u_0)_H = 0.$$

Then note that

$$
\begin{aligned}
&(\partial_H \psi_\varepsilon(u_{0,\varepsilon}) - v_0, u_{0,\varepsilon} - u_0)_H\\
&\quad = (\partial_H \psi_\varepsilon(u_{0,\varepsilon}) - v_0, J_\varepsilon u_{0,\varepsilon} - u_0)_H + (\partial_H \psi_\varepsilon(u_{0,\varepsilon}) - v_0, u_{0,\varepsilon} - J_\varepsilon u_{0,\varepsilon})_H\\
&\quad \geq \varepsilon|\partial_H \psi_\varepsilon(u_{0,\varepsilon})|_H^2 - \varepsilon(v_0, \partial_H \psi_\varepsilon(u_{0,\varepsilon}))_H\\
&\quad \geq \frac{\varepsilon}{2}|\partial_H \psi_\varepsilon(u_{0,\varepsilon})|_H^2 - \frac{\varepsilon}{2}|v_0|_H^2.
\end{aligned}
$$

Thus we find that

$$|u_{0,\varepsilon} - u_0|_H^2 + \frac{\sqrt{\varepsilon}}{2}|\partial_H \psi_\varepsilon(u_{0,\varepsilon})|_H^2 \leq \frac{\sqrt{\varepsilon}}{2}|v_0|_H^2.$$

Hence letting $\varepsilon \to +0$, we can deduce that

$$u_{0,\varepsilon} \to u_0 \quad \text{strongly in } H.$$

Furthermore, since

$$|\partial_H \psi_\varepsilon(u_{0,\varepsilon})|_H \leq |v_0|_H,$$

it follows that

$$J_\varepsilon u_{0,\varepsilon} \to u_0 \quad \text{strongly in } H.$$

On the other hand, (7) gives

$$\partial_H \psi_\varepsilon(u_{0,\varepsilon}) \to v_0 \quad \text{strongly in } H.$$

Now, noting that $\partial_H \psi_\varepsilon(u_{0,\varepsilon}) \in \partial_H \psi(J_\varepsilon u_{0,\varepsilon})$ and recalling the identity (4), we see that

u0e-FM
$$\psi^*(\partial_H \psi_\varepsilon(u_{0,\varepsilon})) = (\partial_H \psi_\varepsilon(u_{0,\varepsilon}), J_\varepsilon u_{0,\varepsilon})_H - \psi(J_\varepsilon u_{0,\varepsilon}). \tag{8}$$

Hence

$$
\begin{aligned}
\limsup_{\varepsilon \to +0} \psi^*(\partial_H \psi_\varepsilon(u_{0,\varepsilon})) &= \lim_{\varepsilon \to +0} (\partial_H \psi_\varepsilon(u_{0,\varepsilon}), J_\varepsilon u_{0,\varepsilon})_H - \liminf_{\varepsilon \to +0} \psi(J_\varepsilon u_{0,\varepsilon})\\
&\leq (v_0, u_0)_H - \psi(u_0) = \psi^*(v_0),
\end{aligned}
$$

which together with the lower semi-continuity of ψ^* yields

$$\lim_{\varepsilon \to +0} \psi^*(\partial_H \psi_\varepsilon(u_{0,\varepsilon})) = \psi^*(v_0).$$

Moreover, recalling (8) [u0e-FM] again, we can also derive that

$$\lim_{\varepsilon \to +0} \psi(J_\varepsilon u_{0,\varepsilon}) = \psi(u_0). \blacksquare$$

5 Construction of Approximate Solutions

const_asol

In this section, we construct a strong solution u_ε of $(\mathrm{CP})_\varepsilon$ on $[0,T]$ such that

$$\begin{aligned}
&u_\varepsilon \in L^p(0,T;V) \cap L^\infty(0,T;H) \cap C_w((0,T];V) \cap W^{1,2}_{loc}((0,T];H),\\
&\partial_H \psi_\varepsilon(u_\varepsilon(\cdot)) \in L^\infty(0,T;H) \cap W^{1,2}_{loc}((0,T];H),\\
&\sqrt{\varepsilon} u_\varepsilon + \partial_H \psi_\varepsilon(u_\varepsilon(\cdot)) \in C_w([0,T];H) \cap W^{1,p'}(0,T;V^*) \cap W^{1,2}_{loc}((0,T];H),\\
&g_\varepsilon \in L^{p'}(0,T;V^*) \cap L^\infty_{loc}((0,T];V^*) \cap L^2_{loc}((0,T];H)
\end{aligned}$$

and u_ε satisfies the following inequalities:

e-ieq-1 (9)

$$\begin{aligned}
&\sup_{t\in[0,T]} |\partial_H \psi_\varepsilon(u_\varepsilon(t))|_H^2\\
&\leq C_T \left(\sqrt{\varepsilon}\psi_\varepsilon(u_{0,\varepsilon}) + |\partial_H \psi_\varepsilon(u_{0,\varepsilon})|_H^2 + \int_0^T |f_\varepsilon(t)|_H^2 dt + C_3 \right),
\end{aligned}$$

e-ieq-2 (10)

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\{ \sqrt{\varepsilon}|u_\varepsilon(t)|_H^2 + |\psi^*(\partial_H \psi_\varepsilon(u_\varepsilon(t)))| \right\} + \int_0^T \varphi^t(u_\varepsilon(t))dt\\
&\leq C_{T,r_0} \Bigg(\sqrt{\varepsilon}|u_{0,\varepsilon}|_H^2 + \psi^*(\partial_H \psi_\varepsilon(u_{0,\varepsilon})) + |\partial_H \psi_\varepsilon(u_{0,\varepsilon})|_H^2 + \int_0^T |f_\varepsilon(t)|_H^2 dt\\
&\quad + \int_0^T |f_\varepsilon(t)|_{V^*}^{p'} dt + \int_0^T \varphi^t(w(t))dt + \int_0^T |w(t)|_V^p dt + 1 \Bigg),
\end{aligned}$$

e-ieq-3 (11)

$$\begin{aligned}
&\sqrt{\varepsilon} \int_0^T t \left| \frac{du_\varepsilon}{dt}(t) \right|_H^2 dt + \sup_{t\in[0,T]} t\varphi^t(u_\varepsilon(t))\\
&\leq C_{T,r_0} \Bigg(\sqrt{\varepsilon}|u_{0,\varepsilon}|_H^2 + \psi^*(\partial_H \psi_\varepsilon(u_{0,\varepsilon})) + |\partial_H \psi_\varepsilon(u_{0,\varepsilon})|_H^2\\
&\quad + \int_0^T t \left\{ C_2^{1/p'} |\dot{a}(t)| + |\dot{b}(t)| \right\} dt + \int_0^T |f_\varepsilon(t)|_H^2 dt\\
&\quad + \int_0^T |f_\varepsilon(t)|_{V^*}^{p'} dt + \sup_{t\in[0,T]} t|f_\varepsilon(t)|_{V^*}^{p'} + \int_0^T t \left| \frac{df_\varepsilon}{dt}(t) \right|_{V^*}^{p'} dt\\
&\quad + \int_0^T \varphi^t(w(t))dt + \int_0^T |w(t)|_V^p dt + 1 \Bigg)\\
&\quad \times \exp\left(\int_0^T \left\{ C_2^{1/p'} |\dot{a}(t)| + |\dot{b}(t)| + 1 \right\} dt \right),
\end{aligned}$$

where $\dot{a} = da/dt$, $\dot{b} = db/dt$, and $w : [0,T] \to V$ is a function given by (Aφ^t) such that $w \in L^p(0,T;V)$ and the mapping $t \mapsto \varphi^t(w(t))$ belongs to $L^1(0,T)$, and C_T (resp. C_{T,r_0}) denotes a constant depending on T (resp. T and r_0) but not on ε.

