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Existence and uniqueness of viscosity solutions for a fully nonlinear parabolic equation associated with the infinity-Laplacian

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Abstract. The existence, uniqueness and regularity of viscosity solutions to the Cauchy-Dirichlet problem are proved for a fully nonlinear parabolic equation of the form $u_t = \Delta_\infty u$, where Δ_∞ denotes the so-called infinity-Laplacian given by $\Delta_\infty u = \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}$. To do so, a coercive regularization of the equation is introduced and barrier function arguments are also employed to verify the equi-continuity of approximate solutions. Furthermore, the Cauchy problem is also studied for (possibly) unbounded initial data satisfying a linear growth condition by using Perron's method.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. In this paper, we study the existence, uniqueness and regularity of viscosity solutions $u = u(x,t)$ in $Q = \Omega \times (0,T)$ for the following Cauchy-Dirichlet problem:

$$u_t = \Delta_\infty u \quad \text{in} \quad Q, \tag{1}$$
$$u = \varphi \quad \text{on} \quad \mathcal{P}Q, \tag{2}$$

where u_t denotes the time-derivative of u, $\mathcal{P}Q$ denotes the parabolic boundary of Q and Δ_∞ stands for the so-called infinity-Laplacian given by

$$\Delta_\infty \phi(x) = \sum_{i,j=1}^N \frac{\partial \phi}{\partial x_i}(x) \frac{\partial \phi}{\partial x_j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x). \tag{3}$$

The infinity-Laplacian is first introduced by Aronsson [2] to investigate the existence of absolutely minimizing Lipschitz extensions (AMLE's for short) of functions g defined only on the boundary $\partial\Omega$ into Ω. According to Jensen's formulation [7], the AMLE of g into Ω means a function $u \in W^{1,\infty}(\Omega)$ satisfying that $u = g$ on $\partial\Omega$ and that for every open subset U of Ω and $\phi \in W^{1,\infty}(U)$, if $u - \phi \in W_0^{1,\infty}(U)$, then

$$|Du|_{L^\infty(U)} \leq |D\phi|_{L^\infty(U)}.$$

In [2], the following elliptic problem is also proposed as an Euler equation for smooth AMLE's.

$$\Delta_\infty u = 0 \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega. \tag{4}$$

In Jensen's celebrated work [7], the existence and uniqueness of AMLE's are rigorously demonstrated under somewhat general assumptions, and moreover, it is also shown that u is a viscosity solution of (4) if and only if u is the AMLE of g. Furthermore, various problems related to the infinity Laplacian, e.g., limiting problems associated with p-Laplacian as $p \to +\infty$, Harnack's inequality, eigenvalue problem, have been studied by many authors. The exciting developments in this field are summarized in Aronsson, Crandall and Juutinen [3].

To the best of the authors' knowledge, parabolic problems associated with the infinity-Laplacian such as (1) have not been studied yet except in [5], [12] and [8]. In Crandall and Wang [5], a characterization of subsolutions to (1) is proposed in terms of the comparison properties with the functions: $\Phi^{bdz}(x,t) := b^3(8/9)^2 t + b(|x-z| + d)^{4/3}$ ($b, d \in \mathbb{R}$, $z \in \mathbb{R}^N$), which also become subsolutions of (1). More precisely, they prove that an upper semicontinuous function $u : Q \to \mathbb{R}$ is a viscosity subsolution in Q of (1) if and only if $u - \Phi^{bdz}$ satisfies the parabolic maximum principle in Q, i.e.,

$$\max_{Q'} \left(u - \Phi^{bdz}\right) = \max_{\mathcal{P}Q'} \left(u - \Phi^{bdz}\right) \quad \text{ for any } \ Q' \subset\subset Q \cap \mathrm{dom}(\Phi^{bdz}),$$

for all $b, d \in \mathbb{R}$ and $z \in \mathbb{R}^N$. In [12], Ôtani proposes a new method of establishing gradient estimates of the form $\sup_{t\in[0,T]} |Du(\cdot,t)|_{L^\infty(\mathbb{R}^N)} \leq C$ for solutions u to the Cauchy problem (1)-(2) with $\Omega = \mathbb{R}^N$, provided that $\varphi = \varphi(x) \in W^{1,\infty}(\mathbb{R}^N)$, by using the L^∞-energy method he developed, which is not used in this paper.

Another type of parabolic equation associated with the infinity-Laplacian is also studied by Juutinen and Kawohl in [8], where they treat the following:

$$u_t = \frac{\Delta_\infty u}{|Du|^2} \quad \text{ in } Q. \tag{5}$$

They investigate the existence and uniqueness of solutions of the Cauchy-Dirichlet problem for (5) with initial-boundary data φ, and moreover, they deal with the Cauchy problem for the case $\Omega = \mathbb{R}^N$ as well. To prove the existence, they introduce approximate problems of the form $(u_{\varepsilon,\delta})_t = \varepsilon\Delta u_{\varepsilon,\delta} + \Delta_\infty u_{\varepsilon,\delta}/(|Du_{\varepsilon,\delta}|^2 + \delta)$, and establish boundary Hölder estimates of their solutions by constructing barrier functions such as $w(x,t) = \varphi(x_0,t_0) + C_1|x - x_0|^\alpha + C_2(t_0 - t)$ with constants C_1, C_2 and $\alpha \in (0,1)$ for each point $(x_0, t_0) \in \partial\Omega \times (0, T)$. Their barrier function argument partially relies on the form:

$$\frac{\Delta_\infty u}{|Du|^2} = \left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \right\rangle.$$

The main purpose of this paper is to investigate the comparison, uniqueness, existence and regularity of viscosity solutions of the Cauchy-Dirichlet problem (1)-(2). Particularly, to prove the existence, we introduce the following approximate problems:

$$(u_\varepsilon)_t = \varepsilon \left(|Du_\varepsilon|^2 + \varepsilon \right) \Delta u_\varepsilon + \Delta_\infty u_\varepsilon \quad \text{in } Q. \tag{6}$$

As in [8], we employ barrier function arguments to establish a priori estimates for solutions of the Cauchy-Dirichlet problems for (6) with initial-boundary data φ. Our proof of the existence is inspired by [8]. Moreover, we also prove the Lipschitz regularity of solutions, provided that φ is Lipschitz continuous in $\overline{Q}$. Furthermore, the existence of solutions to the Cauchy problem for (1) with $\Omega = \mathbb{R}^N$ is also studied for unbounded initial data satisfying boundedness conditions of $D\varphi$ and $D^2\varphi$ by using Perron's method.

