Periodic solutions for doubly nonlinear evolution equations

Akagi, Goro
Ulisse, Stefanelli

(Citation)
Journal of Differential Equations,251(7):1790-1812

(Issue Date)
2011-10-01

(Resource Type)
journal article

(Version)
Accepted Manuscript

(URL)
https://hdl.handle.net/20.500.14094/90001968

Periodic solutions for doubly nonlinear evolution equations

Goro Akagi* and Ulisse Stefanelli†

Abstract

We discuss the existence of periodic solution for the doubly nonlinear evolution equation $A(u'(t)) + \partial\phi(u(t)) \ni f(t)$ governed by a maximal monotone operator A and a subdifferential operator $\partial\phi$ in a Hilbert space H. As the corresponding Cauchy problem cannot be expected to be uniquely solvable, the standard approach based on the Poincaré map may genuinely fail. In order to overcome this difficulty, we firstly address some approximate problems relying on a specific approximate periodicity condition. Then, periodic solutions for the original problem are obtained by establishing energy estimates and by performing a limiting procedure. As a by-product, a structural stability analysis is presented for the periodic problem and an application to nonlinear PDEs is provided.

Keywords: Doubly nonlinear evolution equations; subdifferential; periodic problem; periodic solution; p-Laplacian.

AMS Mathematics Subject Classification (2010): 35K55.

1 Introduction

This paper is concerned with the existence of periodic solutions for the abstract doubly nonlinear equation

$$A(u'(t)) + \partial\phi(u(t)) \ni f(t). \tag{1. 1}$$

*Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-shi, Saitama 337-8570, Japan. E-mail: `g-akagi@shibaura-it.ac.jp`

†Istituto di Matematica Applicata e Tecnologie Infromatiche, Consiglio Nazionale delle Ricerche, v. Ferrata 1, I-27100 Pavia, Italy. E-mail: `ulisse.stefanelli@imati.cnr.it`

Here, $u : t \in [0,T] \mapsto H$ is a trajectory in the Hilbert space H and $u' = \mathrm{d}u/\mathrm{d}t$, A is a maximal monotone operator in H, $\partial\phi$ denotes the subdifferential of a proper, lower-semicontinuous, and convex functional $\phi : H \to [0,\infty]$, and $f \in L^2(0,T;H)$.

The abstract equation (1. 1) stems as a suitable variational formulation of doubly nonlinear PDEs of the form

$$\gamma\left(\frac{\partial u}{\partial t}\right) - \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) = f \tag{1. 2}$$

where γ is maximal monotone in $\mathbb{R}$, $p > 1$ and f is given. In case γ is linearly growing and p is sufficiently apart from 1, our analysis entails in particular the existence of periodic solutions to the latter. The Reader finds some details in this concern in Section 6.

Equation (1. 1) is well studied from the point of view of existence for the related Cauchy problem. Indeed, results in this direction can be traced back at least to Senba [37] and Arai [7]. Later on the problem has been considered also by Colli & Visintin [20] in Hilbert spaces and Colli [18] in Banach spaces. Besides existence, the Cauchy problem has also been considered from the point of view of structural stability [2], perturbations and long-time behavior [3, 4, 8, 34, 35, 36], and variational characterization of solutions [6, 5, 38, 32, 40]. The interest in the study of periodic solutions is in particular to be considered as a further step toward the comprehension of long-time dynamics and bifurcation phenomena.

The aim of this paper is to address equation (1. 1) under the periodic boundary condition

$$u(0) = u(T).$$

To the best of our knowledge, this periodic problem has never been solved before. Moreover, it is clearly quite more delicate with respect to the correspondent Cauchy problem. Consider for instance the ordinary differential equation

$$u'(t) + \partial I_{[1,2]}(u'(t)) + u(t) \ni f(t) \quad \text{in } \mathbb{R}$$

where $I_{[1,2]}$ is the classical indicator function of the interval $[1,2]$ (namely $I_{[1,2]}(x) = 0$ if $x \in [1,2]$ and $I_{[1,2]} = \infty$ elsewhere). This equation is reduced to the abstract form of (1. 1) in $H = \mathbb{R}$, and then, one can check all the assumptions for A and $\partial\phi$ of this paper except only a linear boundedness of A (see (A2) later). As u' is constrained to be greater than 1 for all times, no periodic solution may exist. On the other hand, the Cauchy problem admits a unique solution for any given initial datum.

A second difficulty arises as, being given the solvability of the Cauchy problem with initial datum u_0, the standard approach to periodicity based on finding a fixed point for the Poincaré map $P : u_0 \mapsto u(T)$ seems here of little use as equation (1. 1) is known to show genuine non-uniqueness. Even by resorting to fixed point tools for multivalued applications, one has to be confronted with the fact that Pu_0 cannot be generally expected to be convex.

Our strategy in order to prove existence of periodic solutions to equation (1. 1) is that of tackling an approximating equation possessing a unique Cauchy solution. This is obtained by replacing A with the strongly monotone operator $\varepsilon I_d + A$ (I_d being the identity in H) and ϕ with its Moreau-Yosida regularization ϕ_ε. In particular, the latter approximating problem features an *approximate periodicity* condition of the form $u(0) = J_\varepsilon u(T)$ where J_ε is the classical resolvent of $\partial\phi$ at level ε. Then, periodic solutions for equation (1. 1) are obtained by passing to the limit as $\varepsilon \to 0$. Let us note that we move in the exact same assumption frame as in the existence theory for the Cauchy problem from [20] plus an extra coercivity assumption for ϕ (usually harmless with respect to applications).

As a by-product of our existence analysis, we devise a structural stability result for the periodic problem. More precisely, by letting ϕ^n and A^n be sequences of convex functionals and maximal monotone operators, respectively, such that ϕ^n and A^n are convergent as $n \to \infty$ in some suitable sense, we prove that the periodic solutions for equation (1. 1) with (A, ϕ) replaced by (A^n, ϕ^n) converge to a periodic solution for (A, ϕ) as $n \to \infty$.

Before moving on, we shall remark that the second type of abstract doubly nonlinear equation, namely,

$$(A(u))' + \partial\phi(u) \ni f, \tag{1. 3}$$

has been already considered by from the point of view of the existence of periodic solutions in [1, 25]. See also [26, 27, 33, 41] for $A = \mathrm{id}$ in the perturbation case $f = f(t, u)$. In [1, 25] the authors cannot exploit directly the Poincaré map within a fixed point procedure and resort in proving the existence of periodic solutions for suitably regularized problems. Their argument is then completed by means of a limit passage.

This is the plan of the paper. Section 2 is devoted to the statement of the main existence result for periodic solutions. The mentioned ε-approximating problem is discussed in Section 3 and a first passage to the limit for $\varepsilon \to 0$ under a stronger coercivity assumption on ϕ is provided in Section 4. Then, Section 5 brings to a general structural stability result from which one can eventually conclude the proof of the existence of periodic solutions in the most general setting. The application of our abstract result to the nonlinear PDE (1. 2) is given in Section 6. Finally, the Appendix contains a technical (Gronwall-like) lemma on differential inequalities which is used for the estimates.

2 Main result and preliminary facts

2.1 Main result

Let H be a real Hilbert space with norm $|\cdot|_H$ and inner product $(\cdot,\cdot)_H$. Let A be a maximal monotone operator in H and let ϕ be a proper (i.e., $\phi \not\equiv \infty$) lower semicontinuous convex functional from H into $[0,\infty]$ with the effective domain $D(\phi) := \{u \in H;\ \phi(u) < \infty\}$. The graph of a maximal monotone operator will always be tacitly identified with the operator itself so that, for instance, the positions $[u,\xi] \in A$ and $u \in D(A)$, $\xi \in A(u)$ are equivalent. The reader shall be referred to the classical monographs [10, 16, 42] as well to the recent [11] for a comprehensive discussion on maximal monotone operator techniques and applications.

Let us consider the periodic problem (P) given by

$$A(u'(t)) + \partial\phi(u(t)) \ni f(t) \ \text{ in } H, \quad 0 < t < T, \tag{2. 1}$$
$$u(0) = u(T), \tag{2. 2}$$

where $\partial\phi$ denotes the subdifferential of ϕ (see §2.2 below for the definition) and f is a given function from $(0,T)$ into H. We are concerned with solutions of (P) given in the following sense:

Definition 2.1 (Strong solutions)**.** *A function $u \in C([0,T];H)$ is said to be a* (*strong*) *solution of* (P) *if the following conditions are all satisfied*:

(i) $u \in W^{1,2}(0,T;H)$ *and* $u(0) = u(T)$;

(ii) $u(t) \in D(\partial\phi)$ *and* $u'(t) \in D(A)$ *for a.e.* $t \in (0,T)$;

(iii) *there exist* $\eta, \xi \in L^2(0,T;H)$ *such that*

$$\eta(t) \in A(u'(t)), \quad \xi(t) \in \partial\phi(u(t)),$$
$$\eta(t) + \xi(t) = f(t) \quad \textit{for a.e.} \quad t \in (0,T). \tag{2. 3}$$

In order to discuss the existence of solutions for (P), let us set up our assumptions.