To this end, we introduce the following approximate problems for $(\mathrm{CP})_\varepsilon$:

$$(\mathrm{CP})_{\varepsilon,\lambda} \quad \begin{cases} \sqrt{\varepsilon}\dfrac{du_{\varepsilon,\lambda}}{dt}(t) + \dfrac{d}{dt}\partial_H\psi_\varepsilon(u_{\varepsilon,\lambda}(t)) + \partial_H\varphi^t_{H,\lambda}(u_{\varepsilon,\lambda}(t)) = f_\varepsilon(t) \ \text{ in } H, \quad 0<t<T, \\ \sqrt{\varepsilon}u_{\varepsilon,\lambda}(+0) + \partial_H\psi_\varepsilon(u_{\varepsilon,\lambda}(+0)) = \sqrt{\varepsilon}u_{0,\varepsilon} + \partial_H\psi_\varepsilon(u_{0,\varepsilon}), \end{cases}$$

where $\partial_H\varphi^t_{H,\lambda}$ denotes the Yosida approximation of $\partial_H\varphi^t_H$. In the rest of this section, for abbreviation, we write u_λ instead of $u_{\varepsilon,\lambda}$.

Put $x_\lambda(t) := \sqrt{\varepsilon}u_\lambda(t) + \partial_H\psi_\varepsilon(u_\lambda(t))$. Then we can find that $(\mathrm{CP})_{\varepsilon,\lambda}$ is equivalent to

$$(\mathrm{CP})'_{\varepsilon,\lambda} \quad \begin{cases} \dfrac{dx_\lambda}{dt}(t) + \partial_H\varphi^t_{H,\lambda} \circ (\sqrt{\varepsilon}I + \partial_H\psi_\varepsilon)^{-1}(x_\lambda(t)) = f_\varepsilon(t) \ \text{ in } H, \quad 0<t<T, \\ x_\lambda(+0) = \sqrt{\varepsilon}u_{0,\varepsilon} + \partial_H\psi_\varepsilon(u_{0,\varepsilon}), \end{cases}$$

where I denotes the identity mapping in H. Since the mapping $t \mapsto \partial_H\varphi^t_{H,\lambda}(u)$ is continuous on $[0,T]$ with values in H for all $u \in H$ (see Lemma 2.9 of [1]) and the mapping $(\sqrt{\varepsilon}I + \partial_H\psi_\varepsilon)^{-1} : H \to H$ is Lipschitz continuous, $(\mathrm{CP})'_{\varepsilon,\lambda}$ possesses a unique strong solution $x_\lambda \in C^1([0,T];H)$ of $(\mathrm{CP})'_{\varepsilon,\lambda}$ on $[0,T]$ (see Theorem 1.4 of [6]). Moreover, it follows immediately that $u_\lambda, \partial_H\psi_\varepsilon(u_\lambda(\cdot)) \in W^{1,\infty}(0,T;H)$, $u_\lambda(+0) = u_{0,\varepsilon}$ and $\partial_H\psi_\varepsilon(u_\lambda(+0)) = \partial_H\psi_\varepsilon(u_{0,\varepsilon})$.

Now, we shall establish a couple of a priori estimates to imply the convergences of u_λ as $\lambda \to +0$. First, by multiplying $(\mathrm{CP})_{\varepsilon,\lambda}$ by $\partial_H\psi_\varepsilon(u_\lambda(t))$, we obtain

$$\begin{aligned} &\sqrt{\varepsilon}\frac{d}{dt}\psi_\varepsilon(u_\lambda(t)) + \frac{1}{2}\frac{d}{dt}|\partial_H\psi_\varepsilon(u_\lambda(t))|^2_H + \Big(\partial_H\varphi^t_{H,\lambda}(u_\lambda(t)), \partial_H\psi_\varepsilon(u_\lambda(t))\Big)_H \\ &= \ (f_\varepsilon(t), \partial_H\psi_\varepsilon(u_\lambda(t)))_H \le \frac{1}{2}|f_\varepsilon(t)|^2_H + \frac{1}{2}|\partial_H\psi_\varepsilon(u_\lambda(t))|^2_H. \end{aligned}$$

Just as in [6, Theorem 4.4], we find that

$$\Big(\partial_H\varphi^t_{H,\lambda}(u_\lambda(t)), \partial_H\psi_\varepsilon(u_\lambda(t))\Big)_H \ \ge \ -C_3.$$

Therefore, integrating this over $(0,t)$, we give

$$\begin{aligned} &\sqrt{\varepsilon}\psi_\varepsilon(u_\lambda(t)) + \frac{1}{2}|\partial_H\psi_\varepsilon(u_\lambda(t))|^2_H \\ &\le \ \sqrt{\varepsilon}\psi_\varepsilon(u_{0,\varepsilon}) + \frac{1}{2}|\partial_H\psi_\varepsilon(u_{0,\varepsilon})|^2_H + C_3 t + \frac{1}{2}\int_0^t |f_\varepsilon(\tau)|^2_H d\tau \\ &\quad + \frac{1}{2}\int_0^t |\partial_H\psi_\varepsilon(u_\lambda(\tau))|^2_H d\tau. \end{aligned}$$

Hence, Gronwall's inequality yields

$$\begin{aligned} &\sup_{\tau\in[0,T]} |\partial_H\psi_\varepsilon(u_\lambda(\tau))|^2_H \\ &\le \ C_T\left\{\sqrt{\varepsilon}\psi_\varepsilon(u_{0,\varepsilon}) + |\partial_H\psi_\varepsilon(u_{0,\varepsilon})|^2_H + \int_0^T |f_\varepsilon(\tau)|^2_H d\tau + C_3\right\}, \end{aligned} \tag{12}$$

where C_T denotes a constant depending on T but not on ε, λ.

Next, we also multiply $(\mathrm{CP})_{\varepsilon,\lambda}$ by $u_\lambda(t)$ to get

$$\frac{\sqrt{\varepsilon}}{2}\frac{d}{dt}|u_\lambda(t)|_H^2 + \left(\frac{d}{dt}\partial_H\psi_\varepsilon(u_\lambda(t)), u_\lambda(t)\right)_H + \left(\partial_H\varphi^t_{H,\lambda}(u_\lambda(t)), u_\lambda(t)\right)_H = (f_\varepsilon(t), u_\lambda(t))_H$$

for a.a. $t \in (0,T)$. Since $\partial_H\psi_\varepsilon(u_\lambda(t)) \in \partial_H\psi(J_\varepsilon u_\lambda(t))$, it follows that $J_\varepsilon u_\lambda(t) \in \partial_H\psi^*(\partial_H\psi_\varepsilon(u_\lambda(t)))$, which implies

$$\begin{aligned}\left(\frac{d}{dt}\partial_H\psi_\varepsilon(u_\lambda(t)), u_\lambda(t)\right)_H &= \left(\frac{d}{dt}\partial_H\psi_\varepsilon(u_\lambda(t)), J_\varepsilon u_\lambda(t)\right)_H + \frac{\varepsilon}{2}\frac{d}{dt}|\partial_H\psi_\varepsilon(u_\lambda(t))|_H^2 \\ &= \frac{d}{dt}\psi^*(\partial_H\psi_\varepsilon(u_\lambda(t))) + \frac{\varepsilon}{2}\frac{d}{dt}|\partial_H\psi_\varepsilon(u_\lambda(t))|_H^2.\end{aligned}$$

On the other hand, by virtue of $(\mathrm{A}\varphi^t)$, we can construct a function $w : [0,T] \to V$ such that

$$w(t) \in D(\varphi^t) \quad \forall t \in [0,T], \quad [t \mapsto \varphi^t(w(t))] \in L^1(0,T), \quad w \in L^p(0,T;V).$$

Hence, since $\partial_H\varphi^t_{H,\lambda}(u_\lambda(t)) \in \partial_H\varphi^t_H(J^t_\lambda u_\lambda(t)) \subset \partial_V\varphi^t(J^t_\lambda u_\lambda(t))$, where J^t_λ denotes the resolvent of $\partial_H\varphi^t_H$, we can obtain, by (A2),

$$\begin{aligned}&\left(\partial_H\varphi^t_{H,\lambda}(u_\lambda(t)), u_\lambda(t)\right)_H \\ &= \left(\partial_H\varphi^t_{H,\lambda}(u_\lambda(t)), u_\lambda(t) - w(t)\right)_H + \left(\partial_H\varphi^t_{H,\lambda}(u_\lambda(t)), w(t)\right)_H \\ &\geq \varphi^t_{H,\lambda}(u_\lambda(t)) - \varphi^t_{H,\lambda}(w(t)) - |\partial_H\varphi^t_{H,\lambda}(u_\lambda(t))|_{V^*}|w(t)|_V \\ &\geq \frac{1}{2}\varphi^t_{H,\lambda}(u_\lambda(t)) - \varphi^t(w(t)) - C\left\{|w(t)|_V^p + r_0\right\}.\end{aligned}$$

