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Maximal monotonicity for the sum of two subdifferential operators in L^p-spaces

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Dedicated to the memory of Professor Yukio Kōmura

September 20, 2010

Abstract

This paper is devoted to providing a sufficient condition for the maximality of the sum of subdifferential operators defined on reflexive Banach spaces and proving the maximal monotonicity in $L^p(\Omega) \times L^{p'}(\Omega)$ of the nonlinear elliptic operator $u \mapsto -\Delta_m u + \beta(u(\cdot))$ with a maximal monotone graph β.

Keyword: Maximal monotone, subdifferential, Banach space, nonlinear elliptic operator

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1 Introduction

Let E and E^* be a real reflexive Banach space and its dual space, respectively, and let $\phi_1, \phi_2 : E \to (-\infty, \infty]$ be proper (i.e., $\phi_1, \phi_2 \not\equiv \infty$) lower semicontinuous convex functionals with the effective domains $D(\phi_i) := \{u \in E;\ \phi_i(u) < \infty\}$ for $i = 1, 2$. Then the subdifferential operator $\partial_E \phi_i : E \to 2^{E^*}$ of ϕ_i is defined by

$$\partial_E \phi_i(u) := \{\xi \in E^*;\ \phi_i(v) - \phi_i(u) \geq \langle \xi, v - u \rangle_E \ \text{ for all } \ v \in D(\phi_i)\},$$

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where $\langle \cdot, \cdot \rangle_E$ denotes the duality pairing between E and E^*, with the domain $D(\partial_E \phi_i) = \{u \in D(\phi_i);\ \partial_E \phi_i(u) \neq \emptyset\}$ for $i = 1, 2$. This paper provides a new sufficient condition for the maximal monotonicity of the sum $\partial_E \phi_1 + \partial_E \phi_2$ in $E \times E^*$ and an application to nonlinear elliptic operators in L^p-spaces.

This paper is motivated by the question of whether the following operator $\mathcal{M}$ is maximal monotone in $L^p(\Omega) \times L^{p'}(\Omega)$ with $p \in [2, \infty)$, $p' = p/(p-1)$ and a bounded domain Ω of $\mathbb{R}^N$:

$$\mathcal{M} : D(\mathcal{M}) \subset L^p(\Omega) \to L^{p'}(\Omega);\ u \mapsto -\Delta_m u + \beta(u(\cdot)), \tag{1.1}$$

where β is a maximal monotone graph in $\mathbb{R}$ such that $\beta(0) \ni 0$, and Δ_m is a modified Laplacian given by

$$\Delta_m u = \nabla \cdot \left(|\nabla u|^{m-2} \nabla u \right), \quad 1 < m < \infty$$

equipped with the homogeneous Dirichlet boundary condition, i.e., $u|_{\partial\Omega} = 0$. The operator $\mathcal{M}$ can be divided into two parts: $u \mapsto -\Delta_m u$ and $u \mapsto \beta(u(\cdot))$, and they are maximal monotone in $L^p(\Omega) \times L^{p'}(\Omega)$. Indeed, set $E = L^p(\Omega)$ and put

$$\phi_1(u) := \begin{cases} \dfrac{1}{m} \displaystyle\int_\Omega |\nabla u(x)|^m dx & \text{if } u \in W_0^{1,m}(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{1.2}$$

$$\phi_2(u) := \begin{cases} \displaystyle\int_\Omega j(u(x)) dx & \text{if } j(u(\cdot)) \in L^1(\Omega), \\ \infty & \text{otherwise,} \end{cases} \tag{1.3}$$

where $j : \mathbb{R} \to (-\infty, \infty]$ is a proper lower semicontinuous convex function such that $\partial j = \beta$. Then ϕ_1 and ϕ_2 are lower semicontinuous and convex in E, and moreover, $\partial_E \phi_1(u)$ and $\partial_E \phi_2(u)$ coincide with $-\Delta_m u$ equipped with $u|_{\partial\Omega} = 0$ and $\beta(u(\cdot))$, respectively. Although every subdifferential operator is maximal monotone, the sum of two subdifferential operators might not be maximal monotone. Hence it is not obvious whether the operator $\mathcal{M} = \partial_E \phi_1 + \partial_E \phi_2$ is maximal monotone in $E \times E^*$ or not.

The maximality for the sum of two maximal monotone operators was well studied in Hilbert space settings (see [6] and [7]). These results were combined with nonlinear semigroup theory founded by Yukio Kōmura [10] in 1967 and developed later by Brézis and many people for the study of nonlinear evolution equations. As for Banach space settings, a couple of sufficient conditions are proposed by Brézis, Crandall and Pazy [8] (see also [9] and [13]). Let A and B be maximal monotone operators from E into E^*. Their results ensure the maximal monotonicity of $A + B$ in $E \times E^*$ if one of the following conditions is at least satisfied:

(i) $D(A) \cap (\mathrm{Int} D(B)) \neq \emptyset$,

(ii) B is dominated by A, i.e., $D(A) \subset D(B)$ and $\|B(u)\|_{E^*} \leq k\|A(u)\|_{E^*} + \ell(|u|_E)$ for all $u \in D(A)$ with $k \in (0,1)$ and a non-decreasing function ℓ in $\mathbb{R}$.

