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Local existence of solutions to some degenerate parabolic equation associated with the p-Laplacian

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Abstract

The existence of local (in time) solutions of the initial-boundary value problem for the following degenerate parabolic equation: $u_t(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t)$, $(x,t) \in \Omega \times (0,T)$, where $2 \le p < q < +\infty$, Ω is a bounded domain in $\mathbf{R}^N$, $f : \Omega \times (0,T) \to \mathbf{R}$ is given and Δ_p denotes the so-called p-Laplacian defined by $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, with an initial data $u_0 \in L^r(\Omega)$ is proved under $r > N(q-p)/p$ without imposing any smallness on u_0 and f. To this end, the above problem is reduced into the Cauchy problem for an evolution equation governed by the difference of two subdifferential operators in a reflexive Banach space, and the theory of subdifferential operator and potential well method are employed to establish energy estimates. Particularly, L^r-estimates of solutions play a crucial role to construct a time-local solution and reveal the dependence of the time interval $[0, T_0]$ in which the problem admits a solution. More precisely, T_0 depends only on $|u_0|_{L^r}$ and f.

1 Introduction

This article is concerned with the existence of solutions of the following initial-boundary value problem for a degenerate parabolic equation:

$$
\text{(P)} \quad \begin{cases}
\dfrac{\partial u}{\partial t}(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\
u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\
u(x,0) = u_0(x) \in L^r(\Omega), & x \in \Omega,
\end{cases}
$$

where $2 \le p, q, r < +\infty$, Δ_p denotes the so-called p-Laplacian given by

$$
\Delta_p u(x) \quad := \quad \mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))
$$

and Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$. Particularly, we address ourselves to the case: $p < q$. It is well known that solutions of (P) possibly blow up in finite time (see, e.g., [10], [18]) if $p < q$, so, in general, one cannot expect the existence of time-global solutions for (P) without imposing any smallness on u_0 and f.

For the case where $p = 2$ and $f \equiv 0$, i.e., the semilinear heat equation, sufficient conditions for the existence of solutions for (P) have already been proposed by many authors; in particular, Weissler [20, 21] and Brézis-Cazenave [8] proved the time-local well-posedness in $L^r(\Omega)$ of (P) with $p = 2$ and $f \equiv 0$ under the following condition:

$$r > N(q-2)/2; \tag{1. 1}$$

moreover, they also dealt with the critical case: $r = N(q-2)/2 > 1$. As for the case where $r < N(q-2)/2$, the ill-posedness of (P) with $p = 2$ and $f \equiv 0$ is proved by [11], [20] and [8], so Weissler's sufficient condition is essentially optimal. Furthermore, Brézis and Cazenave [8] also investigated the dependence of the interval $[0, T_0]$ in which (P) admits a solution on initial data. More precisely, if (1. 1) holds true (resp. $r = N(q-2)/2 > 1$), then for any bounded set (resp. compact set) B in $L^r(\Omega)$, one can take $T_0 = T_0(B) > 0$ such that for every $u_0 \in B$, there exists a solution of (P) with $p = 2$ and $f \equiv 0$ on $[0, T_0]$. These results and the latest developments in this field are briefly and usefully summarized in Section 3.1 of [17].

Studies on the well-posedness of the semilinear heat equation such as [20, 21], [11] and [8] rely on the reduction of (P) with $p = 2$ and $f \equiv 0$ to the following integral equation:

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}\left(|u|^{q-2}u(s)\right) ds \tag{1. 2}$$

and decay estimates for the heat semi-group $e^{t\Delta}$ and the well-known contraction mapping principle. Moreover, energy estimates also play an important role in studies of asymptotic behaviors of solutions as well as those of the well-posedness.

On the other hand, for the case where $p \neq 2$, some of major tools described above could not be applied to the degenerate equation (P). Particularly, the approach based on the integral equation (1. 2) is no longer valid. However, energy method is still effective, so the notion of subdifferential operators, which is a generalized one of Fréchet derivative for non-smooth convex functionals and enables us to take account of the energy structure of (P), is often employed to verify the existence of solutions for (P) (see, e.g., [14], [12], [15], [19, §3.10], [3]).

For every $u_0 \in L^\infty(\Omega)$, one can prove the local (in time) existence and the uniqueness of solutions for (P) without imposing any restriction on the growth order q of the blow-up term; indeed, replacing the blow-up term $|u|^{q-2}u(x,t)$ by $g_M(u(x,t))$, where $M := |u_0|_{L^\infty} + 1$ and $g_M : \mathbf{R} \to \mathbf{R}$ is given as follows:

$$g_M(s) := \begin{cases} M^{q-1} & \text{if } s > M, \\ |s|^{q-2}s & \text{if } |s| \le M, \\ -M^{q-1} & \text{if } s < -M, \end{cases}$$

since the mapping $v \mapsto g_M(v(\cdot))$ becomes Lipschitz continuous in $L^2(\Omega)$, one can construct a unique time-global solution of (P) with $|u|^{q-2}u(x,t)$ replaced by $g_M(u(x,t))$ in virtue of the standard theory of evolution equation; furthermore, the unique solution coincides with a solution of the original problem (P) time-locally, since the function $t \mapsto |u(t)|_{L^\infty}$ is right-continuous at $t = 0$ and $M = |u_0|_{L^\infty} + 1$.

As for the case: $u_0 \in W_0^{1,p}(\Omega)$, which is (possibly) unbounded in Ω, Ishii [12] and Ôtani [15] proved the local (in time) existence of a solution u satisfying $\Delta_p u(t), |u|^{q-2}u(t) \in$

$L^2(\Omega)$ of (P) under the condition:

$$(1.\ 3) \qquad q \ < \ p^*/2+1,$$

where p^* denotes the so-called Sobolev's critical exponent, by developing their abstract theories on evolution equations governed by subdifferential operators in Hilbert spaces. The condition (1. 3) is sufficient also for the compactness of the operator $u \mapsto |u|^{q-2}u$ from $W_0^{1,p}(\Omega)$ into $L^2(\Omega)$.

In particular, if $p = 2$, then the existence of time-local solutions for (P) with $p = 2$ can be also proved in [15] under the so-called subcritical growth condition $q < 2^*$ in Sobolev's sense by virtue of the elliptic estimate for the Laplacian: $|u|_{H^2} \leq C(|\Delta u|_{L^2} + |u|_{L^2})$. On the other hand, Tsutsumi [18] proved the existence of a time-global solution u satisfying $\Delta_p u(t), |u|^{q-2}u(t) \in W^{-1,p'}(\Omega)$ of (P) with $f \equiv 0$ for enough small initial data u_0 in $W_0^{1,p}(\Omega)$ under the subcritical growth condition in Sobolev's sense:

$$(1.\ 4) \qquad q \ < \ p^*$$

for every $p \in [2, +\infty)$ by using Galerkin's method. Hence one can expect that (P) admits a time-local solution under (1. 4) also for general p. However, the theory developed in [12] and [15] could not be enough to prove so, because of the lack of the knowledge of elliptic estimates for the nonlinear p-Laplace operator Δ_p.

In [3], they developed the theory of evolution equations governed by subdifferential operators in reflexive Banach spaces and applied their theory to (P); then they succeeded to verify the existence of a time-local solution u satisfying $\Delta_p u(t), |u|^{q-2}u(t) \in W^{-1,p'}(\Omega)$ for (P) with $u_0 \in W_0^{1,p}(\Omega)$ under (1. 4) for general p.

On the other hand, as for $u_0 \in L^r(\Omega)$, there seems to be few results for the local existence (see [15] for the case of $r = 2$). In this paper, we shall prove that for all $u_0 \in L^r(\Omega)$, there exists $T_0 > 0$ depending only on $|u_0|_{L^r}$ and f such that (P) admits a solution on $[0, T_0]$ under the following:

$$(1.\ 5) \qquad r \ > \ N(q-p)/p$$

without imposing any smallness on u_0 and f. It is noteworthy that Weissler's result ($u_0 \in L^r(\Omega)$, $p = 2$ and $f \equiv 0$) in [20], the result on the case where $u_0 \in L^\infty(\Omega)$ described above, and Akagi-Ôtani's result ($u_0 \in W_0^{1,p}(\Omega)$) in [3] could be regarded as special cases of our result, since (1. 5) with $p = 2$ is just (1. 1), and (1. 4) is equivalent to (1. 5) with $r = p^*$. Furthermore, since T_0 depends only on $|u_0|_{L^r}$, we can immediately observe that the maximal existence time T_{max} of solutions for (P) is finite if and only if $\lim_{t \to T_{max}} |u(t)|_{L^r} = +\infty$. These results could play an important role in studying asymptotic behaviors of solutions for (P).

To prove this, we reduce (P) to the Cauchy problem for an evolution equation governed by the difference of two subdifferential operators in a reflexive Banach space as in [3] and also employ the potential well method (see, e.g., [12] and its references) to confine its solutions within a closed ball in $L^r(\Omega)$ and establish energy estimates. More precisely, the energy functional

$$J(u) \ := \ \frac{1}{p}\int_\Omega |\nabla u(x)|^p dx - \frac{1}{q}\int_\Omega |u(x)|^q dx$$

defined on $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ is not bounded below; however, the sum $J(u) + I_K(u)$, where I_K denotes the indicator function over a ball in $L^r(\Omega)$, turns to be coercive in $W_0^{1,p}(\Omega) \cap L^r(\Omega)$ for r satisfying (1. 5). The potential well method could be one of advantages of our approach based on the subdifferential operator theory, since this method requires the notion of the derivatives of non-smooth functionals.

Furthermore, L^r-estimates of approximate solutions for (P) will be also established to construct a time-local solution of (P) with an initial data $u_0 \in L^r(\Omega)$. In [12], [15] and [3], they could not take account of L^r-estimates of approximate solutions, because of the simplicity of their frameworks, so they could not extract enough precise information to prove local existence under (1. 5) for the case: $u_0 \in L^r(\Omega)$. Such L^r-estimates also play an important role to reveal the dependence of T_0 on $|u_0|_{L^r}$ and f.

Our main result will be stated in the next section. Section 3 provides some preliminaries to be used later, and in Section 4, we give a proof of our main result. Finally, in the appendix, some results related to the functional analysis will be given to be used in Section 4.