Moreover, by (A1) and (A2),

A:e:fu (13)
$$\begin{aligned}(f_\varepsilon(t), u_\lambda(t))_H &= (f_\varepsilon(t), J^t_\lambda u_\lambda(t))_H + \lambda(f_\varepsilon(t), \partial_H\varphi^t_{H,\lambda}(u_\lambda(t)))_H \\ &\leq C\{|f_\varepsilon(t)|_{V^*}^{p'} + \lambda|f_\varepsilon(t)|_V^p + r_0\} + \frac{1}{4}\varphi^t_{H,\lambda}(u_\lambda(t)).\end{aligned} \tag{13}$$

Thus we have

$$\begin{aligned}&\frac{\sqrt{\varepsilon}}{2}\frac{d}{dt}|u_\lambda(t)|_H^2 + \frac{d}{dt}\psi^*(\partial_H\psi_\varepsilon(u_\lambda(t))) + \frac{\varepsilon}{2}\frac{d}{dt}|\partial_H\psi_\varepsilon(u_\lambda(t))|_H^2 + \frac{1}{4}\varphi^t_{H,\lambda}(u_\lambda(t)) \\ &\leq \varphi^t(w(t)) + C\left\{|w(t)|_V^p + r_0\right\} + C\{|f_\varepsilon(t)|_{V^*}^{p'} + \lambda|f_\varepsilon(t)|_V^p + r_0\}.\end{aligned}$$

Integrating this over $(0,t)$, we have

$$\begin{aligned}&\frac{\sqrt{\varepsilon}}{2}|u_\lambda(t)|_H^2 + \psi^*(\partial_H\psi_\varepsilon(u_\lambda(t))) + \frac{\varepsilon}{2}|\partial_H\psi_\varepsilon(u_\lambda(t))|_H^2 + \frac{1}{4}\int_0^t \varphi^\tau_{H,\lambda}(u_\lambda(\tau))d\tau \\ &\leq \frac{\sqrt{\varepsilon}}{2}|u_{0,\varepsilon}|_H^2 + \psi^*(\partial_H\psi_\varepsilon(u_{0,\varepsilon})) + \frac{\varepsilon}{2}|\partial_H\psi_\varepsilon(u_{0,\varepsilon})|_H^2 \\ &\quad + C\int_0^T \left\{\varphi^\tau(w(\tau)) + |w(\tau)|_V^p + |f_\varepsilon(\tau)|_{V^*}^{p'} + \lambda|f_\varepsilon(\tau)|_V^p + r_0\right\}d\tau.\end{aligned}$$

Here we notice that

$$\psi^*(\partial_H \psi_\varepsilon(u_\lambda(t))) \ \geq \ -C\left\{|\partial_H \psi_\varepsilon(u_\lambda(t))|_H + 1\right\}$$

(see Proposition 2.1 of [3, Chap. II]). Thus

$$\begin{aligned} \text{(14)}\quad & \sup_{\tau\in[0,T]} \left\{\sqrt{\varepsilon}|u_\lambda(\tau)|_H^2 + |\psi^*(\partial_H \psi_\varepsilon(u_\lambda(\tau)))|\right\} + \int_0^T \varphi^\tau(u_\lambda(\tau))d\tau \\ & \leq \ C_{T,r_0}\left(\sup_{\tau\in[0,T]} |\partial_H \psi_\varepsilon(u_\lambda(\tau))|_H + \sqrt{\varepsilon}|u_{0,\varepsilon}|_H^2 + \psi^*(\partial_H\psi_\varepsilon(u_{0,\varepsilon})) \right. \\ & \quad + \varepsilon|\partial_H\psi_\varepsilon(u_{0,\varepsilon})|_H^2 + \int_0^T \varphi^\tau(w(\tau))d\tau + \int_0^T |w(\tau)|_V^p d\tau \\ & \quad \left. + \int_0^T |f_\varepsilon(\tau)|_{V^*}^{p'} d\tau + \lambda \int_0^T |f_\varepsilon(\tau)|_V^p d\tau + 1\right), \end{aligned}$$

where C_{T,r_0} denotes a constant depending on T and r_0 but not on ε and λ.

Furthermore, multiply $(\mathrm{CP})_{\varepsilon,\lambda}$ by $du_\lambda(t)/dt$. Then we have

$$\sqrt{\varepsilon}\left|\frac{du_\lambda}{dt}(t)\right|_H^2 + \left(\frac{d}{dt}\partial_H\psi_\varepsilon(u_\lambda(t)), \frac{du_\lambda}{dt}(t)\right)_H + \left(\partial_H \varphi_{H,\lambda}^t(u_\lambda(t)), \frac{du_\lambda}{dt}(t)\right)_H \ = \ \left(f_\varepsilon(t), \frac{du_\lambda}{dt}(t)\right)_H.$$

We here note the following facts: since $\partial_H\psi_\varepsilon$ is monotone in H,

$$\left(\frac{d}{dt}\partial_H\psi_\varepsilon(u_\lambda(t)), \frac{du_\lambda}{dt}(t)\right)_H \ \geq \ 0,$$

and moreover, by Lemma 2.12 of [1], the t-smoothness condition $(\mathrm{A}\varphi^t)$ implies

$$\begin{aligned} & \left|\left(\partial_H\varphi_{H,\lambda}^t(u_\lambda(t)), \frac{du_\lambda}{dt}(t)\right)_H - \frac{d}{dt}\varphi_{H,\lambda}^t(u_\lambda(t))\right| \\ & \leq \ |\dot{a}(t)||\partial_H\varphi_{H,\lambda}^t(u_\lambda(t))|_{V^*}\left\{\varphi_{H,\lambda}^t(u_\lambda(t)) + r_0\right\}^{1/p} + |\dot{b}(t)|\left\{\varphi_{H,\lambda}^t(u_\lambda(t)) + r_0\right\}, \end{aligned}$$

which together with (A2) yields

$$\begin{aligned} & \left|\left(\partial_H\varphi_{H,\lambda}^t(u_\lambda(t)), \frac{du_\lambda}{dt}(t)\right)_H - \frac{d}{dt}\varphi_{H,\lambda}^t(u_\lambda(t))\right| \\ & \leq \ \left\{C_2^{1/p'}|\dot{a}(t)| + |\dot{b}(t)|\right\}\left\{\varphi_{H,\lambda}^t(u_\lambda(t)) + r_0\right\}. \end{aligned}$$

Thus we can deduce that

$$\begin{aligned} & \sqrt{\varepsilon}\left|\frac{du_\lambda}{dt}(t)\right|_H^2 + \frac{d}{dt}\varphi_{H,\lambda}^t(u_\lambda(t)) \\ & \leq \ \left\{C_2^{1/p'}|\dot{a}(t)| + |\dot{b}(t)|\right\}\left\{\varphi_{H,\lambda}^t(u_\lambda(t)) + r_0\right\} + \left(f_\varepsilon(t), \frac{du_\lambda}{dt}(t)\right)_H. \end{aligned}$$

Moreover, multiplying both sides by t, we get

$$\begin{aligned}
&\sqrt{\varepsilon} t \left|\frac{du_\lambda}{dt}(t)\right|_H^2 + \frac{d}{dt}\left\{t\varphi_{H,\lambda}^t(u_\lambda(t))\right\} - \varphi_{H,\lambda}^t(u_\lambda(t)) \\
&\leq \left\{C_2^{1/p'}|\dot{a}(t)| + |\dot{b}(t)|\right\}\left\{t\varphi_{H,\lambda}^t(u_\lambda(t)) + tr_0\right\} \\
&\quad + \frac{d}{dt}\left\{t(f_\varepsilon(t), u_\lambda(t))_H\right\} - (f_\varepsilon(t), u_\lambda(t))_H - \left(t\frac{df_\varepsilon}{dt}(t), u_\lambda(t)\right)_H .
\end{aligned}$$