(A1) There exist constants $C_1 > 0$ and $C_2 \geq 0$ such that

$$C_1|u|_H^2 \leq (\eta, u)_H + C_2 \quad \text{for all} \ \ [u, \eta] \in A.$$

(A2) There exists a constant $C_3 \geq 0$ such that

$$|\eta|_H \leq C_3(|u|_H + 1) \quad \text{for all} \ \ [u, \eta] \in A.$$

(A3) For any $\lambda \in \mathbb{R}$, the set $\{u \in D(\phi);\ \phi(u) + |u|_H \leq \lambda\}$ is compact in H.

(A4) $\partial\phi$ is *coercive* in the following sense: there exists $z_0 \in D(\phi)$ such that

$$\liminf_{\substack{|u|_H \to \infty \\ [u,\xi] \in \partial\phi}} \frac{(\xi, u - z_0)_H}{|u|_H} = \infty. \tag{2. 4}$$

(A5) $f \in L^2(0, T; H)$.

Our main result reads,

Theorem 2.2 (Existence of periodic solutions)**.** *Assume that* (A1)–(A5) *are satisfied. Then* (P) *admits at least one solution.*

Let us provide some remarks on the coercivity condition (A4). At first, note that the condition (A4) can be equivalently rewritten as the following: there exists $z_0 \in D(\phi)$ such that for any $\delta > 0$ it follows that

$$|u|_H \leq \delta(\xi, u - z_0)_H + C_\delta \quad \text{for all} \ \ [u, \xi] \in \partial\phi \tag{2. 5}$$

with some constant $C_\delta \geq 0$ (see Proposition B.1). Moreover, let us stress that assumption (A4) follows when there exist $p > 1$ and $C_4 > 0$ such that

$$C_4|u|_H^p \leq \phi(u) + 1 \quad \text{for all } u \in D(\phi). \tag{2. 6}$$

Indeed, let $z_0 \in D(\phi)$ and $[u, \xi] \in \partial\phi$. Then by the definition of subdifferentials, we observe that

$$(\xi, u - z_0)_H \geq \phi(u) - \phi(z_0) \geq C_4|u|_H^p - \phi(z_0) - 1,$$

which implies (A4).

The Cauchy problem for equation (2. 1) was studied by Colli-Visintin [20], and the existence of solutions was proved for any initial datum $u_0 \in D(\phi)$ under (A1)–(A3) and (A5).

Note that the (simpler) equation

$$u'(t) + \partial\phi(u(t)) \ni f(t) \ \text{ in } H, \quad 0 < t < T,$$

(which corresponds to equation (2. 1) in case A is the identity mapping) has been proved to admit periodic solutions under (A4) and (A5) in [16]. Moreover, no periodic solution may exist when the coercivity (A4) does not hold (e.g., the ODE given by $H = \mathbb{R}$, $\partial\phi(u) \equiv 0$, $f(t) = t$).

We close this subsection by a proposition on the uniqueness of periodic solutions in a special case.

Proposition 2.3 (Uniqueness of periodic solutions)**.** *Assume that $\partial\phi$ is linear and A is strictly monotone. Then any two solutions u_1, u_2 of* (P) *satisfy*

$$u_1(t) = u_2(t) + v \quad \textit{for all} \ \ t \in [0, T]$$

with some $v \in D(\partial\phi)$ satisfying $\partial\phi(v) \ni 0$.

Proof. Let u_1 and u_2 be solutions for (P). Then, by taking the difference of the equations and testing it by $u_1'(t) - u_2'(t)$ in H, we see that

$$(\eta_1(t) - \eta_2(t), u_1'(t) - u_2'(t))_H + \frac{\mathrm{d}}{\mathrm{d}t}\phi(u_1(t) - u_2(t)) = 0,$$

where $\eta_i \in L^2(0, T; H)$ belongs to $A(u_i'(\cdot))$ for $i = 1, 2$, almost everywhere in time. The integration of both sides over $(0, T)$ and the periodicity condition (2. 2) yield

$$\int_0^T (\eta_1(t) - \eta_2(t), u_1'(t) - u_2'(t))_H \,\mathrm{d}t = 0,$$

which implies

$$(\eta_1(t) - \eta_2(t), u_1'(t) - u_2'(t))_H = 0 \quad \text{ for a.e. } t \in (0, T).$$

Since A is strictly monotone, we deduce that $u_1'(t) = u_2'(t)$ for a.e. $t \in (0, T)$. Hence one can write $u_1(t) = u_2(t) + v$ with some constant $v \in D(\partial\phi)$. By taking the difference between equations (2. 1) for u_1 and u_2 and using the fact that $\partial\phi$ is linear, we obtain $\partial\phi(v) \ni 0$. □

The assumptions frame of Proposition 2.3 basically corresponds to the uniqueness proof for the Cauchy problem for relation (2. 1) from [18]. Some alternative set of sufficient conditions for uniqueness for (2. 1) are presented in [34, §11.1.3] and [35]. The latter conditions seem however not to be directly applicable to the periodic problem.

As concerns doubly nonlinear equations of the type of (1. 3), uniqueness for the Cauchy problem is already discussed in [17, 21]. Moreover, uniqueness of periodic solutions is proved by [27] in a specific setting, where periodic solutions satisfy an order property.

Henceforth, we shall use the same symbol C in order to denote any non-negative constant depending on data and, in particular, independent of ε. The value of the constant C may change from line to line.

2.2 Preliminaries

We refer the reader to [16] for the definition and fundamental properties of maximal monotone operators in Hilbert spaces. Here let us give some preliminary materials on subdifferentials, their resolvents, and Yosida approximations and Moreau-Yosida regularizations of convex functionals (proofs can be found in [16] as well).

Let ϕ be a proper lower semicontinuous convex functional from H into $[0,\infty]$ with $D(\phi) := \{u \in H;\ \phi(u) < \infty\}$. Then the subdifferential operator $\partial\phi : H \to 2^H$ for ϕ is defined as follows:

$$\partial\phi(u) := \{\xi \in H;\ \phi(v) - \phi(u) \geq (\xi, v-u)_H \ \text{ for all } v \in D(\phi)\}$$

with the domain $D(\partial\phi) := \{u \in D(\phi);\ \partial\phi(u) \neq \emptyset\}$. Since $\partial\phi$ is maximal monotone in H, for $\varepsilon > 0$, one can define the *resolvent* $J_\varepsilon : H \to D(\partial\phi)$ and the *Yosida approximation* $(\partial\phi)_\varepsilon : H \to H$ of $\partial\phi$ by

$$J_\varepsilon := (I_d + \varepsilon\partial\phi)^{-1}, \quad (\partial\phi)_\varepsilon := (I_d - J_\varepsilon)/\varepsilon,$$

where I_d stands for the identity mapping of H. Furthermore, for $\varepsilon > 0$, the *Moreau-Yosida regularization* $\phi_\varepsilon : H \to [0,\infty)$ of ϕ is given by

$$\phi_\varepsilon(u) := \inf_{v\in H}\left\{\frac{1}{2\varepsilon}|u-v|_H^2 + \phi(v)\right\} \quad \text{for all } u \in H. \tag{2. 7}$$

The following proposition provides some classical properties of ϕ_ε.

Proposition 2.4 (Moreau-Yosida regularization)**.** *The Moreau-Yosida regularization ϕ_ε is a Fréchet differentiable convex functional from H into $\mathbb{R}$. Moreover, the infimum in* (2. 7) *is attained by $J_\varepsilon u$, where J_ε denotes the resolvent of $\partial\phi$, i.e.,*

$$\phi_\varepsilon(u) = \frac{1}{2\varepsilon}|u - J_\varepsilon u|_H^2 + \phi(J_\varepsilon u) = \frac{\varepsilon}{2}|(\partial\phi)_\varepsilon(u)|_H^2 + \phi(J_\varepsilon u).$$

Furthermore, the following holds.

(i) $\partial(\phi_\varepsilon) = (\partial\phi)_\varepsilon$, *where* $\partial(\phi_\varepsilon)$ *is the subdifferential* (*Fréchet derivative*) *of* ϕ_ε.

(ii) $\phi(J_\varepsilon u) \leq \phi_\varepsilon(u) \leq \phi(u)$ *for all* $u \in H$ *and* $\varepsilon > 0$.

(iii) $\phi_\varepsilon(u) \to \phi(u)$ *as* $\varepsilon \to 0_+$ *for all* $u \in H$.

Finally, let us recall the chain rule for subdifferentials.

Proposition 2.5 (Chain rule for subdifferentials). *Let* $u \in W^{1,2}(0,T;H)$ *be such that* $u(t) \in D(\partial\phi)$ *for a.e.* $t \in (0,T)$. *Assume that there exists* $\xi \in L^2(0,T;H)$ *such that* $\xi(t) \in \partial\phi(u(t))$ *for a.e.* $t \in (0,T)$. *Then the function* $t \mapsto \phi(u(t))$ *is absolutely continuous on* $[0,T]$. *Moreover, the set*

$$\mathcal{I} := \{t \in [0,T];\ u(t) \in D(\partial\phi),\ u \text{ and } \phi(u(\cdot)) \text{ are differentiable at } t\}$$

has full Lebesgue measure and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(u(t)) = (h, u'(t))_H \quad \text{for every } h \in \partial\phi(u(t)) \text{ and } t \in \mathcal{I}.$$

3 Approximate problems

We shall firstly focus on the case that ϕ satisfies a stronger coercivity requirement, specifically

(A6) There exists a constant $\rho > 0$ such that $\rho|u|_H^2 \leq \phi(u)$ for all $u \in D(\phi)$.