Here we write $\|C\|_{E^*} := \inf\{|c|_{E^*};\ c \in C\}$ for each non-empty subset C of E^*. Furthermore, in case B is a subdifferential operator, the following condition (iii) also ensures the maximal monotonicity of $A+B$, and this fact is proved in [7] when $E = E^* = H$ is a Hilbert space; however, it can be naturally extended to a Banach space setting.

(iii) $B = \partial_E \phi$ with a proper, lower semicontinuous convex function $\phi : E \to (-\infty, +\infty]$, and

$$\phi(J_\lambda u) \leq \phi(u) + C\lambda \quad \text{for} \ \ u \in D(\phi) \text{ and } \lambda > 0, \tag{1.4}$$

where J_λ denotes the resolvent of A in E.

Here the resolvent $J_\lambda : E \to D(A)$ is given such that $u_\lambda := J_\lambda u$ is a unique solution of $F_E(u_\lambda - u) + A(u_\lambda) \ni 0$, where F_E stands for the duality mapping between E and E^*, for each $u \in E$.

However, these results could not be applicable directly to our setting for (1.1). As for (i), neither $D(\partial_E \phi_1)$ nor $D(\partial_E \phi_2)$ might have any interior points in E $(= L^p(\Omega))$. Condition (ii) cannot be checked unless an appropriate growth condition is imposed on β. Condition (iii) is available for the case that $p = 2$, because the duality mapping F_E of $E = L^2(\Omega)$ is the identity and the resolvent J_λ for $\partial_E \phi_2$ has a simple representation formula,

$$(J_\lambda u)(x) = (1 + \lambda\beta)^{-1}(u(x)) \quad \text{for a.e. } x \in \Omega, \tag{1.5}$$

which enables us to check (1.4). However, it is somewhat difficult to check (1.4) for the case that $p \neq 2$. Actually, the relation between the resolvents of $\partial_E \phi_2$ and β is unclear, since the duality mapping F_E is severely nonlinear whenever $p \neq 2$ (see (3.2) below).

In this paper we propose a new sufficient condition for the maximality of $\partial_E \phi_1 + \partial_E \phi_2$ in $E \times E^*$ such that the representation formula (1.5) in $L^2(\Omega)$ can be effectively used in applications to nonlinear elliptic operators such as (1.1). More precisely, we introduce a Hilbert space H as a pivot space of the triplet $E \hookrightarrow H \equiv H^* \hookrightarrow E^*$ and an extension ϕ_2^H of ϕ_2 to H, and moreover, we give a sufficient condition for the maximality in terms of the resolvent and the Yosida approximation for $\partial_H \phi_2^H$.

The treatment of the operator $\mathcal{M}$ in $L^p(\Omega)$ with $p \neq 2$ is required from recent studies on severely nonlinear problems such as generalized Allen-Cahn equations of the form

$$|u_t|^{p-2}u_t - \Delta_m u + \beta(u) + g(u) \ni f \quad \text{in } \Omega \times (0,\infty), \tag{1.6}$$
$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.7}$$
$$u(\cdot,0) = u_0 \quad \text{in } \Omega \tag{1.8}$$

with a non-monotone function $g : \mathbb{R} \to \mathbb{R}$. The main difficulty of treating (1.6) arises from the nonlinearity in u_t. To avoid this, one often chooses $E = L^p(\Omega)$ as a base space of analysis, since the mapping $u \mapsto |u|^{p-2}u$ from E into E^* has fine properties. Moreover, (1.6)–(1.8) can be reduced into the Cauchy problem for the following evolution equation in $E^* = L^{p'}(\Omega)$:

$$\partial_E \psi(u'(t)) + \partial_E \phi(u(t)) + g(u(\cdot,t)) \ni f(t) \quad \text{in } E^*,$$

by putting

$$\psi(u) = \frac{1}{p}\int_\Omega |u(x)|^p dx$$

and by setting $\phi = \phi_1 + \phi_2$ with ϕ_1, ϕ_2 defined by (1.2), (1.3), provided that $\partial_E\phi_1 + \partial_E\phi_2$ is maximal monotone in $E \times E^*$. The existence and the large-time behavior of global solutions for this abstract evolution equation have been studied by the author (see [1]).

Let us briefly discuss several related works. Condition (i), which had been proposed by Rockafellar [13], was generalized by Attouch, Riahi and Théra [3] (see, e.g., [14] for recent developments). Condition (ii) was generalized by [16]. Moreover, these problems have been also studied in general Banach space settings (see, e.g., Borwein [5], Voisei [15] and references therein). Furthermore, Attouch, Baillon and Théra [2] proposed the notion of *variational sum* of maximal monotone operators in Hilbert spaces. On the other hand, perturbation problems were also studied in the theory of m-accretive operators in Banach spaces (see [11, 12] and references therein).