2 Main result

To state our main result, we set up notation: the Hölder conjugate of $p \in (1, +\infty)$ is denoted by p', that is, $1/p + 1/p' = 1$; moreover, we write $C_w([a,b]; X)$ for the set of all weakly continuous functions on $[a,b]$ with values in a set X; furthermore,

$$W^{-1,p'}(\Omega) + L^{r'}(\Omega) \quad := \quad \{u_1 + u_2; u_1 \in W^{-1,p'}(\Omega),\ u_2 \in L^{r'}(\Omega)\},$$

which coincides with the dual space of $W_0^{1,p}(\Omega) \cap L^r(\Omega)$.

Throughout the present paper, we denote by C and ℓ a non-negative constant and a non-decreasing function from $[0, +\infty)$ into itself, respectively, which do not depend on the elements of the corresponding space or set and may vary from line to line.

Now our main result is stated as follows:

Theorem 1 *Let $p, q, r \in [2, +\infty)$ be such that $p < q$ and suppose that*

$$r \quad > \quad N(q-p)/p. \tag{2. 1}$$

Then for every $u_0 \in L^r(\Omega)$ and $f \in W^{1,p'}(0,T; W^{-1,p'}(\Omega) + L^{r'}(\Omega)) \cap L^{1+\gamma}(0,T; L^r(\Omega))$ with $\gamma > 0$ (resp. $\gamma = 0$), there exist a non-increasing function $T_* : [0,+\infty) \times [0,+\infty) \to (0,T]$ (resp. $T_f : [0,+\infty) \to (0,T]$) independent of T, u_0 and f (resp. T and u_0) and at least one function $u \in C_w([0,T_0]; L^r(\Omega))$ with $T_0 := T_*(|u_0|_{L^r}, \int_0^T |f(t)|_{L^r}^{1+\gamma} dt)$ (resp. $T_0 := T_f(|u_0|_{L^r})$) such that

$$\begin{aligned}
&u \in C([0,T_0]; L^2(\Omega)) \cap L^p(0,T_0; W_0^{1,p}(\Omega)) \cap L^q(\Omega \times (0,T_0)),\\
&|u|^{(r-2)/p} u \in L^p(0,T_0; W^{1,p}(\Omega)), \quad |u|^{q-2} u \in L^{q'}(\Omega \times (0,T_0)),\\
&\Delta_p u \in L^{p'}(0,T_0; W^{-1,p'}(\Omega)), \quad du/dt \in L^{q'}(0,T_0; W^{-1,p'}(\Omega) + L^{r'}(\Omega)),\\
&t^{1/p} u \in C_w([0,T_0]; W_0^{1,p}(\Omega)), \quad t^{1/2}(du/dt) \in L^2(\Omega \times (0,T_0))
\end{aligned}$$

and for every $v \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$,

$$\begin{aligned}
\int_\Omega \frac{\partial u}{\partial t}(x,t) v(x) dx + \int_\Omega |\nabla u|^{p-2} \nabla u(x,t) \cdot \nabla v(x) dx - \int_\Omega |u|^{q-2} u(x,t) v(x) dx&\\
= \int_\Omega f(x,t) v(x) dx \quad \text{for a.e. } t \in (0,T_0)&
\end{aligned}$$

and u satisfies the initial condition

$$u(\cdot,t) \to u_0 \quad \textit{strongly in } L^r(\Omega) \textit{ as } t \to +0.$$

Remark 1 The assumption (2. 1) is equivalent to the following:

$$q \quad < \quad \frac{N+r}{N}p. \tag{2. 2}$$

Now if $r = p^*$ or $r = q$, then (2. 2) is equivalent to the condition $q < p^*$, where p^* stands for Sobolev's critical exponent given by $p^* := Np/(N-p)$ if $p < N$; $p^* = +\infty$ if $p \geq N$.

3 Preliminaries

In order to prove Theorem 1, we review some of the standard facts on subdifferential operators. We first give a definition of subdifferential operators $\partial_X \phi$ of functionals ϕ in a reflexive Banach space X.

Definition 1 *Let $\phi \in \Phi(X) := \{\varphi : X \to (-\infty, +\infty]; \varphi$ is lower semicontinuous convex and $\varphi \not\equiv +\infty\}$. Then the effective domain $D(\phi)$ and the subdifferential operator $\partial_X \phi : X \to 2^{X^*}$ of ϕ are given by*

$$\begin{aligned} D(\phi) &:= \{u \in X; \phi(u) < +\infty\}, \\ \partial_X \phi(u) &:= \{\xi \in X^*; \phi(v) - \phi(u) \geq \langle \xi, v-u \rangle_X \quad \forall v \in D(\phi)\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_X$ denotes the duality pairing between X and X^*, with the domain $D(\partial_X \phi) := \{u \in D(\phi); \partial_X \phi(u) \neq \emptyset\}$.

It is well known that every subdifferential operator becomes maximal monotone. Moreover, let H be a Hilbert space whose dual space H^* is identified with H. Then the subdifferential operator $\partial_H \phi : H \to 2^H$ of $\phi \in \Phi(H)$ can be written by

$$\partial_H \phi(u) \quad = \quad \{\xi \in H; \phi(v) - \phi(u) \geq (\xi, v-u)_H \quad \forall v \in D(\phi)\},$$

where $(\cdot,\cdot)_H$ denotes the inner product of H, and also becomes a maximal monotone operator from H into 2^H.

Furthermore, the Moreau-Yosida regularization ϕ_λ of $\phi \in \Phi(H)$ is defined as follows.

$$\phi_\lambda(u) \quad := \quad \inf_{v \in H} \left\{ \frac{1}{2\lambda}|u-v|_H^2 + \phi(v) \right\} \quad \forall u \in H, \ \forall \lambda > 0.$$

The following proposition provides some useful properties of Moreau-Yosida regularizations.

Proposition 1 *For every $\phi \in \Phi(H)$, the Moreau-Yosida regularization ϕ_λ of ϕ is convex and Fréchet differentiable in H, and its derivative $\partial_H(\phi_\lambda)$ coincides with the Yosida approximation $(\partial_H \phi)_\lambda$ of $\partial_H \phi$. Furthermore, the following properties are all satisfied.*

$$\phi_\lambda(u) = \frac{1}{2\lambda}|u - J_\lambda^\phi u|_H^2 + \phi(J_\lambda^\phi u) \quad \forall u \in H, \ \forall \lambda > 0, \tag{3. 1}$$

$$\phi(J_\lambda^\phi u) \leq \phi_\lambda(u) \leq \phi(u) \quad \forall u \in H, \ \forall \lambda > 0, \tag{3. 2}$$

$$\phi(J_\lambda^\phi u) \uparrow \phi(u) \quad \textit{as } \lambda \to +0 \quad \forall u \in H, \tag{3. 3}$$

where J_λ^ϕ denotes the resolvent of $\partial_H \phi$.

In order to deal with evolution equations, we often employ the following type of chain rule for subdifferential operators.

Proposition 2 *Let $\phi \in \Phi(X)$, let $p \in (1,+\infty)$ and let $u \in W^{1,p}(0,T;X)$ be such that $u(t) \in D(\partial_X \phi)$ for a.e. $t \in (0,T)$. Moreover, suppose that there exists $g \in L^{p'}(0,T;X^*)$ such that $g(t) \in \partial_X \phi(u(t))$ for a.e. $t \in (0,T)$. Then the function $t \mapsto \phi(u(t))$ is differentiable for a.e. $t \in (0,T)$; moreover, for every section $f(t) \in \partial_X \phi(u(t))$,*

$$\frac{d}{dt}\phi(u(t)) = \left\langle f(t), \frac{du}{dt}(t) \right\rangle_X \quad \text{for a.e. } t \in (0,T).$$

Let Ω be a bounded domain in $\mathbf{R}^N$, let $u \in L^2(\Omega)$ and let α be a maximal monotone graph in $\mathbf{R}^2$. Here we discuss the representation of $\alpha(u(\cdot)) : \Omega \to 2^{\mathbf{R}}$ in the form of the subdifferential $\partial_{L^2}\Theta(u)$ of some functional Θ defined on $L^2(\Omega)$. Since every maximal monotone graph in $\mathbf{R}^2$ becomes cyclic monotone (see Example 1 of [4, p. 60]), there exists a function $\theta \in \Phi(\mathbf{R})$ such that $\partial_{\mathbf{R}}\theta = \alpha$. Moreover, we have:

Proposition 3 *Let Ω be a bounded domain in $\mathbf{R}^N$ and let $\theta \in \Phi(\mathbf{R})$. Define $\Theta : L^2(\Omega) \to (-\infty,+\infty]$ as follows:*

$$\Theta(u) := \begin{cases} \displaystyle\int_\Omega \theta(u(x))dx & \text{if } u \in L^2(\Omega) \text{ and } \theta(u(\cdot)) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let J_λ^Θ and j_λ^θ denote the resolvents of $\partial_{L^2}\Theta$ and $\partial_{\mathbf{R}}\theta$, respectively. Then the following properties are all satisfied for all $\lambda > 0$:

(1) *$\Theta \in \Phi(L^2(\Omega))$.*

(2) *For all $f, u \in L^2(\Omega)$, it follows that $f \in \partial_{L^2}\Theta(u)$ if and only if $f(x) \in \partial_{\mathbf{R}}\theta(u(x))$ for a.e. $x \in \Omega$.*

(3) *For all $u \in L^2(\Omega)$, $(J_\lambda^\Theta u)(x) = j_\lambda^\theta(u(x))$ for a.e. $x \in \Omega$.*

(4) *For every $s \in [1,+\infty]$, if $u, v \in L^s(\Omega) \cap L^2(\Omega)$, then $J_\lambda^\Theta u$ and $\partial_{L^2}\Theta_\lambda(u)$ belong to $L^s(\Omega) \cap L^2(\Omega)$ and*

$$|J_\lambda^\Theta u - J_\lambda^\Theta v|_{L^s} \le |u-v|_{L^s}, \quad |\partial_{L^2}\Theta_\lambda(u) - \partial_{L^2}\Theta_\lambda(v)|_{L^s} \le \frac{2}{\lambda}|u-v|_{L^s}.$$

(5) *For every $p \in (1,+\infty]$, if $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$, then $J_\lambda^\Theta u$ belongs to $W^{1,p}(\Omega) \cap L^2(\Omega)$ and $|\nabla J_\lambda^\Theta u|_{L^p} \le |\nabla u|_{L^p}$.*

(6) *If $\partial_{\mathbf{R}}\theta(0) \ni 0$, then for every $p \in (1,+\infty)$, it follows that $J_\lambda^\Theta 0 = 0$, $J_\lambda^\Theta u \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ for all $u \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$.*

Proof of Proposition 3 For the proof of (1) and (2), we refer to [4, p. 61], so we give a proof only for (3)-(6). From the definition of J_λ^Θ, it follows that $J_\lambda^\Theta u + \lambda\partial_{L^2}\Theta(J_\lambda^\Theta u) \ni u$ for every $u \in L^2(\Omega)$. Hence, by virtue of (2), we have

$$(J_\lambda^\Theta u)(x) + \lambda\partial_{\mathbf{R}}\theta((J_\lambda^\Theta u)(x)) \ni u(x) \quad \text{for a.e. } x \in \Omega.$$

Therefore, from the definition of j_λ^θ, we obtain (3). Moreover, since $|j_\lambda^\theta(u(x)) - j_\lambda^\theta(v(x))| \le |u(x)-v(x)|$ for a.e. $x \in \Omega$ and all $u, v \in L^s(\Omega) \cap L^2(\Omega)$, we can obtain (4).