Note that the latter is stronger than (A4). Moreover, (A6) entails

$$\frac{\rho}{2\rho\varepsilon + 1}|u|_H^2 \leq \phi_\varepsilon(u) \quad \text{for all } u \in H \text{ and } \varepsilon > 0, \tag{3. 1}$$

where ϕ_ε denotes the Moreau-Yosida regularization of ϕ. In this section we construct solutions for the following approximate problems $(\mathrm{P})_\varepsilon$ for $\varepsilon > 0$:

$$\varepsilon u'(t) + A(u'(t)) + \partial\phi_\varepsilon(u(t)) \ni f_\varepsilon(t) \text{ in } H, \quad 0 < t < T, \tag{3. 2}$$

$$u(0) = J_\varepsilon u(T), \tag{3. 3}$$

where $\partial\phi_\varepsilon := \partial(\phi_\varepsilon) = (\partial\phi)_\varepsilon$, J_ε stands for the resolvent of $\partial\phi$ and (f_ε) is an approximate sequence in $L^\infty(0,T;H)$ such that

$$f_\varepsilon \to f \quad \text{strongly in } L^2(0,T;H). \tag{3. 4}$$

To this end, we also introduce the corresponding Cauchy problem $(\mathrm{C})_\varepsilon$, i.e., (3. 2) with the initial condition,

$$u(0) = u_0 \in H. \tag{3. 5}$$

The existence and the uniqueness of solutions for $(\mathrm{C})_\varepsilon$ can be proved, since (3. 2) is equivalently rewritten by

$$u'(t) = \mathcal{L}_\varepsilon(t, u(t)) := (\varepsilon I_d + A)^{-1} \left(f_\varepsilon(t) - \partial \phi_\varepsilon(u(t)) \right) \quad \text{in } H, \quad 0 < t < T$$

and $\mathcal{L}_\varepsilon(t, \cdot)$ is Lipschitz continuous in H. Hence one can define a single-valued mapping $P_\varepsilon : H \to H$ by

$$P_\varepsilon : u_0 \mapsto J_\varepsilon u(T),$$

where u is the unique solution of $(\mathrm{C})_\varepsilon$ with the initial data u_0.

The main result of this section is the following.

Theorem 3.1 (Existence of approximate solutions)**.** *Assume that* (A1)–(A3) *and* (A6) *are satisfied. Then for each* $\varepsilon > 0$*, problem* $(\mathrm{P})_\varepsilon$ *admits a solution* u_ε*.*

In order to prove this theorem, it suffices to find a fixed point u_0^* of the mapping P_ε. Indeed, let $u_0^* = P_\varepsilon u_0^*$ and u^* be the unique solution for the Cauchy problem $(\mathrm{C})_\varepsilon$ with $u_0 = u_0^*$. Then, by the definition of P_ε, we observe that

$$J_\varepsilon u^*(T) = P_\varepsilon u_0^* = u_0^* = u^*(0).$$

In order to find a fixed point of P_ε, we shall prove the following two lemmas and employ Schauder's fixed point theorem.

Lemma 3.2 (P_ε is a self-mapping)**.** *There exists a constant* $R > 0$ *such that* P_ε *is a self-mapping on the set*

$$B_R := \{u \in D(\phi);\ \phi(u) \le R\},$$

that is, $P_\varepsilon(B_R) \subset B_R$*.*

Proof. By testing (3. 2) by $u'(t)$ and using (A1) and Proposition 2.5, we have

$$\begin{aligned} \varepsilon |u'(t)|_H^2 + C_1 |u'(t)|_H^2 - C_2 + \frac{\mathrm{d}}{\mathrm{d}t} \phi_\varepsilon(u(t)) &\le (f_\varepsilon(t), u'(t))_H \\ &\le C |f_\varepsilon(t)|_H^2 + \frac{C_1}{2} |u'(t)|_H^2, \end{aligned}$$

which implies

$$\varepsilon|u'(t)|_H^2 + \frac{C_1}{2}|u'(t)|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(u(t)) \leq C(|f_\varepsilon(t)|_H^2 + 1) \qquad (3.\ 6)$$

for a.e. $t \in (0, T)$. Moreover, we test (3. 2) by $u(t)$ and exploit (A2) and (3. 1) in order to get

$$\begin{aligned} \phi_\varepsilon(u(t)) &\leq \phi_\varepsilon(0) + (\partial\phi_\varepsilon(u(t)), u(t))_H \\ &= \phi_\varepsilon(0) + (f_\varepsilon(t), u(t))_H - (\varepsilon u'(t), u(t))_H - (\eta(t), u(t))_H \\ &\leq C_\varepsilon\left(|f_\varepsilon(t)|_H^2 + |u'(t)|_H^2 + 1\right) + \frac{1}{2}\phi_\varepsilon(u(t)), \end{aligned}$$

where $\eta := f_\varepsilon - \varepsilon u' - \partial\phi_\varepsilon(u(\cdot)) \in A(u'(\cdot))$ almost everywhere in time and C_ε is a constant depending on ε. Thus we obtain

$$\phi_\varepsilon(u(t)) \leq 2C_\varepsilon\left(|f_\varepsilon(t)|_H^2 + |u'(t)|_H^2 + 1\right) \qquad (3.\ 7)$$

for a.e. $t \in (0, T)$. Hence, by multiplying (3. 7) by some suitably small constant and adding it to (3. 6), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(u(t)) + \alpha_\varepsilon\phi_\varepsilon(u(t)) \leq \beta_\varepsilon$$

where the two constants $\alpha_\varepsilon, \beta_\varepsilon > 0$ possibly depend on ε, provided $f_\varepsilon \in L^\infty(0, T; H)$. Therefore, by virtue of Proposition A.1 one can take a constant $R \geq \beta_\varepsilon/\alpha_\varepsilon > 0$ such that

$$\phi_\varepsilon(u(T)) \leq R \quad \text{if } \phi_\varepsilon(u_0) \leq R,$$

which together with the fact that $\phi(J_\varepsilon u) \leq \phi_\varepsilon(u) \leq \phi(u)$, (see Proposition 2.4) also implies

$$\phi(J_\varepsilon u(T)) \leq R \quad \text{if } \phi(u_0) \leq R.$$

Consequently, we deduce that P_ε maps the set B_R into itself. □

Lemma 3.3 (Continuity of P_ε in H)**.** *The mapping P_ε is continuous in H.*

Proof. Let $u_{0,n}, u_0 \in H$ be such that $u_{0,n} \to u_0$ strongly in H and let u_n and u be the unique solutions for $(\mathrm{C})_\varepsilon$ with initial data $u_{0,n}$ and u_0, respectively. Subtract (3. 2) for u_n from that for u and put $w_n := u - u_n$. We have

$$\varepsilon w_n'(t) + A(u'(t)) - A(u_n'(t)) \ni -\partial\phi_\varepsilon(u(t)) + \partial\phi_\varepsilon(u_n(t)).$$

By testing the latter by $w_n'(t)$ and exploiting the $1/\varepsilon$-Lipschitz continuity of Yosida approximations (see [16]), we obtain

$$\varepsilon|w_n'(t)|_H^2 \leq (-\partial\phi_\varepsilon(u(t)) + \partial\phi_\varepsilon(u_n(t)), w_n'(t))_H \leq \frac{1}{\varepsilon}|w_n(t)|_H|w_n'(t)|_H$$

for a.e. $t \in (0,T)$. Noting that $(\mathrm{d}/\mathrm{d}t)|w_n(t)|_H \leq |w_n'(t)|_H$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}|w_n(t)|_H \leq \frac{1}{\varepsilon^2}|w_n(t)|_H \quad \text{for a.e. } t \in (0,T),$$

which together with Gronwall's inequality yields

$$|w_n(T)|_H \leq |u_0 - u_{0,n}|_H e^{T/\varepsilon^2} \to 0.$$

Therefore $u_n(T) \to u(T)$ strongly in H as $n \to \infty$, and hence, P_ε is continuous in H, since J_ε is non-expansive in H. □

Now, we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. We note by (A3) and (A6) that the set B_R is compact in H. Moreover, B_R is convex because of the convexity of ϕ. Therefore combining Lemmas 3.2 and 3.3 and applying Schauder's fixed point theorem to $P_\varepsilon : B_R \to B_R$, we can take a fixed point $u_0^* \in B_R$ such that $P_\varepsilon u_0^* = u_0^*$. This completes the proof of Theorem 3.1. □

4 Estimates and limiting procedure

In this section we establish a priori estimates for solutions u_ε of $(\mathrm{P})_\varepsilon$ and finally derive the convergence of u_ε to a solution u of (P) as $\varepsilon \to 0$ under the stronger coercivity condition (A6).

Theorem 4.1 (Existence of periodic solutions under (A6))**.** *Assume that* (A1)–(A3)*,* (A5)–(A6) *are satisfied. Then problem* (P) *admits at least one solution.*

Let u_ε be a solution of $(\mathrm{P})_\varepsilon$. We shall firstly present a useful inequality stemming from the fact that $u_\varepsilon(0) = J_\varepsilon u_\varepsilon(T)$.