This paper is composed of five sections. In the next section, we state our main results on the comparison, uniqueness, existence and Lipschitz regularity of viscosity solutions of the Cauchy-Dirichlet problem (1)-(2). Sections 3 and 4 are devoted to proofs of the existence and regularity results, respectively. Moreover, Section 5 is concerned with the Cauchy problem for (1) with $\Omega = \mathbb{R}^N$.

Notation: Throughout of this paper, we use the following notation: $Q = \Omega \times (0, T)$, $\mathcal{S}Q = \partial\Omega \times (0, T)$, $\mathcal{B}Q = \Omega \times \{0\}$, $\mathcal{C}Q = \partial\Omega \times \{0\}$, $\mathcal{P}Q = \mathcal{S}Q \cup \mathcal{B}Q \cup \mathcal{C}Q$,

$$\phi_t = \frac{\partial \phi}{\partial t}, \quad D_i = \frac{\partial}{\partial x_i}, \quad D = (D_1, D_2, \ldots, D_N), \quad D^2_{ij} = \frac{\partial^2}{\partial x_i \partial x_j},$$

and D^2 denotes the $N \times N$ matrix whose (i, j)-th element is D^2_{ij}. Furthermore, we also use the Einstein summation convention, where we sum over repeated Greek indices. As for the definitions of function spaces such as $C^{2,1}$, H^α and $H^{\ell,\ell/2}$ and (semi-)norms, we refer the reader to [10, pp. 7-8]. Moreover, we denote by $Lip(Q)$ the class of Lipschitz continuous functions in Q, and we simply denote by $|\cdot|_\infty$ the sup-norm in the corresponding space if no confusion arises.

2 Main Results

Priori to state our main results, we give a couple of definitions to be used. First, we define

$$P(s, p, X) := p_i p_j X_{ij} - s, \quad (s, p, X) \in \mathbb{R} \times \mathbb{R}^N \times S^N,$$

where S^N denotes the set of all symmetric $N \times N$ matrices. We are then concerned with viscosity solutions of (1) given in the following.

Definition 2.1 *Let Ω be a domain in $\mathbb{R}^N$ and let $Q = \Omega \times (0,T)$. A function $u \in USC(Q)$ is said to be a viscosity subsolution in Q of* (1) *if*

$$P(\phi_t(\hat{x},\hat{t}), D\phi(\hat{x},\hat{t}), D^2\phi(\hat{x},\hat{t})) \geq 0$$

for all $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^{2,1}(Q)$ satisfying $u - \phi$ attains its local maximum at $(\hat{x},\hat{t})$.

A function $u \in LSC(Q)$ is said to be a viscosity supersolution in Q of (1) *if*

$$P(\phi_t(\hat{x},\hat{t}), D\phi(\hat{x},\hat{t}), D^2\phi(\hat{x},\hat{t})) \leq 0$$

for all $(\hat{x},\hat{t}) \in Q$ and $\phi \in C^{2,1}(Q)$ satisfying $u - \phi$ attains its local minimum at $(\hat{x},\hat{t})$.

Moreover, $u \in C(Q)$ is said to be a viscosity solution in Q of (1) *if it is both a viscosity subsolution and a viscosity supersolution in Q of* (1).

Furthermore, viscosity solutions of the Cauchy-Dirichlet problem (1)-(2) are defined as follows:

Definition 2.2 *A function $u \in USC(\overline{Q})$ (resp., $LSC(\overline{Q})$) is said to be a viscosity subsolution (resp., supersolution) in Q of* (1)-(2) *if u is a viscosity subsolution (resp., supersolution) in Q of* (1), *$u \leq \varphi$ (resp., $u \geq \varphi$) on $\mathcal{P}Q$. Furthermore, $u \in C(\overline{Q})$ is a viscosity solution in Q of* (1)-(2) *if it is both a viscosity subsolution and a viscosity supersolution in Q of* (1)-(2).

Applying Theorem 8.2 of [4], we can derive immediately the comparison principle for (1)-(2), and moreover, it also provides the continuous dependence on initial-boundary data and the uniqueness of solution.

Theorem 2.3 (Comparison and uniqueness) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in USC(\overline{Q})$ and $v \in LSC(\overline{Q})$ be a viscosity sub- and supersolution in $Q = \Omega \times (0,T)$ of* (1), *respectively, such that $u \leq v$ on $\mathcal{P}Q$. Then $u \leq v$ in Q.*

In particular, let $\varphi_1, \varphi_2 \in C(\overline{Q})$ and let u_1 and u_2 be viscosity solutions in Q of (1)-(2) *with the initial-boundary data φ_1 and φ_2, respectively. Then it follows that*

$$\sup_{(x,t)\in Q} |u_1(x,t) - u_2(x,t)| \leq \sup_{(x,t)\in \mathcal{P}Q} |\varphi_1(x,t) - \varphi_2(x,t)|, \tag{7}$$

which also implies the uniqueness of solution.

Proof of Theorem 2.3 Due to Theorem 8.2 of [4], the comparison part follows immediately. Now, let u_1 and u_2 be viscosity solutions of (1)-(2) with initial-boundary data φ_1 and φ_2, respectively, and put $w^{\pm}(x,t) := u_2(x,t) \pm \sup_{(x,t)\in\mathcal{P}Q} |\varphi_1(x,t) - \varphi_2(x,t)|$. Then the functions w^- and w^+ become a viscosity sub- and supersolution of (1)-(2) with φ replaced by φ_1, respectively. Thus we have

$$w^- \leq u_1 \leq w^+ \text{ in } Q,$$

which implies (7). In particular, if $\varphi_1 = \varphi_2$ on $\mathcal{P}Q$, then the uniqueness of solution follows. ∎

As for the existence of solution, we first introduce a uniform *exterior sphere condition* defined in the following.

$$(8)\begin{cases} \text{For all } x_0 \in \partial\Omega, \text{ there exists } y_0 \in \mathbb{R}^N \text{ such that } |x_0 - y_0| = R \text{ and} \\ \{x \in \mathbb{R}^N; |x - y_0| < R\} \cap \Omega = \emptyset \text{ for some positive constant } R \text{ independent of } x_0. \end{cases}$$

This condition is employed only for the construction of approximate solutions in classical sense. Now, our result reads:

THEOREM 2.4 (Existence) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $Q = \Omega \times (0,T)$. Suppose that* (8) *is satisfied. Then, for every $\varphi \in C(\overline{Q})$, the Cauchy-Dirichlet problem* (1)-(2) *admits a viscosity solution $u \in C(\overline{Q})$ in Q such that*

$$\sup_{(x,t)\in Q} |u(x,t)| \le \sup_{(x,t)\in \mathcal{P}Q} |\varphi(x,t)|. \tag{9}$$

In particular, if φ is Lipschitz continuous on $\overline{Q}$, then we can also prove the Lipschitz continuity of the solution u in Q for the Cauchy-Dirichlet problem (1)-(2).