Furthermore, we also observe that $|j^\theta_\lambda(u(x+h)) - j^\theta_\lambda(u(x))| \le |u(x+h) - u(x)|$ for a.e. $x \in \Omega$ and every $h \in \mathbf{R}^N$ satisfying $x + h \in \Omega$; hence we can derive (5) from (3). As for the case where $\partial_{\mathbf{R}}\theta(0) \ni 0$, it is obvious that $j^\theta_\lambda 0 = 0$, which implies $J^\Theta_\lambda 0 = 0$ and $|(J^\Theta_\lambda u)(x)| \le |u(x)|$ for a.e. $x \in \Omega$. Now let $u \in W^{1,p}_0(\Omega) \cap L^2(\Omega)$ and take a sequence u_n in $C^\infty_0(\Omega)$ such that $u_n \to u$ strongly in $W^{1,p}(\Omega) \cap L^2(\Omega)$. Then we can deduce from (4) and (5) that $J^\Theta_\lambda u_n \to J^\Theta_\lambda u$ strongly in $L^p(\Omega) \cap L^2(\Omega)$ and weakly in $W^{1,p}(\Omega)$. Thus (6) follows from the fact that $\mathrm{supp} J^\Theta_\lambda u_n \subset \mathrm{supp} u_n \subset\subset \Omega$. ∎

4 Proof of main result

Let $V := W^{1,p}_0(\Omega) \cap L^r(\Omega)$ and $H := L^2(\Omega)$ be equipped with the norms: $|\cdot|_V := (|\cdot|^2_{L^r} + |\nabla \cdot|^2_{L^p})^{1/2}$ and $|\cdot|_H := |\cdot|_{L^2}$. Then since $r \ge 2$, we observe that $V \subset H \equiv H^* \subset V^*$ with densely defined and continuous natural injections.

Remark 2 If $r \le q$, then it follows from (2. 2) that $q < p^*$; hence $q < \max\{r, p^*\}$. Therefore V is compactly embedded in $L^q(\Omega)$.

Moreover, define $\varphi, \psi : V \to [0, +\infty)$ in the following

$$\varphi(u) := \frac{1}{p} \int_\Omega |\nabla u(x)|^p dx, \quad \psi(u) := \frac{1}{q} \int_\Omega |u(x)|^q dx \quad \forall u \in V.$$

It then follows that $\varphi, \psi \in C^1(V; \mathbf{R})$, and $\partial_V \varphi$ and $\partial_V \psi$ coincide with $-\Delta_p u$ equipped with the homogeneous Dirichlet boundary condition $u|_{\partial\Omega} = 0$ and $|u|^{q-2}u$, respectively, in V^* under (2. 2). Thus (P) is rewritten as the following Cauchy problem:

$$(\mathrm{CP}) \quad \begin{cases} \dfrac{du}{dt}(t) + \partial_V \varphi(u(t)) - \partial_V \psi(u(t)) = f(t) \ \text{ in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

First we assume $u_0 \in D(\varphi)$ and $f \in C^1([0, T]; V)$. Define $\phi \in \Phi(H)$ by

$$(4.\ 1) \qquad \phi(u) := \begin{cases} \dfrac{1}{r} \displaystyle\int_\Omega |u(x)|^r dx & \text{if } u \in L^r(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

and let $\sigma := \phi(u_0) + 1$. Moreover, we introduce $\varphi^\sigma \in \Phi(V)$ given by

$$\varphi^\sigma(u) := \begin{cases} \varphi(u) & \text{if } u \in K, \\ +\infty & \text{otherwise}, \end{cases}$$

where $K := \{v \in V; \phi(v) \le \sigma\}$. Then we can easily obtain that $D(\varphi^\sigma) = K \subset D(\varphi)$ and $D(\partial_V \varphi^\sigma) = K \subset D(\partial_V \varphi)$; moreover, Theorem 2.2 of [9] ensures that $\partial_V \varphi^\sigma(u) = \partial_V \varphi(u) + \partial_V I_K(u)$ for all $u \in D(\partial_V \varphi^\sigma)$, where I_K denotes the indicator function over K. Here we deal with the following auxiliary problem instead of (CP).

$$(\mathrm{CP})^\sigma \quad \begin{cases} \dfrac{du}{dt}(t) + \partial_V \varphi^\sigma(u(t)) - \partial_V \psi(u(t)) \ni f(t) \ \text{ in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases}$$

To construct a solution of $(\mathrm{CP})^\sigma$, we define the extensions $\overline{\varphi}^\sigma, \overline{\psi} \in \Phi(H)$ of φ^σ and ψ, respectively, given by

$$\overline{\varphi}^\sigma(u) \quad := \quad \begin{cases} \varphi^\sigma(u) & \text{if } \ u \in V, \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$\overline{\psi}(u) \quad := \quad \begin{cases} \dfrac{1}{q}\displaystyle\int_\Omega |u(x)|^q dx & \text{if } \ u \in L^q(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We then observe that

$$(4.\ 2) \qquad \begin{cases} D(\overline{\varphi}^\sigma) = D(\varphi^\sigma), \quad D(\partial_H \overline{\varphi}^\sigma) \subset D(\partial_V \varphi^\sigma), \\ \partial_H \overline{\varphi}^\sigma(u) \subset \partial_V \varphi^\sigma(u) \quad \forall u \in D(\partial_H \overline{\varphi}^\sigma), \end{cases}$$

$$(4.\ 3) \qquad \begin{cases} \overline{\psi}(u) = \psi(u) \quad \forall u \in V, \quad D(\partial_H \overline{\psi}) \cap V \subset D(\partial_V \psi), \\ \partial_H \overline{\psi}(u) \subset \partial_V \psi(u) \quad \forall u \in D(\partial_H \overline{\psi}) \cap V. \end{cases}$$

Furthermore, let us introduce the following approximate problems in H:

$$(\mathrm{CP})^\sigma_\lambda \quad \begin{cases} \dfrac{du_\lambda}{dt}(t) + \partial_H \overline{\varphi}^\sigma(u_\lambda(t)) - \partial_H \overline{\psi}_\lambda(u_\lambda(t)) \ni f(t) \ \text{ in } H, \quad 0 < t < T, \\ u_\lambda(0) = u_0, \end{cases}$$

where $\overline{\psi}_\lambda$ denotes the Moreau-Yosida regularization of $\overline{\psi}$, for $\lambda > 0$. Then $\partial_H \overline{\psi}_\lambda$ coincides with the Yosida approximation $(\partial_H \overline{\psi})_\lambda$ of $\partial_H \overline{\psi}$, so $\partial_H \overline{\psi}_\lambda$ becomes Lipschitz continuous in $L^r(\Omega)$ as well as in H (see Proposition 3). Thus there exists a unique solution $u_\lambda \in C_w([0,T]; V) \cap W^{1,2}(0,T;H)$ of $(\mathrm{CP})^\sigma_\lambda$ on $[0,T]$ such that

$$\sup_{t \in [0,T]} \phi(u_\lambda(t)) \le \sigma, \quad v_\lambda := |u_\lambda|^{(r-2)/p} u_\lambda \in L^p(0,T; W^{1,p}(\Omega))$$

and the function $t \mapsto \overline{\varphi}^\sigma(u_\lambda(t))$ is absolutely continuous on $[0,T]$ (see Appendix B for more details).

Lemma 1 *It follows that*

$$(4.\ 4) \qquad \psi(u) \quad \le \quad \ell(\phi(u))\{\varphi(u) + 1\}^{1-\varepsilon} \quad \forall u \in D(\varphi) \cap D(\phi)$$

for some $\varepsilon \in (0,1]$.

Proof of Lemma 1 For the case where $q \le r$, we can easily see

$$\psi(u) \quad \le \quad C\phi(u)^{q/r} \quad \forall u \in D(\phi),$$

since Ω is bounded. On the other hand, for the case where $r < q$, by Remark 2, we have $q < p^*$. Hence, by Gagliardo-Nirenberg's inequality, it follows that $|u|_{L^q} \le C|\nabla u|^\theta_{L^p} |u|^{1-\theta}_{L^r}$, where $\theta \in (0,1)$ is given by

$$(4.\ 5) \qquad \frac{1}{q} \quad = \quad \frac{N-p}{Np}\theta + \frac{1-\theta}{r}.$$

Moreover, noting that (2. 2) implies

$$(4.\ 6)\quad \theta q \;=\; (q-r)/\left(1-\frac{N-p}{Np}r\right) < \left(\frac{N+r}{N}p-r\right)/\left(1-\frac{N-p}{Np}r\right) = p,$$

we can deduce that

$$\psi(u) \;\le\; C|\nabla u|_{L^p}^{\theta q}|u|_{L^r}^{(1-\theta)q} \le C\varphi(u)^{\theta q/p}\phi(u)^{(1-\theta)q/r}$$

and $0 < \theta q/p < 1$. ∎

By grace of the above lemma, we get

$$(4.\ 7)\qquad \psi(u) \le \ell(\phi(u))\{\varphi(u)+1\}^{1-\varepsilon} \le \frac{1}{2}\varphi(u) + \ell(\sigma) \quad \forall u \in D(\varphi^\sigma).$$

Moreover, it is easily seen that

$$(4.\ 8)\qquad |u|_V^p \;\le\; C\{\varphi^\sigma(u) + \sigma^{p/r}\} \quad \forall u \in D(\varphi^\sigma).$$

We can also derive the compactness of $\partial_V\psi$ in the following sense. Let $M \ge 0$ and let $\{u_n\}$ be a sequence in $W^{1,2}(0,T;H)\cap L^\infty(0,T;V)$ such that

$$\int_0^T \left|\frac{du_n}{dt}(t)\right|_H^2 dt + \sup_{t\in[0,T]}\varphi^\sigma(u_n(t)) \;\le\; M \quad \forall n \in \mathbf{N}.$$