In §2, we first propose an abstract framework on the maximality for the sum of two subdifferential operators in Banach spaces. Moreover, in §3, we also establish an estimate for the nonlinear elliptic operator $\mathcal{M}$ in $L^r(\Omega)$ with $r \in (1,\infty)$ to check a sufficient condition presented in §2. The final section is devoted to an application to the nonlinear elliptic operator $\mathcal{M}$.

2 Maximality for the sum of subdifferential operators

This section is devoted to our sufficient condition for the maximality of the sum of two subdifferential operators in an abstract form.

Let us first briefly recall a couple of notion associated with maximal monotone operators and subdifferential operators in Hilbert spaces (see, e.g., [6, 7] for more details).

Let $A : H \to H$ be a maximal monotone operator with the domain $D(A) := \{u \in H;\ A(u) \neq \emptyset\}$. The *resolvent* $J_\lambda : H \to D(A)$ of A is defined by $J_\lambda := (I + \lambda A)^{-1}$ with the identity mapping $I : H \to H$, and moreover, the *Yosida approximation* $A_\lambda : H \to H$ of A is given by $A_\lambda := (I - J_\lambda)/\lambda$. Then $A_\lambda(u) \in A(J_\lambda u)$ for all $u \in H$.

Let $\varphi : H \to (-\infty, \infty]$ be a proper lower semicontinuous convex functional. The *Moreau-Yosida regularization* of φ is defined by

$$\varphi_\lambda(u) := \inf_{v \in H} \left\{ \frac{1}{2\lambda}|u - v|_H^2 + \varphi(v) \right\} = \frac{1}{2\lambda}|u - J_\lambda u|_H^2 + \varphi(J_\lambda u) \quad \text{for } u \in H,$$

where J_λ denotes the resolvent for $\partial_H \varphi$. We note that the subdifferential operator of φ_λ coincides with the Yosida approximation of $\partial_H \varphi$, that is, $\partial_H(\varphi_\lambda) = (\partial_H \varphi)_\lambda$, so we simply write $\partial_H \varphi_\lambda$. Moreover, $\varphi(J_\lambda u) \leq \varphi_\lambda(u) \leq \varphi(u)$ for all $u \in H$.

Now, our result reads,

THEOREM 2.1. *Let E be a strictly convex reflexive Banach space with a strictly convex dual space E^* and let H be a Hilbert space whose dual space is identified with itself such that*

$$E \hookrightarrow H \equiv H^* \hookrightarrow E^* \tag{2.1}$$

with continuous and densely defined canonical injections. Let ϕ_1 and ϕ_2 be proper, lower semicontinuous and convex functions from E into $(-\infty, \infty]$ such that $D(\phi_1) \cap D(\phi_2) \neq \emptyset$. Assume that

(A1) *there exists an extension ϕ_2^H of ϕ_2 to H such that ϕ_2^H is lower semicontinuous and convex in H.*

Let J_λ^H stand for the resolvent of $\partial_H \phi_2^H$, and let $\phi_{2,\lambda}^H$ denote the Moreau-Yosida regularization of ϕ_2^H and A_λ^H the Yosida approximation of $\partial_H \phi_2^H$. Suppose that

(A2) *let* $[u_\lambda, \xi_\lambda] \in \partial_E \phi_1$ *be such that* $|\xi_\lambda + A^H_\lambda(u_\lambda)|_{E^*}$, $\phi_1(u_\lambda)$, $\phi^H_{2,\lambda}(u_\lambda)$ *and* $|u_\lambda|_E$ *are bounded as* $\lambda \to +0$. *Then* $J^H_\lambda u_\lambda \in E$, *and moreover,* $|J^H_\lambda u_\lambda|_E$ *and* $|A^H_\lambda(u_\lambda)|_{E^*}$ *are also bounded.*

Then $\partial_E\phi_1 + \partial_E\phi_2$ *is maximal monotone in* $E \times E^*$.

Proof. Since E^* is strictly convex, the duality mapping F_E between E and E^* is single-valued and demicontinuous (i.e., strongly-weakly continuous). Moreover, we can assume $\phi_1, \phi_2 \geq 0$ without any loss of generality (see Proposition 2.1 of [4, Chap. II]). The monotonicity of $\partial_E\phi_1 + \partial_E\phi_2$ follows immediately. To prove the maximality in $E \times E^*$, it suffices to show that the range of $F_E + \partial_E\phi_1 + \partial_E\phi_2$ coincides with E^* (see, e.g., Theorem 1.2 of [4, Chap. II]).