Lemma 4.2. *It holds that $\phi_\varepsilon(u_\varepsilon(0)) \leq \phi_\varepsilon(u_\varepsilon(T))$.*

Proof. By (ii) of Proposition 2.4, since $u_\varepsilon(0) = J_\varepsilon u_\varepsilon(T)$, it follows that

$$\phi_\varepsilon(u_\varepsilon(0)) \le \phi(u_\varepsilon(0)) = \phi(J_\varepsilon u_\varepsilon(T)) \le \phi_\varepsilon(u_\varepsilon(T)).$$

□

Next, we are in the position of establishing the following estimate.

Lemma 4.3. *There exists a constant C independent of ε such that*

$$\int_0^T |u_\varepsilon'(t)|_H^2 \mathrm{d}t \le C. \tag{4. 1}$$

Proof. Test (3. 2) by $u_\varepsilon'(t)$. Then, inequality (3. 6) follows with $u = u_\varepsilon$. Hence, by integrating both sides over $(0,T)$, we deduce that

$$\begin{aligned}\varepsilon \int_0^T |u_\varepsilon'(t)|_H^2 \mathrm{d}t + \frac{C_1}{2}\int_0^T |u_\varepsilon'(t)|_H^2 \mathrm{d}t + \phi_\varepsilon(u_\varepsilon(T)) \\ \le \phi_\varepsilon(u_\varepsilon(0)) + C\left(\int_0^T |f_\varepsilon(t)|_H^2 \mathrm{d}t + 1\right),\end{aligned}$$

which, together with Lemma 4.2, implies (4. 1). □

Moving from estimate (4. 1) we deduce by (A2) and a comparison in equation (3. 2) that

$$\int_0^T |\eta_\varepsilon(t)|_H^2 \mathrm{d}t \ \le \ C, \tag{4. 2}$$

$$\int_0^T |\partial\phi_\varepsilon(u_\varepsilon(t))|_H^2 \mathrm{d}t \ \le \ C, \tag{4. 3}$$

where $\eta_\varepsilon := f_\varepsilon - \varepsilon u_\varepsilon' - \partial\phi_\varepsilon(u_\varepsilon(\cdot)) \in A(u_\varepsilon'(\cdot))$ almost everywhere in time. Moreover, we also note that, by (4. 1),

$$\int_0^T \left|\frac{\mathrm{d}}{\mathrm{d}t} J_\varepsilon u_\varepsilon(t)\right|_H^2 \mathrm{d}t \le \int_0^T |u_\varepsilon'(t)|_H^2 \,\mathrm{d}t \le C, \tag{4. 4}$$

since J_ε is non-expansive in H (see [16]).

Let us get to a crucial estimate, namely an ordinary differential inequality for $\phi_\varepsilon(u_\varepsilon(t))$.

Lemma 4.4. *Let $y_\varepsilon(t) := \phi_\varepsilon(u_\varepsilon(t))$. Then,*

$$\frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t}(t) + y_\varepsilon(t) \leq g_\varepsilon(t) \quad \text{for a.e.} \quad t \in (0,T)$$

where the functions g_ε are uniformly bounded in $L^1(0,T)$ for $\varepsilon \in (0,1)$.

Proof. Recall again that

$$\varepsilon|u'_\varepsilon(t)|_H^2 + \frac{C_1}{2}|u'_\varepsilon(t)|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(u_\varepsilon(t)) \leq C(|f_\varepsilon(t)|_H^2 + 1). \tag{4. 5}$$

On the other hand, by testing (3. 2) by $u_\varepsilon(t) - v_0$ with any $v_0 \in D(\phi)$ and using (A2) and (3. 1), we have, for any $\varepsilon \in (0,1)$,

$$\begin{aligned} \phi_\varepsilon(u_\varepsilon(t)) &\leq \phi_\varepsilon(v_0) + (\partial\phi_\varepsilon(u_\varepsilon(t)), u_\varepsilon(t) - v_0)_H \\ &\leq \phi(v_0) + (f_\varepsilon(t) - \varepsilon u'_\varepsilon(t) - \eta_\varepsilon(t), u_\varepsilon(t) - v_0)_H \\ &\leq C\left(|f_\varepsilon(t)|_H^2 + \varepsilon^2|u'_\varepsilon(t)|_H^2 + |u'_\varepsilon(t)|_H^2 + 1\right) + \frac{1}{2}\phi_\varepsilon(u_\varepsilon(t)), \end{aligned}$$

which yields

$$\phi_\varepsilon(u_\varepsilon(t)) \leq 2C\left(|f_\varepsilon(t)|_H^2 + \varepsilon^2|u'_\varepsilon(t)|_H^2 + |u'_\varepsilon(t)|_H^2 + 1\right) \tag{4. 6}$$

with some constant $C \geq 0$ independent of ε. By adding (4. 6) to (4. 5), we have

$$\frac{\mathrm{d}y_\varepsilon}{\mathrm{d}t}(t) + y_\varepsilon(t) \leq C(|f_\varepsilon(t)|_H^2 + |u'_\varepsilon(t)|_H^2 + 1) =: g_\varepsilon(t)$$

with some constant $C > 0$ independent of ε. Moreover, by (3. 4) and (4. 1), the functions g_ε are bounded in $L^1(0,T)$ for all $\varepsilon \in (0,1)$. □

Hence, by Proposition A.2 and Lemma 4.2 the following estimates follow immediately.

Lemma 4.5. *There exists a constant C independent of ε such that*

$$\sup_{t\in[0,T]} \phi_\varepsilon(u_\varepsilon(t)) \leq C. \tag{4. 7}$$

By (A6)*, it also holds that*

$$\sup_{t\in[0,T]} |J_\varepsilon u_\varepsilon(t)|_H \leq \sqrt{C/\rho}. \tag{4. 8}$$

From these a priori estimates, by extracting a sequence $\varepsilon_n \to 0$, which will be also denoted by ε below, we can derive convergences of u_ε.

Lemma 4.6. *There exist $u \in W^{1,2}(0,T;H)$ and $\eta, \xi \in L^2(0,T;H)$ such that*

$$
\begin{aligned}
J_\varepsilon u_\varepsilon &\to u && \text{strongly in } C([0,T];H), && (4.\ 9)\\
u_\varepsilon &\to u && \text{strongly in } C([0,T];H), && (4.\ 10)\\
& && \text{weakly in } W^{1,2}(0,T;H), && (4.\ 11)\\
\eta_\varepsilon &\to \eta && \text{weakly in } L^2(0,T;H), && (4.\ 12)\\
\partial\phi_\varepsilon(u_\varepsilon(\cdot)) &\to \xi && \text{weakly in } L^2(0,T;H). && (4.\ 13)
\end{aligned}
$$

Moreover, $[u(t), \xi(t)] \in \partial\phi$ for a.e. $t \in (0,T)$.

Proof. By (A3) and Lemma 4.5, the family $(J_\varepsilon u_\varepsilon(t))_{\varepsilon\in(0,1)}$ is precompact in H for each $t > 0$. By estimate (4. 4), the function $t \mapsto J_\varepsilon u_\varepsilon(t)$ is equicontinuous in H on $[0,T]$ for all $\varepsilon > 0$. Therefore Ascoli's compactness lemma yields the uniform convergence (4. 9). Moreover, by exploiting the bound (4. 7), we deduce that

$$\sup_{t\in[0,T]} |u_\varepsilon(t) - J_\varepsilon u_\varepsilon(t)|_H^2 \le 2\varepsilon \sup_{t\in[0,T]} \phi_\varepsilon(u_\varepsilon(t)) \le C\varepsilon \to 0,$$

which leads us to obtain the strong convergence (4. 10). Furthermore, the weak convergences (4. 11), (4. 12) and (4. 13) follow from the estimates (4. 1), (4. 2) and (4. 3), respectively. Finally, from the demiclosedness of maximal monotone operators, one can infer from the convergence (4. 9) and (4. 13) that $u(t) \in D(\partial\phi)$ and $\xi(t) \in \partial\phi(u(t))$ for a.e. $t \in (0,T)$. $\square$

Next, let us prove the periodicity of u.

Lemma 4.7. *It holds that $u(0) = u(T)$.*

Proof. Since both $u_\varepsilon(t)$ and $J_\varepsilon u_\varepsilon(t)$ converge to $u(t)$ strongly in H (uniformly) for all $t \in [0,T]$, we deduce by $u_\varepsilon(0) = J_\varepsilon u_\varepsilon(T)$ that $u(0) = u(T)$. $\square$

We finally check that $\eta(t) \in A(u'(t))$ for a.e. $t \in (0,T)$.