By Remark 2 and (4. 8), the Ascoli compactness lemma (see, e.g., [16]) ensures

$$u_{n'} \;\to\; u \quad \text{strongly in } C([0,T];L^q(\Omega))$$

for some subsequence $\{n'\}$ of $\{n\}$; therefore, since $\partial_V\psi(u_{n'}(\cdot)) = |u_{n'}|^{q-2}u_{n'}$, it follows that

$$\partial_V\psi(u_{n'}(\cdot)) \;\to\; \partial_V\psi(u(\cdot)) \quad \text{strongly in } C([0,T];L^{q'}(\Omega)).$$

Furthermore, let J_λ denote the resolvent of $\partial_H\overline{\psi}$. By Proposition 3, we can then verify that

$$(4.\ 9)\qquad \phi(J_\lambda u) \le \phi(u) \le \sigma, \quad \varphi(J_\lambda u) \le \varphi(u) \quad \forall u \in D(\varphi^\sigma).$$

Hence we conclude that

$$(4.\ 10)\qquad \varphi^\sigma(J_\lambda u) \;\le\; \varphi^\sigma(u) \quad \forall u \in D(\varphi^\sigma).$$

Therefore, as in the proof of Theorem 1 of [3], we can construct a solution u of $(\mathrm{CP})^\sigma$ on $[0,T]$; indeed, multiplying $(\mathrm{CP})^\sigma_\lambda$ by $du_\lambda(t)/dt$ and integrating this over $(0,t)$, by Proposition 2, we can deduce from (4. 7) and (4. 8) that

$$(4.\ 11)\qquad \int_0^T \left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt + \sup_{t\in[0,T]}\varphi^\sigma(u_\lambda(t)) \;\le\; C;$$

moreover, (4. 8) and (4. 11) imply

$$(4.\ 12)\qquad \sup_{t\in[0,T]}|u_\lambda(t)|_V \;\le\; C;$$

furthermore, (4. 10) and (4. 11) yield (4. 11) with $u_\lambda(t)$ replaced by $J_\lambda u_\lambda(t)$; thus we have the following convergences by taking a subsequence of $\{\lambda\}$, which will be written by the same letter $\{\lambda\}$, if necessary:

$$
\text{(4. 13)} \qquad \begin{cases} u_\lambda \to u, \ J_\lambda u_\lambda \to u & \text{weakly star in } L^\infty(0,T;V), \\ & \text{weakly in } W^{1,2}(0,T;H), \\ \partial_H \overline{\psi}_\lambda(u_\lambda(\cdot)) \to \partial_V \psi(u(\cdot)) & \text{strongly in } C([0,T];L^{q'}(\Omega)), \\ g_\lambda \to g \in \partial_V \varphi^\sigma(u(\cdot)) & \text{weakly in } L^2(0,T;V^*), \end{cases}
$$

where $g_\lambda(t) := f(t) - du_\lambda(t)/dt + \partial_H \overline{\psi}_\lambda(u_\lambda(t)) \in \partial_H \overline{\varphi}^\sigma(u_\lambda(t))$.

Now we establish a further a priori estimate for $\phi(u(t))$ by multiplying $(\mathrm{CP})^\sigma_\lambda$ by $\partial_H \phi_\mu(u_\lambda(t))$, where ϕ_μ denotes the Moreau-Yosida regularization of ϕ and $\mu > 0$. Then we see

$$
\text{(4. 14)} \qquad \begin{aligned} & \frac{d}{dt}\phi_\mu(u_\lambda(t)) + (g_\lambda(t), \partial_H \phi_\mu(u_\lambda(t)))_H \\ & = (\partial_H \overline{\psi}_\lambda(u_\lambda(t)), \partial_H \phi_\mu(u_\lambda(t)))_H + (f(t), \partial_H \phi_\mu(u_\lambda(t)))_H. \end{aligned}
$$

Here we prepare a couple of lemmas.

Lemma 2 *For all $u \in D(\partial_V \varphi)$, it follows that*

$$
J^\phi_\mu u \in D(\partial_V \varphi), \quad \partial_H \phi_\mu(u) \in V, \quad v_\mu := |J^\phi_\mu u|^{(r-2)/p} J^\phi_\mu u \in W^{1,p}(\Omega),
$$

where J^ϕ_μ denotes the resolvent of $\partial_H \phi$; in particular, if $u \in D(\partial_V \varphi^\sigma)$, then

$$
\text{(4. 15)} \qquad \alpha |\nabla v_\mu|^p_{L^p} \le \langle g, \partial_H \phi_\mu(u) \rangle \quad \forall g \in \partial_V \varphi^\sigma(u)
$$

for some positive constant α independent of μ.

Proof of Lemma 2 By Proposition 3, we notice that $J^\phi_\mu u \in V = D(\partial_V \varphi)$ for all $u \in V$. Hence we see that $\partial_H \phi_\mu(u) = (u - J^\phi_\mu u)/\mu \in V$ for any $u \in V$.

On the other hand, let $w \in D(\partial_V \varphi) \cap D(\partial_H \phi)$ be such that $\partial_H \phi(w) \in V$. Then since $\partial_H \phi(w) = |w|^{r-2} w$, we have the following formal computation:

$$
\text{(4. 16)} \qquad \begin{aligned} & \langle \partial_V \varphi(w), \partial_H \phi(w) \rangle_V \\ & = \int_\Omega |\nabla w(x)|^{p-2} \nabla w(x) \cdot \nabla \left(|w(x)|^{r-2} w(x) \right) dx \\ & = (r-1) \int_\Omega |\nabla w(x)|^p |w(x)|^{r-2} dx \\ & = (r-1) \int_\Omega \left| |w(x)|^{(r-2)/p} \nabla w(x) \right|^p dx \\ & = (r-1) \left(\frac{p}{r+p-2} \right)^p \int_\Omega \left| \nabla \left(|w(x)|^{(r-2)/p} w(x) \right) \right|^p dx. \end{aligned}
$$

Thus we can verify that $v_\mu := |J^\phi_\mu u|^{(r-2)/p} J^\phi_\mu u \in W^{1,p}(\Omega)$ for every $u \in V$, since $J^\phi_\mu u \in D(\partial_V \varphi)$ and $\partial_H \phi(J^\phi_\mu u) = \partial_H \phi_\mu(u) \in V$ (see also Appendix A for rigorous derivation).

Furthermore, if $u \in D(\partial_V \varphi^\sigma) = K$, then, by Propositions 1 and 3, $J_\mu^\phi u \in K$. Hence we have, for all $u \in D(\partial_V \varphi^\sigma)$ and $g \in \partial_V \varphi^\sigma(u)$,

$$
\begin{aligned}
(4.\ 17) \qquad \langle g, \partial_H \phi_\mu(u) \rangle &\geq \frac{1}{\mu} \left\{ \varphi^\sigma(u) - \varphi^\sigma(J_\mu^\phi u) \right\} \\
&= \frac{1}{\mu} \left\{ \varphi(u) - \varphi(J_\mu^\phi u) \right\} \\
&\geq \frac{1}{\mu} \langle \partial_V \varphi(J_\mu^\phi u), u - J_\mu^\phi u \rangle_V \\
&= \langle \partial_V \varphi(J_\mu^\phi u), \partial_H \phi_\mu(u) \rangle_V.
\end{aligned}
$$

Therefore combining (4. 17) with (4. 16), we can derive (4. 15). ∎

Lemma 3 *There exists $\varepsilon \in (0, 1]$ such that*

$$
\begin{aligned}
(4.\ 18) \qquad & (\partial_H \overline{\psi}_\lambda(u), \partial_H \phi_\mu(u))_H \leq \ell(\phi(u)) \{ |\nabla v|_{L^p}^p + 1 \}^{1-\varepsilon} \\
& \qquad \forall u \in D(\phi) \ \textit{satisfying} \ v := |u|^{(r-2)/p} u \in W^{1,p}(\Omega).
\end{aligned}
$$

Proof of Lemma 3 Noting that $|(J_\lambda u)(x)| \leq |u(x)|$ and $|(J_\mu^\phi u)(x)| \leq |u(x)|$, we have

$$
\begin{aligned}
& (\partial_H \overline{\psi}_\lambda(u), \partial_H \phi_\mu(u))_H \\
&= \int_\Omega |(J_\lambda u)(x)|^{q-2} (J_\lambda u)(x) |(J_\mu^\phi u)(x)|^{r-2} (J_\mu^\phi u)(x) dx \\
&\leq \int_\Omega |u(x)|^{q+r-2} dx.
\end{aligned}
$$

Now, by (2. 2), we see that

$$
q + r - 2 \ < \ \frac{N+r}{N} p + r - 2 = \left(1 + \frac{p}{N} \right) r + p - 2.
$$

On the other hand, if $p < N$, then we observe that

$$
\begin{aligned}
\frac{r+p-2}{p} p^* &= \frac{N}{N-p} r + \frac{N}{N-p} (p-2) \\
&= \left(1 + \frac{p}{N-p} \right) r + \frac{N}{N-p} (p-2),
\end{aligned}
$$

which implies $\rho := p(q+r-2)/(r+p-2) < p^*$.