We note that A^H_λ is Lipschitz continuous in H, in particular, bounded and hemicontinuous from E into E^*. Hence $\partial_E\phi_1 + A^H_\lambda$ becomes maximal monotone in $E \times E^*$ (see Corollary 1.1 of [4, Chap. II]), and therefore, for each $f \in E^*$, there exists $u_\lambda \in D(\partial_E\phi_1)$ such that

$$F_E(u_\lambda) + \xi_\lambda + A^H_\lambda(u_\lambda) = f \quad \text{and} \quad \xi_\lambda \in \partial_E\phi_1(u_\lambda). \tag{2.2}$$

Let us take $v \in D(\phi_1) \cap D(\phi_2)$. Multiply (2.2) by $u_\lambda - v$ and note that $A^H_\lambda = \partial_H(\phi^H_{2,\lambda})$. It then follows that

$$\frac{1}{2}|u_\lambda|^2_E - \frac{1}{2}|v|^2_E + \phi_1(u_\lambda) - \phi_1(v) + \phi^H_{2,\lambda}(u_\lambda) - \phi^H_{2,\lambda}(v) \leq \langle f, u_\lambda - v\rangle. \tag{2.3}$$

Since $\phi^H_{2,\lambda}(v) \leq \phi^H_2(v) = \phi_2(v)$, we have

$$|F_E(u_\lambda)|_{E^*} = |u_\lambda|_E \leq C, \quad \phi_1(u_\lambda) \leq C \quad \text{and} \quad \phi^H_{2,\lambda}(u_\lambda) \leq C, \tag{2.4}$$

and hence,

$$|\xi_\lambda + A^H_\lambda(u_\lambda)|_{E^*} \leq C.$$

Furthermore, by (A2),

$$|J^H_\lambda u_\lambda|_E \leq C \quad \text{and} \quad |A^H_\lambda(u_\lambda)|_{E^*} \leq C,$$

which also implies

$$|\xi_\lambda|_{E^*} \leq C.$$

Therefore since E and E^* are reflexive, we can take a sequence $\lambda_n \to +0$ such that

$$u_{\lambda_n} \to u \quad \text{weakly in } E, \tag{2.5}$$
$$F_E(u_{\lambda_n}) \to u^* \quad \text{weakly in } E^*, \tag{2.6}$$
$$J^H_{\lambda_n} u_{\lambda_n} \to \hat{u} \quad \text{weakly in } E, \tag{2.7}$$
$$A^H_{\lambda_n}(u_{\lambda_n}) \to \eta \quad \text{weakly in } E^*, \tag{2.8}$$
$$\xi_{\lambda_n} \to \xi \quad \text{weakly in } E^* \tag{2.9}$$

with some $u, \hat{u} \in E$ and $u^*, \eta, \xi \in E^*$. Hence $u^* + \xi + \eta = f$. Moreover,

$$|u_\lambda - J_\lambda^H u_\lambda|_{E^*} = \lambda |A_\lambda^H(u_\lambda)|_{E^*} \leq \lambda C \to 0.$$

Thus we obtain $u = \hat{u}$. By (2.4) and the (weak) lower semicontinuity of ϕ_1 and ϕ_2, we observe $u \in D(\phi_1) \cap D(\phi_2)$.

Multiply (2.2) with $\lambda = \lambda_n$ by $u_{\lambda_n} - u$ and pass to the limit as $\lambda_n \to 0$. Then

$$\begin{aligned}\limsup_{\lambda_n \to 0} \langle F_E(u_{\lambda_n}), u_{\lambda_n} \rangle + \liminf_{\lambda_n \to 0} \phi_1(u_{\lambda_n}) + \liminf_{\lambda_n \to 0} \phi_2(J_{\lambda_n}^H u_{\lambda_n}) \\ \leq \langle u^*, u \rangle + \phi_1(u) + \phi_2(u).\end{aligned}$$

Hence by the weak lower semicontinuity of ϕ_1 and ϕ_2 in E, it follows from (2.5) and (2.7) that

$$\limsup_{\lambda_n \to 0} \langle F_E(u_{\lambda_n}), u_{\lambda_n} \rangle \leq \langle u^*, u \rangle.$$

Therefore, since F_E is maximal monotone in $E \times E^*$, by Lemma 1.2 of [8] it also follows that

$$u^* = F_E(u) \quad \text{and} \quad \langle F_E(u_{\lambda_n}), u_{\lambda_n} \rangle \to \langle u^*, u \rangle.$$

Furthermore, we can similarly obtain

$$\xi \in \partial_E \phi_1(u) \quad \text{and} \quad \langle \xi_{\lambda_n}, u_{\lambda_n} \rangle \to \langle \xi, u \rangle.$$

Now it remains to prove $\eta \in \partial_E \phi_2(u)$. Note that

$$\begin{aligned}\langle A_{\lambda_n}^H(u_{\lambda_n}), J_{\lambda_n}^H u_{\lambda_n} \rangle &\leq \langle A_{\lambda_n}^H(u_{\lambda_n}), u_{\lambda_n} \rangle \\ &= \langle f, u_{\lambda_n} \rangle - \langle F_E(u_{\lambda_n}), u_{\lambda_n} \rangle - \langle \xi_{\lambda_n}, u_{\lambda_n} \rangle \\ &\to \langle f - u^* - \xi, u \rangle = \langle \eta, u \rangle.\end{aligned}$$

Recalling by (A1) that $A_{\lambda_n}^H(u_{\lambda_n}) \in \partial_H \phi_2^H(J_{\lambda_n}^H u_{\lambda_n}) \subset \partial_E \phi_2(J_{\lambda_n}^H u_{\lambda_n})$ and employing Lemma 1.2 of [8], we conclude by (2.7) and (2.8) that $\eta \in \partial_E \phi_2(u)$. Consequently, $F_E(u) + \partial_E \phi_1(u) + \partial_E \phi_2(u) \ni f$, and therefore $\partial_E \phi_1 + \partial_E \phi_2$ is maximal monotone in $E \times E^*$. □

Remark 2.2. (i) One can replace (A2) of Theorem 2.1 by

(A2)′ Let $f \in E^*$ and let u_λ be solutions of approximate problems (2.2). Then $J_\lambda^H u_\lambda \in E$, and moreover, $|J_\lambda^H u_\lambda|_E$ and $|A_\lambda^H(u_\lambda)|_{E^*}$ are bounded as $\lambda \to 0$.