THEOREM 2.5 (Lipschitz regularity) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let u be a viscosity solution of the Cauchy-Dirichlet problem* (1)-(2) *with $\varphi \in Lip(\overline{Q})$. Then there exists a constant L depending only on $|\varphi|_\infty$, $|D\varphi|_\infty$, $|\varphi_t|_\infty$,* diam(Ω) *such that*

$$|u(x,t) - u(y,t)| \le L|x-y| \quad \textit{for all } x, y \in \Omega \textit{ and } t \in (0,T), \tag{10}$$

where diam(Ω) *stands for the diameter of Ω, i.e.,* $\mathrm{diam}(\Omega) = \sup_{x\in\Omega}\sup_{y\in\Omega}|x-y|$.

In Section 4, we also provide a couple of variants of the above regularity result.

3 Existence (Proof of Theorem 2.4)

In this section, we give a proof of Theorem 2.4, which is concerned with the existence of viscosity solutions of the Cauchy-Dirichlet problem (1)-(2). First, we deal with the case $\varphi \in H^{2+\alpha,1+\alpha/2}(\overline{Q})$ for some $\alpha \in (0,1)$. We then introduce the following approximation of (1)-(2) for all $\varepsilon \in (0,1)$.

$$(u_\varepsilon)_t = \varepsilon\left(|Du_\varepsilon|^2 + \varepsilon\right)\Delta u_\varepsilon + \Delta_\infty u_\varepsilon \quad \text{in} \quad Q, \tag{11}$$

$$u_\varepsilon = \varphi \quad \text{on} \quad \mathcal{P}Q. \tag{12}$$

Define $a_{ij}^\varepsilon \in C^\infty(\mathbb{R}^N)$ and $P_\varepsilon \in C(\mathbb{R} \times \mathbb{R}^N \times S^N)$ by

$$a_{ij}^\varepsilon(p) := \varepsilon(|p|^2 + \varepsilon)\delta_{ij} + p_i p_j, \quad i,j = 1,2,\dots,N, \quad p \in \mathbb{R}^N$$

and

$$P_\varepsilon(s,p,X) := a_{ij}^\varepsilon(p)X_{ij} - s, \quad (s,p,X) \in \mathbb{R} \times \mathbb{R}^N \times S^N.$$

Then, (11) is rewritten into

$$P_\varepsilon((u_\varepsilon)_t(x,t), Du_\varepsilon(x,t), D^2u_\varepsilon(x,t)) = 0, \quad (x,t) \in Q.$$

Moreover, we observe that

$$\varepsilon(|p|^2+\varepsilon)|\xi|^2 \le a_{ij}^\varepsilon(p)\xi_i\xi_j \le \left\{\varepsilon(|p|^2+\varepsilon)+|p|^2\right\}|\xi|^2$$

for all $\xi \in \mathbb{R}^N$, and furthermore,

$$\left|\frac{\partial a_{ij}^\varepsilon}{\partial p_k}\right|(1+|p|)^3 \le C(1+|p|)^4, \quad i,j,k = 1,2,\ldots,N.$$

Thus, since Ω satisfies (8), Theorem 4.4 of [10, Chap. VI, p. 560] ensures that the Cauchy-Dirichlet problems (11)-(12) admits a classical solution $u_\varepsilon \in C(\overline{Q}) \cap H^{2+\alpha,1+\alpha/2}(Q)$.

We now establish a priori estimates for solutions u_ε of the Cauchy-Dirichlet problems (11)-(12) for each $\varepsilon \in (0,1)$. To derive the convergence of u_ε as $\varepsilon \to +0$, by grace of the stability of viscosity solutions, it suffices to obtain a Hölder estimate for u_ε on $\overline{Q}$, which implies the precompactness of u_ε in $C(\overline{Q})$. The following lemma is concerned with an L^∞-estimate for u_ε.

LEMMA 3.1 (L^∞-estimate) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0,T)$ of the Cauchy-Dirichlet problem* (11)-(12) *with $\varphi \in C(\overline{Q})$. Then we have*

$$|u|_\infty \;\le\; |\varphi|_\infty.$$

PROOF OF LEMMA 3.1 The function $w^+(x,t) \equiv |\varphi|_\infty$ (resp., $w^-(x,t) \equiv -|\varphi|_\infty$) becomes a classical supersolution (resp., subsolution) in Q of (11)-(12), so the classical comparison principle (see, e.g., Theorem 9.1 of [11, p. 213]) implies that $|u|_\infty \le |\varphi|_\infty$. ∎

We have several steps to establish Hölder estimates for u_ε in Q. The first step is concerned with a Lipschitz estimate for $u_\varepsilon(x,\cdot)$ at $t=0$ (see Lemma 3.2), and the second step yields a Lipschitz estimate at any $t \in (0,T)$ (see Lemma 3.3). In the third step, we estimate a Hölder constant of $u_\varepsilon(\cdot,t)$ on $\partial\Omega$ (see Lemma 3.4). Hence these three steps imply a boundary Hölder estimate on $\mathcal{P}Q$ (see Lemma 3.5). Finally, we derive a global Hölder estimate for u_ε in Q from the boundary Hölder estimate (see Lemma 3.6). Our derivations of these estimates are due to the well-known barrier function argument, and we also employ the translation invariance of the equation (11) to extend Lipschitz and Hölder estimates established only on the boundary, e.g., $t=0$, $\partial\Omega$, $\mathcal{P}Q$, as in [9].

LEMMA 3.2 (Lipschitz estimate for $u_\varepsilon(x,\cdot)$ at $t=0$) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0,T)$ of the Cauchy-Dirichlet problem* (11)-(12) *with $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that*

$$|u(x,t) - \varphi(x,0)| \le M_1 t \quad \textit{for all } t \in (0,T) \textit{ and } x \in \Omega, \tag{13}$$

where $M_1 := 2(|D\varphi|_\infty^2 + 1)|D^2\varphi|_\infty + |\varphi_t|_\infty$.