Furthermore, integrate this over $(0,t)$. It then follows that

$$\begin{aligned}
&\sqrt{\varepsilon}\int_0^t \tau\left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + t\varphi_{H,\lambda}^t(u_\lambda(t)) \\
&\leq \int_0^t \varphi_{H,\lambda}^\tau(u_\lambda(\tau))d\tau + \int_0^t \left\{C_2^{1/p'}|\dot{a}(\tau)| + |\dot{b}(\tau)|\right\}\left\{\tau\varphi_{H,\lambda}^\tau(u_\lambda(\tau)) + \tau r_0\right\} d\tau \\
&\quad + t(f_\varepsilon(t), u_\lambda(t))_H - \int_0^t (f_\varepsilon(\tau), u_\lambda(\tau))_H d\tau - \int_0^t \left(\tau\frac{df_\varepsilon}{d\tau}(\tau), u_\lambda(\tau)\right)_H d\tau.
\end{aligned}$$

Here, as in (13), we get, by (A1) and (A2),

$$\begin{aligned}
&t(f_\varepsilon(t), u_\lambda(t))_H - \int_0^t (f_\varepsilon(\tau), u_\lambda(\tau))_H d\tau - \int_0^t \left(\tau\frac{df_\varepsilon}{d\tau}(\tau), u_\lambda(\tau)\right)_H d\tau \\
&= t(f_\varepsilon(t), J_\lambda^t u_\lambda(t))_H + \lambda t(f_\varepsilon(t), \partial_H\varphi_{H,\lambda}^t(u_\lambda(t)))_H \\
&\quad - \int_0^t (f_\varepsilon(\tau), J_\lambda^\tau u_\lambda(\tau))_H d\tau - \lambda\int_0^t (f_\varepsilon(\tau), \partial_H\varphi_{H,\lambda}^\tau(u_\lambda(\tau)))_H d\tau \\
&\quad - \int_0^t \left(\tau\frac{df_\varepsilon}{d\tau}(\tau), J_\lambda^\tau u_\lambda(\tau)\right)_H d\tau - \lambda\int_0^t \left(\tau\frac{df_\varepsilon}{d\tau}(\tau), \partial_H\varphi_{H,\lambda}^\tau(u_\lambda(\tau))\right)_H d\tau \\
&\leq C\left\{t|f_\varepsilon(t)|_{V^*}^{p'} + \lambda t|f_\varepsilon(t)|_V^p + \int_0^t |f_\varepsilon(\tau)|_{V^*}^{p'} d\tau + \lambda\int_0^t |f_\varepsilon(\tau)|_V^p d\tau + \int_0^t \tau\left|\frac{df_\varepsilon}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau\right. \\
&\quad \left. + \lambda\int_0^t \tau\left|\frac{df_\varepsilon}{d\tau}(\tau)\right|_V^p d\tau + (T+T^2)r_0\right\} + \frac{1}{2}t\varphi_{H,\lambda}^t(u_\lambda(t)) + \int_0^t (1+\tau)\varphi_{H,\lambda}^\tau(u_\lambda(\tau))d\tau.
\end{aligned}$$

Now, applying Gronwall's inequality, we can derive from (14) that

$$\begin{aligned}
&\sqrt{\varepsilon}\int_0^T \tau\left|\frac{du_\lambda}{d\tau}(\tau)\right|_H^2 d\tau + \sup_{\tau\in[0,T]} \tau\varphi^\tau(u_\lambda(\tau)) \\
&\leq C_{T,r_0}\left(\sqrt{\varepsilon}|u_{0,\varepsilon}|_H^2 + \psi^*(\partial_H\psi_\varepsilon(u_{0,\varepsilon})) + |\partial_H\psi_\varepsilon(u_{0,\varepsilon})|_H^2\right. \\
&\quad + \int_0^T \tau\left\{C_2^{1/p'}|\dot{a}(\tau)| + |\dot{b}(\tau)|\right\} d\tau + \int_0^T |f_\varepsilon(\tau)|_H^2 d\tau + \int_0^T |f_\varepsilon(\tau)|_{V^*}^{p'} d\tau \\
&\quad + \lambda\int_0^T |f_\varepsilon(\tau)|_V^p d\tau + \int_0^T \varphi^\tau(w(\tau))d\tau + \int_0^T |w(\tau)|_V^p d\tau \\
&\quad + \sup_{\tau\in[0,T]} \tau|f_\varepsilon(\tau)|_{V^*}^{p'} + \lambda\sup_{\tau\in[0,T]} \tau|f_\varepsilon(\tau)|_V^p + \int_0^T \tau\left|\frac{df_\varepsilon}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau
\end{aligned} \tag{15}$$

$$+\lambda \int_0^T \tau \left| \frac{df_\varepsilon}{d\tau}(\tau) \right|_V^p d\tau + 1 \Bigg)$$

$$\times \exp\left(\int_0^T \left\{ C_2^{1/p'} |\dot{a}(\tau)| + |\dot{b}(\tau)| + 1 \right\} d\tau \right).$$

Now it follows from (14), (A1) and (A2) that

$$\int_0^T |J_\lambda^t u_\lambda(t)|_V^p dt \leq C, \tag{16}$$

$$\int_0^T \left| \partial_H \varphi_{H,\lambda}^t (u_\lambda(t)) \right|_{V^*}^{p'} dt \leq C. \tag{17}$$

Moreover, recalling the equation in $(\mathrm{CP})_{\varepsilon,\lambda}$, we have

$$\int_0^T \left| \frac{d}{dt} \left\{ \sqrt{\varepsilon} u_\lambda(t) + \partial_H \psi_\varepsilon(u_\lambda(t)) \right\} \right|_{V^*}^{p'} dt \leq C. \tag{18}$$

From these a priori estimates, we can take a sequence λ_n such that the following convergences hold true as $\lambda_n \to +0$:

$$\partial_H \psi_\varepsilon(u_{\lambda_n}(\cdot)) \to v \qquad \text{weakly in } L^q(0,T;H), \tag{19}$$

$$u_{\lambda_n} \to u \qquad \text{weakly in } L^q(0,T;H), \tag{20}$$

$$\left[t \mapsto J_{\lambda_n}^t u_{\lambda_n}(t) \right] \to u \qquad \text{weakly in } L^p(0,T;V), \tag{21}$$

$$\left[t \mapsto \partial_H \varphi_{H,\lambda_n}^t (u_{\lambda_n}(t)) \right] \to g \qquad \text{weakly in } L^{p'}(0,T;V^*), \tag{22}$$

$$\sqrt{\varepsilon} u_{\lambda_n} + \partial_H \psi_\varepsilon(u_{\lambda_n}(\cdot)) \to \sqrt{\varepsilon} u + v \qquad \text{weakly in } W^{1,p'}(0,T;V^*) \tag{23}$$

for enough large number $q > 1$. Here we used the fact that (17) implies that the function $t \mapsto (u_\lambda(t) - J_\lambda^t u_\lambda(t))$ converges to 0 strongly in $L^{p'}(0,T;V^*)$ as $\lambda \to +0$.

Since (A4) ensures that H is compactly embedded in V^*, it follows that $\{\sqrt{\varepsilon} u_\lambda(t) + \partial_H \psi_\varepsilon(u_\lambda(t))\}_{\lambda \in (0,1]}$ forms a precompact subset in V^* for every $t \in [0,T]$. Further, by (18), the function $t \mapsto \sqrt{\varepsilon} u_\lambda(t) + \partial_H \psi_\varepsilon(u_\lambda(t))$ is equi-continuous in $C([0,T];V^*)$ for all $\lambda \in (0,1]$. Therefore, by Ascoli's compactness lemma,

$$\sqrt{\varepsilon} u_{\lambda_n} + \partial_H \psi_\varepsilon(u_{\lambda_n}(\cdot)) \to \sqrt{\varepsilon} u + v \qquad \text{strongly in } C([0,T];V^*). \tag{24}$$

Moreover, by (19), we can obtain

$$\|v\|_{L^q(0,T;H)} \leq \liminf_{\lambda_n \to +0} \|\partial_H \psi_\varepsilon(u_{\lambda_n}(\cdot))\|_{L^q(0,T;H)} \leq \sup_{\lambda \in (0,1]} \sup_{t \in [0,T]} |\partial_H \psi_\varepsilon(u_\lambda(t))|_H T^{1/q},$$

which together with (12) implies $\|v\|_{L^q(0,T;H)} \leq C$, where C denotes a constant independent of q. From the arbitrariness of q, we can deduce that $v \in L^\infty(0,T;H)$. Just as in the same way, we can also derive $u \in L^\infty(0,T;H)$.