(ii) Brézis, Crandall and Pazy [8] also presented a sufficient and necessary condition for the maximality of the sum $A+B$ of two maximal monotone operators in $E \times E^*$ with $D(A) \cap D(B) \neq \emptyset$. More precisely, let u_λ be a solution of

$$F_E(u_\lambda) + \xi_\lambda + B_\lambda(u_\lambda) = f \quad \text{and} \quad \xi_\lambda \in A(u_\lambda), \tag{2.10}$$

where $J_\lambda : E \to D(A)$ and $B_\lambda : E \to E^*$ are the resolvent and the Yosida approximation, i.e., $B_\lambda(u) = F_E(u - J_\lambda u)/\lambda$, respectively, of $B : E \to E^*$. Then $A + B$ is maximal monotone if and only if for each $f \in E^*$, $|B_\lambda(u_\lambda)|_{E^*}$ is bounded as $\lambda \to 0$ (it also yields the sufficient conditions (i)–(iii) in Section 1). Here we also note that the boundedness of $|J_\lambda u_\lambda|_E$ automatically follows from that of $|u_\lambda|_E$ as $\lambda \to 0$, since $J_\lambda : E \to D(A)$ is bounded.

(iii) The extension of Theorem 2.1 to the sum of maximal monotone operators $A, B : E \to E^*$ is still open and seems to be not obvious. Indeed, the derivations of $u^* = F_E(u)$, $\xi \in \partial_E \phi_1(u)$ and $\eta \in \partial_E \phi_2(u)$ in our proof essentially rely on the structure of subdifferential operators, and moreover, a technique used in [8], where the maximality of $A + B$ is treated, could not be applied directly to our approximation (2.2).

3 An estimate for maximal monotone graphs in $L^r(\Omega)$

In order to check (A2) of Theorem 2.1 for the case that $E = L^p(\Omega)$ and ϕ_1, ϕ_2 are given by (1.2), (1.3) respectively, we shall establish an estimate in $L^r(\Omega)$ for the nonlinear elliptic operator

$$u \mapsto -\Delta_m u + \beta(u(\cdot))$$

with a maximal monotone graph β in $\mathbb{R}$. It will be used to derive the boundedness for $|A^H_\lambda(u_\lambda)|_{E^*}$ as $\lambda \to 0$ for solutions u_λ of (2.2).

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^N$ and let $m, r \in (1, \infty)$. Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone operator in $\mathbb{R}$ such that $\beta(0) \ni 0$, and let $\beta^\circ(s)$ denote the minimal section of $\beta(s)$, i.e., $|\beta^\circ(s)| = \min\{|\xi|;\ \xi \in \beta(s)\}$. Let $u \in W^{1,m}_0(\Omega)$ and $\eta \in L^r(\Omega)$ be such that $\Delta_m u \in L^r(\Omega)$ and $\eta(x) \in \beta(u(x))$ for a.e. $x \in \Omega$. Then it follows that*

$$|\beta^\circ(u(\cdot))|_{L^r(\Omega)} \leq |-\Delta_m u + \eta|_{L^r(\Omega)}. \tag{3.1}$$

In case $r = 2$, Estimate (3.1) has already been established in an abstract formulation (see, e.g., [7]), and its proof relies on a structure intrinsic to Hilbert spaces. However, for $r \neq 2$, it seems to be unknown. We prove (3.1) for general $r \in (1, \infty)$ by finding a necessary and sufficient condition for the maximal monotonicity of the composition of two monotone operators in $\mathbb{R}$ and by exploiting an explicit formula of a duality mapping between $L^r(\Omega)$ and $L^{r'}(\Omega)$.

Let us define a mapping $F_r : L^r(\Omega) \to L^{r'}(\Omega)$ by

$$F_r(v) := \begin{cases} \dfrac{|v|^{r-2}v(\cdot)}{|v|_{L^r(\Omega)}^{r-2}} & \text{if } \ v \in L^r(\Omega) \setminus \{0\}, \\ 0 & \text{if } \ v = 0. \end{cases} \tag{3.2}$$

Then F_r satisfies

$$|F_r(v)|_{L^{r'}(\Omega)} = |v|_{L^r(\Omega)} \ \text{ and } \ \langle F_r(v), v\rangle_{L^r(\Omega)} = |v|^2_{L^r(\Omega)} \quad \text{for all } \ v \in L^r(\Omega).$$

Hence F_r works as a duality mapping between $L^r(\Omega)$ and $L^{r'}(\Omega)$.