Lemma 4.8. *It holds that $[u'(t), \eta(t)] \in A$ for a.e. $t \in (0,T)$.*

Proof. Test $\eta_\varepsilon(t) \in A(u'_\varepsilon(t))$ by $u'_\varepsilon(t)$ and integrate this over $(0,T)$. We obtain

$$
\begin{aligned}
&\int_0^T (\eta_\varepsilon(t), u'_\varepsilon(t))_H \mathrm{d}t\\
&= \int_0^T (f_\varepsilon(t), u'_\varepsilon(t))_H \mathrm{d}t - \varepsilon \int_0^T |u'_\varepsilon(t)|_H^2 \mathrm{d}t - \int_0^T (\partial\phi_\varepsilon(u_\varepsilon(t)), u'_\varepsilon(t))_H \mathrm{d}t\\
&\le \int_0^T (f_\varepsilon(t), u'_\varepsilon(t))_H \mathrm{d}t,
\end{aligned}
$$

since we deduce by Lemma 4.2 that

$$-\int_0^T (\partial\phi_\varepsilon(u_\varepsilon(t)), u_\varepsilon'(t))_H \mathrm{d}t = -\phi_\varepsilon(u_\varepsilon(T)) + \phi_\varepsilon(u_\varepsilon(0)) \le 0.$$

Therefore, as $\phi(u(0)) = \phi(u(T))$, it holds by convergence (3. 4) that

$$\limsup_{\varepsilon\to 0} \int_0^T (\eta_\varepsilon(t), u_\varepsilon'(t))_H \mathrm{d}t \le \int_0^T (f(t) - \xi(t), u'(t))_H \, \mathrm{d}t.$$

Consequently, by Proposition 2.5 of [16], we conclude that $u'(t) \in D(A)$ and $\eta(t) \in A(u'(t))$ for a.e. $t \in (0,T)$. □

Combining these lemmas, we have proved the conclusion of Theorem 4.1.

5 Structural stability

In the last section, we proved the existence of solutions for (P) under (A6), a stronger coercivity condition of ϕ. In order to replace (A6) by the weaker condition (A4), we shall establish a structural stability result for solutions of (P). Indeed, for $n \in \mathbb{N}$, define a functional $\phi^n : H \to [0,\infty]$ by

$$\phi^n(u) := \phi(u) + \frac{1}{2n}|u|_H^2 \quad \text{for } u \in H, \tag{5. 1}$$

which converges to ϕ (precisely, in the sense of Mosco on H) as $n \to \infty$. Then, ϕ^n complies with (A6) as well as all assumptions of Theorem 4.1. Hence, we have the existence of solutions u_n for (P) with ϕ replaced by ϕ^n. If one can obtain a structural stability result, i.e., the convergence of u^n to a solution of (P) as $n \to \infty$, our proof of Theorem 2.2 will be completed.

We shall work here in a more general setting of possibly independent interest. Let (A^n) be a sequence of maximal monotone operators in H and let (ϕ^n) be a sequence of proper lower semicontinuous convex functionals from H into $[0,\infty]$. Assume that

$$A^n \to A \qquad \text{in the sense of graph on } H, \tag{5. 2}$$

$$\phi^n \to \phi \qquad \text{in the sense of Mosco on } H \tag{5. 3}$$

as $n \to \infty$. Here we recall the definitions of *graph-convergence* and *Mosco-convergence*.

Definition 5.1 (Graph-convergence and Mosco-convergence). *Let H be a Hilbert space. Let (A^n) be a sequence of maximal monotone operators in H and let (ϕ^n) be a sequence of proper lower semicontinuous convex functionals from H into $[0, T]$.*

(i) *The sequence (A^n) is said to graph-converge to a maximal monotone operator $A : H \to H$ (or $A^n \to A$ in the sense of graph on H) if for any $[u, \xi] \in A$ and $n \in \mathbb{N}$, there exists $[u_n, \xi_n] \in A^n$ such that*

$$u_n \to u \quad \text{strongly in } H, \quad \xi_n \to \xi \quad \text{strongly in } H \quad \text{as } n \to \infty.$$

(ii) *The sequence (ϕ^n) is said to Mosco-converge to a proper lower semicontinuous convex functional $\phi : H \to [0, \infty]$ ($\phi^n \to \phi$ in the sense of Mosco on H) if the following* (a), (b) *hold:*

(a) *(Liminf condition) Let $u_n \to u$ weakly in H as $n \to \infty$. Then*

$$\liminf_{n \to \infty} \phi^n(u_n) \geq \phi(u).$$

(b) *(Existence of recovery sequences) For every $u \in D(\phi)$, there exists a recovery sequence (u_n) in H such that $u_n \to u$ strongly in H and $\phi^n(u_n) \to \phi(u)$.*

Remark 5.2. Let $\phi, \phi^n : H \to [0, \infty]$ be proper lower-semicontinuous and convex. Then it is known that $\partial\phi^n$ graph-converges to $\partial\phi$ if ϕ^n Mosco-converges to ϕ. We refer the reader to Theorem 3.66 of [9].

Now, let us consider the following periodic problems $(\mathrm{P})^n$:

$$A^n(u'(t)) + \partial\phi^n(u(t)) \ni f_n(t) \quad \text{in } H, \quad 0 < t < T, \tag{5. 4}$$
$$u(0) = u(T), \tag{5. 5}$$

where (f_n) is a sequence in $L^2(0, T; H)$ such that

$$f_n \to f \quad \text{strongly in } L^2(0, T; H) \text{ as } n \to \infty. \tag{5. 6}$$

The main result of this section is concerned with the asymptotic behavior of solutions u_n for $(\mathrm{P})^n$ as $n \to \infty$.

Theorem 5.3 (Structural Stability). *In addition to* (5. 2), (5. 3), *and* (5. 6), *assume that* (A1) *and* (A2) *are satisfied with $A = A^n$ and constants independent of n. Moreover, suppose that*

(A3)′ *every sequence* (u_n) *is precompact in* H *whenever* $\phi^n(u_n) + |u_n|_H$ *is bounded as* $n \to \infty$*,*

(A4)′ *for each* $n \in \mathbb{N}$ *large enough, there exists* $z_{0,n} \in D(\phi^n)$ *such that for any* $\delta > 0$ *sufficiently small it holds that*

$$|u|_H \le \delta(\xi, u - z_{0,n})_H + C_\delta \quad \textit{for all } [u,\xi] \in \partial\phi^n$$

with some constant $C_\delta \ge 0$ *independent of* n*. Moreover,* $|z_{0,n}|_H \le B_1$ *for all* $n \in \mathbb{N}$ *with some constant* $B_1 \ge 0$*.*

Let u_n *be a solution of* $(\mathrm{P})^n$*. Then, there exist a subsequence* (n') *of* (n) *and a solution* u *of* (P) *such that*

$$u_{n'} \to u \qquad \textit{weakly in } W^{1,2}(0,T;H),$$
$$\textit{strongly in } C([0,T];H) \quad \textit{as } n' \to \infty.$$

Before proceeding to a proof, we set up a couple of lemmas.

Lemma 5.4. *Assume that* $(\mathrm{A4})'$ *is satisfied and set* $\mathcal{H} := L^2(0,T;H)$ *with* $\|u\|_{\mathcal{H}} := (\int_0^T |u(t)|_H^2 \mathrm{d}t)^{1/2}$ *for* $u \in \mathcal{H}$*. Let* (u_n) *be a sequence in* $W^{1,2}(0,T;H)$ *such that* $u_n(t) \in D(\partial\phi^n)$ *for a.e.* $t \in (0,T)$*. Let* (ξ_n) *be a sequence in* $\mathcal{H}$ *such that*

$$\xi_n(t) \in \partial\phi^n(u_n(t)) \ \textit{for a.e. } t \in (0,T) \quad \textit{and} \quad \|\xi_n\|_{\mathcal{H}} \le B_2 \ \textit{for all } n \in \mathbb{N}$$

with some constant $B_2 \ge 0$*. Then there exists a constant* C *independent of* n *such that*

$$\|u_n\|_{\mathcal{H}} \le C\left(\|u_n'\|_{\mathcal{H}} + 1\right) \quad \textit{for all } n \in \mathbb{N}.$$

Proof. By $(\mathrm{A4})'$, it follows that

$$|u_n(t)|_H \le \delta(\xi_n(t), u_n(t) - z_{0,n})_H + C_\delta \quad \text{for a.e. } t \in (0,T).$$

Integrating this over $(0,T)$, we find that

$$\begin{aligned}\int_0^T |u_n(t)|_H \mathrm{d}t &\le \delta \int_0^T (\xi_n(t), u_n(t) - z_{0,n})_H \mathrm{d}t + C_\delta T \\ &\le \delta \|\xi_n\|_{\mathcal{H}} \left(\|u_n\|_{\mathcal{H}} + |z_{0,n}|_H T^{1/2}\right) + C_\delta T \\ &\le \delta B_2 \left(\|u_n\|_{\mathcal{H}} + B_1 T^{1/2}\right) + C_\delta T. \qquad (5.\ 7)\end{aligned}$$

On the other hand, by Hölder's and Sobolev's inequalities for vector-valued functions, there is a constant $M \geq 0$ such that

$$\|u\|_{\mathcal{H}} \leq M \left(\int_0^T |u(t)|_H \mathrm{d}t + \|u'\|_{\mathcal{H}} \right) \quad \text{for all } u \in W^{1,2}(0,T;H). \tag{5. 8}$$

Hence combining this with (5. 7) and taking $\delta > 0$ sufficiently small, one can deduce that

$$\int_0^T |u_n(t)|_H \mathrm{d}t \leq C \left(\|u_n'\|_{\mathcal{H}} + 1 \right)$$

with some constant C independent of n. □

The next lemma is well known (one can prove this lemma as in Proposition 3.59 of [9] with slight modifications).