Now let $v := |u|^{(r-2)/p} u$. Then we note that $|u(x)|^{q+r-2} = |v(x)|^\rho$ and $|u(x)|^r = |v(x)|^{pr/(r+p-2)}$. Hence observing that $1 < pr/(r+p-2) < \rho$ and using Gagliardo-Nirenberg's inequality, since $|v|_{W^{1,p}} \leq C(|\nabla v|_{L^p} + |v|_{L^{pr/(r+p-2)}})$, we obtain

$$
|v|_{L^\rho} \leq C |v|_{W^{1,p}}^\theta |v|_{L^{pr/(r+p-2)}}^{1-\theta} \leq C |\nabla v|_{L^p}^\theta |v|_{L^{pr/(r+p-2)}}^{1-\theta} + C |v|_{L^{pr/(r+p-2)}}
$$

with

$$
\theta \ := \ \left(\frac{r+p-2}{pr} - \frac{r+p-2}{p(q+r-2)} \right) \Big/ \left(\frac{r+p-2}{pr} - \frac{N-p}{Np} \right).
$$

Thus we assure that

$$\text{(4. 19)} \qquad \begin{aligned} \int_\Omega |u(x)|^{q+r-2} dx &= |v|^\rho_{L^\rho} \\ &\le C\left\{ |\nabla v|^{\theta\rho}_{L^p} |v|^{(1-\theta)\rho}_{L^{pr/(r+p-2)}} + |v|^\rho_{L^{pr/(r+p-2)}} \right\} \\ &= C\left\{ |\nabla v|^{\theta\rho}_{L^p} |u|^{(1-\theta)(q+r-2)}_{L^r} + |u|^{q+r-2}_{L^r} \right\}. \end{aligned}$$

Moreover, we remark that (2. 2) yields

$$\begin{aligned} \theta\rho &= (q-2)/\left(\frac{r+p-2}{p} - \frac{N-p}{Np} r \right) \\ &< \left(\frac{N+r}{N} p - 2 \right) / \left(\frac{r+p-2}{p} - \frac{N-p}{Np} r \right) = p, \end{aligned}$$

which together with (4. 19) proves (4. 18). ∎

Now let $\lambda > 0$ be fixed. By Lemmas 2 and 3, it follows from (4. 2) and (4. 14) that

$$\text{(4. 20)} \qquad \begin{aligned} &\frac{d}{dt}\phi_\mu(u_\lambda(t)) + \alpha |\nabla v_{\lambda,\mu}(t)|^p_{L^p} \\ &\quad \le \ell(\sigma) \left\{ |\nabla v_\lambda(t)|^p_{L^p} + 1 \right\}^{1-\varepsilon} + (f(t), \partial_H \phi_\mu(u_\lambda(t)))_H, \end{aligned}$$

where $v_{\lambda,\mu} := |J^\phi_\mu u_\lambda|^{(r-2)/p} J^\phi_\mu u_\lambda$, for a.e. $t \in (0,T)$. Here we notice that

$$\begin{aligned} (f(t), \partial_H \phi_\mu(u_\lambda(t)))_H &\le |f(t)|_{L^r} |\partial_H \phi_\mu(u_\lambda(t))|_{L^{r'}} \\ &\le C |f(t)|_{L^r} \phi(J^\phi_\mu u_\lambda(t))^{1/r'} \le C\sigma^{1/r'} |f(t)|_{L^r}. \end{aligned}$$

Integrating this over $(0,t)$, since $v_\lambda = |u_\lambda|^{(r-2)/p} u_\lambda \in L^p(0,T; W^{1,p}(\Omega))$, we get

$$\text{(4. 21)} \quad \begin{aligned} &\phi_\mu(u_\lambda(t)) + \alpha \int_0^t |\nabla v_{\lambda,\mu}(\tau)|^p_{L^p} d\tau \\ &\quad \le \phi_\mu(u_0) + \ell(\sigma) \int_0^t \left\{ |\nabla v_\lambda(\tau)|^p_{L^p} + 1 \right\}^{1-\varepsilon} d\tau + C\sigma^{1/r'} \int_0^t |f(\tau)|_{L^r} d\tau \end{aligned}$$

for all $t \in [0,T]$. Therefore recalling that $\phi_\mu(u_0) \le \phi(u_0)$ and taking a subsequence if necessary, we deduce that

$$\text{(4. 22)} \qquad v_{\lambda,\mu} \to w_\lambda \qquad \text{weakly in } L^p(0,T; W^{1,p}(\Omega))$$

for some $w_\lambda \in L^p(0,T; W^{1,p}(\Omega))$ as $\mu \to +0$. Here we also noticed that $|v_{\lambda,\mu}(t)|_{L^p} \le |v_\lambda(t)|_{L^p}$ for all $\mu > 0$. Now, by Proposition 1, we see that

$$\text{(4. 23)} \qquad \frac{1}{2\mu} |u_\lambda(t) - J^\phi_\mu u_\lambda(t)|^2_H = \phi_\mu(u_\lambda(t)) - \phi(J^\phi_\mu u_\lambda(t)) \le \sigma,$$

which implies that $J^\phi_\mu u_\lambda \to u_\lambda$ strongly in $C([0,T]; H)$ as $\mu \to +0$. Hence, by (4. 22), we can assure that $w_\lambda = v_\lambda = |u_\lambda|^{(r-2)/p} u_\lambda$. Moreover, we get

$$\int_0^t |\nabla v_\lambda(\tau)|^p_{L^p} d\tau \le \liminf_{\mu \to +0} \int_0^t |\nabla v_{\lambda,\mu}(\tau)|^p_{L^p} d\tau.$$

Thus passing to the limit in (4. 21) as $\mu \to +0$ and applying Young's inequality, we have

$$\begin{aligned}(4.\ 24)\qquad & \phi(u_\lambda(t)) + \frac{\alpha}{2}\int_0^t |\nabla v_\lambda(\tau)|_{L^p}^p d\tau \\ & \leq \ \phi(u_0) + t\ell(\sigma) + t\frac{\alpha}{2} + C\sigma^{1/r'}\int_0^T |f(\tau)|_{L^r} d\tau\end{aligned}$$

for all $t \in [0,T]$.

In order to pass to the limit in (4. 24) as $\lambda \to +0$, we notice the following fact; since V is compactly embedded in $L^q(\Omega)$, by Ascoli's compactness lemma, it follows from (4. 13) that

$$(4.\ 25)\qquad u_\lambda \to u \quad \text{strongly in } C([0,T];L^q(\Omega)).$$

So now letting $\lambda \to +0$, since $\phi(u_\lambda(t)) \leq \sigma$ for all $t \in [0,T]$ and $2(r+p-2)/p \leq r$, we can deduce from (4. 24) that

$$\begin{aligned} u_\lambda \to u \qquad & \text{weakly star in } L^\infty(0,T;L^r(\Omega)), \\ v_\lambda \to v := |u|^{(r-2)/p}u \qquad & \text{weakly star in } L^\infty(0,T;H), \\ & \text{weakly in } L^p(0,T;W^{1,p}(\Omega)).\end{aligned}$$

Therefore we conclude that

$$\begin{aligned}(4.\ 26)\qquad & \phi(u(t)) + \frac{\alpha}{2}\int_0^t |\nabla v(\tau)|_{L^p}^p d\tau \\ & \leq \ \phi(u_0) + t\ell(\sigma) + t\frac{\alpha}{2} + C\sigma^{1/r'}\int_0^t |f(\tau)|_{L^r} d\tau,\end{aligned}$$

which also gives $\limsup_{t\to+0}\phi(u(t)) \leq \phi(u_0)$. Furthermore, since u belongs to $C([0,T];H)$, the lower semi-continuity of ϕ implies that $\liminf_{t\to+0}\phi(u(t)) \geq \phi(u_0)$. Therefore, by the uniform convexity of $L^r(\Omega)$, we can verify that

$$(4.\ 27)\qquad u(t) \to u_0 \qquad \text{strongly in } L^r(\Omega) \quad \text{as } t \to +0.$$

Now, for the case where $\gamma > 0$, take a non-increasing function $T_* : [0,+\infty) \times [0,+\infty) \to (0,T]$; $(x,y) \mapsto T_*(x,y)$ independent of T, u_0 and f such that

$$T_*(x,y)\left\{\ell(x+1) + \frac{\alpha}{2}\right\} + C(x+1)^{1/r'}T_*(x,y)^{\gamma/(1+\gamma)}y^{1/(1+\gamma)} \ \leq \ \frac{1}{2}.$$

For the case where $\gamma = 0$, i.e., $f \in L^1(0,T;L^r(\Omega))$, then we can choose a non-increasing function $T_f : [0,+\infty) \to (0,T]$; $x \mapsto T_f(x)$, which depends on f but not on T and u_0, such that

$$T_f(x)\left\{\ell(x+1) + \frac{\alpha}{2}\right\} + C(x+1)^{1/r'}\int_0^{T_f(x)} |f(\tau)|_{L^r} d\tau \ \leq \ \frac{1}{2}.$$

Moreover, since $\sigma = \phi(u_0) + 1$, it follows that

$$(4.\ 28)\qquad \sup_{t\in[0,T_0]} \phi(u(t)) \ < \ \sigma,$$

where T_0 is given by

$$T_0 := T_* \left(\phi(u_0), \int_0^T |f(\tau)|_{L^r}^{1+\gamma} d\tau \right) > 0 \quad \text{or} \quad T_0 := T_f(\phi(u_0)).$$

Now we claim that $\partial_V \varphi^\sigma(u(t)) = \partial_V \varphi(u(t))$ for a.e. $t \in (0, T_0)$ to verify that u is a solution of (CP) on $[0, T_0]$. Actually, since $\phi(u(t)) < \sigma$ for all $t \in [0, T_0]$, we can deduce that $\partial_V I_K(u(t)) = \{0\}$, which implies $\partial_V \varphi^\sigma(u(t)) = \partial_V \varphi(u(t))$ for a.e. $t \in (0, T_0)$. Therefore we conclude that u becomes a solution of (CP) on $[0, T_0]$.

Before proceeding to the next step, we establish further estimates for u to be used later. Multiply $(\mathrm{CP})_\lambda^\sigma$ by $u_\lambda(t)$ and integrate this over $(0, t)$. Then we get, by (4. 8),

$$\begin{aligned}
&\frac{1}{2}|u_\lambda(t)|_H^2 + \int_0^t \overline{\varphi}^\sigma(u_\lambda(\tau)) d\tau \\
&\quad \le \frac{1}{2}|u_0|_H^2 + \int_0^t |\partial_H \overline{\psi}_\lambda(u_\lambda(\tau))|_{L^{q'}} |u_\lambda(\tau)|_{L^q} d\tau \\
&\qquad + \int_0^T |f(\tau)|_{V^*} |u_\lambda(\tau)|_V d\tau \\
&\quad \le \frac{1}{2}|u_0|_H^2 + C \int_0^t \psi(u_\lambda(\tau)) d\tau \\
&\qquad + C \int_0^T |f(\tau)|_{V^*}^{p'} d\tau + \frac{1}{2} \int_0^t \overline{\varphi}^\sigma(u_\lambda(\tau)) d\tau + \frac{T}{2} \sigma^{p/r}.
\end{aligned}$$

Hence, by Lemma 1, as in (4. 7), we can deduce that

$$\begin{aligned}
&\sup_{t \in [0,T]} |u_\lambda(t)|_H^2 + \int_0^T \overline{\varphi}^\sigma(u_\lambda(\tau)) d\tau \\
&\quad \le C \left(|u_0|_H^2 + T\ell(\sigma) + \int_0^T |f(t)|_{V^*}^{p'} dt \right).
\end{aligned} \tag{4. 29}$$