We next discuss a necessary and sufficient condition for the maximal monotonicity of the composition of two monotone operators in $\mathbb{R}$.

LEMMA 3.2. *Let $\alpha, \beta : \mathbb{R} \to 2^{\mathbb{R}}$ be monotone operators in $\mathbb{R}$ and define a mapping $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ by*

$$\gamma(s) = \alpha \circ \beta(s) := \{\eta \in \alpha(\xi);\ \xi \in \beta(s) \cap D(\alpha)\}$$

with the domain $D(\gamma) = \{s \in D(\beta);\ \beta(s) \cap D(\alpha) \neq \emptyset\}$. Then γ is maximal monotone in $\mathbb{R}$ if and only if $\beta^{-1} + \alpha$ is surjective in $\mathbb{R}$.

Proof. We first prove the monotonicity of γ. Let $s_1, s_2 \in D(\gamma)$ be such that $s_1 \leq s_2$. For all $\eta_1 \in \gamma(s_1)$ and $\eta_2 \in \gamma(s_2)$, by the definition of γ, there exist $\xi_1, \xi_2 \in D(\alpha)$ such that $\eta_i \in \alpha(\xi_i)$ and $\xi_i \in \beta(s_i)$ for $i = 1, 2$. Since $s_1 \leq s_2$ and β is monotone, we have $\xi_1 \leq \xi_2$, which together with the monotonicity of α implies $\eta_1 \leq \eta_2$. Thus γ is monotone in $\mathbb{R}$.

Assume that $\beta^{-1} + \alpha$ is surjective in $\mathbb{R}$. To prove the maximality, for each $f \in \mathbb{R}$, it suffices to find $s \in D(\gamma)$ such that $s + \gamma(s) \ni f$. From assumption, we can take $\xi \in D(\alpha) \cap D(\beta^{-1})$ such that $\beta^{-1}(\xi) + \alpha(\xi) \ni f$. Then we can take $s \in D(\beta)$ such that $\xi \in \beta(s)$ and $s + \alpha(\xi) \ni f$, that is, $s + \gamma(s) \ni f$. Thus γ is maximal monotone in $\mathbb{R}$.

Conversely, assume that γ is maximal monotone, that is, for any $f \in \mathbb{R}$, there is $[s, \eta] \in \gamma$ such that $s + \eta = f$. From the definition of γ, we can take $\xi \in D(\alpha)$ such that $\eta \in \alpha(\xi)$ and $\xi \in \beta(s)$. Hence since $s \in \beta^{-1}(\xi)$, we have $\beta^{-1}(\xi) + \alpha(\xi) \ni f$, which implies the surjectivity of $\beta^{-1} + \alpha$. $\square$

We further prepare the following lemma.

LEMMA 3.3. *Let $m, r \in (1, \infty)$ and let $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone operator in $\mathbb{R}$ such that $\gamma(0) \ni 0$. Then it follows that*

$$\int_\Omega -\Delta_m u(x)\gamma^\circ(u(x))dx \geq 0$$

for all $u \in W_0^{1,m}(\Omega)$ satisfying $\Delta_m u \in L^r(\Omega)$ and $\gamma^\circ(u(\cdot)) \in L^{r'}(\Omega)$.

Proof. Let γ_λ and ρ_λ be the Yosida approximation and the resolvent of γ respectively. Let $u \in W_0^{1,m}(\Omega)$ be such that $\Delta_m u \in L^r(\Omega)$ and $\gamma^\circ(u(\cdot)) \in L^{r'}(\Omega)$. Then we find that $|\rho_\lambda(u(x+h)) - \rho_\lambda(u(x))| \leq |u(x+h) - u(x)|$ for $x \in \Omega$ and $h \in \mathbb{R}^N$ satisfying $x + h \in \Omega$, since ρ_λ is non-expansive. Moreover, we see $\rho_\lambda(0) = 0$, because $\gamma(0) \ni 0$. Hence we have

$$|\nabla \rho_\lambda(u(x))| \leq |\nabla u(x)| \quad \text{for a.e. } x \in \Omega \tag{3.3}$$

and $\rho_\lambda(u(\cdot)) \in W_0^{1,m}(\Omega)$. Furthermore, we can deduce that $\gamma_\lambda(u(\cdot)) \in L^{r'}(\Omega)$ from the fact that $|\gamma_\lambda(u(x))| \leq |\gamma^\circ(u(x))|$ and $\gamma^\circ(u(\cdot)) \in L^{r'}(\Omega)$.