PROOF OF LEMMA 3.2 Put $w^{\pm}(x,t) = \varphi(x,0) \pm M_1 t$ and observe that

$$\begin{aligned} & P(w_t^+(x,t), Dw^+(x,t), D^2 w^+(x,t)) \\ = \ & -M_1 + a_{ij}^{\varepsilon}(D\varphi(x,0)) D_{ij}^2 \varphi(x,0) \\ \leq \ & -M_1 + \varepsilon(|D\varphi|_\infty^2 + \varepsilon)|D^2\varphi|_\infty + |D\varphi|_\infty^2 |D^2\varphi|_\infty \leq 0 \end{aligned}$$

for all $(x,t) \in Q$. Moreover, if $(x,t) \in \mathcal{P}Q$, then

$$\begin{aligned} w^+(x,t) &= \varphi(x,0) + M_1 t \\ &= \varphi(x,t) - \varphi(x,t) + \varphi(x,0) + M_1 t \\ &\geq \varphi(x,t) - |\varphi_t|_\infty t + M_1 t \geq \varphi(x,t). \end{aligned}$$

We can also deduce that $P(w_t^-(x,t), Dw^-(x,t), D^2 w^-(x,t)) \geq 0$ for all $(x,t) \in Q$ and $w^- \leq \varphi$ on $\mathcal{P}Q$. Therefore, the classical comparison principle ensures that $w^- \leq u \leq w^+$ in Q. Hence we obtain (13). ∎

By using the translation invariance of the equations (11) and the above lemma, we can derive a Lipschitz estimate for $u_\varepsilon(x,\cdot)$ in $(0,T)$.

LEMMA 3.3 (Lipschitz estimate for $u_\varepsilon(x,\cdot)$ in $(0,T)$) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0,T)$ of the Cauchy-Dirichlet problem* (11)-(12) *with $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that*

$$|u(x,t) - u(x,s)| \leq M_1 |t-s| \quad \text{for all } t,s \in (0,T) \text{ and } x \in \Omega, \tag{14}$$

where $M_1 = 2(|D\varphi|_\infty^2 + 1)|D^2\varphi|_\infty + |\varphi_t|_\infty$.

PROOF OF LEMMA 3.3 Let $h \in (-T,T)$ be fixed and set $Q_h = \Omega \times (h, T+h)$. Putting $v(x,t) = u(x,t-h)$, we see that v remains to be a solution in Q_h of (11)-(12) with φ replaced by $\varphi(\cdot, \cdot - h)$. Hence, by Lemma 3.2, we infer that

$$|v(x,t) - u(x,t)| \leq M_1 |h| \quad \text{for all } (x,t) \in \mathcal{B}(Q \cap Q_h).$$

Here we used the fact that $t = \max\{0,h\}$ if $(x,t) \in \mathcal{B}(Q \cap Q_h)$. Thus we can derive $u \leq v + M_1|h|$ on $\mathcal{B}(Q \cap Q_h)$. Moreover, if $(x,t) \in \mathcal{S}(Q \cap Q_h)$, then we see that $(x,t) \in \mathcal{S}Q$, which implies that

$$\begin{aligned} v(x,t) + M_1|h| &= u(x,t-h) + M_1|h| \\ &= \varphi(x,t-h) + M_1|h| \\ &\geq \varphi(x,t) = u(x,t). \end{aligned}$$

Therefore, since $u(x,t) \leq v(x,t) + M_1|h|$ for all $(x,t) \in \mathcal{P}(Q \cap Q_h)$ and $v + M_1|h|$ also becomes a supersolution in $Q \cap Q_h$ of (11), it follows that $u \leq v + M_1|h|$ in $Q \cap Q_h$. Repeating the above argument with $v + M_1|h|$ replaced by $v - M_1|h|$, we can deduce that $v - M_1|h| \leq u \leq v + M_1|h|$ in $Q \cap Q_h$, which also gives $|u(x,t) - u(x,t-h)| \leq M_1|h|$ for all $(x,t) \in Q \cap Q_h$. Furthermore, using the arbitrariness of h, we can verify (14). ∎

We next proceed to the third step, where a Hölder estimate is derived for $u_\varepsilon(\cdot,t)$ on $\partial\Omega$.

Lemma 3.4 (Hölder estimate for $u_\varepsilon(\cdot,t)$ on $\partial\Omega$) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $\alpha\in(0,1)$ and $R>0$ be fixed. Let $u\in C(\overline{Q})\cap C^{2,1}(Q)$ be a classical solution in $Q=\Omega\times(0,T)$ of the Cauchy-Dirichlet problem* (3.1)-(3.2) *with $\varphi\in C(\overline{Q})$ satisfying*

$$|u_t|_\infty<\infty \quad \textit{and} \quad \langle\varphi\rangle^\alpha_{x,Q}:=\sup\left\{\frac{|\varphi(x,t)-\varphi(y,t)|}{|x-y|^\alpha}; x,y\in\Omega, x\neq y, t\in[0,T]\right\}<\infty.$$

Then there exists a number M_2 depending only on $|\varphi|_\infty$, $|\varphi_t|_\infty$, $\langle\varphi\rangle^\alpha_{x,Q}$, N, α and R such that

$$\begin{aligned}&|u(x,t)-\varphi(x_0,t_0)|\le M_2(|x-x_0|^\alpha+t_0-t)\\ &\quad \textit{for all } (x_0,t_0)\in\mathcal{S}Q,\ x\in\Omega\cap B_R(x_0) \textit{ and } t\in(\max\{0,t_0-1\},t_0),\\ &\qquad \textit{where } B_R(x_0):=\{x\in\mathbb{R}^N; |x-x_0|<R\}.\end{aligned}$$

In particular, the same conclusion also follows with $\Omega\cap B_R(x_0)$ replaced by Ω by choosing $R>0$ enough large.

Proof of Lemma 3.4 Let $(x_0,t_0)\in\mathcal{S}Q$ and $\alpha\in(0,1)$ be fixed and define

$$w^+(x,t)=\varphi(x_0,t_0)+\kappa|x-x_0|^\alpha+\rho(t_0-t)$$

for all $x\in B_R(x_0):=\{x\in\mathbb{R}^N; |x-x_0|<R\}$ and all $t<t_0$ with positive constants κ and ρ which will be determined later. Observing that

$$\begin{aligned}&w^+_t(x,t)=-\rho,\quad D_iw^+(x,t)=\kappa\alpha|x-x_0|^{\alpha-2}(x-x_0)_i,\\ &D^2_{ij}w^+(x,t)=\kappa\alpha(\alpha-2)|x-x_0|^{\alpha-4}(x-x_0)_i(x-x_0)_j+\kappa\alpha|x-x_0|^{\alpha-2}\delta_{ij},\end{aligned}$$

we then see that

$$\Delta_\infty w^+(x,t) \;=\; (\kappa\alpha)^3(\alpha-1)|x-x_0|^{3\alpha-4}.$$

Thus, it follows that

$$\begin{aligned}&-w^+_t(x,t)+a^\varepsilon_{ij}(Dw^+(x,t))D^2_{ij}w^+(x,t)\\ &\quad=\rho+(\kappa\alpha)^3\left\{\varepsilon(\alpha-2+N)+\alpha-1\right\}|x-x_0|^{3\alpha-4}\\ &\qquad+\varepsilon^2\kappa\alpha(\alpha-2+N)|x-x_0|^{\alpha-2}.\end{aligned}$$