Moreover, since $\sqrt{\varepsilon} u + \partial_H \psi_\varepsilon(u(\cdot)) \in L^\infty(0,T;H) \cap W^{1,p'}(0,T;V^*) \subset C_w([0,T];H)$, it follows that

$$\sqrt{\varepsilon} u(t) + \partial_H \psi^\varepsilon(u(t)) \to \sqrt{\varepsilon} u_{0,\varepsilon} + \partial_H \psi^\varepsilon(u_{0,\varepsilon}) \qquad \text{strongly in } V^* \text{ and weakly in } H$$

as $t \to +0$.

On the other hand, let $\delta > 0$ be fixed. Then, it follows from (15) that

$$\sqrt{\varepsilon}\int_\delta^T \left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt + \sup_{t\in[\delta,T]}\left\{|J_\lambda^t u_\lambda(t)|_V^p + \varphi_\lambda^t(u_\lambda(t))\right\} \ \leq \ \frac{C}{\delta}, \tag{25}$$

which together with (A2) yields

$$\sup_{t\in[\delta,T]} |\partial_H \varphi_{H,\lambda}^t(u_\lambda(t))|_{V^*}^{p'} \ \leq \ \frac{C}{\delta}. \tag{26}$$

Further, since $\partial_H \psi_\varepsilon$ is Lipschitz continuous in H with the Lipschitz constant $2/\varepsilon$, we can also verify that

$$\int_\delta^T \left|\frac{d}{dt}\partial_H \psi_\varepsilon(u_\lambda(t))\right|_H^2 dt \ \leq \ \frac{C}{\delta \varepsilon^{5/2}},$$

which together with (CP) gives

$$\int_\delta^T \left|\partial_H \varphi_{H,\lambda}^t(u_\lambda(t))\right|_H^2 dt \ \leq \ C\left(\int_0^T |f_\varepsilon(t)|_H^2 dt + \frac{1}{\delta} + \frac{1}{\delta\varepsilon^{5/2}}\right).$$

Moreover, by Lemma 2.9 of [1], the function $t \mapsto J_\lambda^t u_\lambda(t)$ becomes equicontinuous in $C([\delta,T];H)$ for all λ. Thus, by (A4), we can take a subsequence λ_n^δ of λ_n such that

$$\begin{aligned}
\left[t \mapsto J_{\lambda_n^\delta}^t u_{\lambda_n^\delta}(t)\right] &\to u \qquad \text{strongly in } C([\delta,T];H),\\
\partial_H \psi_\varepsilon(u_{\lambda_n^\delta}(\cdot)) &\to v \qquad \text{weakly in } W^{1,2}(\delta,T;H),\\
\left[t \mapsto \partial_H \varphi_{H,\lambda_n^\delta}^t(u_{\lambda_n^\delta}(t))\right] &\to g \qquad \text{weakly in } L^2(\delta,T;H).
\end{aligned}$$

Moreover, by (25) and (26), we can also verify that $u \in L^\infty(\delta,T;V)$ and $g \in L^\infty(\delta,T;V^*)$. The demiclosedness of maximal monotone operators and Proposition 2.1 of [11] ensure that $g(t) \in \partial_H \varphi_H^t(u(t))$ for a.a. $t \in (\delta,T)$.

Furthermore, noting that

$$\begin{aligned}
\sup_{t\in[\delta,T]} |J_\lambda^t u_\lambda(t) - u_\lambda(t)|_H^2 &\leq 2\lambda \sup_{t\in[\delta,T]} \varphi_{H,\lambda}^t(u_\lambda(t))\\
&\leq 2\lambda C/\delta \to 0 \quad \text{as } \lambda \to +0,
\end{aligned}$$

we deduce that

$$u_{\lambda_n^\delta} \to u \qquad \text{strongly in } C([\delta,T];H). \tag{27}$$

Then, since $\partial_H \psi_\varepsilon$ is Lipschitz continuous, it follows that $\partial_H \psi_\varepsilon(u_{\lambda_n^\delta}(\cdot)) \to v$ strongly in $C([\delta,T];H)$ as $\lambda_n^\delta \to +0$ and $v(t) = \partial_H \psi_\varepsilon(u(t))$ for all $t \in [\delta,T]$.

From the arbitrariness of δ, we conclude that

$$\begin{aligned}
&u \in L^\infty_{loc}((0,T];V) \cap W^{1,2}_{loc}((0,T];H) \subset C_w((0,T];V),\\
&v \in W^{1,2}_{loc}((0,T];H), \quad g \in L^\infty_{loc}((0,T];V^*) \cap L^2_{loc}((0,T];H),\\
&v(t) = \partial_H \psi_\varepsilon(u(t)) \quad \forall t \in (0,T], \quad g(t) \in \partial_H \varphi_H^t(u(t)) \ \text{ for a.a. } t \in (0,T).
\end{aligned}$$

Furthermore, we can derive (9), (10) and (11) by passing to the limit in (12), (14) and (15) as $\lambda_n \to 0$.

Finally, we prepare an inequality to derive the convergence of solutions for $(\mathrm{CP})_\varepsilon$ as $\varepsilon \to +0$. To do so, we define the functional $\phi^\varepsilon : H \to [0, +\infty)$ by

$$\phi^\varepsilon(u) \quad := \quad \frac{\sqrt{\varepsilon}}{2}|u|_H^2 + \psi_\varepsilon(u) \quad \forall u \in H.$$

Then, it can be easily seen that $\partial_H \phi^\varepsilon = \sqrt{\varepsilon} I + \partial_H \psi_\varepsilon$. We see that

$$\begin{aligned}
&\int_0^T \left(\partial_H \varphi_{H,\lambda}^t(u_\lambda(t)), J_\lambda^t u_\lambda(t)\right)_H dt \\
&\quad = \int_0^T \left(\partial_H \varphi_{H,\lambda}^t(u_\lambda(t)), u_\lambda(t)\right)_H dt - \lambda \int_0^T \left|\partial_H \varphi_{H,\lambda}^t(u_\lambda(t))\right|_H^2 dt \\
&\quad \leq \int_0^T (f_\varepsilon(t), u_\lambda(t))_H \, dt - \int_0^T \left(\frac{d}{dt}\partial_H \phi^\varepsilon(u_\lambda(t)), u_\lambda(t)\right)_H dt \\
&\quad = \int_0^T (f_\varepsilon(t), u_\lambda(t))_H \, dt - (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_\lambda(T))) + (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_{0,\varepsilon})).
\end{aligned}$$

Then, noting that

$$\liminf_{\lambda_n^\delta \to 0} (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_{\lambda_n^\delta}(T))) \quad \geq \quad (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u(T))),$$

we can deduce that

$$\begin{aligned}
&\limsup_{\lambda_n^\delta \to 0} \int_0^T \left\langle \partial_H \varphi_{H,\lambda_n^\delta}^t(u_{\lambda_n^\delta}(t)), J_{\lambda_n^\delta}^t u_{\lambda_n^\delta}(t)\right\rangle_V dt \\
&\quad \leq \int_0^T (f_\varepsilon(t), u(t))_H \, dt - (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u(T))) + (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_{0,\varepsilon})).
\end{aligned}$$

Here we claim that

$$(\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u(T))) - (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_{0,\varepsilon})) \quad \geq \quad \int_0^T \left\langle \frac{d}{dt}\partial_H \phi^\varepsilon(u(t)), u(t) \right\rangle dt.$$