Multiplying $-\Delta_m u$ by $\gamma_\lambda(u(\cdot))$ and integrating this over Ω, we obtain

$$\begin{aligned} \int_\Omega -\Delta_m u(x)\gamma_\lambda(u(x))dx &= \int_\Omega -\Delta_m u(x)\frac{u(x) - \rho_\lambda(u(x))}{\lambda}dx \\ &\geq \frac{\phi_1(u) - \phi_1(\rho_\lambda(u(\cdot)))}{\lambda}, \end{aligned}$$

where $\phi_1(u) = (1/m)\int_\Omega |\nabla u(x)|^m dx$. Therefore since (3.3) yields $\phi_1(u) \geq \phi_1(\rho_\lambda(u(\cdot)))$, we deduce that

$$\int_\Omega -\Delta_m u(x)\gamma_\lambda(u(x))dx \geq 0.$$

Letting $\lambda \to +0$ and noting that

$$\gamma_\lambda(u(x)) \to \gamma^\circ(u(x)), \quad |\gamma_\lambda(u(x))| \leq |\gamma^\circ(u(x))| \quad \text{for a.e. } x \in \Omega$$

and $\gamma^\circ(u(\cdot)) \in L^{r'}(\Omega)$, we infer that $\gamma_\lambda(u(\cdot)) \to \gamma^\circ(u(\cdot))$ strongly in $L^{r'}(\Omega)$. Thus we conclude that

$$\int_\Omega -\Delta_m u(x)\gamma^\circ(u(x))dx \geq 0.$$

This completes our proof. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $u \in W_0^{1,m}(\Omega)$ and $\eta \in L^r(\Omega)$ be such that $\Delta_m u \in L^r(\Omega)$ and $\eta(x) \in \beta(u(x))$ for a.e. $x \in \Omega$. Then $\beta^\circ(u(\cdot))$ also belongs to $L^r(\Omega)$. Put $v = \beta^\circ(u(\cdot))$ and multiply $-\Delta_m u + \eta$ by $F_r(v)$. Here we note that

$$\langle F_r(v), -\Delta_m u\rangle_{L^r(\Omega)} = \frac{1}{|v|_{L^r(\Omega)}^{r-2}} \int_\Omega -\Delta_m u(x)|v|^{r-2}v(x)dx.$$

Put $\alpha(s) := |s|^{r-2}s$. Then α is continuous and coercive in $\mathbb{R}$. Moreover, since β^{-1} is maximal monotone, $\beta^{-1} + \alpha$ becomes surjective. Hence by Lemma 3.2, the composite mapping $\gamma = \alpha \circ \beta$ is maximal monotone in $\mathbb{R}$. Therefore Lemma 3.3 yields

$$\int_\Omega -\Delta_m u(x)\gamma^\circ(u(x))dx \geq 0.$$

Here the minimal section $\gamma^\circ(s)$ of $\gamma(s)$ coincides with $\alpha(\beta^\circ(s))$ for all $s \in D(\gamma)$. Thus by Lemma 3.3 it follows that $\langle F_r(v), -\Delta_m u\rangle_{L^r(\Omega)} \geq 0$. Furthermore, we observe

$$\langle F_r(v), \eta\rangle_{L^r(\Omega)} = \frac{1}{|v|_{L^r(\Omega)}^{r-2}} \int_\Omega |v|^{r-2}v(x)\eta(x)dx \geq |v|_{L^r(\Omega)}^2,$$

because $v(x)\eta(x) \geq |v(x)|^2$ for a.e. $x \in \Omega$. Hence we have

$$\langle F_r(v), -\Delta_m u + \eta\rangle_{L^r(\Omega)} \geq |v|_{L^r(\Omega)}^2,$$

which implies (3.1). □

4 Application to nonlinear elliptic operators

In this section, we apply the preceding results to investigate the maximality of the operator $\mathcal{M}$ defined by (1.1).

THEOREM 4.1. *Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone graph satisfying $\beta(0) \ni 0$ and let $j : \mathbb{R} \to (-\infty, \infty]$ be a primitive function of β, i.e., $\beta = \partial j$. Suppose that*

$$1 < m < \infty \quad \textit{and} \quad 2 \leq p < \infty. \tag{4.1}$$

Let $E = L^p(\Omega)$ and let $\phi_1, \phi_2 : E \to (-\infty, \infty]$ be defined by (1.2), (1.3) respectively. Then $\mathcal{M} = \partial_E\phi_1 + \partial_E\phi_2$ is maximal monotone in $E \times E^*$. In addition, if β is single-valued, then it holds that

$$|\partial_E\phi_2(u)|_{E^*} \leq |\partial_E(\phi_1 + \phi_2)(u)|_{E^*} \quad \textit{for all} \ \ u \in D(\partial_E\phi_1) \cap D(\partial_E\phi_2).$$

REMARK 4.2. By this theorem, $\mathcal{M} = \partial_E\phi_1 + \partial_E\phi_2$ coincides with $\partial_E\phi$, where $\phi = \phi_1 + \phi_2$. Indeed, one can easily check that $\partial_E\phi_1 + \partial_E\phi_2 \subset \partial_E(\phi_1 + \phi_2)$ in terms of their graphs. Hence if $\partial_E\phi_1 + \partial_E\phi_2$ is maximal monotone in $E \times E^*$, then $\partial_E\phi_1 + \partial_E\phi_2 = \partial_E(\phi_1 + \phi_2)$.