Here taking ε enough small such that

$$\varepsilon(\alpha-2+N)+\alpha-1<\frac{1}{2}(\alpha-1),$$

we have

$$\begin{aligned}&-w^+_t(x,t)+a^\varepsilon_{ij}(Dw^+(x,t))D^2_{ij}w^+(x,t)\\ &<\rho+\frac{(\kappa\alpha)^3}{2}(\alpha-1)|x-x_0|^{3\alpha-4}+\varepsilon^2\kappa\alpha(\alpha-2+N)|x-x_0|^{\alpha-2}\\ &=\rho+\kappa\alpha|x-x_0|^{\alpha-2}\left\{\frac{(\kappa\alpha)^2}{2}(\alpha-1)|x-x_0|^{2\alpha-2}+\varepsilon^2(\alpha-2+N)\right\}\\ &\le\rho+\kappa\alpha|x-x_0|^{\alpha-2}\left\{\frac{(\kappa\alpha)^2}{2}(\alpha-1)R^{2\alpha-2}+\varepsilon^2(\alpha-2+N)\right\},\end{aligned}$$

where we used the fact that $|x-x_0|<R$. Hence

$$\frac{(\kappa\alpha)^2}{2}(\alpha-1)R^{2\alpha-2}+\varepsilon^2(\alpha-2+N)\le\frac{(\kappa\alpha)^2}{4}(\alpha-1)R^{2\alpha-2},$$

provided that $\kappa\ge 1$ and ε is enough small such that

$$\frac{\alpha^2}{4}(\alpha-1)R^{2\alpha-2}+\varepsilon^2(\alpha-2+N)\le 0.$$

Thus,

$$\begin{aligned}
&-w_t^+(x,t)+a_{ij}^\varepsilon(Dw^+(x,t))D_{ij}^2w^+(x,t)\\
&\le\ \rho+\frac{(\kappa\alpha)^3}{4}(\alpha-1)R^{2\alpha-2}|x-x_0|^{\alpha-2}\\
&\le\ \rho+\frac{(\kappa\alpha)^3}{4}(\alpha-1)R^{3\alpha-4}.
\end{aligned}$$

Therefore, taking κ enough large such that $4\rho\le(\kappa\alpha)^3(1-\alpha)R^{3\alpha-4}$, we conclude that

$$-w_t^+(x,t)+a_{ij}^\varepsilon(Dw^+(x,t))D_{ij}^2w^+(x,t)\le 0$$

for all $x\in B_R(x_0)\cap\Omega$ and all $t<t_0$.

We next prove that $w^+\ge u$ on $\mathcal{P}((B_R(x_0)\cap\Omega)\times(t_0-1,t_0))$ for the case that $t_0>1$. To do so, we divide our proof to the following three cases:

(i) Let $x\in(\partial B_R(x_0))\cap\Omega$ and $t<t_0$ be fixed. From the fact that $|x-x_0|=R$, we then see that

$$w^+(x,t)=\varphi(x_0,t_0)+\kappa R^\alpha+\rho(t_0-t)\ge\varphi(x_0,t_0)+\kappa R^\alpha\ge|\varphi|_\infty\ge u(x,t),$$

provided that $\kappa\ge 2|\varphi|_\infty/R^\alpha$.

(ii) Let $x\in B_R(x_0)\cap\partial\Omega$ and $t<t_0$ be fixed. Since $\varphi(x,t)=u(x,t)$, it follows that

$$w^+(x,t)=u(x,t)+\varphi(x_0,t_0)-u(x,t)+\kappa|x-x_0|^\alpha+\rho(t_0-t)\ge u(x,t),$$

provided that $\kappa\ge\langle\varphi\rangle_{x,Q}^\alpha$ and $\rho\ge|\varphi_t|_\infty$.

(iii) Let $x\in B_R(x_0)\cap\Omega$ and let $t=t_0-1$ be fixed. Then

$$w^+(x,t)=\varphi(x_0,t_0)+\kappa|x-x_0|^\alpha+\rho\ge\varphi(x_0,t_0)+\rho\ge|\varphi|_\infty\ge u(x,t),$$

provided that $\rho\ge 2|\varphi|_\infty$.

Now as for the case where $t_0<1$, we employ $(B_R(x_0)\cap\Omega)\times(0,t_0)$ instead of the cylinder used in the last case. Then it is easily seen that, for $x\in B_R(x_0)\cap\Omega$ and $t=0$,

$$w^+(x,0)=\varphi(x_0,t_0)+\kappa|x-x_0|^\alpha+\rho t_0\ge\varphi(x,0)=u(x,0),$$

provided that $\kappa\ge\langle\varphi\rangle_{x,Q}^\alpha$ and $\rho\ge|\varphi_t|_\infty$.

Therefore the comparison principle ensures that

$$u \leq w^+ \quad \text{on} \ \ \overline{B_R(x_0) \cap \Omega} \times [\max\{0, t_0 - 1\}, t_0].$$

Repeating the same argument with the function $w^-(x,t) := \varphi(x_0, t_0) - \kappa|x - x_0|^\alpha - \rho(t_0 - t)$, we can also obtain $w^- \leq u$ on $\overline{B_R(x_0) \cap \Omega} \times [\max\{0, t_0 - 1\}, t_0]$. Consequently, we can deduce that

$$|u(x,t) - \varphi(x_0, t_0)| \leq \kappa|x - x_0|^\alpha + \rho(t_0 - t)$$

for all $(x_0, t_0) \in \mathcal{S}Q$ and $x \in B_R(x_0) \cap \Omega$ and $t \in [\max\{0, t_0 - 1\}, t_0]$. ▮

Thus, Lemmas 3.3 and 3.4 imply the following:

LEMMA 3.5 (Hölder estimate on $\mathcal{P}Q$) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $\alpha \in (0, 1)$. Suppose that* (8) *is satisfied. Let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0, T)$ of the Cauchy-Dirichlet problem* (11)-(12) *with $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that*

(15) $|u(x,t) - \varphi(x_0, t_0)| \leq M_3 \left(|x - x_0|^\alpha + |t - t_0|\right)$ *for all* $(x_0, t_0) \in \mathcal{P}Q$ *and* $(x,t) \in Q$,

where $M_3 = M_1 + M_2 + \langle\varphi\rangle^{(\alpha)}_{x,Q}$.