Indeed, for an arbitrary $\tau \in (0, T]$, since $\partial_H \phi^\varepsilon(u(\cdot)) \in W_{loc}^{1,2}((0,T]; H)$, $u \in L^2(0,T;H)$ and $u(t) \in \partial_H (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u(t)))$ for a.a. $t \in (\tau, T)$, the function $t \mapsto (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u(t)))$ becomes absolutely continuous on $[\tau, T]$ (see, e.g., [6]). Therefore

$$(\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u(T))) - (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u(\tau))) \quad = \quad \int_\tau^T \left(\frac{d}{dt}\partial_H \phi^\varepsilon(u(t)), u(t)\right)_H dt \tag{28}$$

for every $\tau > 0$. Noting that

$$\begin{aligned}
&\partial_H \phi^\varepsilon(u(\tau)) \to \partial_H \phi^\varepsilon(u_{0,\varepsilon}) \quad \text{weakly in } H \text{ as } \tau \to +0, \\
&\partial_H \phi^\varepsilon(u(\cdot)) \in W^{1,p'}(0,T;V^*), \quad u \in L^p(0,T;V),
\end{aligned}$$

we can derive that

$$
\begin{aligned}
(29)\qquad &(\phi^\varepsilon)^*(\partial_H\phi^\varepsilon(u(T))) - (\phi^\varepsilon)^*(\partial_H\phi^\varepsilon(u_{0,\varepsilon}))\\
&\geq\ (\phi^\varepsilon)^*(\partial_H\phi^\varepsilon(u(T))) - \liminf_{\tau\to+0}(\phi^\varepsilon)^*(\partial_H\phi^\varepsilon(u(\tau)))\\
&=\ \lim_{\tau\to+0}\int_\tau^T \left\langle \frac{d}{dt}\partial_H\phi^\varepsilon(u(t)), u(t)\right\rangle dt\\
&=\ \int_0^T \left\langle \frac{d}{dt}\partial_H\phi^\varepsilon(u(t)), u(t)\right\rangle dt,
\end{aligned}
$$

which implies our claim, and therefore

$$
\begin{aligned}
&\limsup_{\lambda_n^\delta\to 0}\int_0^T \left\langle \partial_H\varphi^t_{H,\lambda_n^\delta}(u_{\lambda_n^\delta}(t)), J^t_{\lambda_n^\delta}u_{\lambda_n^\delta}(t)\right\rangle_V dt\\
&\leq\ \int_0^T \left\langle f_\varepsilon(t) - \frac{d}{dt}\partial_H\phi^\varepsilon(u(t)), u(t)\right\rangle_V dt = \int_0^T \langle g(t), u(t)\rangle_V\, dt.
\end{aligned}
$$

By using the monotonicity of $\partial_V\varphi^t$, we have

$$
\limsup_{\lambda_n^\delta\to 0}\int_0^T \left\langle \partial_H\varphi^t_{H,\lambda_n^\delta}(u_{\lambda_n^\delta}(t)), J^t_{\lambda_n^\delta}u_{\lambda_n^\delta}(t)\right\rangle_V dt = \int_0^T \langle g(t), u(t)\rangle_V\, dt.
$$

Thus we obtain

$$
\begin{aligned}
(30)\qquad &\int_0^T \langle g(t), u(t)\rangle_V\, dt\\
&\leq\ \int_0^T (f_\varepsilon(t), u(t))_H\, dt - (\phi^\varepsilon)^*(\partial_H\phi^\varepsilon(u(T))) + (\phi^\varepsilon)^*(\partial_H\phi^\varepsilon(u_{0,\varepsilon})).
\end{aligned}
$$

6 Proof of Theorem 3.2

Now, we shall derive from (9), (10) and (11) the convergences of solutions $(u_\varepsilon, \partial_H\psi_\varepsilon(u_\varepsilon(\cdot)))$ of $(\mathrm{CP})_\varepsilon$ as $\varepsilon\to+0$. By taking a sequence ε_n, by (A1) and (A2), we can easily obtain

$$
\begin{aligned}
(31)\qquad & \partial_H\psi_{\varepsilon_n}(u_{\varepsilon_n}(\cdot)) \to v && \text{weakly star in } L^\infty(0,T;H),\\
(32)\qquad & u_{\varepsilon_n} \to u && \text{weakly in } L^p(0,T;V),\\
(33)\qquad & g_{\varepsilon_n} \to g && \text{weakly in } L^{p'}(0,T;V^*),\\
(34)\qquad & \sqrt{\varepsilon_n}u_{\varepsilon_n} + \partial_H\psi_{\varepsilon_n}(u_{\varepsilon_n}(\cdot)) \to v && \text{weakly in } W^{1,p'}(0,T;V^*).
\end{aligned}
$$

On the other hand, it also follows from (10) that

$$
(35)\qquad \sqrt{\varepsilon_n}u_{\varepsilon_n}(t)\to 0 \qquad \text{strongly in } H \text{ uniformly for all } t\in[0,T],
$$

and furthermore, by repeating the same argument as in the proof of (24), we can deduce that

$$
(36)\qquad \sqrt{\varepsilon_n}u_{\varepsilon_n} + \partial_H\psi_{\varepsilon_n}(u_{\varepsilon_n}(\cdot)) \to v \qquad \text{strongly in } C([0,T];V^*),
$$

which implies

$$\partial_H \psi_{\varepsilon_n}(u_{\varepsilon_n}(t)) \to v(t) \qquad \text{strongly in } V^* \text{ uniformly for all } t \in [0,T]. \tag{37}$$

Let $t \in (0,T]$ be fixed. By (9), we can take a subsequence $\varepsilon_{t,n}$ of ε_n such that

$$\partial_H \psi_{\varepsilon_{t,n}}(u_{\varepsilon_{t,n}}(t)) \to v(t) \qquad \text{weakly in } H.$$

On the other hand, observing that

$$\begin{aligned}
&|v(t) - v_0|_{V^*} \\
&\quad \le \ |v(t) - \{\sqrt{\varepsilon_n} u_{\varepsilon_n}(t) + \partial_H \psi_{\varepsilon_n}(u_{\varepsilon_n}(t))\}|_{V^*} \\
&\qquad + |\sqrt{\varepsilon_n} u_{\varepsilon_n}(t) + \partial_H \psi_{\varepsilon_n}(u_{\varepsilon_n}(t)) - \{\sqrt{\varepsilon_n} u_{0,\varepsilon_n} + \partial_H \psi_{\varepsilon_n}(u_{0,\varepsilon_n})\}|_{V^*} \\
&\qquad\quad + |\sqrt{\varepsilon_n} u_{0,\varepsilon_n} + \partial_H \psi_{\varepsilon_n}(u_{0,\varepsilon_n}) - v_0|_{V^*} \\
&\quad \le \ \sup_{\tau \in [0,T]} |v(\tau) - \{\sqrt{\varepsilon_n} u_{\varepsilon_n}(\tau) + \partial_H \psi_{\varepsilon_n}(u_{\varepsilon_n}(\tau))\}|_{V^*} \\
&\qquad + C t^{1/p} + |\sqrt{\varepsilon_n} u_{0,\varepsilon_n} + \partial_H \psi_{\varepsilon_n}(u_{0,\varepsilon_n}) - v_0|_{V^*} \to C t^{1/p} \ \text{ as } \varepsilon_n \to +0,
\end{aligned}$$

we can conclude that $v(t) \to v_0$ strongly in V^* and weakly in H as $t \to +0$.

Furthermore, by (A1), (A3) and (10), we find that

$$\int_0^T |J_\varepsilon u_\varepsilon(t)|_V^p \, dt \ \le \ C. \tag{38}$$

Thus

$$J_{\varepsilon_n} u_{\varepsilon_n} \to u \qquad \text{weakly in } L^p(0,T;V). \tag{39}$$

Therefore, it follows that

$$\begin{aligned}
&\int_0^T (\partial_H \psi_{\varepsilon_n}(u_{\varepsilon_n}(t)), J_{\varepsilon_n} u_{\varepsilon_n}(t))_H \, dt \\
&\quad \le \ \int_0^T \langle \sqrt{\varepsilon_n} u_{\varepsilon_n}(t) + \partial_H \psi_{\varepsilon_n}(u_{\varepsilon_n}(t)), J_{\varepsilon_n} u_{\varepsilon_n}(t) \rangle_V \, dt \\
&\qquad + \sqrt{\varepsilon_n} \sup_{t \in [0,T]} |u_{\varepsilon_n}(t)|_H \int_0^T |J_{\varepsilon_n} u_{\varepsilon_n}(t)|_H dt \\
&\quad \to \ \int_0^T \langle v(t), u(t) \rangle_V \, dt = \int_0^T (v(t), u(t))_H \, dt,
\end{aligned} \tag{40}$$

which ensures that $v(t) \in \partial_H \psi(u(t))$ for a.a. $t \in (0,T)$.