Lemma 5.5. *Let (A^n) be a sequence of maximal monotone operators in H such that A^n graph-converges to a maximal monotone operator A. Let $[v_n, \eta_n] \in A^n$ be such that*

$$v_n \to v \quad \textit{and} \quad \eta_n \to \eta \quad \textit{weakly in } H,$$
$$\limsup_{n \to \infty} (\eta_n, v_n)_H \leq (\eta, v)_H.$$

Then $[v, \eta] \in A$ and $(\eta_n, v_n)_H \to (\eta, v)_H$.

Now, we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. By testing equation (5. 4) by $u_n'(t)$ and using (A1) with C_1, C_2 independent of n, we deduce that

$$\frac{C_1}{2} |u_n'(t)|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t} \phi^n(u_n(t)) \leq C_2 + C|f_n(t)|_H^2 \quad \text{for a.e. } t \in (0,T). \tag{5. 9}$$

Integrating this over $(0,T)$ and using periodicity (5. 5), we have

$$\int_0^T |u_n'(t)|_H^2 \mathrm{d}t \leq C, \tag{5. 10}$$

which, together with (A2) with C_3 independent of n, implies

$$\int_0^T |\eta_n(t)|_H^2 \mathrm{d}t \leq C, \tag{5. 11}$$

where $\eta_n(t)$ denotes a section of $A^n(u_n'(t))$ as in equation (2. 3). Moreover, by comparison in equation (5. 4), it follows that

$$\int_0^T |\xi_n(t)|_H^2 \mathrm{d}t \le C, \tag{5. 12}$$

where $\xi_n := f_n - \eta_n$ is a section of $\partial\phi^n(u_n(\cdot))$. Therefore, by Lemma 5.4, we have

$$\int_0^T |u_n(t)|_H^2 \mathrm{d}t \le C, \tag{5. 13}$$

which together with the estimate (5. 10) entails that (u_n) is bounded in $W^{1,2}(0,T;H)$. Hence, $\sup_{t\in[0,T]} |u_n(t)|_H \le C$.

We next prove the uniform boundedness of $\phi^n(u_n(t))$ on $[0,T]$. Let $v_0 \in D(\phi)$. Then, one can take $v_{0,n} \in D(\phi^n)$ such that $\phi^n(v_{0,n}) \to \phi(v_0)$ and $v_{0,n} \to v_0$ strongly in H, since $\phi^n \to \phi$ in the sense of Mosco. We observe that

$$\begin{aligned}\phi^n(u_n(t)) &\le \phi^n(v_{0,n}) + (\xi_n(t), u_n(t) - v_{0,n})_H \\ &= \phi^n(v_{0,n}) + (f_n(t), u_n(t) - v_{0,n})_H - (\eta_n(t), u_n(t) - v_{0,n})_H \\ &\le \phi^n(v_{0,n}) + C\left(|f_n(t)|_H^2 + |u_n'(t)|_H^2 + |u_n(t)|_H^2 + |v_{0,n}|_H^2 + 1\right)\end{aligned}$$

with a constant C independent of n. Hence, by adding the latter to inequality (5. 9) we deduce that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\phi^n(u_n(t)) + \phi^n(u_n(t)) \\ &\le g_n(t) := \phi^n(v_{0,n}) + C\left(|f_n(t)|_H^2 + |u_n'(t)|_H^2 + |u_n(t)|_H^2 + |v_{0,n}|_H^2 + 1\right)\end{aligned}$$

with a constant $C \ge 0$ independent of n. From already obtained estimates we notice that (g_n) is uniformly bounded in $L^1(0,T)$. Therefore, by Proposition A.2 we conclude that

$$\sup_{t\in[0,T]} \phi^n(u_n(t)) \le \left(\frac{1}{T} + 1\right) \int_0^T |g_n(t)| \mathrm{d}t \le C. \tag{5. 14}$$

By (A3)$'$, we note that $(u_n(t))$ is precompact in H for all $t \in [0,T]$. Hence one can derive the following convergences:

$$\begin{aligned} u_n \to u \quad & \text{weakly in } W^{1,2}(0,T;H), \\ & \text{strongly in } C([0,T];H), \\ \xi_n \to \xi \quad & \text{weakly in } L^2(0,T;H), \\ \eta_n \to \eta \quad & \text{weakly in } L^2(0,T;H). \end{aligned}$$

Here we also obtain $u(0) = u(T)$, since $u_n(0) = u_n(T)$. From Lemma 5.5, one can prove $\xi(t) \in \partial\phi(u(t))$ and $\eta(t) \in A(u(t))$ for a.e. $t \in (0,T)$ by a similar argument to that of Sect. 4. Thus u solves (P) and it completes our proof. □

Eventually, let us complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Define a sequence (ϕ^n) of functionals from H into $[0,\infty]$ by

$$\phi^n(u) := \phi(u) + \frac{1}{2n}|u|_H^2 \quad \text{for} \quad u \in H$$

with $D(\phi^n) = D(\phi)$. Then ϕ^n satisfies (A6). Hence by Theorem 4.1, one deduce that (P) with ϕ replaced by ϕ^n admits a strong solution u_n. We can easily check that $\phi^n \to \phi$ in the sense of Mosco on H as $n \to \infty$, and moreover, we derive (A3)$'$ and (B4)$'$ with $z_{0,n} \equiv z_0$, since ϕ complies with (A3) and (A4). Therefore due to Theorem 5.3, u_n converges to u as $n \to \infty$ and the limit u solves (P). □

6 Application to PDEs

This section is devoted to a typical application of the preceding abstract theory. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We are concerned with the following periodic problem:

$$\gamma(u_t) - \Delta_p u \ni f \qquad \text{in } \ \Omega \times (0,T), \tag{6. 1}$$

$$u = 0 \qquad \text{on } \ \partial\Omega \times (0,T), \tag{6. 2}$$

$$u(\cdot,0) = u(\cdot,T) \qquad \text{in } \ \Omega, \tag{6. 3}$$

where $u_t = \partial u/\partial t$, γ is a maximal monotone graph in $\mathbb{R}^2$, $f = f(x,t)$ is a given function, and Δ_p is the so-called p-Laplace operator given by

$$\Delta_p\phi(x) := \nabla \cdot \left(|\nabla\phi(x)|^{p-2}\nabla\phi(x)\right), \quad 1 < p < \infty.$$

Inclusions of the type of (6. 1) may arise in connection with phase transitions [12, 15, 19, 24, 28, 29], gas flow through porous media [39], and damage processes [13, 14, 22, 23, 31]. In the limiting case of graphs α being 0-homogeneous (which is however not covered by our analysis) this kind of equation may stem in elastoplasticity, brittle fractures, ferroelectricity, and general rate-independent systems [30]. Let us remark that relation (6. 1) stems as the *gradient flow* in $H = L^2(\Omega)$ of the *(complementary) energy* functional ϕ given by

$$\phi(u,t) := \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{p}|\nabla u(x)|^p - f(x,t)u(x)\right) \mathrm{d}x & \text{if } \ u \in W_0^{1,p}(\Omega), \\ \infty & \text{otherwise} \end{cases}$$

with respect to the metric structure induced by the *dissipation* functional

$$F(u(x)) := \begin{cases} \displaystyle\int_\Omega \widehat{\alpha}(u(x))\mathrm{d}x & \text{if } \ \widehat{\alpha}(u(\cdot)) \in L^1(\Omega), \\ \infty & \text{otherwise} \end{cases}$$

for $\widehat{\alpha}' = \alpha$. Indeed, by taking variations in $L^2(\Omega)$, inclusion (6. 1) is equivalent to the kinetic relation

$$\partial F(u_t) + \partial_u \phi(u,t) \ni 0$$

which represents the balance between the system of conservative ($\partial_u\phi(u,t)$, respectively) and dissipative actions ($\partial F(u_t)$, respectively) in the physical system. The question of the periodic solvability of inclusion (6. 1) has hence a clear applicative interest, especially in connection with the study of long-time behavior of the above-mentioned physical systems in case of periodic external actions.

In order to state our result, we assume that

(H1) There exist constants $C_5 > 0$, $C_6 \geq 0$ such that

$$C_5|s|^2 \leq gs + C_6 \quad \text{for all } \ [s,g] \in \gamma.$$

(H2) There exists a constant $C_7 \geq 0$ such that

$$|g| \leq C_7(|s|+1) \quad \text{for all } \ [s,g] \in \gamma.$$

Remark 6.1. Assumptions (H1) and (H2) allow γ to be degenerate and multivalued. Indeed, γ_1 and γ_2 given below satisfy the assumptions:

$$\gamma_1(s) = \begin{cases} s & \text{if } \ s < 0, \\ 0 & \text{if } \ 0 \leq s \leq 1, \\ s-1 & \text{if } \ 1 < s. \end{cases} \qquad \gamma_2(s) = \begin{cases} s & \text{if } \ s < 0, \\ [0,1] & \text{if } \ s = 0, \\ s+1 & \text{if } \ 0 < s. \end{cases}$$

Furthermore, one can check (H1) and (H2) if γ satisfies these assumptions only for $|s| \geq R$ with some constant $R > 0$ and $\gamma \in W^{1,\infty}(-R,R)$.