PROOF OF LEMMA 3.5 For the case: $(x_0, t_0) \in \mathcal{S}Q$, by virtue of Lemmas 3.3 and 3.4,

$$\begin{aligned} |u(x_0, t_0) - u(x,t)| &\leq |u(x_0, t_0) - u(x, t_0)| + |u(x, t_0) - u(x,t)| \\ &\leq M_2|x_0 - x|^\alpha + M_1|t_0 - t|. \end{aligned}$$

For the case: $(x_0, t_0) \in \mathcal{B}Q$, that is, $t_0 = 0$, by Lemma 3.2, we also have

$$\begin{aligned} |u(x_0, 0) - u(x,t)| &\leq |\varphi(x_0, 0) - \varphi(x, 0)| + |\varphi(x, 0) - u(x,t)| \\ &\leq \langle\varphi\rangle^{(\alpha)}_{x,Q}|x_0 - x|^\alpha + M_1 t. \end{aligned}$$

Hence (15) follows. ▮

Now, we extend the above Hölder estimate on the parabolic boundary $\mathcal{P}Q$ into the parabolic domain Q in the following lemma, which is derived from Theorem 6 of [9], but for the completeness, we give a proof.

LEMMA 3.6 (Global Hölder estimate) *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and let $\alpha \in (0, 1)$. Suppose that* (8) *is satisfied. Let $u \in C(\overline{Q}) \cap C^{2,1}(Q)$ be a classical solution in $Q = \Omega \times (0, T)$ of the Cauchy-Dirichlet problem* (11) *and* (12) *with $\varphi \in C^{2,1}(\overline{Q})$. Then it follows that*

(16) $$|u(x,t) - u(y,s)| \leq M_3 \left(|x - y|^\alpha + |t - s|\right) \text{ for all } (x,t), (y,s) \in Q,$$

where $M_3 = M_1 + M_2 + \langle\varphi\rangle^{(\alpha)}_{x,Q}$.

Proof of Lemma 3.6 Let $h := (h_x, h_t) \in \mathbb{R}^N \times \mathbb{R}$ be fixed and let $Q + h := \{(x,t) \in \mathbb{R}^{N+1}; (x - h_x, t - h_t) \in Q\}$. Moreover, put $v(x,t) = u(x - h_x, t - h_t)$. We then find that v still remains to be a solution in $Q + h$ of (11) and (12) with φ replaced by $\varphi(\cdot - h_x, \cdot - h_t)$. Then, by Lemma 3.5, we can assure that, for $(x,t) \in \mathcal{P}\{Q \cap (Q+h)\}$, $|v(x,t) - u(x,t)| \leq M_3 |h|_{\alpha,1}$, where $|h|_{\alpha,1} := |h_x|^\alpha + |h_t|$; hence, $v - M_3|h|_{\alpha,1} \leq u \leq v + M_3|h|_{\alpha,1}$ on $\mathcal{P}\{Q \cap (Q+h)\}$. Furthermore, since $v \pm M_3|h|_{\alpha,1}$ also become a super- and subsolution in $Q \cap (Q+h)$ of (11), the classical comparison theorem ensures that $v - M_3|h|_{\alpha,1} \leq u \leq v + M_3|h|_{\alpha,1}$ in $Q \cap (Q+h)$. Using the arbitrariness of h, we can verify (16). ∎

By virtue of the global Hölder estimate for u_ε in Lemma 3.6 and Ascoli-Arzela's compactness theorem, taking a sequence $\varepsilon_n \to +0$, we can deduce that

$$u_{\varepsilon_n} \to u \quad \text{uniformly on } \overline{Q} \tag{17}$$

as $\varepsilon_n \to +0$. We also note that

$$P_\varepsilon(s,p,X) \to P(s,p,X) \ \text{ as } \varepsilon \to +0, \ \text{ for all } (s,p,X) \in \mathbb{R} \times \mathbb{R}^N \times S^N.$$

Therefore, the stability of viscosity solution (see, e.g., Section 6 of [4]) ensures that the limit u becomes a viscosity solution of (1)-(2).

Now we proceed to the case $\varphi \in C(\overline{Q})$. By virtue of Weierstrass's approximation theorem (see, e.g., 1.29 Corollary of [1, p. 10]), we can take an approximate sequence $\varphi_n \in H^{2+\alpha,1+\alpha/2}(\overline{Q})$ such that $\varphi_n \to \varphi$ uniformly on $\overline{Q}$. Hence, due to the last case, there exists a viscosity solution u_n of (1)-(2) with φ replaced by φ_n. Moreover, by Theorem 2.3,

$$\sup_{(x,t)\in Q} |u_n(x,t) - u_m(x,t)| \leq \sup_{(x,t)\in \mathcal{P}Q} |\varphi_n(x,t) - \varphi_m(x,t)| \to 0$$

as $n, m \to +\infty$. Thus (u_n) forms a Cauchy sequence in $C(\overline{Q})$, so $u_n \to u$ uniformly on $\overline{Q}$. Therefore, from the stability of viscosity solution, u also becomes a viscosity solution of (1)-(2) with the initial data $\varphi \in C(\overline{Q})$. Furthermore, as in Lemma 3.1, (9) follows immediately. This completes our proof of Theorem 2.4. ∎

4 Lipschitz Regularity

In this section, a couple of regularity results are established for solutions of the Cauchy-Dirichlet problem (1)-(2); particularly, we give a proof of Theorem 2.5, which ensures the Lipschitz regularity of solutions of (1)-(2) with $\varphi \in Lip(\overline{Q})$. For this purpose, in the following lemma, we first verify a boundary estimate with a bound L by constructing a barrier function. It is noteworthy that the barrier function used in our proof can not be applied to the approximate problems (11)-(12) in the last section.