Now, we shall prove that $g(t) \in \partial_V \varphi^t(u(t))$ for a.a. $t \in (0,T)$. By (30), we see that

$$\begin{aligned}
&\int_0^T \langle g_\varepsilon(t), u_\varepsilon(t) \rangle_V \, dt \\
&\quad \le \ \int_0^T \langle f_\varepsilon(t), u_\varepsilon(t) \rangle_V \, dt - (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_\varepsilon(T))) + (\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_{0,\varepsilon})).
\end{aligned}$$

We here notice that

$$
\begin{aligned}
&(\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_\varepsilon(T))) \\
&= (\partial_H \phi^\varepsilon(u_\varepsilon(T)), u_\varepsilon(T))_H - \phi^\varepsilon(u_\varepsilon(T)) \\
&= \frac{\sqrt{\varepsilon}}{2}|u_\varepsilon(T)|_H^2 + (\partial_H \psi_\varepsilon(u_\varepsilon(T)), u_\varepsilon(T))_H - \psi_\varepsilon(u_\varepsilon(T)) \\
&= \frac{\sqrt{\varepsilon}}{2}|u_\varepsilon(T)|_H^2 + (\partial_H \psi_\varepsilon(u_\varepsilon(T)), J_\varepsilon u_\varepsilon(T))_H + \frac{\varepsilon}{2}\left|\partial_H \psi_\varepsilon(u_\varepsilon(T))\right|_H^2 - \psi(J_\varepsilon u_\varepsilon(T)) \\
&\geq \psi^*(\partial_H \psi_\varepsilon(u_\varepsilon(T))).
\end{aligned}
$$

Further, by Lemma L:u0e 4.1, we see

$$
\begin{aligned}
&(\phi^\varepsilon)^*(\partial_H \phi^\varepsilon(u_{0,\varepsilon_n})) \\
&= \frac{\sqrt{\varepsilon}}{2}|u_{0,\varepsilon_n}|_H^2 + (\partial_H \psi_\varepsilon(u_{0,\varepsilon_n}), J_\varepsilon u_{0,\varepsilon_n})_H + \frac{\varepsilon}{2}\left|\partial_H \psi_\varepsilon(u_{0,\varepsilon_n})\right|_H^2 - \psi(J_\varepsilon u_{0,\varepsilon_n}) \\
&\to (v_0, u_0)_H - \psi(u_0) = \psi^*(v_0).
\end{aligned}
$$

Therefore we can derive

$$
\begin{aligned}
&\limsup_{\varepsilon_{T,n}\to 0} \int_0^T \left\langle g_{\varepsilon_{T,n}}(t), u_{\varepsilon_{T,n}}(t)\right\rangle_V dt \\
&\leq \int_0^T \langle f(t), u(t)\rangle_V \, dt - \psi^*(v(T)) + \psi^*(v_0).
\end{aligned}
$$

Now, from the definition of subdifferentials, for an arbitrary $\tau > 0$, we notice that

$$
\begin{aligned}
&\int_\tau^{T-h} |\psi^*(v(t+h)) - \psi^*(v(t))|^{p'} \, dt \\
&\leq \sup_{t\in[\tau,T]} |u(t)|_V^{p'} \int_\tau^{T-h} |v(t+h) - v(t)|_{V^*}^{p'} dt \\
&\leq \sup_{t\in[\tau,T]} |u(t)|_V^{p'} C h^{p'},
\end{aligned}
$$

and therefore $\psi^*(v(\cdot))$ is absolutely continuous on $[\tau, T]$. Moreover, we prepare the following proposition (see the end of this section for their proofs).

:LFt-chain **Proposition 6.1** *Let $\phi \in \Phi(H)$ and let u be a V^*-valued absolutely continuous function on $[0, T]$ such that $u(t) \in D(\partial_H \phi)$ for a.a. $t \in (0, T)$ and $\phi(u(\cdot))$ is differentiable for a.a. $t \in (0, T)$. Then, for any $g \in L^1(0, T; V)$ satisfying $g(t) \in \partial_H \phi(u(t))$ for a.a. $t \in (0, T)$, it follows that*

LFt-chain (41)
$$
\frac{d}{dt}\phi(u(t)) = \left\langle \frac{du}{dt}(t), g(t) \right\rangle_V \quad \text{for a.a. } t \in (0, T).
$$

Hence repeating the same argument as in lc:phi*-abconti:1 (28) and lc:phi*-abconti:2 (29), we obtain

$$
\begin{aligned}
&\limsup_{\varepsilon_{T,n}\to 0} \int_0^T \left\langle g_{\varepsilon_{T,n}}(t), u_{\varepsilon_{T,n}}(t)\right\rangle_V dt \\
&\leq \int_0^T \left\langle f(t) - \frac{dv}{dt}(t), u(t) \right\rangle_V dt = \int_0^T \langle g(t), u(t)\rangle_V \, dt,
\end{aligned}
$$

which together with Proposition 2.1 of [11] ensures that $g(t) \in \partial_V \varphi^t(u(t))$ for a.a. $t \in (0,T)$. This completes our proof. ∎

We end this section with a proof of Proposition 6.1.

Proof of Proposition 6.1 We can take

$$I \;:=\; \{t \in [0,T];\ \phi(u(\cdot)) \text{ and } u \text{ are differentiable at } t, \text{ and } u(t) \in D(\partial_H \phi)\}$$

such that $|[0,T] \setminus I| = 0$. Now let $t_0 \in I$ be fixed and let h_n^+ be a sequence in $(0,+\infty)$ such that $t_0 + h_n^+ \in [0,T]$ and $u(t_0 + h_n^+) \in D(\partial_H \phi)$. Moreover, let $g(t_0) \in \partial_H \phi(u(t_0))$. Then we have:

$$\phi(u(t_0 + h_n^+)) - \phi(u(t_0)) \;\geq\; (g(t_0), u(t_0 + h_n^+) - u(t_0))_H.$$

Dividing both sides by $h_n^+ > 0$, we see that

$$\frac{\phi(u(t_0 + h_n^+)) - \phi(u(t_0))}{h_n^+} \;\geq\; \frac{\langle g(t_0), u(t_0 + h_n^+) - u(t_0)\rangle_V}{h_n^+}.$$

Further letting $h_n^+ \to +0$, we obtain

$$\frac{d}{dt}\phi(u(t_0)) \;\geq\; \left\langle g(t_0), \frac{du}{dt}(t_0) \right\rangle_V.$$

Repeating the same argument with a sequence $h_n^- \in (-\infty, 0)$, we can also deduce that

$$\frac{d}{dt}\phi(u(t_0)) \;\leq\; \left\langle g(t_0), \frac{du}{dt}(t_0) \right\rangle_V.$$

Therefore, for every $t \in I$, (41) holds true. ∎

7 Applications

In this section, we give a typical example of PDEs to which our abstract theory can be applied.

Let Ω be a bounded domain in $\mathbb{R}^N$ with compact smooth boundary $\partial\Omega$ and let α be a (possible multi-valued) maximal monotone mapping in $\mathbb{R}$. Now, we consider the equation:

$$\text{(42)} \qquad \begin{cases} \dfrac{\partial v}{\partial t}(x,t) - \operatorname{div} \mathbf{a}(x,t,\nabla u(x,t)) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ v(x,t) \in \alpha(u(x,t)), & (x,t) \in \Omega \times (0,T), \end{cases}$$

where $f : \Omega \times (0,T) \to \mathbb{R}$ is a given function and $\mathbf{a}(\cdot,\cdot,\cdot)$ is a function from $\Omega \times [0,T] \times \mathbb{R}^N$ into $\mathbb{R}^N$ satisfying

(H0) There exists a function $A : \Omega \times [0,T] \times \mathbb{R}^N \to \mathbb{R}$ such that $A(x,t,\cdot)$ is convex and Fréchet differentiable in $\mathbb{R}^N$ and, $\mathbf{a}(x,t,\cdot)$ coincides with the Fréchet derivative $\partial A(x,t,\cdot)$ of $A(x,t,\cdot)$ for a.e. $x \in \Omega$ and all $t \in [0,T]$;

(H1) The function $\mathbf{a}(\cdot,t,\mathbf{p})$ is measurable in Ω for all $(t,\mathbf{p}) \in [0,T] \times \mathbb{R}^N$, and $\mathbf{a}(x,t,\cdot)$ is continuous in $\mathbb{R}^N$ for a.e. $x \in \Omega$ and all $t \in [0,T]$.

Moreover, we impose the boundary condition (43) and the initial condition (44) on (42).

A:bc
$$u(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T), \tag{43}$$

A:ic
$$v(x,0)=v_0(x), \quad x\in\Omega. \tag{44}$$

Here we are concerned with weak solutions defined below.