Furthermore, multiplying $(\mathrm{CP})_\lambda^\sigma$ by $t(du_\lambda(t)/dt)$ and noting that

$$\begin{aligned}
\left(f(t), t\frac{du_\lambda}{dt}(t) \right)_H &= \frac{d}{dt} \{ t(f(t), u_\lambda(t))_H \} - (f(t), u_\lambda(t))_H \\
&\quad - t \left(\frac{df}{dt}(t), u_\lambda(t) \right)_H,
\end{aligned}$$

we have

$$\begin{aligned}
&t \left| \frac{du_\lambda}{dt}(t) \right|_H^2 + \frac{d}{dt} \{ t\overline{\varphi}^\sigma(u_\lambda(t)) \} - \overline{\varphi}^\sigma(u_\lambda(t)) \\
&\quad \le \frac{d}{dt} \left\{ t\overline{\psi}_\lambda(u_\lambda(t)) \right\} - \overline{\psi}_\lambda(u_\lambda(t)) \\
&\qquad + \frac{d}{dt} \{ t(f(t), u_\lambda(t))_H \} - (f(t), u_\lambda(t))_H - t \left(\frac{df}{dt}(t), u_\lambda(t) \right)_H.
\end{aligned}$$

Hence integrate this over $(0, t)$. It then follows from (4. 7) and (4. 8) that

$$\int_0^t \tau \left| \frac{du_\lambda}{d\tau}(\tau) \right|_H^2 d\tau + t\overline{\varphi}^\sigma(u_\lambda(t)) + \int_0^t \overline{\psi}_\lambda(u_\lambda(\tau)) d\tau$$

$$
\begin{aligned}
&\leq \; t\overline{\psi}_\lambda(u_\lambda(t)) + \int_0^t \overline{\varphi}^\sigma(u_\lambda(\tau))d\tau \\
&\quad +t(f(t),u_\lambda(t))_H - \int_0^t (f(\tau),u_\lambda(\tau))_H d\tau - \int_0^t \tau\left(\frac{df}{d\tau}(\tau),u_\lambda(\tau)\right)_H d\tau \\
&\leq \; \frac{t}{2}\overline{\varphi}^\sigma(u_\lambda(t)) + C\int_0^T \overline{\varphi}^\sigma(u_\lambda(\tau))d\tau \\
&\quad +C\sup_{\tau\in[0,T]} \tau|f(\tau)|_{V^*}^{p'} + \frac{t}{4}\overline{\varphi}^\sigma(u_\lambda(t)) + C\int_0^t |f(\tau)|_{V^*}^{p'}d\tau \\
&\quad +\int_0^T \left|\tau\frac{df}{d\tau}(\tau)\right|_{V^*}^{p'} d\tau + T\ell(\sigma).
\end{aligned}
$$

Therefore, by virtue of (4. 29), we can assert that

$$
\begin{aligned}
\text{(4. 30)} \qquad &\int_0^T t\left|\frac{du_\lambda}{dt}(t)\right|_H^2 dt + \sup_{t\in[0,T]} t\overline{\varphi}^\sigma(u_\lambda(t)) \\
&\leq \; C\left(|u_0|_H^2 + T\ell(\sigma) + \int_0^T |f(t)|_{V^*}^{p'}dt + \int_0^T \left|t\frac{df}{dt}(t)\right|_{V^*}^{p'} dt\right).
\end{aligned}
$$

Here we have used the fact (see [2]) that

$$
\sup_{t\in[0,T]} t|f(t)|_{V^*}^{p'} \;\leq\; C\left(\int_0^T |f(t)|_{V^*}^{p'}dt + \int_0^T \left|t\frac{df}{dt}(t)\right|_{V^*}^{p'} dt\right).
$$

Then letting $\lambda \to +0$, by (4. 13) and (4. 25), we can obtain

$$
\begin{aligned}
\text{(4. 31)} \qquad &\sup_{t\in[0,T]} |u(t)|_H^2 + \int_0^T \varphi^\sigma(u(\tau))d\tau \\
&\leq \; C\left(|u_0|_H^2 + T\ell(\sigma) + \int_0^T |f(t)|_{V^*}^{p'}dt\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(4. 32)} \qquad &\int_0^T t\left|\frac{du}{dt}(t)\right|_H^2 dt + \sup_{t\in[0,T]} t\varphi^\sigma(u(t)) \\
&\leq \; C\left(|u_0|_H^2 + T\ell(\sigma) + \int_0^T |f(t)|_{V^*}^{p'}dt + \int_0^T \left|t\frac{df}{dt}(t)\right|_{V^*}^{p'} dt\right).
\end{aligned}
$$

Secondly, we deal with the case where $u_0 \in L^r(\Omega)$ and $f \in W^{1,p'}(0,T;V^*) \cap L^{1+\gamma}(0,T;L^r(\Omega))$. To do this, we take approximate sequences $u_{0,n} \in D(\varphi)$ and $f_n \in C^1([0,T];V)$ such that $u_{0,n} \to u_0$ strongly in $L^r(\Omega)$ and $f_n \to f$ strongly in $W^{1,p'}(0,T;V^*) \cap L^{1+\gamma}(0,T;L^r(\Omega))$. Moreover, let $\sigma := \phi(u_0) + 2$ and remark that $\phi(u_{0,n}) \leq \phi(u_0) + 1$ and $\int_0^T |f_n(t)|_{L^r}^{1+\gamma}dt \leq \int_0^T |f(t)|_{L^r}^{1+\gamma}dt + 1$ for enough large n. Furthermore, there exists $h \in L^{1+\gamma}(0,T)$ such that $|f_{n'}(t)|_{L^r} \leq h(t)$ for a.e. $t \in (0,T)$ and all n' for some subsequence n' of n, which will be denoted briefly by n.

Hence we can construct solutions u_n of (CP) with initial data $u_{0,n}$ and the forcing term f_n, which will be denoted by $(\mathrm{CP})_n$ in the rest of this section, on $[0,T_0]$ such that

$$
\text{(4. 33)} \qquad \sup_{t\in[0,T_0]} \phi(u_n(t)) < \phi(u_{0,n}) + 1 \leq \sigma
$$

for some $T_0 > 0$ independent of n, by recalling the first step. Actually, for the case $\gamma > 0$, it is obvious that $T_*(\phi(u_{0,n}), \int_0^T |f_n(t)|_{L^r}^{1+\gamma} dt) \geq T_*(\phi(u_0)+1, \int_0^T |f(t)|_{L^r}^{1+\gamma} dt+1) > 0$. As for the case: $\gamma = 0$, since $|f_n(t)|_{L^r} \leq h(t)$, we can choose $T_h : [0, +\infty) \to (0, T]$ such that $T_h(x)\{\ell(x+1)+\alpha/2\}+C(x+1)^{1/r'} \int_0^{T_h(x)} |h(\tau)|d\tau \leq 1/2$ and $T_{f_n}(x) \geq T_h(x) > 0$ for all $x \in [0, +\infty)$ and n. Thus we can take $T_0 > 0$ uniformly with respect to n.

Now we shall establish a priori estimates for u_n and derive convergences of u_n as $n \to +\infty$. First, by recalling (4. 31) and (4. 32), we have

$$\sup_{t\in[0,T_0]} |u_n(t)|_H + \int_0^{T_0} |\nabla u_n(t)|_{L^p}^p dt \leq C, \tag{4. 34}$$

$$\int_0^{T_0} t \left|\frac{du_n}{dt}(t)\right|_H^2 dt + \sup_{t\in[0,T_0]} t|\nabla u_n(t)|_{L^p}^p dt \leq C. \tag{4. 35}$$

Moreover, since $\partial_V \varphi(u_n(t)) = -\Delta_p u_n(t)$, we can also derive

$$\int_0^{T_0} |\partial_V \varphi(u_n(\tau))|_{W^{-1,p'}}^{p'} d\tau \leq C \tag{4. 36}$$

from (4. 34). Furthermore, by Lemma 1, we observe

$$\begin{aligned}\int_0^{T_0} |\partial_V \psi(u_n(\tau))|_{L^{q'}}^{q'} d\tau &\leq C \int_0^{T_0} \psi(u_n(\tau))d\tau \\ &\leq \ell(\sigma) \int_0^{T_0} \{\varphi(u_n(\tau)) + 1\}^{1-\varepsilon} d\tau \leq C.\end{aligned}$$

Thus since $L^{q'}(\Omega)$ is continuously embedded in V^*, we get, by $(\mathrm{CP})_n$,

$$\int_0^{T_0} \left|\frac{du_n}{dt}(t)\right|_{V^*}^{q'} dt \leq C. \tag{4. 37}$$

Now, by virtue of (4. 26) with $u = u_n$, $v = v_n := |u_n|^{(r-2)/p} u_n$, $u_0 = u_{0,n}$ and $f = f_n$, it follows that

$$\int_0^{T_0} |\nabla v_n(\tau)|_{L^p}^p d\tau \leq C. \tag{4. 38}$$

Moreover, since $2(r + p - 2)/p \leq r$, it follows from (4. 33) that v_n is bounded in $L^\infty(0, T_0; H)$.

From these a priori estimates, we can obtain the following convergences by taking a subsequence of $\{n\}$, which will be also denoted by the same letter $\{n\}$, if necessary:

(4. 39)	$u_n \to u$	weakly star in $L^\infty(0, T_0; L^r(\Omega))$,
(4. 40)		weakly in $L^p(0, T_0; V)$,
(4. 41)	$t^{1/p}u_n \to t^{1/p}u$	weakly star in $L^\infty(0, T_0; W_0^{1,p}(\Omega))$,
(4. 42)	$v_n \to v$	weakly star in $L^\infty(0, T_0; H)$,
(4. 43)		weakly in $L^p(0, T_0; W^{1,p}(\Omega))$,
(4. 44)	$du_n/dt \to du/dt$	weakly in $L^{q'}(0, T_0; V^*)$,
(4. 45)	$t^{1/2}(du_n/dt) \to t^{1/2}(du/dt)$	weakly in $L^2(0, T_0; H)$,
(4. 46)	$\partial_V \varphi(u_n(\cdot)) \to g$	weakly in $L^{p'}(0, T_0; W^{-1,p'}(\Omega))$,
(4. 47)	$\partial_V \psi(u_n(\cdot)) \to h$	weakly in $L^{q'}(0, T_0; L^{q'}(\Omega))$.