Lemma 4.1 *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let $u \in C(\overline{Q})$ be a viscosity solution of* (1)-(2) *with $\varphi \in C(\overline{Q})$ and let $r > 0$. Suppose that there exist a constant λ_0 and a function $\omega \in C^2((0,r])$ such that $\omega(+0) = 0$, $\omega'(s) > 0$ and $\omega''(s) \leq 0$ for all $s > 0$, $\omega_0 := \sup_{s\in(0,r]} \omega'(s)s < +\infty$ and*

$$|\varphi(x,t) - \varphi(x_0,t_0)| \leq \omega(|x - x_0|) + \lambda_0(t_0 - t)$$

for all $(x,t) \in Q$*,* $(x_0,t_0) \in \mathcal{S}Q$ *satisfying* $t \le t_0$ *and* $|x - x_0| \le r$*. Then there exists a constant* $C_1 = C_1(r, |\varphi|_\infty, \lambda_0, \omega_0, \omega'(r), \omega(r)) \ge 1$ *such that*

$$|u(x,t) - \varphi(x_0,t_0)| \le C_1\omega(|x - x_0|) + \lambda_0(t_0 - t) \tag{18}$$
$$\text{for all } (x_0,t_0) \in \mathcal{S}Q \text{ and } (x,t) \in Q \text{ satisfying } |x - x_0| \le r \text{ and } t \le t_0.$$

Proof of Lemma 4.1 Let $(x_0,t_0) \in \mathcal{S}Q$ be fixed and let $N := \{(x,t) \in \mathbb{R}^{N+1}; |x - x_0| < r, t < t_0\}$. Moreover, define $w^\pm(x,t) := \varphi(x_0,t_0) \pm \{C_1\omega(|x-x_0|) + \lambda_0(t_0 - t) - |x - x_0|^2\}$, where C_1 will be determined later. Then we see that

$$\begin{aligned}
&P(w_t^+(x,t), Dw^+(x,t), D^2w^+(x,t)) \\
&\le \ \lambda_0 - 2\left(C_1\frac{\omega'(|x-x_0|)}{|x-x_0|} - 2\right)^2 |x - x_0|^2 \\
&\le \ \lambda_0 - 2C_1^2(\omega'(r))^2 + 8C_1\omega_0 \quad \text{ for all } (x,t) \in Q \cap N.
\end{aligned}$$

Hence, $P(w_t^+(x,t), Dw^+(x,t), D^2w^+(x,t)) \le 0$, provided that $\lambda_0 - 2C_1^2(\omega'(r))^2 + 8C_1\omega_0 \le 0$. Moreover, if $(x,t) \in \mathcal{P}Q \cap N$, then since $\omega(s) = \int_0^s \omega'(\tau)d\tau \ge \omega'(r)s$, we see that $w^+(x,t) \ge \varphi(x,t) - \varphi(x,t) + \varphi(x_0,t_0) + \{C_1 - r(\omega'(r))^{-1}\}\omega(|x-x_0|) + \lambda_0(t_0-t) \ge \varphi(x,t) + \{C_1 - r(\omega'(r))^{-1} - 1\}\omega(|x - x_0|) \ge \varphi(x,t) = u(x,t)$, provided that $C_1 \ge r(\omega'(r))^{-1} + 1$. Furthermore, if $(x,t) \in \mathcal{S}N$, then $w^+(x,t) = \varphi(x_0,t_0) + C_1\omega(r) + \lambda_0(t_0 - t) - r^2 \ge |\varphi|_\infty \ge u(x,t)$, provided that $C_1\omega(r) \ge r^2 + 2|\varphi|_\infty$.

Thus taking $C_1 = C_1(r, |\varphi|_\infty, \lambda_0, \omega_0, \omega'(r), \omega(r))$ enough large such that

$$\lambda_0 - 2C_1^2(\omega'(r))^2 + 8C_1\omega_0 \le 0 \ \text{ and } \ C_1 \ge \max\left(\frac{r}{\omega'(r)} + 1, \frac{r^2 + 2|\varphi|_\infty}{\omega(r)}\right),$$

we observe that $P(w_t^+, Dw^+, D^2w^+) \le 0$ in $Q \cap N$ and $w^+ \ge u$ on $\mathcal{P}(Q \cap N)$. Moreover, repeating the same argument for w^-, we have $P(w_t^-, Dw^-, D^2w^-) \ge 0$ in $Q \cap N$ and $w^- \le u$ on $\mathcal{P}(Q \cap N)$. Hence Theorem 2.3 ensures that $w^- \le u \le w^+$ in $N \cap Q$. Therefore, we obtain (18). ∎

Now, we prove the global Lipschitz estimate in Theorem 2.5 by using the boundary Lipschitz estimate and the comparison principle obtained in Lemma 4.1 and Theorem 2.3, respectively.

Proof of Theorem 2.5 Let $h \in \mathbb{R}^N$ be fixed. We then notice that $v(x,t) := u(x - h,t)$ remains to be a viscosity solution in $\tilde{Q}_h := \{(x,t) \in \mathbb{R}^{N+1}; x - h \in \Omega, t \in (0,T)\}$ of (1)-(2) with φ replaced by $\varphi(\cdot - h, \cdot)$. Moreover, if $(x,t) \in \mathcal{S}(Q \cap \tilde{Q}_h)$, we then deduce that $|v(x,t) - u(x,t)| \le L|h|$, where $L = C_1|D\varphi|_\infty$ and C_1 is a constant appeared in (18), by using Lemma 4.1 with $r = \mathrm{diam}(\Omega) := \sup_{x\in\Omega}\sup_{y\in\Omega}|x - y|$, $\omega(s) = |D\varphi|_\infty s$ and $\lambda_0 = |\varphi_t|_\infty$, so $v - L|h| \le u \le v + L|h|$ on $\mathcal{S}(Q \cap \tilde{Q}_h)$. Further, if $(x,t) \in \mathcal{B}(Q \cap \tilde{Q}_h)$, i.e., $t = 0$, then $v(x,0) + L|h| = \varphi(x - h, 0) + L|h| \ge \varphi(x,0) = u(x,0)$ and $v(x,0) - L|h| \le u(x,0)$. Thus, since $v - L|h| \le u \le v + L|h|$ on $\mathcal{P}(Q \cap \tilde{Q}_h)$ and $v \pm L|h|$ also become viscosity solutions of (1), the Theorem 2.3 implies that $v - L|h| \le u \le v + L|h|$ in $Q \cap \tilde{Q}_h$. Therefore we have

$$|u(x-h,t) - u(x,t)| \le L|h| \quad \text{ for all } (x,t) \in Q \cap \tilde{Q}_h.$$

From the arbitrariness of h, we can obtain (10). ∎

REMARK 4.2 As for the case $\varphi \in C^{2,1}(\overline{Q})$, one can also verify the Lipschitz regularity in time of viscosity solutions in Q of (1)-(2), that is,

$$(19) \qquad |u(x,t) - u(x,s)| \leq M_4|t-s| \quad \text{for all } x \in \Omega,\ t,s \in (0,T)$$

with $M_4 = |D\varphi|_\infty^2 |D^2\varphi|_\infty + |\varphi_t|_\infty$, by repeating the similar argument as in proofs of Lemmas 3.2 and 3.3.

Furthermore, we give another type of regularity result in the following.