Our result reads,

Theorem 6.2 (Existence of periodic solutions for a nonlinear PDE)**.** *Assume that* $f \in L^2(0,T;L^2(\Omega))$ *and* (H1), (H2) *are satisfied. Moreover, suppose that* $p > 2N/(N+2)$. *Then problem* (6. 1)–(6. 3) *admits at least one solution.*

Proof. We set $H = L^2(\Omega)$ with the norm $|\cdot|_H := |\cdot|_{L^2}$ and define ϕ by

$$\phi(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u(x)|^p \mathrm{d}x & \text{if } \ u \in W_0^{1,p}(\Omega), \\ \infty & \text{else.} \end{cases}$$

Moreover, we let $A : H \to H$ be given by

$$A(u) := \{\eta \in H; \ \eta(x) \in \gamma(u(x)) \ \text{ for a.e. } x \in \Omega\}$$

with the domain $D(A) := \{u \in H; \ A(u) \neq \emptyset\}$. Then, we observe that $\partial\phi(u) = -\Delta_p u$ with the homogeneous Dirichlet boundary condition in H, and A is maximal monotone in H. Thus (6. 1)–(6. 3) is reduced to (P). Now, (A1) and (A2) follow immediately from (H1) and (H2). Moreover, since $W_0^{1,p}(\Omega)$ is compactly embedded in $L^2(\Omega)$ by $2N/(N+2) < p$, every sublevel set of ϕ is compact in H. Hence (A3) holds true. Furthermore, thanks to the Sobolev-Poincaré inequality,

$$|u|_H \leq C_p \|\nabla u\|_{L^p(\Omega)} \ \text{ for all } \ u \in W_0^{1,p}(\Omega), \quad \text{provided that } \ p \geq 2N/(N+2),$$

we see that

$$\frac{(\partial\phi(u), u)_H}{|u|_H} = \frac{\|\nabla u\|_{L^p(\Omega)}^p}{|u|_H} \geq C_p^{-p} |u|_H^{p-1} \quad \text{for all } \ u \in D(\partial\phi),$$

which implies (A4) with $z_0 = 0$. Therefore, by applying Theorem 2.2, we conclude that the system (6. 1)–(6. 3) admits at least one solution. □

As for uniqueness we have the following.

Proposition 6.3. *In case* $p = 2$ *and* γ *is strictly monotone, the periodic solution for* (6. 1)–(6. 3) *is unique.*

Proof. Along with the same positions as in the proof of Theorem 6.1, if $p = 2$, then $\partial\phi = -\Delta$ is linear and $(\partial\phi)^{-1}(0) = 0$. Furthermore, A is strictly monotone, since γ is so. Hence by Proposition 2.3 the solution of the system (6. 1)–(6. 3) is unique. □

The above well-posedness results can be generalized in a number of different directions. At first, the results for equation (6. 1) can be extended to more general elliptic operators by replacing the p-Laplacian $-\Delta_p u$ by

$$-\nabla \cdot \boldsymbol{m}(x, \nabla u) + \partial I_\kappa(u)$$

where the function $\boldsymbol{m} = \boldsymbol{m}(x, \boldsymbol{p}) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable in x and differentiable and maximal monotone in $\boldsymbol{p}$ and the indicator function I_κ over some closed interval $\kappa \subset \mathbb{R}$. Indeed, the statement of Theorem 2.2 can be extended by assuming that $\boldsymbol{m}(x, \cdot)$ admits a strongly coercive primitive function $P(x, \cdot)$ and by setting

$$\phi(u) := \int_\Omega P(x, \nabla u(x))dx + I_K(u),$$

where I_K denotes the indicator function over the set $K := \{u \in L^2(\Omega);\ u(x) \in \kappa$ for a.e. $x \in \Omega\}$.

Furthermore, we can treat doubly nonlinear systems such as

$$\boldsymbol{\alpha}(\boldsymbol{u}_t) - \Delta_p \boldsymbol{u} \ni \boldsymbol{f} \quad \text{in } \Omega \times (0, T), \tag{6. 4}$$

where $\boldsymbol{u} = (u_1, u_2, \ldots, u_m) : \Omega \times (0, T) \to \mathbb{R}^m$ and $\boldsymbol{\alpha}$ is a maximal monotone operator in $\mathbb{R}^m$ with linear growth. Here, the vectorial p-Laplacian $\Delta_p \boldsymbol{u}$ is defined by

$$\Delta_p \boldsymbol{u} = (\Delta_p u_1, \Delta_p u_2, \ldots, \Delta_p u_m).$$

Again, the results of Theorem 2.2 can be extended in order to cover (6. 4) by letting $H = (L^2(\Omega))^m$ and

$$\phi(\boldsymbol{u}) := \frac{1}{p} \sum_{i=1}^m \int_\Omega |\nabla u_i(x)|^p dx.$$

A Some tools on ordinary differential inequalities

In this appendix we provide two types of estimates for solutions to ordinary differential inequalities. Let us start by recalling without proof an elementary estimate, for the sake of completeness.

Proposition A.1. *Let $T > 0$ and let $y : [0, T] \to \mathbb{R}$ be an absolutely continuous function satisfying*

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \alpha y(t) \le \beta \quad \textit{for a.e.} \quad t \in (0, T) \tag{A. 1}$$

with constants $\alpha > 0$ and $\beta \geq 0$. Then it follows that

$$\sup_{t\in[0,T]} y(t) \leq R \quad \textit{if} \quad y(0) \leq R$$

for any constant $R \geq \beta/\alpha$.

The following proposition is exploited in the limiting procedure for solutions of $(\mathrm{P})_\varepsilon$ as $\varepsilon \to \infty$ in Section 5 (see also [33]).

Proposition A.2. *Let $T > 0$ and let $y : [0, T] \to [0, \infty)$ be an absolutely continuous function satisfying*

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) + \alpha y(t) \leq g(t) \quad \textit{for a.e.} \quad t \in (0, T), \tag{A. 2}$$

$$y(0) \leq y(T), \tag{A. 3}$$

where α is a positive constant and $g \in L^1(0, T)$. Then it follows that

$$\sup_{t\in[0,T]} y(t) \leq \left(\frac{1}{\alpha T} + 1\right) \int_0^T |g(t)|\mathrm{d}t.$$

Proof. By integrating the ordinary differential inequality (A.2) and using condition (A. 3), we have

$$y(T) + \alpha \int_0^T y(\tau)\mathrm{d}\tau \leq y(0) + \int_0^T |g(\tau)|\mathrm{d}\tau \leq y(T) + \int_0^T |g(\tau)|\mathrm{d}\tau,$$

which implies

$$\int_0^T y(\tau)\mathrm{d}\tau \leq \frac{1}{\alpha} \int_0^T |g(\tau)|\mathrm{d}\tau =: M.$$

Let $t_{\min}$ and $t_{\max}$ be a minimizer and a maximizer of $y = y(t)$, respectively. Then we find that

$$y(t_{\min})T \leq \int_0^T y(\tau)\mathrm{d}\tau \leq M,$$

which gives $y(t_{\min}) \leq M/T$.

In case $t_{\min} \leq t_{\max}$, by integrating inequality (A. 2) over $(t_{\min}, t_{\max})$, we observe that

$$y(t_{\max}) \leq y(t_{\min}) + \int_0^T |g(\tau)|\mathrm{d}\tau \leq \frac{M}{T} + \int_0^T |g(\tau)|\mathrm{d}\tau.$$

In case $t_{\min} > t_{\max}$, the integration of (A. 2) over $(0, t_{\max})$ yields

$$y(t_{\max}) \leq y(0) + \int_0^{t_{\max}} g(\tau)\mathrm{d}\tau.$$

Moreover, the integration of (A. 2) over $(t_{\min}, T)$ gives

$$y(T) \leq y(t_{\min}) + \int_{t_{\min}}^{T} g(\tau)\mathrm{d}\tau.$$

Since $y(0) \leq y(T)$, we conclude that

$$y(t_{\max}) \leq y(t_{\min}) + \int_0^T |g(\tau)|\mathrm{d}\tau \leq \frac{M}{T} + \int_0^T |g(\tau)|\mathrm{d}\tau.$$

Thus we have proved this proposition. □

Remark A.3. By repeating a similar argument to the above, one can also verify the following: if an absolutely continuous function $y : [0, T] \to \mathbb{R}$ satisfies

$$y'(t) \leq g(t) \quad \text{for a.e.} \quad t \in (0, T), \qquad \int_0^T y(\tau)\mathrm{d}\tau \leq M, \quad y(0) \leq y(T),$$

it then follows that

$$\sup_{t \in [0,T]} y(t) \leq \frac{M}{T} + \int_0^T |g(\tau)|\mathrm{d}\tau.$$

B Equivalence of coercivity conditions

In this section we prove the equivalence of two coercivity conditions.