Hence we can also deduce that $u \in C_w([0,T_0]; L^r(\Omega)) \cap C((0,T_0]; H)$. Moreover, since V and $L^r(\Omega)$ are compactly embedded in $L^q(\Omega)$ and V^*, respectively, we can deduce that

$$u_n \to u \qquad \text{strongly in } L^p(0,T_0; L^q(\Omega)) \cap C([0,T_0]; V^*), \tag{4. 48}$$

which together with (4. 42) implies $v = |u|^{(r-2)/p}u$. Moreover, it follows from (4. 37) and (4. 48) that $u(t) \to u_0$ strongly in V^* as $t \to +0$.

Next, we shall prove that $\partial_V \psi(u(t)) = h(t)$ for a.e. $t \in (0,T_0)$. To this end, we divide our proof into two cases. For the case where $r < q$, as in the proof of Lemma 1, we can deduce from (4. 6) that

$$\begin{aligned} &\int_0^{T_0} |u_n(t) - u(t)|_{L^q}^q dt \\ &\leq C \left(\int_0^{T_0} |\nabla u_n(t) - \nabla u(t)|_{L^p}^p dt \right)^{\theta q/p} \left(\int_0^{T_0} |u_n(t) - u(t)|_{L^r}^{(1-\theta)q\nu} dt \right)^{1/\nu}, \end{aligned}$$

where θ is given by (4. 5) and $\nu := p/(p - \theta q)$. Moreover, by (4. 33) and (4. 48), we can assure that

$$u_n \to u \qquad \text{strongly in } L^{(1-\theta)q\nu}(0,T_0; L^r(\Omega)). \tag{4. 49}$$

Hence (4. 34) and (4. 49) imply

$$u_n \to u \qquad \text{strongly in } L^q(0,T_0; L^q(\Omega)). \tag{4. 50}$$

Here we note that $\partial_V \psi(u) = \partial_{L^q} \psi_{L^q}(u)$ if $u \in V$, where $\psi_{L^q} : L^q(\Omega) \to [0,+\infty)$ is defined by $\psi_{L^q}(u) := (1/q) \int_\Omega |u(x)|^q dx$ for all $u \in L^q(\Omega)$. Therefore, on account of the demiclosedness of $\partial_{L^q} \psi_{L^q}$ in $L^q(\Omega) \times L^{q'}(\Omega)$ and Proposition 1.1 of [13], we can assert that $h(t) = \partial_V \psi(u(t))$ for a.e. $t \in (0,T_0)$. For the case where $q \leq r$, (4. 50) follows immediately from (4. 33) and (4. 48). Hence we can also verify that $h(t) = \partial_V \psi(u(t))$ for a.e. $t \in (0,T_0)$.

Now, in order to show that $g(t) = \partial_V \varphi(u(t))$ for a.e. $t \in (0,T_0)$, by (4. 48), we take a set $I \subset (0,T_0)$ such that $u_n(s) \to u(s)$ strongly in $L^q(\Omega)$ for all $s \in I$ and $|(0,T_0) \setminus I| = 0$. Hence multiply $\partial_V \varphi(u_n(t))$ by $u_n(t)$ and integrate this over (s,T_0) for an arbitrary $s \in I$. It then follows that

$$\begin{aligned} &\int_s^{T_0} \langle \partial_V \varphi(u_n(t)), u_n(t) \rangle dt \\ &= \int_s^{T_0} \langle f_n(t), u_n(t) \rangle dt + \int_s^{T_0} \langle \partial_V \psi(u_n(t)), u_n(t) \rangle dt \\ &\quad - \frac{1}{2} |u_n(T_0)|_H^2 + \frac{1}{2} |u_n(s)|_H^2. \end{aligned}$$

Hence letting $n \to +\infty$, since (4. 45) ensures $u \in W^{1,2}(s,T_0; H)$, we have

$$\begin{aligned} &\limsup_{n \to +\infty} \int_s^{T_0} \langle \partial_V \varphi(u_n(t)), u_n(t) \rangle dt \\ &\leq \int_s^{T_0} \langle f(t), u(t) \rangle dt + \int_s^{T_0} \langle \partial_V \psi(u(t)), u(t) \rangle dt \\ &\quad - \frac{1}{2} |u(T_0)|_H^2 + \frac{1}{2} |u(s)|_H^2 = \int_s^{T_0} \langle g(t), u(t) \rangle dt. \end{aligned}$$

Therefore it follows from (4. 40) and (4. 46) that $g(t) = \partial_V \varphi(u(t))$ for a.e. $t \in (s, T_0)$. From the arbitrariness of s and the fact that $|(0, T_0) \setminus I| = 0$, we conclude that $g(t) = \partial_V \varphi(u(t))$ for a.e. $t \in (0, T_0)$.

Finally, we check the initial condition $u(0) = u_0$ in the sense of $L^r(\Omega)$. To do this, we recall (4. 26) with $u = u_n$, $v = v_n$, $u_0 = u_{0,n}$ and $f = f_n$ and pass to the limit as $n \to +\infty$. It then follows that

$$\phi(u(t)) \quad \leq \quad \phi(u_0) + t\left\{\ell(\sigma) + \frac{\alpha}{2}\right\} + C\sigma^{1/r'} \int_0^t |f(\tau)|_{L^r} d\tau$$

for all $t \in [0, T_0]$, which implies that $\limsup_{t \to +0} \phi(u(t)) \leq \phi(u_0)$. Hence, since ϕ is weakly lower semi-continuous in $L^r(\Omega)$, we conclude that

$$\phi(u(t)) \to \phi(u_0) \qquad \text{as } t \to +0.$$

Therefore $u(t) \to u_0$ strongly in $L^r(\Omega)$ as $t \to +0$; moreover, $u \in C([0, T_0]; H)$. Thus we complete the proof. ▮

A Rigorous calculation of (4. 16)

In this section, we provide a rigorous proof of (4. 16), that is,

Proposition 4 *Let Ω be a (possibly unbounded) domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and suppose that*

(A. 1) $$2 \leq r < +\infty, \quad 2 \leq p < +\infty.$$

For every $u \in W^{1,p}(\Omega)$ satisfying $|u|^{r-2}u \in W^{1,p}(\Omega)$, it follows that $v := |u|^{(r-2)/p}u$ belongs to $W^{1,p}(\Omega)$ and

(A. 2) $$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla \left(|u(x)|^{r-2}u(x)\right) dx = \alpha \int_\Omega |\nabla v(x)|^p dx,$$

where $\alpha := (r-1)p^p/(r+p-2)^p > 0$.

On the other hand, for any $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, it is well known that $|u|^{r-2}u$ also belongs to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$; moreover, it follows that

(A. 3) $$\partial_{x_i} \left(|u(x)|^{r-2}u(x)\right) \quad = \quad (r-1)|u(x)|^{r-2} \partial_{x_i} u(x),$$

where $\partial_{x_i} := \partial/\partial x_i$ (see, e.g., Proposition IX.5 of [5]). However, for all $u \in W^{1,p}(\Omega)$ satisfying $|u|^{r-2}u \in W^{1,p}(\Omega)$, it would not be obvious whether (A. 3) holds true or not, and (A. 3) will be required to prove Proposition 4. To verify (A. 3) for every $u \in W^{1,p}(\Omega)$ satisfying $|u|^{r-2}u \in W^{1,p}(\Omega)$, we introduce a non-decreasing function $\zeta_n \in C^1(\mathbf{R})$ characterized by

$$\zeta_n(s) = \begin{cases} s & \text{if } |s| \leq n, \\ n+1 & \text{if } s \geq n+2, \\ -(n+1) & \text{if } s \leq -(n+2), \end{cases} \qquad |\zeta_n(s)| \leq |s|, \quad |\zeta_n'(s)| \leq 1 \quad \forall s \in \mathbf{R},$$

and prepare the following.

Lemma 4 *Let* $p \in [1, +\infty)$ *and let* $u \in L^p(\Omega)$ *and put* $u_n = \zeta_n(u(\cdot))$. *Then* $u_n \in L^p(\Omega) \cap L^\infty(\Omega)$ *and*

$$u_n \to u \qquad \textit{strongly in } L^p(\Omega) \textit{ as } n \to +\infty.$$

In particular, if $u \in W^{1,p}(\Omega)$, *then* $u_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and*

$$u_n \to u \qquad \textit{strongly in } W^{1,p}(\Omega) \textit{ as } n \to +\infty.$$

Proof of Lemma 4 Let $\omega_n := \{x \in \Omega; |u(x)| \geq n\}$ and let $u_n(x) := \zeta_n(u(x))$. We then find that $u_n \in L^p(\Omega) \cap L^\infty(\Omega)$. Moreover, it follows that

$$\int_\Omega |u_n(x) - u(x)|^p dx \;=\; \int_{\omega_n} |u_n(x) - u(x)|^p dx \leq 2^p \int_{\omega_n} |u(x)|^p dx$$

as $n \to +\infty$, since $|\omega_n| \to 0$. In particular, if $u \in W^{1,p}(\Omega)$, then since $\partial_{x_i} u_n(x) = \zeta_n'(u(x))\partial_{x_i} u(x)$ (see also Proposition IX.5 of [5]), we have

$$\begin{aligned}
\int_\Omega |\nabla u_n(x) - \nabla u(x)|^p dx &= \int_\Omega |\zeta_n'(u(x))\nabla u(x) - \nabla u(x)|^p dx \\
&= \int_{\omega_n} |\zeta_n'(u(x))\nabla u(x) - \nabla u(x)|^p dx \\
&\leq 2^p \int_{\omega_n} |\nabla u(x)|^p dx \to 0
\end{aligned}$$

as $n \to +\infty$. Therefore $u_n \to u$ strongly in $W^{1,p}(\Omega)$. ∎

Now we have:

Lemma 5 *Suppose that* (A. 1) *is satisfied and let* $u \in W^{1,p}(\Omega)$ *be such that* $|u|^{r-2}u \in W^{1,p}(\Omega)$. *Then it follows that*

$$\partial_{x_i}\left(|u(x)|^{r-2}u(x)\right) \;=\; (r-1)|u(x)|^{r-2}\partial_{x_i} u(x).$$

Proof of Lemma 5 Since $u_n = \zeta_n(u(\cdot)) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, we notice that

$$\partial_{x_i}\left(|u_n(x)|^{r-2}u_n(x)\right) \;=\; (r-1)|u_n(x)|^{r-2}\partial_{x_i} u_n(x).$$

Thus we get, for every $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&\int_\Omega |u_n(x)|^{r-2}u_n(x)\partial_{x_i}\varphi(x)dx \\
&\quad= -(r-1)\int_\Omega |u_n(x)|^{r-2}\left(\partial_{x_i} u_n(x)\right)\varphi(x)dx.
\end{aligned} \tag{A. 4}$$