DEFINITION 7.1 *A pair of functions $(u,v):\Omega\times(0,T)\to\mathbf{R}^2$ is said to be a weak solution of the initial-boundary value problem $\{(42),(43),(44)\}$ on $[0,T]$ if the following (45)-(47) are all satisfied.*

A:def-1
$$v(x,t)\in\alpha(u(x,t)) \quad \textit{for a.e. } (x,t)\in\Omega\times(0,T), \tag{45}$$

A:def-2
$$\left\langle \frac{dv}{dt}(\cdot,t),\phi\right\rangle_{W_0^{1,p}}+\int_\Omega \mathbf{a}(x,t,\nabla u(x,t))\cdot\nabla\phi(x)dx=\int_\Omega f(x,t)\phi(x)dx, \tag{46}$$
$$\textit{for a.a. } t\in(0,T) \textit{ and all } \phi\in W_0^{1,p}(\Omega),$$

A:def-3
$$v(\cdot,t)\to v_0 \quad \textit{strongly in } W^{-1,p'}(\Omega) \textit{ and weakly in } L^2(\Omega) \textit{ as } t\to+0. \tag{47}$$

Now, we introduce the following assumptions for some $p\in(1,+\infty)$ and $m\in L^1(\Omega)$.

(H2) There exist $b\in W^{1,1}(0,T)$ and $\delta>0$ such that
$A(x,t,\mathbf{p})-A(x,s,\mathbf{p})\le|b(t)-b(s)|\{A(x,s,\mathbf{p})+m(x)\}$ for a.e. $x\in\Omega$
and all $\mathbf{p}\in\mathbb{R}^N$ and $t,s\in[0,T]$ satisfying $|t-s|<\delta$;

(H3) There exists a constant C_4 such that
$|\mathbf{p}|^p\le C_4A(x,t,\mathbf{p})+m(x)$ for a.e. $x\in\Omega$ and all $(t,\mathbf{p})\in[0,T]\times\mathbb{R}^N$;

(H4) There exists a constant C_5 such that
$|\mathbf{a}(x,t,\mathbf{p})|^{p'}\le C_5A(x,t,\mathbf{p})+m(x)$ for a.e. $x\in\Omega$ and all $(t,\mathbf{p})\in[0,T]\times\mathbb{R}^N$;

(H5) If $|\mathbf{p}|\le|\mathbf{q}|$, then $A(x,t,\mathbf{p})\le A(x,t,\mathbf{q})$ for a.e. $x\in\Omega$ and all $t\in[0,T]$.

REMARK 7.2 Set $A(x,t,\mathbf{p})=k(x,t)|\mathbf{p}|^p$ for a given function $k(x,t)\in W^{1,1}(0,T;L^\infty(\Omega))$. Then $\mathbf{a}(x,t,\mathbf{p})=k(x,t)p|\mathbf{p}|^{p-2}\mathbf{p}$, and we can easily verify that (H0), (H1) and (H3)-(H5) are all satisfied; moreover, we can also infer (H3) from the fact that

$$\begin{aligned} A(x,t,\mathbf{p})-A(x,s,\mathbf{p}) &\le |k(x,t)-k(x,s)||\mathbf{p}|^p \\ &\le |b(t)-b(s)||\mathbf{p}|^p, \end{aligned}$$

where $b(t)=|dk(\cdot,t)/dt|_{L^\infty(\Omega)}$. In particular, if $k\equiv1/p$, then div $\mathbf{a}(x,t,\nabla u(x))$ coincides with $\Delta_pu(x)$, where Δ_p stands for the well-known p-Laplace operator.

To verify the existence of solutions for the initial-boundary value problem $\{(42),(43),(44)\}$, we set $V=W_0^{1,p}(\Omega)$ and $H=L^2(\Omega)$ with the norms

$$|u|_V:=|\nabla u|_{L^p(\Omega)} \quad \text{and} \quad |u|_H:=|u|_{L^2(\Omega)}$$

and put

A:def-phi (48)
$$\varphi^t(u) = \int_\Omega A(x, t, \nabla u(x))dx$$

A:def-psi (49)
$$\psi(u) = \begin{cases} \int_\Omega \mathcal{A}(u(x))dx & \text{if } \mathcal{A}(u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{A}$ denotes a primary function of α, i.e., $\alpha = \partial_{\mathbf{R}}\mathcal{A}$. Here we note the fact that every maximal monotone mapping in $\mathbb{R}$ becomes cyclic monotone, so there exists a function $\mathcal{A}$ such that $\partial_{\mathbf{R}}\mathcal{A} = \alpha$.

It then follows that $\varphi^t \in \Phi(V)$ and $\psi \in \Phi(H)$ (see [6]). We can then rewrite the initial-boundary value problem $\{(42),(43),(44)\}$ to the abstract Cauchy problem (CP). Indeed, $\partial_V \varphi^t(u)$ coincides with $-\mathrm{div}\ \mathbf{a}(\cdot, t, \nabla u(\cdot))$ with the boundary condition (43) in the sense of distribution, and $\partial_H \psi(u)$ coincides with $\alpha(u(\cdot))$ in $L^2(\Omega)$.

Remark 7.3 Since $\mathcal{A} \in \Phi(\mathbb{R})$, we can take $\xi_0, C_0 \in \mathbb{R}$ such that $\tilde{\mathcal{A}}(s) := \mathcal{A}(s) + \xi_0 s + C_0 \geq 0$. Hence, if ψ is unbounded below, we then replace ψ with $\tilde{\psi}$ given as in (49) with $\tilde{\mathcal{A}}$. In order to show the existence of solutions for $\{(42),(43),(44)\}$, by virtue of Remark 3.3, it suffices to prove the existence of solutions for (CP) with ψ replaced by $\tilde{\psi}$.

Now, we check (Aφ^t), (A1) - (A4) to apply the preceding abstract theory to the initial-boundary value problem. Let $t_0 \in [0, T]$ and $x_0 \in D(\varphi^{t_0})$ be fixed. Then, by (H3), we take a function $x \equiv x_0$ to get

$$\varphi^t(x(t)) - \varphi^{t_0}(x_0) \leq |\beta(t) - \beta(s)| \left\{ \int_\Omega A(x, t_0, x_0(x))dx + |m|_{L^1} \right\},$$

which implies (Aφ^t). Furthermore, by (H2) and (H3), it is obvious that (Aφ^t) and (A1) hold true. Let $[u, g] \in \partial_V \varphi^t$. Then we see, by (H4),

$$\begin{aligned} \langle g, z \rangle_V &= \int_\Omega \mathbf{a}(x, t, \nabla u(x)) \cdot \nabla z(x)dx \\ &\leq |\mathbf{a}(\cdot, t, \nabla u(\cdot))|_{L^{p'}} |\nabla z|_{L^p} \\ &\leq \left(\int_\Omega \{C_5 A(x, t, \nabla u(x, t)) + m(x)\} dx \right)^{1/p'} |z|_V, \quad \forall z \in V, \end{aligned}$$

which yields (A2). Furthermore, since $j_\lambda := (1 + \lambda\alpha)^{-1} : \mathbb{R} \to \mathbb{R}$ is non-expansive, we get $|\nabla j_\lambda u(x)| \leq |\nabla u(x)|$, so (A3) follows immediately from (H5). Moreover, V is embedded compactly in H if $2N/(N+2) < p$. Consequently, by Theorem 3.2, we have:

A:T:pde-ex Theorem 7.4 *Let* $p \in (2N/(N+2), +\infty)$ *and suppose that* (H0)-(H5) *hold. Then for all* $f \in L^{p'}(0, T; W^{-1,p'}(\Omega)) \cap L^2(0, T; L^2(\Omega))$ *and* $u_0 \in L^2(\Omega)$ *satisfying* $t(df/dt) \in L^{p'}(0, T; W^{-1,p'}(\Omega))$ *and*

$$v_0 \in \left\{ w \in L^2(\Omega); \ \text{there exists } u_0 \in L^2(\Omega) \text{ such that } w(x) \in \alpha(u_0(x)) \ \text{ for a.e. } x \in \Omega \right\},$$

the initial-boundary value problem $\{(42), (43), (44)\}$ *admits at least one weak solution* (u, v) *satisfying*

$$u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty_{loc}((0, T]; W_0^{1,p}(\Omega)),$$
$$v \in C_w([0, T]; L^2(\Omega)) \cap W^{1,p'}(0, T; W^{-1,p'}(\Omega)).$$

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