Proposition B.1 (Equivalence of coercivity conditions)**.** *Let ϕ be a proper lower semicontinuous convex functional from H into $[0, \infty]$ and let $z_0 \in D(\phi)$. Then the following two conditions are equivalent*:

(i) $\displaystyle \liminf_{\substack{|u|_H \to \infty \\ [u,\xi] \in \partial\phi}} \frac{(\xi, u - z_0)_H}{|u|_H} = \infty.$

(ii) *For any $\delta > 0$, there exists a constant $C_\delta \geq 0$ such that*

$$|u|_H \leq \delta(\xi, u - z_0)_H + C_\delta \quad \text{for all} \quad [u, \xi] \in \partial\phi.$$

Proof. We first show that (i) implies (ii). Assume on the contrary that one can take $\delta_0 > 0$ such that for all $n \in \mathbb{N}$ there exists $[u_n, \xi_n] \in \partial\phi$ satisfying

$$|u_n|_H > \delta_0(\xi_n, u_n - z_0)_H + n.$$

Then it follows that

$$|u_n|_H \geq -\delta_0\phi(z_0) + n \to \infty$$

and

$$\frac{(\xi_n, u_n - z_0)_H}{|u_n|_H} < \frac{1}{\delta_0}.$$

Hence, by passing to the limit as $n \to \infty$, we derive

$$\liminf_{n\to\infty} \frac{(\xi_n, u_n - z_0)_H}{|u_n|_H} \leq \frac{1}{\delta_0} < \infty,$$

which contradicts (i).

Let us next prove that (ii) implies (i). From (ii), one can immediately deduce for $u \neq 0$ that

$$\frac{1}{\delta} \leq \frac{(\xi, u - z_0)_H}{|u|_H} + \frac{C_\delta}{\delta|u|_H} \quad \text{for all} \quad \delta > 0 \quad \text{and} \quad [u, \xi] \in \partial\phi.$$

Taking a liminf in both sides as $|u|_H \to \infty$, we deduce that

$$0 < \frac{1}{\delta} \leq \liminf_{\substack{|u|_H\to\infty \\ [u,\xi]\in\partial\phi}} \frac{(\xi, u - z_0)_H}{|u|_H}.$$

Hence by letting $\delta \to 0_+$, we complete our proof. $\square$

Acknowledgements

The first author gratefully acknowledges the kind hospitality and support of the IMATI-CNR in Pavia, Italy, where this research was initiated, and the supports of Short-Term Mobility Program 2010 of CNR, of the SIT grant and of the Grant-in-Aid for Young Scientists (B) # 22740093, MEXT. The second author is partly supported by FP7-IDEAS-ERC-StG Grant # 200947 (BioSMA).

References

[1] Aizicovici, S. and Hokkanen, V.-M., Doubly nonlinear periodic problems with unbounded operators, J. Math. Anal. Appl. **292** (2004), 540–557.

[2] Aizicovici, S. and Yan, Q., Convergence theorems for abstract doubly nonlinear differential equations, Panamer. Math. J. **7** (1997), 1–17.

[3] Akagi, G., Doubly nonlinear evolution equations with non-monotone perturbations in reflexive Banach spaces, to appear in J. Evol. Eq.

[4] Akagi, G., Global attractors for doubly nonlinear evolution equations with non-monotone perturbations, J. Diff. Eq. **250** (2011), 1850–1875.

[5] Akagi, G. and Stefanelli, U., Weighted energy-dissipation functionals for doubly nonlinear evolution, J. Funct. Anal. **260** (2011), 2541–2578.

[6] Akagi, G. and Stefanelli, U., A variational principle for doubly nonlinear evolution, Appl. Math. Lett. **23** (2010), 1120–1124.

[7] Arai, T., On the existence of the solution for $\partial\varphi(u'(t))+\partial\psi(u(t)) \ni f(t)$, *J. Fac. Sci. Univ. Tokyo Sec. IA Math.* **26** (1979), 75–96.

[8] Aso, M., Frémond, M. and Kenmochi, N., Phase change problems with temperature dependent constraints for the volume fraction velocities, Nonlinear Anal. **60** (2005), 1003–1023.

[9] Attouch, H., *Variational convergence for functions and operators*, Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[10] Barbu, V., *Nonlinear semigroups and differential equations in Banach spaces*, Noordhoff, Leyden, 1976.

[11] Barbu, V., *Nonlinear differential equations of monotone types in Banach spaces*, Springer Monographs in Mathematics, Springer, New York, 2010.

[12] Blanchard, D., Damlamian, A. and Ghidouche, H., A nonlinear system for phase change with dissipation, Differential Integral Equations **2** (1989), 344–362.

[13] Bonetti, E. and Schimperna, G., Local existence for Frémond's model of damage in elastic materials, Contin. Mech. Thermodyn. **16** (2004), 319–335.

[14] Bonetti, E., Schimperna, G. and Segatti, A., On a doubly non linear model for the evolution of damaging in viscoelastic materials, J. Diff. Eq. **218** (2005), 91–116.

[15] Bonfanti, G., Frémond, M., and Luterotti, F., Global solution to a nonlinear system for irreversible phase changes, Adv. Math. Sci. Appl. **10** (2000), 1–24.

[16] Brézis, H., *Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, Math Studies, Vol.5, North-Holland, Amsterdam/New York, 1973.

[17] Carrillo, J., Entropy solutions for nonlinear degenerate problems, Arch. Ration. Mech. Anal. **147** (1999), 269–361.

[18] Colli, P., On some doubly nonlinear evolution equations in Banach spaces, Japan J. Indust. Appl. Math. **9** (1992), 181–203.

[19] Colli, P., Luterotti, F., Schimperna, G. and Stefanelli U., Global existence for a class of generalized systems for irreversible phase changes, NoDEA Nonlinear Differential Equations Appl. **9** (2002), 255–276.

[20] Colli, P. and Visintin, A., On a class of doubly nonlinear evolution equations, Comm. Partial Differential Equations **15** (1990), 737–756.

[21] Di Benedetto, E. and Showalter, R., Implicit degenerate evolution equations and applications, SIAM J. Math. Anal. **12** (1981), 731–751.

[22] Frémond, M., Kuttler, K. and Shillor, M., Existence and uniqueness of solutions for a dynamic one-dimensional damage model, J. Math. Anal. Appl. **229** (1999), 271–294.

[23] Frémond, M., Kuttler, K., Nedjar, B. and Shillor, M., One-dimensional models of damage, Adv. Math. Sci. Appl. **8** (1998), 541–570.

[24] Frémond, M. and Visintin, A., Dissipation dans le changement de phase. Surfusion. Changement de phase irréversible, C. R. Acad. Sci. Paris Sér. II Méc. Phys. Chim. Sci. Univers Sci. Terre **301** (1985), 1265–1268.

[25] Hokkanen, V.-M., Existence of a periodic solution for implicit nonlinear equations, Differential Integral Equations **9** (1996), 745–760.

[26] Kenmochi, N. and Kubo, M., Periodic solutions to a class of nonlinear variational inequalities with time-dependent constraints, Funkcial. Ekvac. **30** (1987), 333–349.

[27] Kenmochi, N. and Kubo, M., Periodic behavior of solutions to parabolic-elliptic free boundary problems, J. Math. Soc. Japan **41** (1989), 625–640.

[28] Luterotti, F., Schimperna, G. and Stefanelli, U., Existence result for a nonlinear model related to irreversible phase changes, Math. Models Methods Appl. Sci. **11** (2001), 809–825.

[29] Luterotti, F., Schimperna, G. and Stefanelli, U., Global solution to a phase-field model with irreversible and constrained phase evolution, Quart. Appl. Math. **60** (2002), 301–316.

[30] Mielke, A., *Evolution of rate-independent systems*, in Handbook of Differential Equations, Evolutionary Equations, C. Dafermos and E. Feireisl (eds), Vol. 2, 461–559 Elsevier, 2005.

[31] Mielke, A. and Roubíček, T., Rate-independent damage processes in nonlinear elasticity, Math. Models Methods Appl. Sci. **16** (2006), 177–209.

[32] Mielke, A. and Stefanelli, U., A discrete variational principle for rate-independent evolution, Adv. Calc. Var. **1** (2008), 4:399–431.

[33] Ôtani, M., Nonmonotone perturbations for nonlinear parabolic equations associated with subdifferential operators, periodic problems, J. Diff. Eq. **54** (1984), 248–273.

[34] Roubíček, T., *Nonlinear partial differential equations with applications*, International Series of Numerical Mathematics, Vol. 153. Birkhäuser Verlag, Basel, 2005.

[35] Schimperna, G., Segatti, A. and Stefanelli, U., Well-posedness and long-time behavior for a class of doubly nonlinear equations, Discrete Contin. Dyn. Syst. **18** (2007), 15–38.

[36] Segatti, A., Global attractor for a class of doubly nonlinear abstract evolution equations, Discrete Contin. Dyn. Syst. **14** (2006), 801–820.

[37] Senba, T., On some nonlinear evolution equation, Funkcial Ekvac. **29** (1986), 243–257.

[38] Stefanelli, U., The Brezis-Ekeland principle for doubly nonlinear equations, SIAM J. Control Optim. **47** (2008), 1615–1642.

[39] Visintin, A., *Models of phase transitions*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 28, Birkhäuser, 1996.

[40] Visintin, A., A new approach to evolution, C. R. Acad. Sci. Paris Sér. I Math. **332** (2001), 233–238.

[41] Vrabie, I.I., Periodic solutions for nonlinear evolution equations in a Banach space, Proc. Amer. Math. Soc. **109** (1990), 653–661.

[42] Zeidler, E., *Nonlinear functional analysis and its applications. II/B. Nonlinear monotone operators.* Springer-Verlag, New York, 1990.