Now we claim the following:

$$|u_n|^{r-2}u_n \to |u|^{r-2}u \qquad \text{strongly in } L^p(\Omega), \tag{A. 5}$$

$$|u_n|^{r-2}\partial_{x_i} u_n \to |u|^{r-2}\partial_{x_i} u \qquad \text{strongly in } L^\rho(\Omega) \tag{A. 6}$$

for some $\rho \geq 1$ as $n \to +\infty$. Indeed, recalling $\omega_n := \{x \in \Omega; |u(x)| \geq n\}$, we obtain

$$\int_\Omega \left| |u_n(x)|^{r-2}u_n(x) - |u(x)|^{r-2}u(x) \right|^p dx \;\leq\; 2^p \int_{\omega_n} |u(x)|^{p(r-1)} dx \to 0.$$

Moreover, by virtue of Lemma 4, taking a subsequence if necessary, we see

$$|u_n(x)|^{r-2}\partial_{x_i}u_n(x) \quad \to \quad |u(x)|^{r-2}\partial_{x_i}u(x) \qquad \text{for a.e. } x \in \Omega.$$

Furthermore, we find that

$$\Big||u_n(x)|^{r-2}\partial_{x_i}u_n(x)\Big| \quad \leq \quad |u(x)|^{r-2}|\partial_{x_i}u(x)| \in L^\rho(\Omega),$$

where $\rho \in [1, +\infty)$ is given by

$$\frac{1}{\rho} = \frac{r-2}{(r-1)p^*} + \frac{1}{p} \leq \frac{1}{p'} + \frac{1}{p} = 1,$$

since the fact that $|u|^{r-2}u \in W^{1,p}(\Omega)$ implies that $u \in L^{(r-1)p^*}(\Omega)$. Thus Lebesgue's dominant convergence theorem ensures (A. 6).

Therefore passing to the limit in (A. 4) as $n \to +\infty$, we conclude that for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega |u(x)|^{r-2}u(x)\partial_{x_i}\varphi(x)dx \quad = \quad -(r-1)\int_\Omega |u(x)|^{r-2}\left(\partial_{x_i}u(x)\right)\varphi(x)dx,$$

which implies that $\partial_{x_i}(|u(x)|^{r-2}u(x)) = (r-1)|u(x)|^{r-2}\partial_{x_i}u(x)$. ∎

Now we proceed to the proof of Proposition 4. First, we have:

Lemma 6 *Suppose that* (A. 1) *is satisfied, and let* $u \in W^{1,p}(\Omega)$ *be such that* $|u|^{r-2}u \in W^{1,p}(\Omega)$. *Moreover, put* $u_n := \zeta_n(u(\cdot))$. *Then it follows that*

$$|u_n|^{r-2}u_n \to |u|^{r-2}u \qquad \textit{strongly in } W^{1,p}(\Omega).$$

Proof of Lemma 6 In the proof of Lemma 5, we have proved that

$$\begin{aligned} &|u_n|^{r-2}u_n \to |u|^{r-2}u \quad \text{strongly in } L^p(\Omega), \\ &\partial_{x_i}\left(|u_n(x)|^{r-2}u_n(x)\right) \to \partial_{x_i}\left(|u(x)|^{r-2}u(x)\right) \qquad \text{for a.e. } x \in \Omega. \end{aligned}$$

Moreover, since, by Lemma 5, $(r-1)|u|^{r-2}\partial_{x_i}u = \partial_{x_i}(|u|^{r-2}u) \in L^p(\Omega)$, it follows that

$$\begin{aligned} \left|\partial_{x_i}\left(|u_n(x)|^{r-2}u_n(x)\right)\right| \quad &= \quad (r-1)|u_n(x)|^{r-2}|\partial_{x_i}u_n(x)| \\ &\leq \quad (r-1)|u(x)|^{r-2}|\partial_{x_i}u(x)| \in L^p(\Omega). \end{aligned}$$

Therefore we can deduce that

$$\partial_{x_i}\left(|u_n|^{r-2}u_n\right) \to \partial_{x_i}\left(|u|^{r-2}u\right) \qquad \text{strongly in } L^p(\Omega). \blacksquare$$

Proof of Proposition 4 We recall the approximate sequence $u_n := \zeta_n(u(\cdot)) \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ of u again and compute

$$\begin{aligned} &\int_\Omega |\nabla u_n(x)|^{p-2}\nabla u_n(x) \cdot \nabla\left(|u_n(x)|^{r-2}u_n(x)\right)dx \\ &= \quad (r-1)\int_\Omega |\nabla u_n(x)|^p|u_n(x)|^{r-2}dx \\ &= \quad (r-1)\int_\Omega \Big||u_n(x)|^{(r-2)/p}\nabla u_n(x)\Big|^p dx \\ &= \quad (r-1)\left(\frac{p}{r+p-2}\right)^p\int_\Omega \left|\nabla\left(|u_n(x)|^{(r-2)/p}u_n(x)\right)\right|^p dx. \end{aligned}$$

Now letting $n \to +\infty$ and noting Lemmas 4 and 6, we can derive

$$\int_\Omega |\nabla u_n(x)|^{p-2}\nabla u_n(x) \cdot \nabla \left(|u_n(x)|^{r-2}u_n(x)\right) dx \\ \to \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla \left(|u(x)|^{r-2}u(x)\right) dx,$$

which implies that $|\nabla(|u_n|^{(r-2)/p}u_n)|_{L^p} \le C$. Moreover, we notice that $p \le r+p-2 \le p(r-1)$ and

$$\left||u_n|^{(r-2)/p}u_n\right|_{L^p} = |u_n|_{L^{r+p-2}}^{(r+p-2)/p} \le |u|_{L^{r+p-2}}^{(r+p-2)/p}.$$

Thus, by Lemma 4, $|u_n|^{(r-2)/p}u_n \to |u|^{(r-2)/p}u$ weakly in $W^{1,p}(\Omega)$. Furthermore, by Lemma 6 with r replaced by $(r-2)/p+2$, we deduce that

$$|u_n|^{(r-2)/p}u_n \to |u|^{(r-2)/p}u \qquad \text{strongly in } W^{1,p}(\Omega).$$

Thus we assure that (A. 2) holds true. ∎

B Regularity of solutions for $(\mathrm{CP})_\lambda^\sigma$

In this section, we discuss the existence, the uniqueness and the regularity of solutions of the approximate problems $(\mathrm{CP})_\lambda^\sigma$ with $u_0 \in D(\varphi^\sigma)$ and $f \in L^2(0,T;H) \cap L^1(0,T;L^r(\Omega))$. The existence part and the uniqueness part can be proved in virtue of Theorem 3.6 and Proposition 3.12 of [6], since $\partial_H\overline{\psi}_\lambda$ is Lipschitz continuous from H into itself. Moreover, the unique solution u of $(\mathrm{CP})_\lambda^\sigma$ belongs to $W^{1,2}(0,T;H)$ and satisfies that $u(t) \in V$, $\phi(u(t)) \le \sigma$ for all $t \in [0,T]$ and the function $t \mapsto \varphi^\sigma(u(t))$ is absolutely continuous on $[0,T]$.

Furthermore, we claim that $|u|^{(r-2)/p}u \in L^p(0,T;W^{1,p}(\Omega)))$. Indeed, multiplying $(\mathrm{CP})_\lambda^\sigma$ by $\partial_H\phi_\mu(u(t))$ with $\mu > 0$, by Proposition 2, we get

$$\text{(B. 1)} \qquad \frac{d}{dt}\phi_\mu(u(t)) + (g(t), \partial_H\phi_\mu(u(t)))_H \\ \le (\partial_H\overline{\psi}_\lambda(u(t)), \partial_H\phi_\mu(u(t)))_H + (f(t), \partial_H\phi_\mu(u(t)))_H,$$

where $g(t) := f(t) - du(t)/dt + \partial_H\overline{\psi}_\lambda(u(t)) \in \partial_H\overline{\varphi}^\sigma(u(t))$, for a.e. $t \in (0,T)$. Then, by Lemma 2, it follows from (4. 2) that

$$\alpha|\nabla v_\mu(t)|_{L^p}^p \le \langle \partial_H\overline{\varphi}^\sigma(u(t)), \partial_H\phi_\mu(u(t))\rangle,$$

where $v_\mu(t) := |J_\mu^\phi u(t)|^{(r-2)/p}J_\mu^\phi u(t)$. Moreover, by Proposition 3, $\partial_H\overline{\psi}_\lambda$ also becomes Lipschitz continuous from $L^r(\Omega)$ into itself; hence, by Proposition 1, we have

$$\left(\partial_H\overline{\psi}_\lambda(u(t)), \partial_H\phi_\mu(u(t))\right)_H \le |\partial_H\overline{\psi}_\lambda(u(t))|_{L^r}|\partial_H\phi_\mu(u(t))|_{L^{r'}} \\ \le C_\lambda\phi(u(t)) \le C_\lambda\sigma$$

for some constant C_λ depending on λ but not on μ. Furthermore, we can also obtain

$$(f(t), \partial_H\phi_\mu(u(t)))_H \le C\sigma^{1/r'}|f(t)|_{L^r}.$$

Combining these facts and integrating (B. 1) over $(0,t)$, by Proposition 1, we can deduce that

$$\text{(B. 2)} \qquad \begin{aligned} &\phi(J_\mu^\phi u(t)) + \alpha \int_0^t |\nabla v_\mu(\tau))|_{L^p}^p d\tau \\ &\leq \phi(u_0) + C_\lambda \sigma T + C\sigma^{1/r'} \int_0^T |f(\tau)|_{L^r} d\tau \quad \forall t \in [0,T]. \end{aligned}$$

Now, by passing to the limit as $\mu \to 0$, as in (4. 23), we can derive

$$J_\mu^\phi u \to u \qquad \text{strongly in } C([0,T];H).$$

Furthermore, since $2(r+p-2)/p \leq r$, it follows from (B. 2) that

$$\begin{aligned} J_\mu^\phi u \to u & \qquad \text{weakly star in } L^\infty(0,T;L^r(\Omega)), \\ v_\mu \to v & \qquad \text{weakly star in } L^\infty(0,T;H), \\ & \qquad \text{weakly in } L^p(0,T;W^{1,p}(\Omega)) \end{aligned}$$

and $v = |u|^{(r-2)/p}u$, which proves the claim.

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