REMARK 4.3. Before proceeding to a proof of Theorem 4.1, let us check the convexity and the lower semicontinuity of ϕ_1, ϕ_2 in E. The convexity is obvious, so let us prove the lower semicontinuity. Since ϕ_1 is convex and continuous in $W_0^{1,m}(\Omega)$, ϕ_1 is weakly lower semicontinuous in $W_0^{1,m}(\Omega)$. The lower semicontinuity in E of ϕ_1 is equivalent to the closedness in E of sublevel sets $[\phi_1 \leq \lambda] := \{u \in E;\ \phi_1(u) \leq \lambda\}$ for any $\lambda \in \mathbb{R}$. Let (u_n) be a sequence in $[\phi_1 \leq \lambda]$ with an arbitrary $\lambda \in \mathbb{R}$ such that $u_n \to u$ strongly in E. Then from the definition of ϕ_1, u_n is bounded in $W_0^{1,m}(\Omega)$. Hence one can take a subsequence (n') of (n) such that $u_{n'} \to u$ weakly in $W_0^{1,m}(\Omega)$. Then since ϕ_1 is weakly lower semicontinuous in $W_0^{1,m}(\Omega)$, we have

$$\phi_1(u) \leq \liminf_{n' \to \infty} \phi_1(u_{n'}) \leq \lambda,$$

which implies $u \in [\phi_1 \leq \lambda]$. Hence ϕ_1 is lower semicontinuous in E. The lower semicontinuity of ϕ_2 in E can be proved by Fatou's lemma as in the case of $p = 2$ (see Example 3, p. 61 of [4]).

Proof of Theorem 4.1. Put $H = L^2(\Omega)$ and notice by (4.1) that (2.1) holds. Define an extension $\phi_2^H : H \to (-\infty, \infty]$ of ϕ_2 by

$$\phi_2^H(u) := \begin{cases} \displaystyle\int_\Omega j(u(x))dx & \text{if } j(u(\cdot)) \in L^1(\Omega), \\ \infty & \text{otherwise} \end{cases} \qquad \text{for } u \in H.$$

Then $\phi_2^H = \phi_2$ on E and ϕ_2^H is convex and lower semicontinuous in H by the convexity and the lower semicontinuity of j in $\mathbb{R}$ (see Example 3, p. 61 of [4]), so (A1) holds.

We next check (A2). Let $[u_\lambda, \xi_\lambda] \in \partial_E\phi_1$ be such that $|\xi_\lambda + A_\lambda^H(u_\lambda)|_{E^*}$, $\phi_1(u_\lambda)$, $\phi_{2,\lambda}^H(u_\lambda)$ and $|u_\lambda|_E$ are bounded as $\lambda \to +0$, where A_λ^H denotes the Yosida approximation of $\partial_H\phi_2^H$. Then we notice by (1.5) that

$$[u, f] \in A_\lambda^H \quad \text{if and only if} \quad f(x) = \beta_\lambda(u(x)) \quad \text{for a.e. } x \in \Omega$$

for all $u, f \in L^2(\Omega)$, where β_λ stands for the Yosida approximation of β. Since β_λ is maximal monotone in $\mathbb{R}$ and $\beta_\lambda(0) = 0$, Theorem 3.1 ensures that

$$|A_\lambda^H(u)|_{E^*} \leq |\partial_E\phi_1(u) + A_\lambda^H(u)|_{E^*} \quad \text{for all } u \in D(\partial_E\phi_1).$$

Thus $|A_\lambda^H(u_\lambda)|_{E^*}$ is bounded as $\lambda \to +0$.

As for the boundedness of $J_\lambda^H u_\lambda$ in E, we note that

$$(J_\lambda^H u)(x) = (1+\lambda\beta)^{-1}(u(x)) \quad \text{for a.e. } x \in \Omega,$$

where J_λ^H is the resolvent for $\partial_H \phi_2^H$, i.e., $J_\lambda^H = (I + \lambda \partial_H \phi_2^H)^{-1}$. Since $(1+\lambda\beta)^{-1}$ is non-expansive in $\mathbb{R}$ and $(1+\lambda\beta)^{-1}(0) = 0$, it follows that $|(J_\lambda^H u)(x)| \leq |u(x)|$ for a.e. $x \in \Omega$. Hence

$$J_\lambda^H u \in E \quad \text{and} \quad |J_\lambda^H u|_E \leq |u|_E \quad \text{for all } u \in E.$$

Therefore $J_\lambda^H u_\lambda$ belongs to E and $J_\lambda^H u_\lambda$ is bounded in E if $|u_\lambda|_E$ is bounded as $\lambda \to 0$. Thus (A2) holds true. Consequently, by Theorem 2.1, we assure that $\partial_E \phi_1 + \partial_E \phi_2$ is maximal monotone in $E \times E^*$.

In case β is single-valued, i.e., $\beta = (\beta)^\circ$, Theorem 3.1 also implies

$$|\partial_E \phi_2(u)|_{E^*} \leq |\partial_E(\phi_1 + \phi_2)(u)|_{E^*} \quad \text{for all } u \in D(\partial_E \phi_1) \cap D(\partial_E \phi_2).$$

We have completed our proof. □

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