COROLLARY 4.3 *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let $u \in C(\overline{Q})$ be a viscosity solution of the Cauchy-Dirichlet problem* (1)-(2) *with $\varphi \in C(\overline{Q})$. Suppose that*

$$\langle \varphi \rangle_{x,Q}^{(\alpha)} := \sup_{t \in [0,T]} \sup \left\{ \frac{|\varphi(x,t) - \varphi(y,t)|}{|x-y|^\alpha}; x,y \in \Omega,\ x \neq y \right\} < +\infty, \quad |\varphi_t|_\infty < +\infty$$

for some $\alpha \in (0,1)$. Then there exists a constant L depending only on α, $|\varphi|_\infty$, $\langle \varphi \rangle_{x,Q}^{(\alpha)}$, $|\varphi_t|_\infty$, diam(Ω) *such that*

$$(20) \qquad |u(x,t) - u(y,t)| \leq L|x-y|^\alpha \quad \textit{for all } x,y \in \Omega \textit{ and } t \in (0,T).$$

PROOF OF LEMMA 4.3 Apply Lemma 4.1 with

$$r = \operatorname{diam}(\Omega),\ \lambda_0 = |\varphi_t|_\infty,\ \omega(s) = \langle \varphi \rangle_{x,Q}^{(\alpha)} s^\alpha.$$

We then observe that $w'(s) = \langle \varphi \rangle_{x,Q}^{(\alpha)} \alpha s^{\alpha-1} > 0$ and $w''(s) = \langle \varphi \rangle_{x,Q}^{(\alpha)} \alpha(\alpha-1)s^{\alpha-2} < 0$ for all $s > 0$. Thus repeating the same argument as in the proof of Theorem 2.5, we can derive (20). ∎

In particular, if $\varphi = \varphi(x)$ is independent of t, we can then establish a sharper estimate. The following lemma will be also employed in §5.

COROLLARY 4.4 *Let Ω be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let $u \in C(\overline{Q})$ be a viscosity solution of the Cauchy-Dirichlet problem* (1)-(2) *with $\varphi \in H^\alpha(\overline{\Omega})$ (resp., $Lip(\overline{\Omega})$) and $\alpha \in (0,1)$ (resp., $\alpha = 1$). Then it follows that*

$$(21) \qquad |u(x,t) - u(y,t)| \leq L|x-y|^\alpha \quad \textit{for all } x,y \in \Omega \textit{ and } t \in (0,T)$$

with $L = \langle \varphi \rangle_\Omega^{(\alpha)} := \sup\{|\varphi(x) - \varphi(y)|/|x-y|^\alpha; x,y \in \Omega, x \neq y\}$ (resp., $L = |D\varphi|_{L^\infty(\Omega)}$).

PROOF OF COROLLARY 4.4 Let $x_0 \in \partial\Omega$ and put $w^\pm(x,t) := \varphi(x_0) \pm L|x-x_0|^\alpha$ with $\alpha \in (0,1)$ (resp., $\alpha = 1$) and $L = \langle \varphi \rangle_\Omega^{(\alpha)}$ (resp., $L = |D\varphi|_{L^\infty(\Omega)}$). It then follows immediately that

$$\pm P(w_t^\pm(x,t), Dw^\pm(x,t), D^2w^\pm(x,t)) \leq 0 \ \text{ for all } (x,t) \in Q.$$

Moreover, we can also derive that $w^+(x,t) = \varphi(x) - \varphi(x) + \varphi(x_0) + L|x-x_0|^\alpha \geq \varphi(x) = u(x,t)$ for all $(x,t) \in \mathcal{P}Q$ and that $w^- \leq u$ on $\mathcal{P}Q$. Therefore, $w^- \leq u \leq w^+$ on $\mathcal{P}Q$. Hence, Theorem 2.3 implies $w^- \leq u \leq w^+$ in Q. Thus it follows that $|u(x,t) - u(x_0,t)| \leq L|x-x_0|^\alpha$ for all $(x,t) \in Q$ and $x_0 \in \partial\Omega$. Moreover, we can extend this boundary estimate into Ω, by repeating the same argument as in the proof of Theorem 2.5. ∎

5 Cauchy Problem

This section concerns the existence and uniqueness of solutions of the Cauchy problem for (1) with $\Omega = \mathbb{R}^N$ and $\varphi = \varphi(x) \in C^2(\mathbb{R}^N)$. Suppose that $\varphi \in C^2(\mathbb{R}^N)$ satisfies

$$\sup_{x\in\mathbf{R}^N} \Big\{|D\varphi(x)| + |D^2\varphi(x)|\Big\} < +\infty. \tag{22}$$

Thus there exists a constant K independent of x such that $|\varphi(x)| \le K(|x|+1)$ for all $x \in \mathbb{R}^N$. Therefore, Theorem 2.1 of [6] ensures the comparison principle for solutions of the Cauchy problem with an initial data φ.

Moreover, let $w^{\pm}(x,t) := \varphi(x) \pm M_1 t$ and obtain

$$\begin{aligned} P(w_t^+, Dw^+, D^2w^+) &= M_1 - D_i\varphi(x)D_j\varphi(x)D_{ij}^2\varphi(x) \\ &\ge M_1 - |D\varphi|_\infty^2|D^2\varphi|_\infty \ge 0, \end{aligned}$$

by taking $M_1 = |D\varphi|_\infty^2|D^2\varphi|_\infty$. We can also deduce that w^- becomes a classical subsolution of (1) in $\mathbb{R}^N \times (0,T)$. Furthermore, $w^{\pm}(\cdot,0) = \varphi$.

Now, define

$$W(x,t) := \sup\Big\{v(x,t);\ w^-(x,t) \le v(x,t) \le w^+(x,t), \quad v \in \mathcal{S}\Big\},$$

where $\mathcal{S}$ denotes the set of all viscosity subsolutions of (1) in $\mathbb{R}^N \times (0,T)$. Then, by virtue of the comparison principle and Perron's method, the restriction of W on $Q_R := \{(x,t) \in \mathbb{R}^N \times (0,T); |x| \le R\}$ becomes a viscosity solution of (1) in Q_R for any $R > 0$. Thus, we can deduce that W becomes a viscosity solution of $P = 0$ in $\mathbb{R}^N \times (0,T)$ from the arbitrariness of R. Furthermore, since $w^- \le W \le w^+$ and $w^-(x,0) = w^+(x,0) = \varphi(x)$, it follows that $W(x,0) = \varphi(x)$. Therefore, W is a viscosity solution of the Cauchy problem in $\mathbb{R}^N \times (0,T)$.

Consequently, we have:

THEOREM 5.1 *Let $Q = \mathbb{R}^N \times (0,T)$ and let $\varphi \in C^2(\mathbb{R}^N)$. Suppose that* (22) *holds. Then there exists a unique viscosity solution $u \in C(\overline{Q})$ in Q of the Cauchy problem* (1)-(2).

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